Thompson, P. W. (2013). In the absence of meaning… . In Leatham, K. (Ed.), *Vital directions for research in mathematics education* (pp. 57-93). New York, NY: Springer.

CHAPTER 4

IN THE ABSENCE OF MEANING…[1,2]

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Abstract: There are many diagnoses of the bad state of U.S. mathematics education, ranging from incoherent curricula to low quality teaching. In this chapter I will address a foundational reason for the many manifestations of failure—a systemic, cultural inattention to mathematical meaning and coherence. The result is teachers' inability to teach for understanding and students' inability to develop personal mathematical meanings that support interest, curiosity, and future learning. In developing this argument I discuss the subtle ways in which actual meanings with which teachers currently teach and actual meanings students currently develop in interaction with instruction contribute to dysfunctional mathematics education. I end by proposing a long-term strategy to address this situation.

I hope to address the issue of meaning in mathematics education in a way that conveys its nature and importance and also that conveys ramifications of addressing this issue for teaching, learning, and research in mathematics education. One ramification is to become aware of how deeply meaningless mathematics teaching and learning are in the United States. We must be aware of the depth of the problem as a prelude to devising solutions for it.

In this chapter I discuss meanings of "meaning", the creation of meaning through teaching, and difficulties that students have in creating mathematical meanings. I hasten to note that an incoherent meaning is a meaning, so please do not read "creating meaning through teaching" as pointing only to rosy outcomes.

I also share some consequences of systemic inattention to mathematical meaning in the United States and a positive outcome of one teacher's attempt to help students build coherent meanings in algebra. My hope is that inspecting these examples will clarify ways that attending to issues of meaning allows us to see problems of mathematics learning as emergent from fundamental cultural orientations as much as from epistemological problems of learning sophisticated ideas. I end with a proposed agenda for how to move forward so that a focus on meaning is central to improving mathematics teaching and learning in the United States.

Meanings of "Meaning"

I have yet to find anyone who finds the phrase *the meaning of "meaning"* odd. They might ask, "What do you mean?" but they do not act as if I've spoken nonsense. What this points to is something that is innately human. Any time that we invoke the idea of meaning we invoke

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[1] Revised version of a talk given at the Senior Scholars in Mathematics Education Lecture Series, Brigham Young University, December 11, 2008.

[2] Preparation of this chapter was supported by National Science Foundation Grants No. EHR-0353470 and DUE-1050595. Any conclusions or recommendations stated here are those of the author and do not necessarily reflect official positions of NSF.

the idea of meaning. The idea of meaning is so deeply recursive that when we talk about issues of meaning we are talking about an intellectual capacity that is unique to humans. The recursive nature of attempts to examine the nature of meaning suggests, in line with Dewey (1910, 1933), that reflection and abstraction are at its core.

Philosophical disputes about the nature of meaning have centered historically around the referential relationship between language and reality. Ogden and Richards (1923/1989) offered their well-known semantic triangle (*Figure 1*), which places referents in the world, but the relationship between a symbol and a referent exists only by way of a person making the association. Putnam (1973, 1975) argued strongly that meanings cannot be characterized by individuals' psychological states. "Meanings just ain't in the head", he famously said (Putnam, 1975, p. 227).

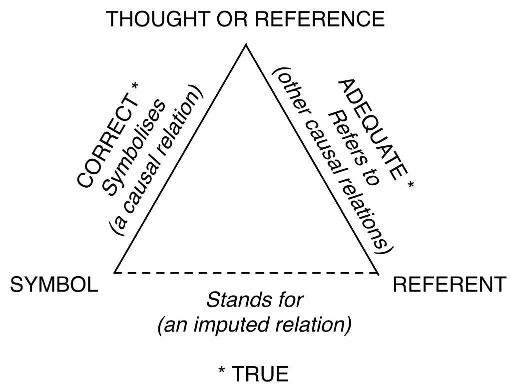


Figure 1. Ogden and Richards' (1923/1989) semiotic triangle.

A second perspective on meaning focuses on what people intend to convey via an utterance, and what people imagine being conveyed as they hear an utterance. Grice (1957) presented an entertaining analysis of this perspective. He first distinguished between natural meanings and nonnatural meanings. A natural meaning, in Grice's usage, is an inference one makes from observing something-in-the-world (e.g., "red spots mean measles"). He was not much interested in this type of meaning, focusing instead on what he called nonnatural meanings—"meaning$_{NN}$"—ideas and ways of thinking that someone intends to convey to someone else and uses signs or symbols to do so. Grice distinguished among three ways that meaning$_{NN}$ can be seen at play in typical uses of "to mean":

> (1) "*A* meant$_{NN}$ something by *x*" is (roughly) equivalent to "*A* intended the utterance of *x* to produce some effect in an audience by means of the recognition of this intention"; and we may add that to ask what *A* meant is to ask for a specification of the intended effect (though, of course, it may not always be possible to get a straight answer involving a "that" clause, for example, "a belief that . . .").
>
> (2) "*x* meant something" is (roughly) equivalent to "Somebody meant$_{NN}$ something by *x*." Here again there will be cases where this will not quite work. I feel inclined to say that (as regards traffic lights) the change to red meant$_{NN}$ that the traffic was to stop; but it would be very unnatural to say, "Somebody (e.g., the Corporation) meant$_{NN}$ by the red-light change that the traffic was to stop." Nevertheless, there seems to be some sort of reference to somebody's intentions.

> (3) "*x* means$_{NN}$ (timeless) that so-and-so" might as a first shot be equated with some statement or disjunction of statements about what "people" (vague) intend (with qualifications about "recognition") to effect by *x*. (Grice, 1957, p. 385)

The significance of Grice's position for mathematics education is that the "mathematics on the page" cannot be the conveyor of meaning. Meanings reside in the minds of the person producing it and the person interpreting it.

Walker Percy, in his famous *Delta Factor* (Percy, 1975a, 1975b), expressed the result of his years-long puzzlement over the nature of man through an analysis of Helen Keller learning the word "water". He began from the perspective of Ogden and Richards' triangle, but later abandoned that approach because of the difficulty he had arguing that Keller had access to a real-world material called "water". Percy realized that Keller's connections were not between water and a sign. Rather, the connections were between experiences that she was having. Instead of the causal relations that Ogden and Richards posited between symbol and thought, Percy insisted that there was no causation at all—that the triangle was irreducible, and that the links in the triangle were all made by Helen. Percy called Keller's irreducible construction an instance of *The Delta Phenomenon.* He reflected upon his attempt to use Ogden and Richards' triangle to capture "what happened" when Keller learned the word "water":

> The longer one thought about the irreducible triangle and its elements and relations, the queerer they got.
>
> Compare Delta Δ phenomenon with the pseudo triangle of Ogden and Richards: buzzer→dog→food. The latter is a pseudo triangle because one needn't think of it as a triangle at all but can conceive the event quite easily as a series of energy exchanges beginning with buzzer and ending in the dog's salivation and approaching food.
>
> But consider the Delta phenomenon in its simplest form. A boy has just come into the naming stage of language acquisition and one day points to a balloon and looks questioningly at his father. The father says, "That's a balloon," or perhaps just, "Balloon."
>
> Here the Delta phenomenon is as simple as Helen's breakthrough in the well-house, the main difference being that the boy is stretching out over months what Helen took by storm in a few hours.
>
> But consider.
>
> Unlike the buzzer-dog-salivation sequence, one runs immediately into difficulty when one tries to locate and specify the Delta elements—balloon (thing), balloon (word), boy (organism).
>
> In a word, my next discovery was bad news. It was the discovery of three mystifying negatives. In the Delta phenomenon it seems: The balloon is not the balloon out there. The word balloon is not the sound in the air. The boy is not the organism boy. (Percy, 1975a, Kindle Locations 661-673).

In other words, Percy saw Keller's construction of meaning as an epiphenomenon, an emergent unification created by Keller's association of her tactile experience of what an observer would call *water pouring over her hand* and the tactile experience of what Anne Sullivan would have called *signing into her hand*. Keller made her experiences whole through an act of *naming*. It was Keller's connecting these experiences that made Sullivan's sign have a referent, and the meaning *within* Keller was irreducible. It had all three components simultaneously.

John Dewey (1910, 1933) considered meaning and understanding to be synonymous, and either to be the product of thinking. His idea of thinking was very special, however. His interest was in what he sometimes called *reflective* thinking. To Dewey, coherence is a characteristic outcome of thinking – thinking leads to “the organization of facts and conditions which, just as they stand, are isolated, fragmentary, and discrepant, the organization being effected through the introduction of connecting links, or middle terms” (Dewey, 1910, p. 79). Dewey also considered thinking to be the primary mechanism for the construction and refinement of meaning: “That thinking both employs and expands notions, conceptions, is then simply saying that in inference and judgment we use meanings, and that this use also corrects and widens them” (Dewey, 1910, p. 125). He also emphasized the role of meaning in human communication:

> It is significant that one meaning of the term *understood* is something so thoroughly mastered, so completely agreed upon, as to be *assumed*; that is to say, taken as a matter of course without explicit statement. The familiar “goes without saying” means “it is understood.” If two persons can converse intelligently with each other, it is because common experience supplies a background of mutual understanding upon which their respective remarks are projected. To dig up and to formulate this common background would be imbecile; it is “understood”, that is, it is silently sup-plied and im-plied as the taken-for-granted.
>
> If, however, the two persons find themselves at cross purposes, it is necessary to dig up and compare the presuppositions, the implied context, on the basis of which each is speaking. The im-plicit is ex-plicit; what was unconsciously assumed is exposed to the light of conscious day.” (Dewey, 1910, p. 214).

Meaning and understanding were synonymous to Piaget, also. But he put it differently than Dewey. Though I know of no place where Piaget said this directly, I agree with Skemp (1961, 1962, 1979) that, to Piaget, “to understand” was synonymous with “to assimilate to a scheme”. Of course, this is entirely unhelpful if we do not know what Piaget meant by a scheme. My understanding of what Piaget meant by “scheme” differs from that proposed by Cobb and Glaersfeld (1983) and by Glasersfeld (1995, 1998). They proposed that, to Piaget, a scheme was a three-part mental structure: a condition that would trigger a scheme, an action or system of actions, and an anticipation of what the action should produce. I believe what Cobb and Glasersfeld described fits better with Piaget called a *schema* of action (Piaget, 1968, p. 11; Piaget & Inhelder, 1969, p. 4). Piaget spoke of a child’s sucking schema, for example. I believe Piaget had larger organizations in mind when he spoke of schemes—organizations of operations, images, schemata, and schemes—that did not have easily identified entry points that might trigger action.[3] I have spoken, for example, of a *rate of change scheme* (Thompson & Thompson, 1996; Thompson, 1994a, 1994c; Thompson & Thompson, 1992) that entails a complex coordination of understandings of quantity, variation, relative change, accumulation, and proportionality. Luis Saldanha and I (2003) wrote about a coordination among understandings of quantity, measure, proportionality, multiplication, and division as comprising a *fraction scheme.* So, in Piaget’s system, *to understand* means *to assimilate to a scheme*, but this is still somewhat unsatisfactory because we need to understand Piaget’s meaning of assimilation.

[3] The concept of scheme is recursive. A scheme can entail other schemes.

Standard meanings of “assimilate” all entail some sense of something being absorbed by something else. As Piaget famously said, “A rabbit that eats a cabbage doesn’t become cabbage; it is the cabbage that becomes rabbit—that’s assimilation. It’s the same thing at the psychological level. Whatever a stimulus is, it is integrated with internal structures” (Bringuier, 1980, p. 42). Piaget’s use of “assimilate” is in a cognitive sense. It does not entail energy transfer. Rather, it emphasized absorption of information. A physical stimulation on a retina creates information that is processed by the nervous system. What looks like absorption is actually imbuement. Montangero and Maurice-Naville (1997, p. 72) quoted Piaget as saying,

> Assimilating an object to a scheme involves giving one or several meanings to this object, and it is this conferring of meanings that implies a more or less complete system of inferences, even when it is simply a question of verifying a fact. In short, we could say that an assimilation is an association accompanied by inference. (Johnckheere, Mandelbrot, & Piaget, 1958, p. 59)

So, to understand is to assimilate to a scheme. But whence schemes? From assimilation. From a Piagetian viewpoint, to construct a meaning is to construct an understanding—a scheme, and to construct a scheme requires applying the same operations of thought repeatedly to understand situations being made meaningful by that scheme. “Assimilation… is the source of schemes.... Assimilation is the operation of integration of which the scheme is the result” (Piaget, 1977, p. 70). Put another way, we construct stable understandings by repeatedly constructing them anew.

Hiebert and Carpenter (Carpenter, 1986; Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986) characterized mathematical understandings similarly to Piaget. They spoke of desirable understandings as rich networks of connections among concepts and procedures. Their characterization of concepts, however, is largely non-cognitive. They did not address how someone thinks to have them and their notion of meaning is static. I find Piaget’s ideas on understanding (meaning) to do more work for thinking about teaching and learning. As Piaget and Garcia (1991) made clear, their notion of meaning is implicative—meaning comes from an assimilation’s implications for further action. Moreover, Piaget’s genetic epistemology entails a rich conception of ways that understandings can be made and how they work in reasoning.

Why Attending to Meaning Matters

In one sense, the issue of meaning is irrelevant to mathematics education—if we accept the current state of mathematics education. It is rare for a mathematics teacher, at least one in the United States, to be concerned with meaning, either intended or conveyed. If we believe the results of TIMSS classroom studies (e.g., Hiebert, Stigler, Jacobs, Givvin, Garnier, Smith, Hillingsworth, Manaster, Wearne, & Gallimore, 2005; Schmidt, Houang, & Cogan, 2002; Schmidt, Wang, & McKnight, 2005; Stigler & Hiebert, 1999), the main goal of most U.S. mathematics teachers is that students learn to perform prescribed procedures. Issues of meaning are largely irrelevant. But if we intend that students develop mathematical understandings that will serve them as creative and spontaneous thinkers outside of school, then issues of meaning are paramount.

I am therefore speaking to educators who are concerned about the conveyance of mathematical meaning. To convey meaning is one of the most important goals toward which teachers can strive. As we think about teaching and the conveyance of mathematical meaning, it will be productive to look for useful ways to imagine how “conveyance” happens. Is meaning on a printed page? Written on a whiteboard? Does it appear on a computer screen? Is meaning conveyed to students by directing their attention to "real world" referents? Each of these stances puts meaning in the world, so that there are "correct" meanings to be had and any meanings that depart from them are incorrect. Simon et al. (2000) characterized this image of conveying mathematical meaning "perception-based" mathematics. They claimed that this is the predominant view of mathematics schoolteachers that teachers expect students to see in mathematical statements what they see. It is "there" for students to take up.

I concur with Cobb (2007) in taking the stance that we should adopt theoretical perspectives only to the extent that they help us do our work. I maintain that any stance that puts meaning outside of individuals is less helpful for purposes of instructional and curricular design, teacher preparation, and professional development than a stance that puts meaning within individuals. This is because most of our efforts in working with students occur at a time when they do not possess mathematical meanings that we hope they will have eventually. Ogden and Richards' triangle is of little use in this case. Their triangle offers no guidance. The meanings that matter at the moment of interacting with students are the meanings that students have, for it is their current meanings that constitute the framework within which they operate and it is their personal meanings that we hope students will transform. Our making explicit the meanings we intend that they have eventually is important, because they comprise our instructional and curricular goals. But those targeted meanings must come to exist within individual students—in the sense that Percy (1975a) described Helen Keller's acquisition of "water"—for us to have succeeded.

There is empirical evidence that the mathematical meanings a teacher possesses matter in regard to what students learn. For example, Branca (1980) studied the communication of semantic structure from teacher to student regarding the content of a unit on operational systems. He studied six high school mathematics teachers over the duration of the instructional unit. At the beginning of the unit there was little resemblance between what teachers meant and what students meant by key terms and phrases. By the end of the study, students' meanings regarding operational systems very much resembled their teacher's—including inappropriate meanings. Teachers' and students' meanings became aligned even about ideas that were not taught. So, even tacit meanings that teachers carry can be conveyed to students. (I address how this might happen in the next section.) Branca's study, however, was tightly focused on definitions and theorems having to do with systems defined by operations on sets, so issues of meaning were central to the subject matter being taught. Examples given later in this chapter show that students develop understandings and ways of thinking about the mathematics they learn even when meaning is not central to the teachers' subject matter. But the understandings that students develop in those settings are not propitious for later learning.

Conveying Meaning Through Teaching

If we maintain the stance that meanings are entirely within individuals, we face the immediate question of how people can appear to learn a meaning from someone else. How shall we explain the seemingly evident fact that teachers *can* convey meanings to students? I find two sources immensely helpful in conceptualizing human communication so that we can speak sensibly about the conveyance of meaning without violating our self-imposed stance that all meanings lay within individuals. The first is Piaget's notion of *intersubjectivity* (Glasersfeld, 1995; Piaget, 1995; Thompson, 2000) and the other is Pask's *conversation theory* (Pask, 1975, 1976; Scott, 2009). Piaget placed great emphasis on the idea of *decentering*, or attempting to adopt a viewpoint that differs from your own. He used the term *intersubjective operations* to describe thoughts that are directed at another. As Glasersfeld (1995) put it, once a child starts to think that another person “thinks like me”, she can then also notice occasions where the person seems not to think like her. This is at the root of what Glasersfeld called “the construction of others” (Glasersfeld, 1995, Ch 6).

Pask’s conversation theory attempted to explain how social interaction can lead to participants’ construction of knowledge. His theory was rather technical, but the important part for this chapter is his concept of a conversation. To Pask (1975, 1976), a conversation is more than face-to-face verbal exchanges confined to a specific place and time. Rather, a conversation involves all the actions entailed in conversants’ attempts to convey and discern meaning. So, a classroom conversation, in Pask’s sense, could include an exchange that involved the teacher introducing an idea, handing out a worksheet, and discussing how she expects students to use it. The teacher’s soliloquy can be considered part of an ongoing conversation, as can students asking questions about a worksheet or about what the teacher expects. It goes without saying that conversations are most productive when each participant is oriented to understand what others have in mind and is oriented to have others understand what he or she intends.

What follows is an amalgam of Piaget’s notion of intersubjective operations and Pask’s conversation theory. The amalgam is necessary simply because neither Piaget nor Pask focused squarely on the construction of mathematical meaning. Pask’s theory was quite technical, and it was more interested in conversations than in participants. His interest in teaching expressed itself largely in the form of adaptive teaching machines that made decisions about problems that a student should work given the student’s performance on prior problems. He paid little attention to imagery and its role in meaning, and he did not consider specific mathematical meanings, such as what it might mean for someone to understand the idea of quantity. On the other hand, Piaget’s theory embraced imagery as a key component of cognitive development (see Thompson, 1994a, 1996), but he was not interested in specific mathematical ideas.

Figure 2 shows Persons A and B attempting to have a meaningful conversation. Person A intends to convey something to Person B. The intention is constituted by a thought that A holds that he wishes B to hold as well. The figure shows A not just considering how to express his thought, but considering how B might interpret A’s utterances and actions. It is worthwhile noting that A’s action towards B is not really towards B. A’s action towards B is towards A’s image of B. In a sophisticated conversation A’s action towards B is not just towards B, but it’s towards B with some understanding of how B might hear A. Likewise, B is doing the same thing. He assimilates A’s utterances, imbuing them with meanings that he would have were he to say the same thing. But B then colors those understandings with what he knows about A’s meanings and according to the extent to which A said something differently than B would have said it to mean what B thinks A means. B then formulates a response to A with the intent of conveying to A what B now has in mind, but B colors his intention with his model of how he thinks A might hear him, where the model is updated by anything he has just learned from attempting to understand A’s utterance. And so on.

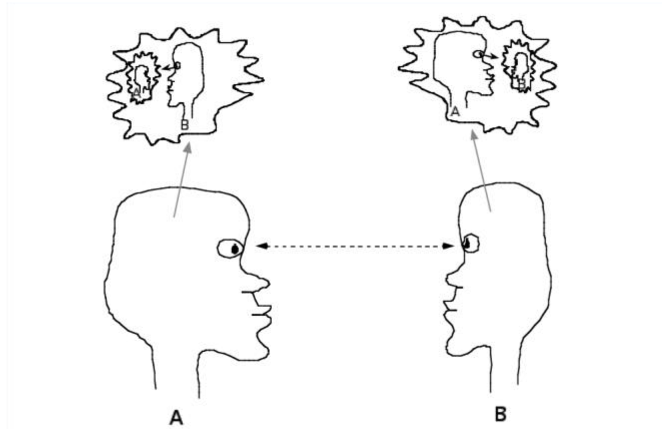


Figure 2. Summary of intersubjective operations involved in the communication of meaning.

The process of mutual interpretation and accommodation described above, which Steffe and I (2000) also called *reciprocal assimilations*, is what I understand Piaget to have meant by the negotiation of meaning. The negotiation is not sitting down and developing a contract, like negotiation of meaning is often portrayed. The negotiations that happen are rarely negotiated explicitly. The negotiations that happen involve each person monitoring the other's responses, comparing them to the responses he anticipated, and then adjusting his model of the other to make better decisions about how to act and what to expect in the future. This, I believe, is what Bauersfeld (1980, 1988) meant by communication as interactions among mutually reflexive systems. Both A and B adjust their understandings of the other's understanding, and possibly adjust their personal understandings in the process. In Piaget's and Glasersfeld's usage, A's and B's conversation enters a state of intersubjectivity when neither of them has a reason to believe that he has misunderstood the other. They may in fact have completely misunderstood each other, but they have not discerned any evidence of such. As Glasersfeld (1995) makes clear, a conversation being in a state of intersubjectivity has no implication for whether the participants' meanings align. Rather, the nature of a conversation that is in a state of intersubjectivity is that neither participant has any reason to believe that he has misunderstood the other. It is important to note that it is a conversation that is in a state of intersubjectivity. It is a category error to say that the participants are in a state of intersubjectivity.

The above description of conversation assumes that all participants really are participants—that they care to understand other participants. If, for some reason, B were to not care what A meant, then there is no conversation. This observation has important implications for teachers' management of classroom conversations: students must intend to discern meaning in order to construct meaning from a conversation. The teacher's guidance in creating an atmosphere where making meaning is valued and expected is central to students' construction of meaning through conversations (Cobb, Boufi, McClain, & Whitenack, 1997). It is also important to distinguish between a conversation of equals and a classroom conversation. The teacher is a very special participant in classroom conversations. A teacher has power and trust in a conversation that students do not have. Teachers can manage conversations; students are rarely positioned to manage a conversation. Teachers can manage a conversation so that students *do* have power and trust, but it is teachers who allow and nurture those opportunities.

Many people take the case of intersubjectivity in a two-participant conversation as being completely unlike a conversation that involves many participants. "It is impossible for each participant to have a model of every other participant's understanding", they might say. This stance, however, misses the essential character of a conversation that is in a state of intersubjectivity. A conversation is in a state of intersubjectivity when it is in a state of equilibrium—when each participant takes for granted that that no one has misunderstood anyone else's understanding. Disagreements do not necessarily puncture a conversation's equilibrium. Two people can disagree with what they discern the other to mean, but if neither person feels that he has misunderstood the other, then the equilibrium persists. In a sense, each person has created an epistemic "other" to which he or she can attach a variety of ways of thinking about the conversation's subject.

We can return now to Branca's (1980) study and the question of how teachers conveyed meaning to students. The teachers needn't have said, "No, no, no, that meaning is wrong, I want you to have this meaning". The teachers probably rarely said anything like this. Rather, by focusing their attention on their meaning of an operational system, students adapted their understandings to fit what they discerned the teacher to have in mind. This would account for students ending with meanings that were compatible with their teachers' meanings even when those meanings were either normatively inappropriate or never discussed explicitly.

The notion of intersubjectivity, as described above, can also give us insight into how *mis*communication happens. The example given below illustrates a conversation that was in a state of intersubjectivity for a relatively long period of time even though the participants had misinterpreted each other quite severely.

Mindi is a ninth-grader enrolled in Algebra I. Her teacher, Sheila, is a participant in a professional development project that emphasizes student-oriented instruction that focuses on supporting students' creation of meaningful, coherent mathematics. Sheila's review of arithmetic at the beginning of the year emphasized meanings and ways of thinking that underlie arithmetical operations (e.g., division as measuring or partitioning, multiplication as multiple copies or as dilation, fractions as a reciprocal relation of relative size, order of operations as a system of conventions that imposes structure on arithmetic and symbolic expressions, and so on). Prior to where this example starts, Sheila had drawn on these meanings of addition, subtraction, multiplication, division, and order of operations to help students build ways to think about expressions and equations. For example, instead of teaching "do the same to both sides" Sheila emphasized inferences one could make about numerical relationships that would allow you to see numerical relationships that were not directly stated. For instance, when discussing what value or values of x makes $x/5 + 15 = 30$ true, she guided students to reason about what the equation is saying:

> If $x/5 + 15$ is 30, then $x/5$ must be 15, because 15 + 15 is 30. Now $x/5$ means "one-fifth of the number represented by x". If one-fifth of the number x is 15, and since there are five 1/5ths of x in x, then x must contain five fifteens, and therefore must be five times as large as 15.

Sheila used diagrams and illustrations to accompany her statements, and she consciously designed her instruction so that students would generate equation-solving methods as abstractions from their experiences of repeated reasoning.

Many of Sheila's students seemed to thrive in the context of her instruction; some did not. Sheila asked that I interview Mindi, a hardworking, bright girl who did well in class until the section on solving linear equations. Several excerpts of that interview follow.

Excerpt 1: Meaning of equations. P: Pat; M: Mindy

1. P: Before we start, could you tell me what pops into your head when you see an equation?
2. M: Well, you are supposed to isolate the variable so that it equals a number or another expression. If it equals another expression, then you try again to isolate the variable so that it equals a number.

In a sense, Mindi's description of her meaning for equations seems quite standard. When you see an equation you think to solve it. Excerpt 2, however, reveals that Mindi did not interpret equations as stating numerical relationships. It reveals the depth of her procedural perspective.

Excerpt 2: *w*/3 = 11. P: Pat; M: Mindy

1. P: I'm not asking you to solve this equation. Instead, just tell me what it says.
2. M: It says that when you divide some number by 3, you get 11.
3. P: Okay. Now, can you tell me what this expression stands for (circles w/3)?
4. M: It stands for a number.
5. P: Any idea what that number is?
6. M: No. I'd have to solve for w. Then I could tell you what w slash three stands for.[4]

Mindi revealed the same way of thinking when discussing $8m - 4 = 8$ (I circled $8m - 4$) and $a/5 + 15 = 30$ (I circled $a/5 + 15$). Thus, it seems safe to say that the meaning of an equation, for Mindi, was that it was a symbolic form that she was expected to act on to end with another form $x = number$. This was confirmed again when she actually solved the equation $8m - 4 = 8$, getting $m = 12/8$ ("12 over 8, which reduces to 3 over 2"). I asked her whether the answer is "12 over 8" or "12 eighths". She stated that she preferred saying "12 over 8" because it made more sense to her than "12 eighths."

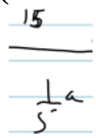
I then asked Mindi to revisit the intermediate result $8m = 12$ that she had written, and which appeared immediately above "$m = 3/2$". I said, "This says that 12 is some number of times as large as 8. Twelve is how many times as large as 8?" No answer. "Is it 25% larger, 100% larger?" Mindi thought that 100% was too much. I then asked, "What is 150% of 8?" Mindi replied that she had difficulty finding 150% of a number. In other words, Mindi's meaning for equations did not entail seeing them as an expression of numerical relationships. She did not see that her answer $m = 3/2$ meant that 12 is 3/2 times as large as 8. Her numerical reasoning did not support seeing equations that way. She could not imbue an equation with meanings drawn from numerical relationships. Her scheme for equations entailed actions for operating on them and little else.

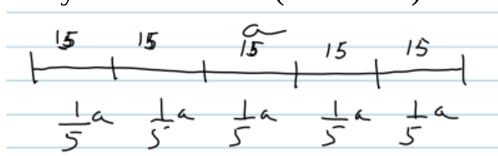
Excerpt 3, below, was about the equation $a/5 + 15 = 30$. It reveals more about Mindi's schemes—not just her scheme for equations, but her schemes for engaging with classroom mathematical instruction. It picks up after I had led Mindi to the conclusion that $a/5$ had to be 15 because 15 + 15 is 30.

Excerpt 3: *a*/5 + 15 = 30. P: Pat; M: Mindy

1. P: You said that *a* divided by 5 is 15. Can you interpret *a*/5 as a fraction?

[4] Sheila's expectation was that students would understand that, in the context of the equation $w/3=11$, that $w/3$ stood for the number 11. Though she expected this interpretation, she never stated it.

2. M: It is 1/5 of *a*.
3. P: Can you draw a diagram to show 1/5 of *a*?
4. M: (Draws the upper part of this diagram. Pat writes "1/5 *a*" below it.)
5. 

6. P: Does this tell you anything about what *a* is?
7. M: No. Not really.
8. P: How many fifths of *a* are in *a*?
9. M: Five.
10. P: Can you draw them? (see below).
11. 

12. [….]
13. P: You figured out that 1/5 of *a* is 15, and therefore that 5 of those things make *a*.
14. M: Yeah.
15. P: Does that make sense?
16. M: Yeah, kind of.
17. P: What's the "kind of" part?
18. M: Well, I just would probably never do it that way, 'cause // it's kind of confusing. Like it kinda makes sense, but it's still kinda confusing.
19. P: Well, notice here (on a quiz she took the prior week) what you wrote. (Mindi had written *a*/5 = 45, and *a* = 45/5). So, even doing it your way there was something confusing about it.
20. M: Yeah.
21. P: So, let me ask you a question. You seem to be reluctant to figure things out, without a rule. Is that right?
22. M: Yeah.
23. P: Why is that?
24. M: I don't know. Well, whenever I do it this way (reasoning with numerical relationships) I feel like I'm doing it wrong. You know, like I mean, with a rule, I can be sure. Because a rule says to do this, I know what I'm supposed to do and I know I'm doing it right, but with this way there is too much room for error. I think.
25. P: Okay. What about all these places where you used rules (pointing to errors she made on her quiz) and…
26. M: Most of them are stupid mistakes. Like here I added 15 instead of subtracting 15. With that one (another error) // a stupid mistake.
27. P: So, how can you avoid stupid mistakes?
28. M: Just by practicing more. Studying more.
29. P: Do you practice a lot?
30. M: No, not really. But I never needed to! Because I'd always just got // done it, like perfect.

31. P: So it's a little more complicated now?
32. M: Yeah, but I'll just practice more and I'm sure I'll do okay.
33. P: Okay. So you have more faith in practicing the rules than you do in practicing reasoning it through?
34. M: Yes.
35. [….]
36. P: Well, this has been very useful for me. I hope it's been useful for you.
37. M: Yeah.
38. P: Is there anything you would like to ask me?
39. M: (Pause.) Ummm. Well, maybe just like // why do you want us to do it this way (reasoning) so much?

It is clear from Excerpt 3 that Mindi had little faith that reasoning was a reliable problem solving technique and that she had much greater trust in using procedures that she had memorized.

In answering Mindi's last question I explained the benefits of reasoning—making fewer mistakes and catching mistakes when you make them. I also explained that by practicing reasoning students often found learning new ideas to be easier. I then asked Mindi about what she did when Sheila tried to teach her to reason about problems. It turned out that Mindi understood the reasoning that Sheila thought she was teaching as just the steps that Sheila wanted Mindi to remember. Mindi found the steps to be confusing—she couldn't remember them. So Mindi waited until Sheila taught "the rule". "The rule" from Sheila's perspective was a generalization of the reasoning she thought she had taught, but from Mindi's perspective it was the meat of the lesson, and Mindi could not understand why Sheila waited so long to tell it.

To summarize, Mindi's scheme for equations (apply procedures to isolate a variable) existed within a larger scheme that anticipated what she should get from instruction (rules) and how she should participate in lessons (remember the rule). Mindi saw her role as remembering steps; the teacher's role was to provide steps. Successful participation, for Mindi, was that she had a clear idea of the rules she was supposed to use. This is the way Mindi assimilated Sheila's actions and utterances—even though Sheila intended her actions and utterances to help students construct meanings. Mindi assimilated Sheila's reasoning steps as "new rules"—rules that were harder for her to remember than the bottom-line rules that she eventually discerned. Despite its dysfunctional nature, the conversation constituted by Sheila-Mindi interactions was in a state of intersubjectivity until Sheila discerned that something was amiss with her understanding of Mindi's understanding.

Sheila was aghast as she listened to Mindi's interview. She had no idea that Mindi was hearing her as she was. Sheila decided to find out whether other students had understood her instruction as Mindi did—as just providing steps they should memorize. She opened up classroom discussions to include several questions I'd asked in Mindi's interview. As a result, she found that Mindi's perspective was common, even among students who Sheila thought had been solving problems from a basis of meanings of equations and meanings of operations.

I suspect that Mindi's predominant experience in mathematics classrooms prior to entering Sheila's had been such that understanding was equated with correct performance, and that classroom conversations, even when everyone thought they were about understanding, were actually about procedures. Once Mindi developed her way of thinking about what mathematics *is,* she then heard her teachers projecting that same way of thinking, and she found no occasion to believe otherwise. Similarly, her teachers thought that Mindi understood what they intended because Mindi performed successfully, and the conversations in which Mindi participated were such that her teachers saw no occasion to believe otherwise.

The final comment to be made here is about teachers' expectations for students' understanding. Branca's (1980) study, mentioned earlier, suggests that teachers' meanings create a space for students' meanings. If a teacher's image of what students are to learn entails weak meanings, or no meanings, then intersubjectivity can be attained with students collectively possessing a wide variety of meanings that fit the discourse, many of which we would identify as problematic. When a teacher's image of what students should learn entails a strong system of meanings, then the space for possible student meanings is much smaller, assuming that teacher and students mutually adapt their understanding of the other. The teacher will find more occasions to discern that students' meanings differ from what she intends, and students will find more occasions to discern that what the teacher has in mind differs from what they understand. If the classroom culture is such that participants expect that noticed differences in meaning should be resolved, then it is more likely that students will develop coherent systems of meaning that guide their mathematical performance.

The discerning reader might object to the previous paragraph—Sheila had strong meanings and yet Mindi's thinking seemed unconstrained by them. How is this possible? It is possible because the conversations that Sheila managed had strong overtones of "what should we *do*", not "what should we *mean*". Sheila told students what they should mean, and then too quickly moved the discussion to how to answer questions in the worksheets "meaningfully". She demonstrated ways to answer questions using the meanings she had told them, but the conversation allowed students to think that she was simply showing them how to answer the questions. Students could safely ignore those occasions when Sheila asked them questions they couldn't answer (e.g., "How does the meaning of division tell us to multiply both sides by 5?"). The conversation's bottom line, in the students' eyes, was that you should multiply both sides by 5.

What Happens in the Absence of Meaning?

The case of Mindi illustrates how a well-meaning teacher who has a fairly strong system of meanings can nevertheless fail to influence a student in the way she intends. Another case, though, is when a teacher does not have a strong system of meanings regarding a particular body of ideas. A teacher with a weak system of meanings for an idea cannot help being vague or confusing when he speaks about ideas, and naturally avoids issues of meaning. However, even if he avoids speaking about ideas explicitly, his actions will be unconstrained by a strong system of meanings, and a conversation's meaning-spaces will have a high probability of entailing many inappropriate possibilities. As Dewey (1910) said, vagueness of meaning is a source of *mis*understanding, *mis*apprehension and of *mis*taking. Confused meanings (i.e., undifferentiated, vague, confounded) are "too gelatinous" to support students' productive analysis and reflection: "Vagueness disguises the unconscious mixing of different meanings, and facilitates the substitution of one for the other" (Dewey, 1910, p. 129ff).

Dewey's point is illustrated by a group of high school teachers who were working together in weekly Professional Learning Community (PLC) meetings that they used to discuss material that they taught in common. At the time of this meeting, January 20, 2006, they were in the midst of teaching a unit on trigonometry to tenth-graders. The current topics were angle and angle measure.

An outside facilitator met with the group in the role of a consultant. The teachers and facilitator were arranged in a semicircle. The camera was about 20 feet away, directly in front of them. Excerpt 4 begins with the facilitator asking, offhandedly, “What is an angle measure?”.

Excerpt 4: Teachers discuss the meaning of angle measure. F: Facilitator; T: Teacher

1. F: So, what is angle measure? You might raise this issue…
2. T1: (Interrupting) What is angle measure? I think that is a good question.
3. F: What is angle measure?
4. T2: It is very different from measuring the length of a side // I had a couple of students who thought they could be the same thing.
5. F: What did you say to them?
6. T2: You can’t do that! They’re not the same thing!
7. T1: So, how would you define it [angle measure]?
8. T3: How do you // how do you define angle measure?
9. T1: The ray sweeps // isn’t the angle created when the terminal ray sweeps from the initial side to the terminal side // so angle measure is defined as what?
10. T3: Are you talking about, then you start getting into that thing of are you talking about arc length?
11. T1: Well, I don’t know. How do you define angle measure?
12. T4: The curvature.
13. T3: (To T1) You mean your initial ray?
14. F: How do you [say], “Angle measure means this.”
15. T3: (Reading from a textbook.) “The measure of angle A is denoted by // The measure of an angle can be approximated with a protractor using units called degrees. For instance” // they don’t ever get into what is a degree.
16. T1: (Reading) An angle consists of two different rays.
17. T3: That’s just defining an angle.
18. T1: It’s the portion of a complete rotation that you take out as the terminal side sweeps (stops).

The teachers’ only meaning for angle measure was to lay a protractor down and read off a number. They realized, however, that reading off a number from a protractor does not explain what an angle measure *is*. The teachers’ meaning for angle measure (or lack thereof) had consequences for students’ learning.

We asked the teachers to give this question to their students: “What are you measuring when we measure an angle?” Students’ responses are summarized in Table 1, which shows that 93% of the students thought that an angle measure was measuring something between the sides, either a distance from one side to the other or an area bounded by the angle’s sides. Only one student said anything related to an arc, and even this answer seems to be oriented toward a distance. We should note that teachers never discussed with students what an angle measure is, or what one measures when measuring an angle. The answers students gave might have been preformed, in the sense that these are meanings that they created prior to their geometry class. However, even if their meanings existed prior to taking geometry, it seems that there was nothing in their experiences within their geometry class to alert them that their particular ways of thinking might be problematic. We must also entertain the possibility that students had never thought about what they were measuring when finding an angle measure. This interpretation seems sensible if their only experience with measuring angles was simply to follow a procedure that employed a protractor.

Table 1
Students' responses to "What are you measuring when you measure an angle?" (n = 110)

Student Response	Percentage
Distance between sides	51%
Distance between labeled points	2%
Shape of the angle (directions of the rays)	3%
Diameter of the angle	2%
"Arc of the angle (how wide it is)"	1%
Area of the angle	42%

Independently of our question in Table 1, teachers created a set of geometry questions and an interview protocol as part of their PLC work, and they each interviewed three students from their respective class. One of the interview questions is given in *Figure 3*. The teachers were to ask students to solve the problem and then were to discuss the students' solutions.

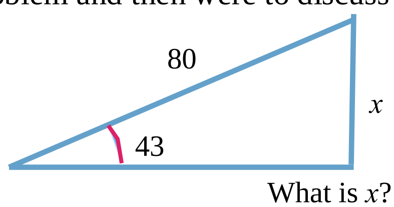


Figure 3. Problem included in teachers' student interview protocol.

The interviews took place in mid-March, 2006 at the end of this particular instructional unit. In their March 26 PLC meeting they discussed students' responses to the interview questions. Excerpt 5 presents the portion of that meeting in which they discussed students' answers to the problem shown in *Figure 3*.

Excerpt 5: Teachers discuss results of student interviews

1. T4: I was really surprised at the interviews. Two of the three students I interviewed really mixed information. They mixed 180 degrees in a triangle // They confused 180 with a side length. They subtracted 180-43 and got 137. Then they subtracted 80 from 137 to get 57 for the other side length.
2. T3: Triangles have to add up to 180.
3. T2: My kids make no distinction between angles and sides.
4. T5: My honors kids today were going to take 360 and subtract a length, and I told them you are mixing angles and lengths! You can't do that!!

I find it remarkable in Excerpt 5 that none of the teachers considered the possibility that the students' confusions were rooted in the teachers' teaching. Why should students *not* confuse (what we take as) angle measures and (what we take as) side lengths when, to the students, numbers rarely have any meaning? By the teachers' own admission in Excerpt 4, they paid no attention to the meaning of an angle measure. Moreover, it is ironic that in the context of complaining that students cannot differentiate between angle measures and side lengths that T3 uttered, "Triangles have to add up to 180." I do not know what it means to add up triangles. If this is the level of precision T3 used in class, then it is no wonder that her students cannot distinguish between angle measures and side lengths.

I suspect that, in the context of classroom instruction, the teachers' students could easily succeed in the moment without paying any attention to the meanings of the numbers that appeared in problems. Within the context of the problems they were working in a particular section, students simply applied the procedure that was being taught at that moment. A number was a number was a number. When different numbers mean different things within the context of one situation, to distinguish between numbers that are side lengths and numbers that are angle measures students must have a system of meanings that keep them separate.

An Example of a Teacher Attending to Meaning

To construct a meaning requires repeatedly constructing and using the operations (ways of thinking) whose organizations constitutes that meaning. The most usable meanings are those that are richly connected with imagery action and that tie into other meanings.

In 2006-2007 I had the pleasure of working with a ninth-grade Algebra I teacher, Augusta, who took seriously the matter of students learning mathematics meaningfully and coherently. She structured the subject over the year so that students would build ways of thinking that would constitute an understanding of algebra that had a clear trajectory for supporting their future learning of calculus. I share a sample drawn from her unit on polynomial functions to illustrate what I have said about constructing a meaning by repeatedly constructing and using the operations (ways of thinking) whose organizations constitutes that meaning. But to make this sample understandable, I must first describe how she prepared students to participate in the conversations about the idea of polynomial function that I will share.

Augusta was conscientious about helping students build meanings that would lend coherence to their algebraic thinking and provide a foundation for later learning. She

- Began with building variation as a way of thinking about quantities changing. Students could *imagine* a quantity changing continuously. Variables varied. Always.
- Built covariation as a way of thinking about two quantities varying simultaneously. Time on a clock varies while a runner runs. The clock doesn't *cause* a runner to run. We simply keep track of how far she has run in relation to how much time has elapsed on the clock.
- Built the idea of function as an invariant relationship between the values of covarying quantities. The perimeter of a circle is always 2π times the length of its radius no matter how we change either (assuming that it remains a circle).
- Built the idea of linear function as a function that has a constant rate of change.
- Built the idea of a graph as having points, where the coordinates of each point tell us the value that each quantity has in relation to the other. Each point provides a "snapshot" of the quantities' covariation.
- Built an understanding of constant rate of change as a relationship between two quantities that are changing simultaneously such that all changes in the value of one quantity are proportional to changes in the value of the other.
- Built an understanding of average rate of change. First, two quantities, A and B, need to change simultaneously, and each has a total change. The average rate of change of Quantity A with respect to Quantity B is that constant rate of change of A with respect

to B that would produce the same change in A in relation to the change in B that actually happened.

Augusta also had an agenda with regard to symbolic facility and representational equivalence. To explain what she did in regard to symbol sense is not important for this example, though I will say more about it later.

Augusta intended that students understand a polynomial function as a function that is the sum of monomial functions (Dugdale, Wagner, & Kibbey, 1992). That is, she wanted them to think of $f(x)=2x^3-x^2+5x+2$ as the sum of $f_1(x)=2x^3$, $f_2(x)=-x^2$, $f_3(x)=5x$, $f_4(x)=2$, and hence that $f(x)=f_1(x)+f_2(x)+f_3(x)+f_4(x)$. She aimed for this understanding so that students could anticipate the behavior of a polynomial function, expressed in standard form, by examining its addends. To develop this way of thinking about polynomial functions, Augusta needed to help students understand the meaning of a sum of two functions. This is the focus of the example I share below—Augusta is introducing the idea of a function that is the sum of two functions.

The sum of two functions *f* and *g* is often defined as $\left(f+g\right)(x)=f(x)+g(x)$.[5] which emphasizes how you would calculate the value of a sum for a given value of *x*. Augusta's aim was that students could also *imagine* the sum of two functions in a way that was non-symbolic, yet true to the definition. She wanted her students to have a *way of thinking* about making a function that is a sum.

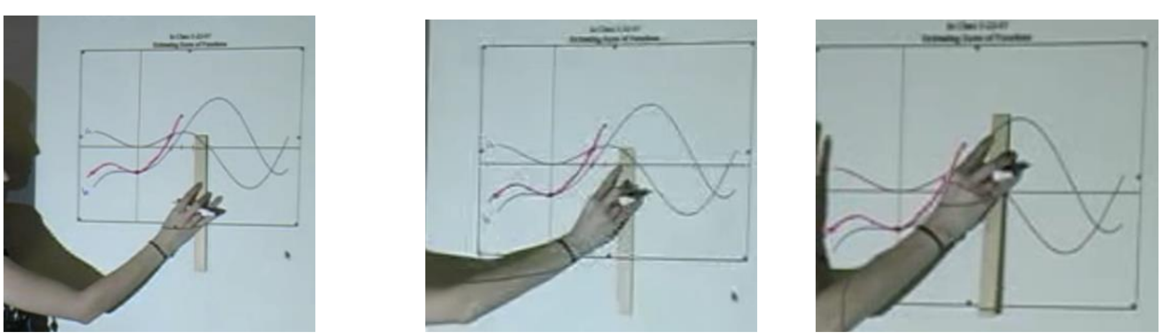
The example enters a lesson at the time that Augusta is displaying the graphs of two functions within the same coordinate system. A special feature of her display is that she has not included any numbers and she designed the functions so that their graphs were unlike anything the students might recognize and be able to name. Her reason for doing this is that she had discovered in the past that when she placed numbers on the axes, students tried to estimate points' coordinates and add them numerically to get a value of the sum function. They then used that number to plot a point, again with great concern for accuracy. When she included numbered axes, students became bogged down trying to be highly accurate and they also often made addition errors. In the process of all this focus on accuracy, they lost the image of combining the values of two functions to get the value of a third.

To draw students' attention away from numbers and accurate placement of points, Augusta gave students blank straightedges (rulers with no markings). She showed them how to use the rulers to estimate the functions' values simply as magnitudes, and to imagine the value of the sum as putting one magnitude on top of the other. Excerpt 6 picks up Augusta's lesson after she has estimated the value of the sum function at several places along the horizontal axis. Students have a copy of the displayed graph and are attempting to replicate Augusta's placement of points on the sum's graph.

[5] The "+" in "($f + g$)" does not mean the same thing as "+" in "$f(x) + g(x)$". The first instance of "+" is part of the function's name; the second instance is the arithmetic operation of addition.

Excerpt 6: Augusta attempts to convey meaning of “sum of two functions”. A: Augusta; S: Student

1. A: Let’s go forward some more. I don’t know how much more, but go forward some more (see Figure 4a). Uhhhhhm. Again, you can use your ruler to help you estimate. How positive is function A?
2. Ss: It’s positive.
3. A: It’s positive. Is it very positive?
4. Ss: A little bit.
5. A: Yeah maybe. It depends on how you scale it. But, it’s about // we can pinch it that much positive (see Figure 4b). So you guys, on your scales, on your graphs can pinch off just how positive the A value, the A function is. (She waits for students to “pinch off” the value of A). What about the B function?
6. S: It’s positive.
7. A: It’s also positive. How can I show with my ruler how I’m going to add this to the value of the B function?
8. A: So that’s how positive A is. Now I need to add that to the value of B. So how so? What do I want to do with this length? To show the sum? // How do you show adding with two lengths?
9. S: Mark another one.
10. A: How, next to it?
11. S: No, down farther
12. A: Yeah! Right on top of it! So, if this is how much positive my A function is, and that’s how positive the B function is, I’m going to take this and ... add it! You can literally think of stacking it. So here, that’s maybe how positive the value of A is. That’s perhaps how positive the value of B is. So their sum? How would you show it?
13. S: It’s bigger.
14. A: How much bigger?
15. Ss: Add ‘em.
16. A: That on top of that. Exactly! It’s that much bigger. So you’re *stacking* these magnitudes now, because they’re positive. You’re literally stacking the lengths that you’re estimating, because they’re positive. So you can still use your ruler to help you pinch, so that’s how positive A is, that’s how positive B is. So *stack* it, and you are actually up... about here (see Figure 4c).



(a) (b) (c)

Figure 4. Augusta indicates (a) a value in the domain of both functions, (b) the value of one function, and (c) that value added to the value of the other function.

In Excerpt 6 we see Augusta employing covariation ("Let's move forward a little bit", Line 1) and thinking in magnitudes ("Pinch off just how much positive it is", Line 5) and thinking of combining magnitudes ("you're literally stacking the lengths", Line 16).

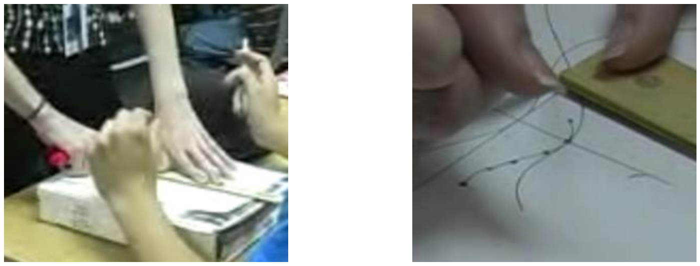
After Excerpt 6, Augusta turned responsibility over to the students to complete sketching the sum function's graph. Her motive for asking students to complete the sketch was that they create the value of the sum function as the result of an action of combining. She wanted students to develop what Dubinsky and Harel (1992) called an *action conception* of a sum function—the image of actually combining the function's values. Students' action conception of a sum function prepared them to develop later what Dubinsky and Harel called a *process conception* of a sum function—the ability to envision the action of summing immediately, focusing on the outcome of the action.

Excerpt 7 captures an interaction between Augusta and a student as he attempts to complete the sketch. Prior to this excerpt, Augusta and the student had a somewhat rambling conversation in which the student expressed his confusion about where to look for the functions' values ("there aren't any numbers") and how to think about adding them.[6]

Excerpt 7: Augusta discusses worksheet with student who is having difficulty

1. A: Where are you looking? Maybe around here somewhere?
2. S: Yeah, down here.
3. A: Pinch off how negative the negative function is.
4. S: That right there.
5. A: That much. How much will the positive lift it?
6. S: It will get lower, won't it? (Appearing to look at the positive function.) Because ... less.
7. A: It will get less negative (looking at the negative function). Yeah, exactly. Right now it is this negative. But since you are adding a positive to it, it will be less negative. How much less negative?
8. S: This much (see *Figure 5*a).
9. A: That much. So keep your finger where how negative it is. And then, lift with me, keep your finger on it, it will get lifted... that much. That much. I mean, this is estimation.
10. S: So this (the value on the negative graph) goes up higher!
11. A: Yeah! It used to be *that* negative, but it will get lifted that much. So take that negative value, and lift it... that much (see *Figure 5*b).
12. S: Oh, I get it.

[6] Part of Augusta's management of this conversation was to anticipate the difficulties students would experience making sense of what she demonstrated during the whole-class discussion of combining function's magnitudes. She anticipated that they would find it odd not to have numbers. Thus, she was not surprised at the student's comment.



(a) (b)

Figure 5. Augusta and student discuss how to interpret "sum of functions" at a value of the domain.

Augusta's language and actions while speaking with this student emphasized building an image of "stacking" function values, the same way of thinking she attempted to convey during the immediately prior whole-class discussion. The payoff of Augusta's emphasis on having students solidify the action of combining two functions came in subsequent lessons. She asked students to imagine the location of points on the graph of a sum of two functions whose graphs were displayed simultaneously as she steadily moved her finger along the horizontal axis. She asked students to imagine the sum's graph "evolving" simultaneously with running through values in the addend functions' domain. Students' eventual ability to rapidly anticipate a visual estimation of the sum functions' values were expressions of their process conception of a sum function and gave them opportunities to solidify an understanding of a function that is the sum of two functions. It paid off further when Augusta came to polynomial functions, where she asked students to envision the behavior of the sum of two or more monomial functions given their prior knowledge of the monomials' graphs.

This example from Augusta's class focused on her attempt to create a meaning for a function that is the sum of two functions. I should point out that Augusta's lesson, which emphasized imagistic meaning, also reflects her year-long struggle with de-emphasizing talk about "what to do". We often discussed the value of stepping back and talking with students about what she intended that they create and, once created, what they had created and what it meant.

I would be remiss not to point out that the meaning of function sum was just one part of a larger scheme that Augusta intended that students build. That scheme entailed their prior work on understanding functions defined as a product of factors and an understanding of producing an equivalent representation by using the distributive property of multiplication over addition. Put more broadly, Augusta's intent was that students see a function's graph as invariant across representations of the function, and to build meaning within each representation by focusing on schemes for imagining a function's behavior. Her long-term instructional design was attentive to what Lehrer and Schauble (2000) described as the inseparable, interrelated development of inscription and meaning. In the case of polynomial functions, she aimed to develop a scheme of meanings that entailed students' abilities to transform one symbolic representation of a polynomial function into other symbolic representations, and that they take for granted that there was something called "the function" (a relationship expressed as a graph) that remained the same. Augusta's approach to having one meaning be invariant across representations of a polynomial goes beyond the issue I raised in (Thompson, 1994b), where I questioned what was then called the "multiple representations" movement.

> I believe that the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of representation. Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us, but instead may be, as Moschkovich, Schoenfeld, and Arcavi (1993) have said, areas of representational activity among which we have built rich and varied connections. It could well be a fiction that there is any interior to our network of connections, that our sense of "common referent" among tables, expressions, and graphs is just an expression of our sense, developed over many experiences, that we can move from one type of representational activity to another, keeping a current situation somehow intact. Put another way, the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance.
>
> I do not make these statements idly, as I was one to jump on the multiple-representations bandwagon early on (Thompson, 1987, 1989), and I am now saying that I was mistaken. I agree with Kaput (1993) that it may be wrongheaded to focus on graphs, expressions, or tables as representations of function, but instead focus on them as representations of something that, from the students' perspective, is *representable*, such as some aspect of a specific situation. The key issue then becomes twofold: (1) To find situations that are sufficiently propitious for engendering multitudes of representational activity and (2) Orient students to draw connections among their representational activities in regard to the situation that engendered them. (Thompson, 1994b, pp. 39-40)

Augusta went beyond the concern I raised in 1994 by first addressing it squarely (developing students' meanings for each form of expression in terms of ways to read it for information about joint variation) and then raising the question of how you could change to another form of expression while retaining the information students discerned originally. In this way, she helped students develop a "subjective sense of invariance" while moving from one representation of polynomial function to another, taking the graph of a function as the "most basic" representation of it.

Lastly, Augusta supported her class conversations with specially designed didactic objects (Thompson, 2002)—displays, diagrams, graphs, mathematical expressions, or class activities that she designed conscientiously to support specific reflective conversations she intended to have with students. For example, the graphs that she used during the function-stacking activity had blank axes and unfamiliar shapes. This design feature enabled Augusta to focus students' attention on function's magnitudes at a common value of their domains instead of on points' coordinates.

Absence of Meaning in Mathematics Education

The preponderance of research on learning mathematics in the United States suggests that my examples of meaningless learning and teaching are far from uncommon and that meaningful instruction is rare. One study in particular stands out—the TIMSS eighth-grade video study (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999; Stigler & Hiebert, 1999). They formed nationally representative samples consisting of 81 U.S. classrooms, 50 Japanese classrooms, and

100 German classrooms. As part of this study a team of U.S. mathematicians and mathematics educators examined the lessons (which were blinded for national identity) with regard to the quality of the lessons:

> They based their judgments on a detailed written description of the content that was altered for each lesson to disguise the country of origin (deleting, for example, references to currency). They completed a number of in-depth analyses, the simplest of which involved making global judgments of the quality of each lesson's content on a three-point scale (Low, Medium, High). Quality was judged according to several criteria, including the coherence of the mathematical concepts across different parts of the lesson, and the degree to which deductive reasoning was included. Whereas 39 percent of the Japanese lessons and 28 percent of the German ones received the highest rating, none of the U.S. lessons received the highest rating. Eighty-nine percent of U.S. lessons received the lowest rating, compared with 11 percent of Japanese lessons. (Stigler *et al.*, 1999, p. iv)

The rarity of meaningful, coherent mathematics instruction in the U.S.—instruction that aims to develop students' mathematical thinking in the sense of Dewey—is very troubling. The rarity with which popular textbooks, both elementary and secondary, and both traditional and reform, attempt to develop mathematics as a coherent system of meanings is also troubling.

What I find more troubling is the rarity of research in mathematics education that takes the issue of mathematical meaning seriously. Research that is ostensibly on knowing or understanding, whether the context is teaching or learning, too often examines performance instead of clarifying the meanings students or teachers have when they perform correctly or the meanings they are working from when they fail to perform correctly. Neither correct performance nor incorrect performance says anything about the nature of a person's system of meanings that expresses itself therein. This is not to say that *no* research considers students' or teachers' meanings. Rather, it is too rare.

Some publications fail to address the issue of meaning even when their titles say it is about meaning. The chapters in Kilpatrick, Hoyles, Skovsmose, and Valero (2005) discuss the many ways that "meaning" is used in mathematics education, but they do not explicate a meaning of "meaning" that does work for designing curriculum or instruction that will improve mathematics learning. Kieran's (2007) review of research on learning and teaching algebra is a case in point. Its subtitle is, "Building meaning for symbols and their manipulation". The article is an astonishing piece of scholarship in the scope of the research it reviews, but by the criteria I've set in this chapter, it fails to say what Kieran or any of the articles she reviews take "meaning" to mean, and the article gives few examples of anyone's thinking that might constitute a meaning for symbols or their manipulation. Moreover, the article is devoid of references to research on quantitative reasoning as a source of meaning for arithmetic and algebra, and its review of research on function completely misses the research on ways of thinking that might constitute various understandings of function. Instead, it focuses on evidence that students find the concept of function, whatever it is, difficult.

Research on calculus learning is another case in point. Research on students' understanding of the derivative (e.g., Clark, Cordero, Cottril, Czarnocha, DeVries, St. John, Tolias, & Vidakovic, 1997; Ferrini-Mundy & Gauadard, 1992; Ferrini-Mundy & Graham, 1994; Heid, 1988; Machín, Rivero, & Santos-Trigo, 2010; Orton, 1983; Sofronos & DeFranco, 2010; White & Mitchelmore, 1996) takes "slope of secant" as a primary meaning of average rate of change (the other is the computation $\Delta y/\Delta x$) and takes "slope of tangent" as a primary meaning of instantaneous rate of change (the other is the limit of average rates of change, where average rate of change is defined as slope of a secant). I am puzzled by the approach of taking "slope of secant" and "slope of tangent" as fundamental meanings for average rate of change and instantaneous rate of change, respectively. Secants and tangents are lines. They are geometric objects. I can easily imagine a thoughtful student asking, for example, "What do lines have to do with speed?" Clearly, there is a complex system of meanings behind thinking of a secant as somehow embodying an average rate of change, and there is an even more complex system of meanings behind taking a tangent as somehow embodying an instantaneous rate of change. I outlined part of that system earlier, when I spoke of a rate of change scheme. However, none of these studies explicates such a system of meanings. Hence they do not investigate them.

Unfortunately, when researchers treat meanings for slope (whose computation students often take as an index of "slantiness"), secant (which students often think of as a piece of wire that is laid across a graph), and tangent (which students often think of as a line that "just touches" a curve) as *primary* meanings, not as emergent meanings, they cannot understand the sources of students' success or failure to learn. Hackworth (1994) drove this point home. She studied 90 first-semester calculus students' understandings of rate of change. Her question was, "What have calculus students, after studying differentiation and derivatives, learned about rate of change?" By her measures, they learned nothing about rate of change. In some instances students understood more about rate of change before receiving instruction than they did after the course.

Carla Stroud (2010), in a follow-up to Hackworth's (1994) study, interviewed 15 students in Calculus 2 and Calculus 3 about their meaning of instantaneous speed. One question was this:

> When the Discovery space shuttle is launched, its speed increases continually until its booster engines separate from the shuttle. During the time it is continually speeding up, the shuttle is <u>*never*</u> moving at a constant speed. What, then, would it mean to say that at precisely 2.15823 seconds after launch the shuttle is traveling at precisely 183.8964 miles per hour? (Hackworth, 1994, p. 108)

Consistent with Hackworth's (1994) findings, the primary meaning held by students in Stroud's study was that of a speedometer. The space shuttle's instantaneous speed 2.15823 seconds after launch is whatever number its speedometer points at. There are two problems with this way of thinking: (1) the space shuttle doesn't have a speedometer, and (2) even if it did, what about the speedometer's design guarantees that it is pointing at the correct number? Some students had a backup way of thinking—you would take the limit of the space shuttle's average speed over smaller and smaller intervals or you would simply differentiate the shuttle's position function. Carla asked, "And how would you do that?" The students presumed that there was some function they could act upon symbolically—and the shuttle's speed would pop out of that.

The area of mathematics education research that is most wanting today regarding attention to meaning is research on teachers' mathematical knowledge for teaching (MKT). First, the verb "to know" is used in this research as a primitive, undefined term. The question of what "to know" means in regard to knowing mathematics is unaddressed. Second, this area is quite taken with the idea that teachers' knowledge, whatever that means, can be categorized (Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; Hill, 2010; Hill, Blunk, Charalambous, Lewis, Phelps, Sleep, & Ball, 2008). I suspect that the desire to create instruments to assess teachers' knowledge is the driving force behind this focus. Item specifications need categories. When you categorize a teacher's knowledge based on an answer to an item, however, your attention is necessarily drawn away from the system of meanings by which the teacher was operating. Assessments that do not address teachers' meanings can be summative, but they cannot be diagnostic. I'll illustrate this point with an example.

Before I can share this example I must review the idea of continuous variation. Castillo-Garsow (2010) identified two ways, in principle, that one can think about continuous variation, what he called "chunky" and "smooth". A conception of continuous variation that is *chunky* is one where someone thinks of a variable varying in discretely-continuous amounts. By "discretely-continuous" I mean that they imagine that the variable varies, but they imagine "next" values and mentally connect the values. The variation comes in one chunk between current and next values. The value of x goes directly from initial to end without passing through the values in between. The values in between current and next values are "there", but the person imagining the variation does not imagine passing through them. A conception of a variable varying smoothly is recursive. One might imagine a "next" value, but does so with the anticipation that the variable varies smoothly between current and next value by varying smoothly between values that exist between current and next (Thompson, 2011).

A ninth-grade algebra teacher, Sandra, was in the midst of teaching a lesson on the point-slope and point-point formulas. She was attempting to use a method that she had just learned which takes a rate-of-change approach to the point-slope formula. The method works like this: Suppose a function has a constant rate of change r. You start by assuring that students have an appropriate meaning of constant rate of change, such as "r is the constant rate of change of y with respect to x" means that however much x changes, y changes r times as much.[7] With this meaning in hand, if you know that a function with a constant rate of change of 1.7 passes through the point (3, 9), then if you decrease the value of x by 3 (i.e., increase it by -3), the function's value will change by 1.7 times -3. Thus, the value of the function at x = 0 is 9 + (1.7)(-3). The function's definition is therefore $y = 1.7x + \left(9 + (1.7)(-3)\right)$. The two-point method follows as a corollary by determining the function's average rate of change between two points and realizing that you now have a situation where a function has a known constant rate of change and its graph passes through a known point. Sandra was excited to try this method with her class.

Sandra worked through several examples using this method to find a function definition when given one point and a rate of change. Things fell apart, though, when she moved to the case of having two points that the function's graph passes through. Excerpt 8 picks up as she discusses the two-point case.

Excerpt 8: Sandra discusses the two-point case

1. S: (Plots the points (3,1) and (7,4) in a coordinate system on the board.) Now we'll look at a something that is a little bit different. Now all we're given is two points, and we're supposed to find the equation for the line that goes through them. Any ideas?
2. (Silence)
3. S: Well, let's notice something. This function goes over 4 and up 3 (sketches segments). So if we do the same thing as before and move x back to 0 we'll know what the y intercept is! So if we go 4 to the left (draws a horizontal segment of length 4 to the left from (3,1). See *Figure 6*.)

[7] The phrase "assure that students have" can be misleading. It does not mean "teach this idea in the 5 minutes before the point-slope lesson." Rather, it means to assure that this meaning of constant rate of change has been the target of instruction over a long period of time, long enough so that students have this meaning and all its entailments.

4. S: (Long pause) We'll pick this up tomorrow. (Pause) Here are some practice problems. Do just the ones with one point.

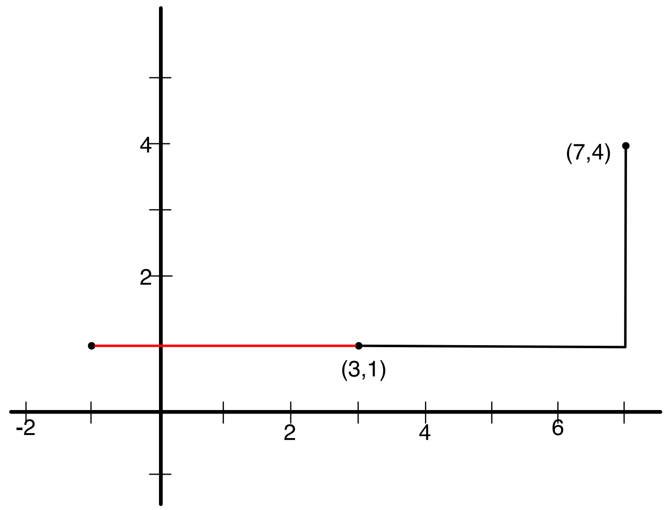


Figure 6. Sandra's boardwork while working the two-point problem.

Though Sandra's difficulty actually began in Line 3, where she described the change as "over 4 and up 3", her entire difficulty resided in her schemes for variation, slope, division, and rate of change.[8] First, she saw the change in x as a chunk. This was unproblematic in the case of one point. However, her chunk in this problem did not place her at $x = 0$ as she wished. Second, her meaning for slope was "rise over run", where rise and run were both chunks. Third, her computation of slope, not evident in this excerpt but made clear later, was of a procedure that produced a number that is an index of a line's "slantiness". Division did not produce a quotient that has the meaning that the dividend is so many times as large as the divisor—3/4 as a slope was not a number that gave a rate of change. It gave a "slantiness". Fourth, her meaning for rate of change entailed neither smooth variation nor proportionality. It was more akin to her meaning of slope—two things changing in chunks. These meanings not only failed to provide Sandra a connection between her current setting (two points) and prior method, they led her down the dead-end path she followed. Had Sandra reasoned proportionally and with smooth continuous variation, she might have said "… over 3/4 of 4 and up 3/4 of 3". That would have given her the graph's y-intercept.

I fail to see how categorizing Sandra's knowledge would enhance our understanding of why her lesson fell apart. With our above understanding of what Sandra knew (i.e., the meanings from which she operated) we are positioned to help her improve. Putting her knowledge in categories like "curricular knowledge", "common mathematical knowledge", or "specialized mathematical knowledge" serves no practical purpose except to see whether her score on a test meets a benchmark. I feel strongly that assessments of MKT must be rooted in developmental theories of MKT. Otherwise, despite being ostensibly rooted in the work teachers do, the assessments will have little explanatory power with regard to *why* teachers do what they do and will have little usefulness in helping them improve what they do.

[8] It is important to notice that I said *schemes*. Sandra's meanings for variation, slope, division, and rate of change did not exist within a single scheme. They were unrelated.

I propose that we develop a new type of assessment, aimed at assessing teachers' *Mathematical Meanings for Teaching* (MMT). The enterprise of developing such assessments might redirect the field's attention to the subtle, yet foundational, role that meanings play in what teachers and students do. It might also redirect the field's attention towards an emphasis on explicating desirable, powerful systems of meanings that we feel students should develop. Lest I be misinterpreted, I hasten to add that the issue of skill would still be paramount. But our conception of skilled performance would change. Our descriptions of students' skilled performance would necessarily entail our intention that it be evidence that they have built powerful, rich, integrated systems of mathematical meanings.

A focus on MMT would also foster the field's conceptualization of bridges among what teachers know (as a system of meanings), how they teach (their orientation to high-quality conversations), what they teach (the meanings that an observer can reasonably imagine that students might construct, over time, from teachers' actions), and what students learn (the meanings they construct).

Assessments of MMT would be more diagnostic than current assessments of MKT. Information from them would be useful for teachers' professional development. I imagine that such instruments would also be useful in designing professional development aimed at improving teachers' ability to help their students learn. Sample items from an assessment of MMT might alert teachers to ways of understanding the ideas they teach that are expected of them and of their students. For example, can teachers explain a meaning of division that gives a meaning of quotient? Are they *inclined* to teach a meaning of division that gives a meaning of quotient? Can they explain that 8 divided by 5 equaling 1.6 means that 8 is 1.6 times as large as 5? Do they think it is important for students to know this? Do they have a coherent system of meanings of multiplication, division, and fractions that allows them to explain that 43 * 18 = 774 means, at once, that 774 is 18 times as large as 43, that 774 is 43 times as large as 18, that 43 is 1/18 of 774, and that 18 is 1/43 of 774? Are they *inclined* to explain those meanings? Do they think these are important meanings for students to have? Are they inclined to ask students questions that force those connections?

An Agenda for Change

My intention in this chapter was to convey the nature of meaning as it relates to mathematics education and the importance of taking meaning as a foundational consideration in mathematics learning, teaching, and instructional design.[9] How, though, might we as a nation bring about changes that resolve the lack of meaningful mathematics in schools and colleges? I draw inspiration from Tucker (2011) to answer how we might move forward with such an agenda for change. Tucker examined the educational policies of Ontario, Finland, Japan, Shanghai, and Singapore to see what policies they either have in place to sustain an excellent educational system or put in place to pull themselves to a level of internationally elite educational systems. He pointed to five areas of policy that are central to elite systems attaining

[9] I did not emphasize issues of curriculum. I agree whole-heartedly with Carlson *et al.* (2010), who argued convincingly that a well-designed curriculum will play a central role in supporting teachers' reconceptualization of the mathematics they teach and will be an essential component in efforts to make meaning central to teaching and learning. It is my experience, however, that teachers' meanings trump curriculum, so I have emphasized teachers' meanings for the purposes of this chapter. On the other hand, there is a large intersection between issues of instructional design and issues of curriculum, so I have not ignored curricular issues entirely.

and sustaining excellence, and he turned each into a set of recommendations to be implemented at the state level. His recommendations are as follows: Benchmark the best, design for quality, design for equity, design for productivity, design for coherence. These categories serve well as organizers for thinking about how to make meaning central to mathematics education.

Benchmark the best

Tucker pointed out that prior to World War II, the U.S. borrowed ideas and practices from other countries at a rapid rate, but after World War II we seemed to think that no one had anything to offer. Recently, there have been several efforts to benchmark international standards (e.g., National Mathematics Advisory Panel, 2008). However, benchmarking standards is like surveying a landscape from 50,000 feet. You might see broad outlines, but you have no sense of the details by which things are made to happen. Elsewhere (Thompson, 2008a), I stated that the National Mathematics Panel Report recommendations read like a table of contents. The Panel did not attend to what it might mean to understand the things in their lists. By that I meant that they paid little heed to how other countries actually implemented their standards and that the Panel ignored the idea that attending carefully to issues of meaning was one way that elite countries attained excellence in mathematics education.

Funding agencies should commission studies to benchmark the systems of mathematical meanings toward which elite systems strive. They should also document how those meanings are achieved and the consequences of achieving them. For example,

- Singapore elementary education targets a deep understanding of speed as rate of change in grades 1-5 (as does Russia). This deep understanding entails ideas of variation, covariation, and proportionality. The Singapore curriculum outline does not state this specifically, but if you examine their texts and instructional guides it leaps at you. Their early attention to speed is later leveraged in developing students' understanding of variable and linear function. What other meanings does Singapore target, how does it build them, and how does it leverage those meanings in students' later learning?
- Japanese elementary education emphasizes whole-number numeration as a systematic way to represent numerical value—to a far great extent than in the United States. In Japan, numerical algorithms arise out of a system of meanings that constitute an understanding of place value. They are not taught as meaningless, memorized notational procedures. What other meanings does Japan target, how does it build them, and how does it leverage those meanings in students' later learning?
- Russian elementary education emphasizes quantity and measurement (as, to a lesser extent, do Singapore and Japan). A deep understanding of measurement entails understanding ratio and proportion. Russians leverage this early learning by intermingling it with the idea of generalization, which necessitates ideas of representation and representational equivalence. What other meanings does Russia target, how does it build them, and how does it leverage those meanings in students' later learning?

Design for Quality

Tucker's (2011) first bullet in this section is, "Get your goals clear, and get public and professional consensus on them" (p. 5). This feat will not be easily accomplished with regard to targeted systems of mathematical meanings, but it is essential. National funding agencies will play an essential role in the effort to clarify systems of meanings, and how they might be expressed skillfully, that mathematics education should take as its primary goals.

The clarification of goals will also address the matter of coherence in the mathematics curriculum. A number of studies have stated boldly that the typical judgment of U.S. mathematics curricula at all levels is that they are conceptually incoherent (Cai, 2010; Oehrtman, Carlson, & Thompson, 2008; Schmidt *et al.*, 2002; Schmidt *et al.*, 2005; Thompson, 2008b; Thompson, Carlson, & Silverman, 2007). A focus on developing coherent meanings will not guarantee coherent curricula, but it surely will increase the likelihood that a curriculum designed to support students' development of a coherent system of meanings will be coherent.

Another aspect of designing for quality is that targeted meanings must be worth having. We must be able to argue that having them will pay off in important ways, either in preparation for life or preparation for future learning. The arguments must be specific in regard to *how* having them will be important. Research will play an essential role in the quest to design for quality, because we often realize the intricacies of a targeted meaning's "payoff", or lack thereof, only in attempting to help students develop it. Research will also play an essential role in identifying and characterizing important meanings that students and teachers should have, and conveying those meanings to parties who can use that information. (Please understand that I use "convey" in the sense that I've described repeatedly in this chapter.)

Design for Equity

Tucker's (2011) emphasis on equity is largely in regard to allocation of resources. He argues that school systems and students should get resources according to their need. Hardest-to-educate students should receive sufficient resources necessary to enable them to attain the high standards set in the quest for quality already described.

With regard to issues of meaning, hardest-to-educate students will be those who are farthest from developing the meanings that we decide are essential. Designing for equity with regard to meaning requires that we identify ways of thinking and systems of meaning that are highly obstructive for constructing the meanings we intend, and then investigating means of support to effect change most efficiently. Just as we do not give steak to someone who is malnourished, we cannot expect someone who reasons additively to participate productively in instruction on making multiplicative comparisons as a foundation for reasoning proportionally.

Design for Productivity

Tucker (2011) characterized designing for productivity in terms of making frequent and timely checks for quality. With regard to a focus on meanings, this translates into helping teachers attend to the meanings that students are actually constructing and adjust instruction appropriately. Research will play a central role in identifying effective ways that teachers can do this and effective ways to help teachers do this. I am not speaking of grand assessment strategies or high-stakes tests, though I anticipate that we will continue to have them—and that their character will change. Rather, the road to students building powerful, coherent, and useful systems of mathematical meaning will be built upon teachers' ability to conduct constant, formative assessments of students' learning.

Design for Coherence

The coherence Tucker (2011) had in mind was that efforts to address the prior four areas cohere, that they complement and draw from each other. The same is surely true for mathematics education. In regard to the issue of moving the field towards making meaning a primary concern, I see several additional ways in which we must design for coherence. The obvious one is designing coherent systems of meanings that we target for student learning. We also must coordinate efforts in instructional design, teacher preparation, and professional development around those meanings. We not only want school students to develop coherent meanings for arithmetic operations, we want future teachers to be prepared to convey and assess them and professional development programs that support continued teacher growth.

It is on the criterion of coherence that Tucker found the greatest strength in elite systems and the greatest fault in the U.S. system. Culture surely plays a large role in both cases. Cultures change over time, but they rarely change abruptly. They continually regenerate themselves through intersubjective operations among their participants. An educational system is slow to change for the same reasons—entering teachers have images of mathematical teaching and learning that they formed as students. Lortie (1975) noted this when explaining why U.S. instruction seemed to change so little—adults who choose to enter teaching developed a deep resonance with their experience of schooling as students. We can leverage Lortie's observation to gain insight into different educational systems' clear differences in teachers' and textbooks' orientations to meaning. Ma (1999) observed Chinese elementary school teachers with the equivalent of high school education displaying what she called profound understandings of the mathematics they taught. U.S. elementary teachers in her study, with college degrees, displayed superficial and fragmented understandings of the mathematics they taught. Why the difference? Because, I suspect, the Chinese teachers developed meanings while school students that they later refined as prospective and practicing teachers, while the U.S. teachers developed unproductive meanings while students—meanings that served them poorly as a foundation for conveying coherent mathematical understandings to their students. Coherence, and incoherence, is inherited as much as it is produced.

Tucker (2011) ended his report by noting that each of these elite educational systems took from 30 to 100 years to transform themselves and that the U.S. should not expect less. Also, he noted that each reform effort involved a high degree of experimentation within its policy frameworks without losing sight of its goals. We must take Tucker's observation to heart—expect that it will take 30 to 100 years of concerted, purposeful effort to transform the U.S. educational system so that coherent meanings are at the core of mathematics teaching, learning, and curriculum. At the same time, we cannot expect to succeed without striving to develop a clear vision of what we mean that meaning be at the core of mathematics teaching, learning, and curriculum.

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