Making the Fundamental Theorem of Calculus Fundamental to *Students'* Calculus[1,2]

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I describe a calculus curriculum (from Project DIRACC)[3]*, based in quantitative reasoning, that puts the FTC at its center. It builds from the ideas that differentials are variables whose values vary and that rate of change and accumulation are two sides of a coin. I will share results from comparisons with traditional curricula, surprising insights into different meanings of rate of change students must have at different places in the curriculum, and the central role of technology in making this approach possible.*

Introduction

Popular U.S. calculus textbooks state the Fundamental Theorem of Calculus (FTC) as: (1) If a function f is continuous over the interval $[a, b]$, the function F defined as $F(x) = \int_a^x f(t)dt$ is an antiderivative of *f,* and (2) If *G* is an antiderivative of *f,* then $\int_a^b f(x)dx = G(b) - G(a)$. To make these statements meaningful, authors build meanings of derivative, antiderivative, and definite integral prior to stating the FTC. But these textbooks' meaning for derivative is slope of a tangent line. Their meaning for integral is area of a region bounded by a curve. The net result is the FTC, in students' experience, is nothing more than a way to compute definite integrals. It adds nothing to their understanding of derivatives or integrals. Derivatives are still slopes and integrals are still areas, and the FTC says nothing about either.

The standard statement of the FTC is not true in general when integrals are areas and derivatives are slopes. The integral $\int_a^b f(x)dx$ gives the area of a region bounded by $x=a$, $x=b$, and $y = f(x)$ only when y and x are ordinate and abscissa in a Cartesian coordinate system. It doesn't work in any other coordinate system. Also, in a polar coordinate system, the graph of $y = mx + b$ is a spiral, not a line, making derivative as "slope of a tangent" unworkable. As I will clarify later, integral as accumulation from rate of change and derivative as rate of change of accumulation are not dependent upon a coordinate system.

The strong tie between integrals, derivatives, and the Cartesian coordinate system creates epistemological obstacles for students just as does tying fractions to pieces of a pie. It gives students meanings for derivatives and integrals that do not generalize and, in fact, create obstacles regarding their application of integrals in non-area settings and rate of change in non-slope settings. Why, then, do textbook authors make the tie between the Cartesian coordinate system and initial ideas of

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[2] Plenary paper to be presented at the *Conference on Calculus in Upper Secondary and Beginning University Mathematics*, Kristiansand, Norway, 6-9 August 2019.

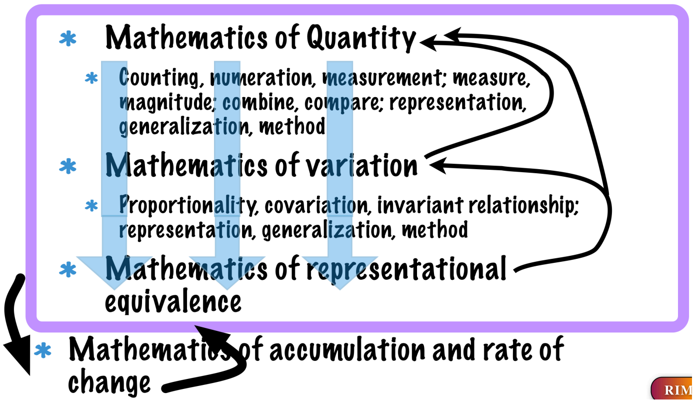
[3] Developing and Investigating a Rigorous Approach to Conceptual Calculus, P. Thompson (PI), F. Milner (co-PI), M. Ashbrook (co-PI). http://patthompson.net/ThompsonCalc

derivatives and integrals so strongly? In my opinion, it is because it allows authors to pretend they are teaching ideas of calculus to students and it allows students to pretend they are learning ideas of calculus.

My goal is that students understand a calculus that is about more than lines, areas and pseudo-connections with quantitative situations. It is that students understand a calculus that arises from their reasoning about quantities and relationships among quantities. To do that, however, requires they reason about quantities and generalize their reasoning. More specifically, I wish students' comprehensions of phenomena to entail ideas of calculus they then represent symbolically.

The aim that students' calculus be rooted in their reasoning about quantities requires me to differentiate between what Les Steffe and I called students' mathematics and a mathematics of students (Steffe & Thompson, 2000). Students' mathematics is the mathematical reality they experience, which is wholly theirs and is unknowable to us in the same way dark matter is unknowable to us. Like with dark matter, the best we can do is make models that fit observations and are consistent with other models. We then use models of students' mathematics and its development (our mathematics of students) to inform our design of instruction and curricula. The calculus I share here arose in that way – from a career of creating models of students' mathematics and its potential for developing into powerful ways of thinking mathematically.

(More here later that explains the themes in the diagram below—how quantitative reasoning can provide the backbone for a mathematics of students throughout their schooling.)



The FTC, quantitatively

To design a calculus that students might create from reasoning about their worlds quantitatively requires that we define our learning goals in these terms. For students to see the FTC as relating rates of change and accumulations students must conceptualize rate of change as a relationship between quantities whose values vary.

A particular scheme for constant rate of change of one quantity with respect to another is the earliest form of the FTC learned by some middle school students (and not possessed by some calculus students). It entails an image of two quantities varying simultaneously so that increments in each are in constant proportion and accumulations of each are in the same proportion. I described this scheme in several earlier publications (Thompson, 1994a, 1994b; Thompson & Thompson, 1992).

Several aspects of holding this scheme warrant comment. The scheme, called FTC-E(arly), entails this imagery:

1) "Same proportion" means same relative size of measured quantities.
2) It is accumulations of quantities that vary. To a student holding the FTC-E scheme, variation implies accumulation.
3) Variation in an accumulation happens by its increments.
4) A student can envision increments happening smoothly or "chunkily" (Castillo-Garsow, Johnson, & Moore, 2013; Saldanha & Thompson, 1998; Thompson, 1994b; Thompson & Carlson, 2017). Envisioning increments happening smoothly is more advanced.

Aspect 3, variation-implies-accumulation, is a crucial component of the FTC-E scheme. Imagine a person running. If you imagine the runner's distance scaling to a larger size, like an arrow becoming longer, you are not imagining distance accumulating. It is just becoming larger. To envision the runner's distance accumulating one must envision the distance covered by each stride added to the runner's distance traveled up to that stride. In this image, the runner's accumulated distance increases with each stride. It is in this way that someone envisions variation in an accumulation happening by its increments.

Aspect 4 distinguishes, in principle, between two ways of envisioning how a quantity's value varies. There is a distinct difference between a student imagining a runner's distance accumulating chunkily and a student imagining it accumulating smoothly. The first student imagines the length of a completed stride being added. The second student imagines the length of a stride in progress. This distinction is important when students face the problem of modeling accumulation symbolically while taking the independent variable as varying continuously.

A far more advanced scheme, called FTC-A(dvanced), gives a later form of the FTC. A student holding the FTC-A scheme coordinates advanced schemes of variation, covariation, and constant rate of change in support of this imagery:

1) Two quantities vary (accumulate) smoothly and simultaneously.
2) They each vary in increments which themselves vary smoothly.
3) Increments can be small enough so, no matter how the accumulations vary, they covary through increments at an essentially constant rate of change with respect to each other.
4) The rate of change of the accumulations with respect to each other is the rate of change of their increments with respect to each other.

The first two aspects of FTC-A entail the idea of function as a relationship between covarying quantities. The third aspect of FTC-A entails the idea of rate of change function—a function whose values give the rate of change of an accumulation at each moment of accumulating. The fourth aspect of FTC-A is where the relationship between accumulation and rate of change is explicit. Seeing the rate of change of an increment as the rate at which an accumulation varies with respect to another quantity is the conceptual heart of the FTC. It is the conceptual equivalent of understanding that an integral's rate of change function is the integrand of an indefinite integral.

The fourth aspect of FTC-A is nontrivial for calculus students. Figure 1 contains an item from Project DIRACC's Calculus 1 Concept Inventory given to 380 students enrolled in traditional or engineering calculus. It aims to have students consider an accumulating distance's rate of change when given information about the overall accumulation (the car's average rate of change over a four-hour period) and its rate of change over a small increment of time after that four-hour period.

> A car left from San Diego heading to New York. The car's average speed for the first 4 hours of the trip was 52 mph. In the next 0.003 hours, the car had an average speed of 71 mph. Which is the best estimate of how fast the car's distance from San Diego was changing at 4 hours after leaving San Diego?

Figure 1: FTC item from Calculus 1 Concept Inventory (© 2018 Arizona Board of Regents)

Options presented to students are below. Comments in brackets are explanations to you.

(a) 52 mph [miles per hour; the car's average speed over the first four hours]
(b) 52.014 mph [the car's average speed over 4.003 hours]
(c) 61.5 mph [the mean of 52 and 71]
(d) 71 mph [the car's average speed over the 0.003 hours immediately after the four-hour period]
(e) Cannot determine [There is insufficient information to answer the question]

Table 1 shows students responses to the item in Figure 1. While no option garnered a high percentage of responses, it is worth noting that 71 mph, the best approximation to the car's speed at an elapsed time of four hours, was the least popular option (13.1%). It is also worth emphasizing that 71mph is the only option consistent with FTC-A, aspect 4.

52 mph	52.014 mph	61.5 mph	**71 mph**	Cannot determine	No Answer
24.4%	17.7%	26.2%	13.1%	16.2%	2.3%

Table 1: Responses to item in Figure 1 from 380 calculus students 13 weeks into a 15-week semester

DIRACC Calculus 1

Traditional, semester-based, university calculus in the U.S. is called Calculus 1, 2, and 3. The content of Calculus 1 traditionally covers differentiation and applications up to optimization and related rates and integration up to the FTC and elementary applications of integrals.[4] The content of Calculus 2 traditionally covers a potpourri of disconnected topics: advanced antidifferentiation and applications of integrals, parametric functions, sequences and series (including Taylor series), and polar coordinates. Calculus 3 covers multivariable and vector calculus.

The DIRACC curriculum covers the content of semester-based Calculus 1 and 2. However, we strived to make it grounded in quantitative reasoning and to develop ideas more coherently than traditional calculus. The coherence we aimed to create rests on what I call *Foundational Problems of Calculus:*

FP-1: You know how fast a quantity varies at every moment; you want to know how much of it there is at every moment.

FP-2: You know how much of a quantity there is at every moment; you want to know how fast it varies at every moment.

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[4] This is also the content of traditional Advanced Placement Calculus BC many students take in high school.

DIRACC Calculus 1

FP-1 and FP-2 are stated at the outset of Calculus 1 and remain thematic throughout the two courses. We anticipated a dialectic between students' development of their FTC-A scheme and their work to understand and respond to FP-1 and then to FP-2.

DIRACC Calculus 1 has these central features. It:

- contains a forward-looking review of pre-calculus ideas.
- aims for students to build meaning and to reason meaningfully (reasoning based on meanings).
- emphasizes convergence, not limits.
- defines quantities so their measures are computable. The goal to compute is an organizing idea.
- aims to support students in building dynamic imagery to accompany their construction of the FTC-A scheme.

Thompson, Byerley, and Hatfield (2013) give a detailed account of the organization and content of DIRACC Calculus 1 (as of 2013). The current version is at http://patthompson.net/ThompsonCalc. In brief, the course evolves in four phases. The aims of each phase are:

Phase 1: Review

This is important; most students do not understand basic ideas or hold productive imagery for them.

- Quantitative reasoning.
- Values of variables vary; strong distinctions among variables, parameters, and constants:
    - If you use a notation to represent the value of a quantity whose value varies, you are using that notation as a *variable*.
    - If you use a notation to represent the value of a quantity whose value never varies (e.g., π or e), you are using that notation as a *constant*.
    - If you use a notation to represent the value of a quantity whose value is constant within a situation, but can have different values across situations, you are using that notation as a *parameter*.
- Differentials are variables; they are the "bits" by which variables vary
- Constant rate of change is defined in terms of a relationship between differentials: $dy = m\,dx$
- Large variations are made of tiny variations.
- Values of functions covary with their arguments.
- Emphasis on using function notation representationally.
- Functions defined in open form are bona fide functions just as functions defined by formulas.
- Introduce idea of "essentially equal to".
- Conceptualize graphs, coordinate systems, and displayed graphs (the display of a statement's graph within a coordinate system).

Phase 2: Accumulation from rate of change

Phase 2 addresses FP-1: You know how fast a quantity varies at every moment; you want to know how much of it there is at every moment.

2.1 Students conceptualize constant rate of change as two quantities covarying so that variations in one are proportional to variations in the other ($dy = m\,dx$). Indeed, we define a differential in a dependent variable only in the context that the two (say, *dy* and *dx*) are related proportionally—they change at a constant rate with respect to each other.

2.2 Students represent approximate rate of change functions built from exact rate of change functions.

2.2.1 An exact rate of change function r_f is a function whose values $r_f(x)$ give the rate of change of an accumulation function *f* at each moment of its argument.

2.2.2 An approximate rate of change function *r* has values *r*(*x*) which are constant over intervals of fixed length Δ*x* and approximate values $r_f(x)$.

2.3 Students represent approximate net accumulation functions built from approximate rate of change functions. Approximate net accumulation accrues over each interval at a constant rate; independent variable varies smoothly. The structure of an approximate net accumulation function *A* is:

Assume the domain of *x* is partitioned in intervals of size Δ*x*, starting at *x* = *a*.

For each value of $x \geq a$,

A(*x*) = (accumulation over complete Δ*x*-intervals between *a* and *x*)

+

(accumulation within Δ*x*-interval containing current value of *x*)

In symbols, this is

$$\text{Eq. 1.}\quad A(x) = \left(\sum_{k=1}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} \left(r_f(a+(k-1)\Delta x \right) \Delta x \right) + r(x)\left(x - \text{left}(x)\right),$$

where $\left\lfloor \dfrac{x-a}{\Delta x} \right\rfloor$ is the number of complete Δ*x*-intervals between *a* and *x*, and left(*x*) is the value of the left end of the Δ*x*-interval containing the current value of *x*. The function called "left" is computed as

$$\text{left}(x) = a + \left\lfloor \frac{x-a}{\Delta x} \right\rfloor \Delta x$$

and the function *r* is defined as $r(x) = r_f(\text{left}(x))$.

It is important I point out that the construction of these function definitions is presented as a problem to students–how to actually compute values of functions we define conceptually.

I should also say that the first term in Equation 1 is not a standard Riemann sum. It has a variable upper limit, and it presumes the value of *x* varies smoothly. A graph of the first term alone (Figure 2, left side) is a step function—the value of $\sum_{k=1}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} \left(r_f(a+(k-1)\Delta x \right) \Delta x$ is constant for all values of *x* in $[a+(k-1)\Delta x, a+k\Delta x)$ because the value of $\left\lfloor \dfrac{x-a}{\Delta x} \right\rfloor$ (the number of complete Δx-intervals between a and x) is constant within the interval. Second, the graph of $A(x)$ (Figure 2, right side) is piecewise linear because the term $r(x)(x - \text{left}(x))$ represents the accumulation-so-far within the Δx-interval containing the current value of x, which varies at the constant rate $r(x)$ within the interval. The term $r(x)$ in $r(x)(x - \text{left}(x))$ is analogous to option (d) in Figure 1.

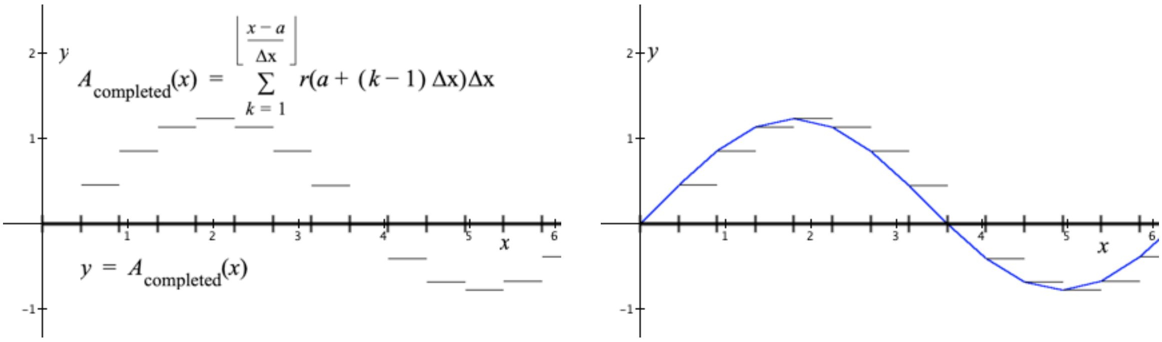

Figure 2. (Left) Graph of net approximate accumulation over completed Δx-intervals as the value of x varies smoothly; (Right) Graph of net approximate accumulation over completed Δx-intervals plus partial net accumulation within Δx-intervals, all as the value of x varies smoothly

2.4 Students define exact net accumulation functions as approximate net accumulation over intervals of size Δx where Δx is so small that making it smaller gives no discernable improvement in approximations.[5] In DIRACC, we represent exact net accumulation in open form as

$$F(x) = \int_a^x r_f(t)dt,$$

where r_f is an exact rate of change function, the value of x varies, t varies from a to x, and dt is a variable that varies through moments of the variable t.[6] Definite integrals are just specific values of exact accumulation functions: $\int_a^b r_f(x)dx$ is simply $F(b)$.

2.5 We use the meaning of accumulation function to establish that if f is an (unknown) accumulation function with (known) exact rate of change $r_f(x)$, and known value $f(a)$, then total accumulation up to the value of x is

$$f(x) = f(a) + \int_a^x r_f(t)dt.$$

In words, $f(a)$ is accumulation up to the value of a (from some unspecified reference point) while $\int_a^x r_f(t)dt$ is net accumulation from a to x. Therefore, $f(x)$, accumulation up to the value of x, is $f(a) + \int_a^x r_f(t)dt$.

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[5] This approach is motivated by Cauchy's meaning of convergence, which emphasizes distance between terms instead of distance from a limit. This is how we avoid dealing with limits. We speak of convergence and terms being computationally indistinguishable from each other.

[6] The terms "exact", "infinitesimal", "moment", "essentially equal to", and ""converges", have special meanings in DIRACC. I'll explain these in my presentation.

While the statement $f(x) = f(a) + \int_a^x r_f(t)dt$ might look to you like the FTC, it is stated to students as a way to represent a complete accumulation function. It is not *stated* as a relationship between accumulation and rate of change. Rather, it *embodies* the relationship between accumulation and rate of change. It is in this way that the FTC becomes *present* in students' thinking long before it is stated for its full import.

2.6 "Solve" many applications using integrals by way of integrating an accumulation's rate of change function. Notice: With the aid of a computer program that "understands" integral notation, none of these applications require finding an antiderivative.

Students engage in all of 2.1 – 2.6 with the aid of a computer graphing program (called *Graphing Calculator*) that allows them to type mathematical statements in standard form to define functions, evaluate functions, and graph functions. Instructors using DIRACC at other institutions use programs like Desmos or Geogebra.

In Phase 2, we both start with concrete settings to build the mathematical ideas and emphasize applications of integrals in physical and social sciences. But none of these applications involve finding antiderivatives. Rather, students focus on conceptualizing quantities involved in situations and modeling their rate of change with respect to one another. Once they have a quantity's rate of change function, they can represent accumulation of that quantity with respect to its independent quantity using open-form integrals—which GC can graph and with which they can evaluate definite integrals. (See Appendix 1.)

Phase 3: Rate of change from accumulation

Phase 3 addresses FP-2: You know how much of a quantity there is at every moment; you want to know how fast it varies at every moment.

3.1 Reconceive "amount" functions—functions whose values give an amount of one quantity in relation to an amount of another—as accumulation functions by envisioning the independent quantity's value varying.

3.2 Connect amount functions to rate of change functions via reconceiving them as accumulation functions. For example, if $f(x) = x^2$ gives the area of a square as it side length *x* varies from 0, then $x^2 = \int_0^x r_f(t)dt$ for some rate of change function r_f.

3.3 Reverse the process developed in Phase 2 to construct a method for deriving exact rate of change functions defined in closed form from exact accumulation functions defined in closed form.

3.4 Notice this brings us full circle. In Phase 2 we started with, for example, $r_f(x) = 2x$ and ended with $f(x) = \int_a^x r_f(t)dt$. In Phase 3 we started with $f(x) = x^2$ and ended with $r_f(x) = 2x$. This means $x^2 = \int_a^x 2tdt$ for some value of *a*. In other words, any time we find a closed-form rate of change function for a closed-form accumulation function, we've found a closed-form definition for an open-form integral. Notice also that this is not yet the "official" FTC—we have not introduced the idea of antiderivative.

3.4.1 Emphasize representational equivalence: Exact accumulation functions and exact rate of change functions can be represented in open form or closed form.

3.5 Standard derivations of rate of change functions from accumulation functions defined in closed form.

3.5.1 Build a library of closed-form definitions of open-form integrals (what will later be called antiderivatives)

Phase 4: Applications of Derivatives

While applications play a central role in Phases 2 and 3, they are more for illustration and for students to engage in repeated reasoning with the conceptual methods being developed. Phase 4 focuses directly on applications.

4.1 Standard applications of rate of change functions (continuing the themes of quantitative reasoning, variation, and covariation): Related rates, optimization, graphical behaviors of functions.

The role of the FTC in this development

I hope it is evident that the FTC-E is at play from late in Phase 1 (differentials and constant rate of change) and the very beginning of Phase 2 (accumulation from rate of change). It segues to the FTC-A in the definition of the exact net accumulation function $\int_a^x r_f(t)dt$ and the representation of exact accumulation as $f(x) = f(a) + \int_a^x r_f(t)dt$. It is restated explicitly as the traditional FTC once rate of change from accumulation is developed and the idea of antiderivative as a closed-form equivalent to an accumulation function defined in open form is crystalized.

The role of technology in this development

DIRACC calculus would be impossible without the incorporation of the technology we use. The aim to make all definitions computable requires something to compute them. Also, defining, and having students define, functions in open form only on paper makes it improbable that they think of those functions as actually computing values of quantities.

Defining functions in open form in GC makes them "alive", just as alive as functions defined in closed form, which in turn enhances their capacity to think of open-form definitions as representing values of quantities. Our use of technology in DIRACC is in many ways aligned with instrumentalism as described by Verillon and its adaptation in mathematics education by Artigue, Drijvers, Trouche, White, and others (Artigue, 2002; Drijvers & Gravemeijer, 2005; Trouche, 2005; White, 2019) and Heideggar's idea of turning artifacts into tools by conceiving them as "ready at hand" in goal-oriented activity (Winograd & Flores, 1986).

There is an important aspect to our use, and students' use, of functions defined in open form. Open-form definitions are, by nature, more reflective of their meaning. Having students answer questions by defining accumulation functions and rate of change functions in open form allows them to think more clearly about what they are representing in the situation as they've conceived it. For example, in traditional calculus courses students spend more time finding an antiderivative in "applying" integrals than they do conceptualizing the situation in which the "problem" is embedded. Allowing students to answer questions about those same situations by identifying a rate of change and using it to define an open-form accumulation function allows them to focus clearly on conceptualizing the situation—conceptualizing quantities that compose it and relationships among them.

(Example: A student is asked to find $\frac{d}{dx}x^{\cos(x)}$. He writes $\cos(x)x^{\cos(x)-1}$. How might he check his answer? In DIRACC, he checks his answer by defining $f(x)=x^{\cos(x)}$ and graphing both $y=\cos(x)x^{\cos(x)-1}$ (his derived rate of change function) and $y=\frac{f(x+0.0001)-f(x)}{0.0001}$ (the average rate of change of $f(x)$ over every interval of length 0.0001 in its domain). When he sees dramatically different graphs (Figure 3), he concludes, because he trusts the open-form definition of approximate rate of change function, that his derived rate of change function is incorrect.)

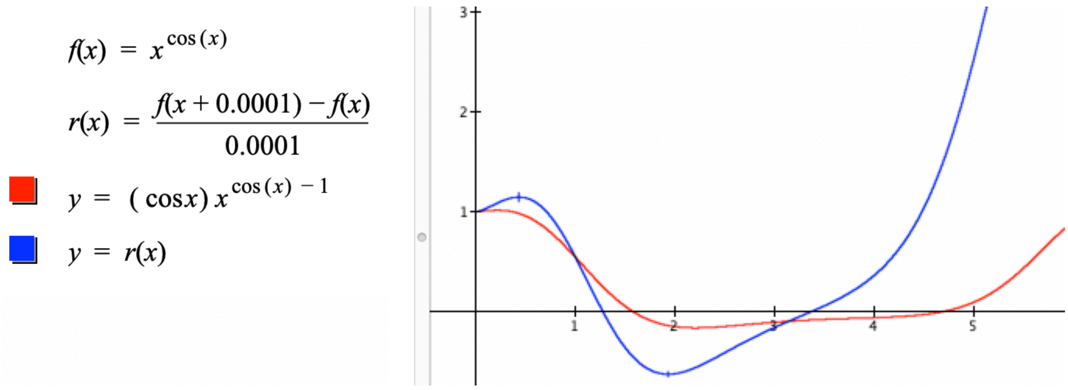

Figure 3. Graphs of student's derived and open-form rate of change functions.

Comparisons of DIRACC, traditional, and engineering Calculus 1 at ASU

Pre-Post Calculus 1 Comparison. The mathematics department at ASU, like many U.S. universities, has different "flavors" of calculus: Calculus for business majors, life science majors, mathematics and science majors, and engineering majors. Business and life science calculus focuses on applications and contain very little concept development. Mathematics/Science and engineering calculus are largely identical in topics, but engineering calculus holds class three hours per week while mathematics/science calculus holds class four hours per week. Within math/science calculus we offered both traditional and DIRACC calculus for three years to compare them; DIRACC is now ASU's calculus for math/science majors.

In Fall 2015 we compared students' performance in DIRACC and traditional calculus using an 11-item pretest at the beginning of the semester, then including the pretest within students' final examination as a posttest. A committee of five people, two teaching traditional Calculus 1, two teaching DIRACC Calculus 1, and the department's director of STEM education, constructed the pretest/posttest. No item was included without unanimous agreement that it assessed a central idea in calculus and addressed it acceptably (see Appendix 2).

Table 2 contains results from the pretest and posttest. The pooled t-test for posttest comparison is $t=11.853, p<0.0001$.

	PreTest		PostTest	
	Mean	StdDev	Mean	StdDev
Traditional $n = 248$	3.18	1.53	4.89	2.45
DIRACC $n = 149$	2.98		7.90	

Table 2. Pre-Post Comparison of Traditional and DIRACC Calculus 1. Possible score = 11.

Calculus 1 Concept Inventory. The construction of two "concept inventories" for semester-based Calculus 1 and 2 was part of the National Science Foundation's charge to DIRACC in their funding award. The Calculus 1 Concept Inventory (C1CI) assessed students' understandings in the areas under the headings Variation and covariation, Function, Modeling/Quantitative Reasoning, Structure sense, Rate of change, Accumulation, and Fundamental Theorem of Calculus. Examples from each area are presented in Appendix 3. Table 3 contains Scheffé comparisons among DIRACC, traditional, and engineering calculus students' scores on the C1CI near the end of their semester. Interpret p-values hesitantly. Students taking the C1CI were recruited from each program's courses with an inducement of $40 each to participate, so sampling was subject to unknown sources of bias.

	Difference	*Std Error*	*p-Value*
DIRACC vs Engineering	3.26	1.72	0.17
DIRACC vs Traditional	0.94	2.24	0.96
Traditional vs Engineering	2.33	1.66	0.37

Table 3. Scheffé comparisons from ANOVA for Calculus 1 Concept Inventory.

Department-required Derivatives Test

The department requires Calculus 1 students to pass a derivatives mastery test to pass the course. Students get three attempts to pass the test. We were unable to get pass rates from traditional and engineering instructors who declined to share them. The department's undergraduate director did share with us that the overall DIRACC pass rate was approximately the same as other programs, and the DIRACC pass rate on first attempt was higher.

DIRACC Calculus 2

The above description of the DIRACC Calculus 1 is necessarily sparse. I'd like to spend more time describing and exemplifying how the themes developed in Calculus 1 lay a foundation for a coherent development of ideas in Calculus 2—which typically stand in splendid isolation from Calculus 1 and from each other.

As I mentioned, traditional Calculus 2 in the U.S. is a potpourri of disconnected topics. My challenge for Calculus 2 was to reconceptualize these topics so they are coherent with ideas of accumulation from rate of change and rate of change from accumulation and coherent with each other. The idea of having students conceptualize differentials as variables was key to developing a coherent Calculus 2.

The DIRACC approach to quantifying regions bounded by graphs and regions within solids of revolution might clarify the power of conceiving differentials as variables. I share these examples instead of equally interesting examples from the sciences because so many people ask how area and volume are treated in the DIRACC curriculum.

Quantifying Regions in the Plane Bounded by Graphs

Computing areas of regions bounded by graphs in the Cartesian coordinate system is the traditional context in which integrals are developed. The idea is to sum areas of rectangles with bases on the horizontal axis and one vertex on the function's graph. This development creates a number of cognitive difficulties for students, including: (1) Area is always positive, so regions above and below the graph must be computed separately. (2) Only definite integrals are addressed; the idea of integral as a function is absent. (3) Integral as area makes students wonder how, for example, computing an area can answer a question about distance, force, or any non-area quantity. (4) Connections between integrals and rate of change are absent.

On the other hand, area of a bounded region is a quantity. It should be possible to apply the DIRACC approach of accumulation from rate of change to the quantification of a planar region bounded by graphs and a spatial region bounded by a surface of revolution.

Suppose a rectangle has height 3 cm and varying base length of x cm. The rate at which the rectangle's area changes with respect to its base length is 3 cm^2/cm. We emphasize to students that even though the rate of change of area with respect to base length has the same *value* as the rectangle's height, the rectangle's height is not the rate at which area changes with respect to base length. The rectangle's height is 3 cm. The rectangle's rate of change of area with respect to base length is 3 cm^2/cm. They are different quantities that have the same value.

More generally, the differential dA of bounded area in the Cartesian coordinate system is the area of a rectangle of height $f(x)$ and base length dx. When the height of a rectangle is $f(x)$ and its base length varies by dx, its rate of change of area with respect to x has the same value as $f(x)$ (see Figure 4).

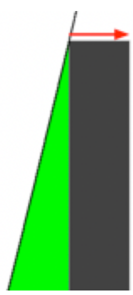

Figure 4. The area of a rectangle with constant height and varying base has a rate of change of area with respect to its base equal in value to the measure of its height.

In DIRACC Calculus 1, we developed that when any quantity has a rate of change function $r_f(x)$, net accumulation in f as the value of t varies from a to x is $A_f(x) = \int_a^x r_f(t)dt$. Therefore, net accumulated area bounded by the graph of $y = f(x)$, $x = a$, and $x = b$ is $\int_a^b f(t)dt$. However, the idea of "net accumulated area" must mean the quantity is *signed* area. Over intervals where $f(x)$ is negative, area accumulates at a negative rate of change, so net change in area will have a negative measure.

The fact that the value of $f(x)$ is the rate of change of signed area with respect to x depends on the graph being displayed in the Cartesian coordinate system. In a polar coordinate system, the rate of change of area bounded by the graph of $r = f(\theta)$ as the value of θ varies is not $f(\theta)$. The differential for the area of a bounded region in a polar coordinate system is a sector of a circle (see Figure 5).

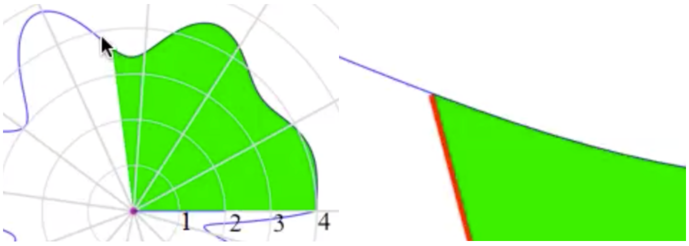

Figure 5. The differential of area bounded by a graph in polar coordinates is area of a sector of a circle with radius $f(\theta)$ and angle measure $d\theta$.

Since dA, the differential in area, changes at a constant rate with respect to $d\theta$, $dA = m\, d\theta$. Moreover, $dA = \pi \dfrac{r^2}{2}$ when $d\theta = \pi$. So, in polar coordinates, $dA = \dfrac{r^2}{2} d\theta$. The exact rate of change function for area bounded by a graph in polar coordinates is therefore $r_A(\theta) = \frac{f(\theta)^2}{2}$ and accumulation of unsigned area bounded by a graph in the polar coordinate system is $\int_a^\theta \dfrac{f(t)^2}{2} dt$.

Unsigned area is not the same as net accumulated area. Net accumulated area is a signed quantity—it must have a negative value when the value of the function is negative. However, $f(t)^2$ is always non-negative. We can adjust to this by multiplying the integrand by the sign of the function's value, giving $A(a,\theta) = \int_a^\theta \left(\operatorname{sgn}(f(t)) \dfrac{f(t)^2}{2} \right) dt$.

Quantifying Regions bounded by surfaces of revolution

In DIRACC we speak of surfaces of revolution and regions they bound rather than speaking of solids of revolution. This decision arose from two sources—our commitment to crafting a dynamic calculus and interviews with students studying traditional developments of volumes of solids of revolution.

In traditional developments, students see textbooks proposing to generate solids by revolving a bounded region in the plane around an axis, then slicing the solid into pieces whose volumes they approximate with disks, washers, or shells. Sometimes textbooks propose to first approximate the region with rectangles before revolving so that the revolution itself generates disks, washers, or shells.

I conducted a Calculus 2 exploratory design experiment in Spring 2017. What this means is I wrote just-in-time modules for upcoming topics, asked probing questions during class, and interviewed students during office visits and volunteers outside of class.

My first pass at solids of revolution resembled traditional approaches with a small difference—treating slices as differentials. What I quickly noticed was that students exhibited what every Calculus 2 instructor witnesses – students were confused about whether to use disks, washers, or shells in setting up their integrals.

Further probing led me to conclude the root of their problem was in their conceptualization of the solid. It was entirely unnatural to think of anything varying within it. It also became evident that students did not see a connection between a method they contemplated and the integral's independent variable. I pondered how to address the two issues simultaneously.

The first modification was to change the metaphor for solids of revolution. Instead of revolving a region to generate a solid, I spoke of revolving the graph of $y = f(x)$ about the y-axis or x-axis to generate a surface (Figure 6a), pointing out that the surface encloses an empty region in space. I then offered the metaphor of "filling" the enclosed region, asking about ways to fill it so we can quantify the region (compute an approximation to its volume; Figure 6b and Figure 6c).

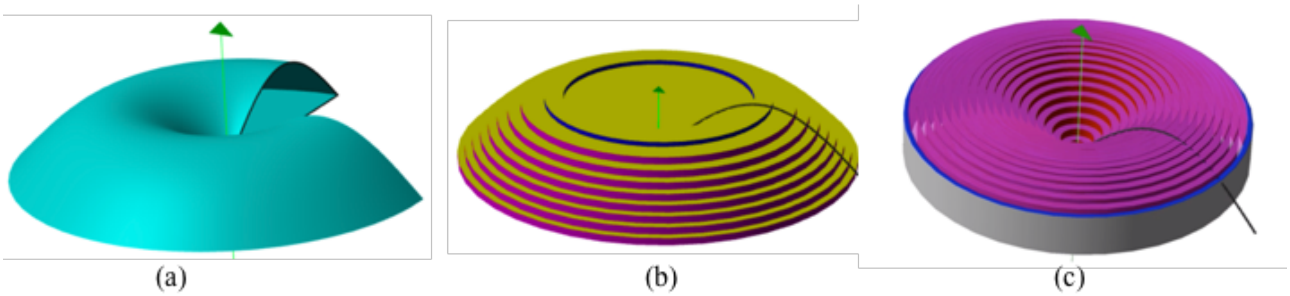

Figure 6. Quantifying a region enclosed by a surface of revolution. Method (b) fills the region with cylinders having constant base and varying height. Method (c) fills the region with cylinders having constant height and varying base. Volume of a blue-edged cylinder is the differential in volume for the respective method of accumulation.

There are three benefits of this approach. First, students distinguish between creating a region in space and making a solid. They make a solid by filling an empty shell after they revolve a graph to make it. Second, they must decide explicitly which variable (original function's dependent or independent variable) will be their independent variable of accumulation. They can choose either, and their choice will determine the kind of cylinders they will use to fill the empty region, which in turn will determine the rate of change of accumulating volume with respect to the accumulation's independent variable. Third, they only need to think of two kinds of cylinders—cylinders having constant base and varying height, or cylinders having constant height and varying base.[7]

In Figure 6b, y is the independent variable of accumulation and the differential of approximate accumulating volume is $dV = (\textit{area of cylinder's base})dy$, which means approximate volume's rate of change function has the same numerical value as the area of the cylinder's base. In Figure 6c, the

[7] We develop a general definition of right cylinder as any geometric figure having congruent top and bottom lying in parallel planes and having sides perpendicular to both.

differential in volume is $dV = 2\pi x f(x)dx$, which means the accumulating volume's rate of change function is $r_V(x) = 2\pi x f(x)$ and volume is $V(a,x) = \int_a^x r_v(t)dt$.

Figure 6 (b and c) also illustrates how this approach helps students clarify decisions they must make about which variable to use as accumulating volume's independent variable. If a student chooses (b), taking y as the accumulation volume's independent variable, she will be forced to use f^{-1} to compute inner and outer radii of her cylinders. In this instance, students can avoid this difficulty by using x as accumulating volume's independent variable. The point is that students consider which variable to take as their independent variable of accumulation apart from which variable is the original function's independent variable. In the traditional approach, students were unaware that volumes they tried to compute had an independent variable.

Motive for these examples

I shared this disquisition on signed area in Cartesian and polar coordinate systems and volume of regions bounded by a surface of revolution to illustrate the coherence gained across topics by the thematic attention to quantities and their variation and by casting differentials as variables. Standard developments of integrals and derivatives based in static geometric arguments presume a timeless, unchanging world from which mathematics derives meaning—a world counter to students' lived experience. This presumption also leads to standard calculus having a strong focus on geometric interpretations of definite integrals as areas and derivatives as slopes of tangents at a point.

I hasten to add that the mathematics students create through school (at least in the U.S.) is their accommodation to a mathematics presented as if nothing varies. Variables' values change by substituting one number for another. Students having this conception of variable cannot use letters to represent quantities' values in dynamic settings, which lends to students' evolving beliefs that mathematics is about symbolic rituals. I address students' felt conflicts between their school mathematics and DIRACC calculus in the next section.

Approximating Integrals and Polynomial Approximations

(I'm unsure whether to include this discussion. While it extends the other two examples in illustrating the coherence of the DIRACC approach, the discussion might add too much to the paper's length.)

Comparisons of DIRACC, Traditional, and Engineering Calculus 2

	Difference	*Std Error*	*p-Value*
DIRACC vs Engineering	3.95	0.90	0.001
DIRACC vs Traditional	2.26	1.10	0.13
Traditional vs Engineering	1.68	0.90	0.18

Table 4. Scheffé comparisons after significant ANOVA for Calculus 2 Concept Inventory.

Students' Reactions to DIRACC Calculus

Related Work (to be completed)

(Rosenthal, 1992)(Strang, 1990, 1991)(Macula, 1995) in some ways foreshadow what I've said about the FTC-E and FTC-A schemes. However, they (1) focus on finding areas under curves, (2) are unconcerned with students' images of variation, and (3) their primary goal is the second form of the FTC [F(b)-F(a)]. Moreover, Strang emphasizes change in area by combining static bits

$$\left(F(x_n)-F(x_{n-1})\right)+\left(F(x_{n-1})-F(x_{n-2})\right)+\cdots+\left(F(x_2)-F(x_1)\right)=F(x_n)-F(x_1)$$

rather than a summation of terms $f(x_i)\Delta x_i$ – the quantities made by a rate of change over an interval of change.

An explosion of focus on and research on accumulation functions in calculus:

(Bressoud, 2009, 2011)(Sealey, 2014; Swidan, 2019; Swidan & Naftaliev, 2019; Swidan & Yerushalmy, 2015) (Kouropatov & Dreyfus, 2014). Point out that they omit the idea of rate of change in their meaning of accumulation.

Comment on Ely's (Ely, 2010, 2017) notions of infinitesimals and "smooth continuous" and difficulty with building rate of change conceptually from them.

Image for presentation – change, with prior changes "evaporating" upon next change.

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Appendix I

Assignment Sheet for Students: Measuring Energy Consumption[8]

Energy is commonly measured in units of Joules (J), which is the work done by a force of one Newton when its point of application moves one meter in the direction of the force. The rate at which energy is used is known as power. Power is measured in Joules/sec (1 Joule/sec is called a Watt). It is also common to measure an amount of energy in kilowatt-hours (kWh), which is the amount of energy used when it is used at the rate of 1000 Joules/sec (1000 Watts) for one hour. A megawatt hour is the amount of energy used when it is used at the rate of one million Watts (one million Joules/sec) for one hour.

Suppose a city consumes electrical energy on a given day at an approximate rate of $r(t)$, where t is a number of hours since midnight and r is defined, in megawatts, as

$$r(t) = 400 - 300\cos\left(\frac{\pi}{12}(t-5)\right),\ 0 \le t$$

a) Fill in the blanks in 5 different ways to make this statement true:

   3 Megawatt-hours = The energy consumed at a rate of __________ Joules/sec for __________ hours.

b) Why is it sensible that $\frac{\pi}{12}(t-5)$ is the argument to cosine in this model of the city's rate of electrical energy consumption? (Examine the graph of r.)

c) Define a function that gives the electrical energy this city will have consumed x hours after midnight on a given day. Explain how your function produces a value that is in the desired units. Does the value of x have a necessary upper bound?

d) Approximately how much electrical energy does this city use in a typical day? In a typical week? In a typical month? In a typical year? Be sure to state quantity's units.

e) This city's electric utility company charges for electricity at the rate of \$0.13 per kilowatt-hour for electricity used between 7:00a and 5:00p, and at the rate of \$0.07 per kilowatt-hour otherwise. What is this city's electrical bill for one day?

f) Burning 1 kg of coal produces about 450 kWh of energy. How many kg of coal are required to meet the energy needs of the city for one day? One year?

g) A wind turbine normally generates electricity at a rate of 200 kW. Approximately how many wind turbines would be required to meet the needs of this city for one day? One year?

h) A medium-sized household typically uses about 300-500 kilowatt-hours of electrical energy in a month during the fall. Residential consumption is typically about 30% of a major city's total electrical energy consumption in that same period. Approximately how large is this city's population?

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[8] Adapted from Briggs & Cochran, Chapter 6.1, Problem 57.

Partial Student Response using GC

☒ $r(t) = 400 - 300\cos\left(\frac{\pi}{12}(t-5)\right)$

This is the rate at which amount of electrical energy that the city has used is changing *t* hours after midnight on a particular day. The value of *r* is a number of megawatts (1 million Joules/sec).

☒ $E(x) = \int_0^x r(t)\mathrm{d}t$

(c) E is a function that gives the amount of electrical energy that the city uses (in megawatt-hours) from midnight to *x* hours after midnight on a particular day. The value of *r* is in megawatts and the value of *dt* is in hours, so the value of *r*(*t*)*dt* for a value of *t* and a really small value of *dt* is in megawatt-hours. So the value of *E* is in megawatt-hours.

The value of *x* can be any positive number (assuming the model remains applicable). For example, because the model is periodic, E(48) would be the electrical energy that the city used in two days from the beginning of the first day.

☒ $E(24) \quad = 9600$

(d) E(24) gives the number of megawatt hours that the city uses in one day (24 hours from midnight on a particular day). According to this model, the city would use 9600 megawatt-hours of energy in one day. Energy used in a week would be E(7*24) megawatt-hours, in a typical (30-day) month the city would use E(30*24) megawatt-hours and in a (365-day) year it would use E(365.25*24) megawatt-hours of electrical energy.

☒ $P_1 = 0.13\times10^3, P_2 = 0.07\times10^3$

P_1 and P_2 are the prices of electricity per megawatt-hour that the electrical company charges during the two different periods of the day.

☒ $P_1\,(E(17) - E(7)) + P_2\,(E(7) + E(24) - E(17)) \quad = 946377$

(e) E(17) - E(7) is the amount of electrical energy that the city uses between 7am and 5pm on a particular day. E(7) is the amount of energy used between midnight and 7am, and E(24) - E(17) is the amount of energy that the city uses between 5pm and midnight. The sum of E(7) and E(24)-E(17) is the amount of energy the city uses at a price of P_2 dollars per meagawatt-hour, and E(17) - E(7) is the amount of energy the city uses at a price of P_1 dollars per megawatt-hour. According to this model, this city spends about $946,000 per day for electricity.

Challenge

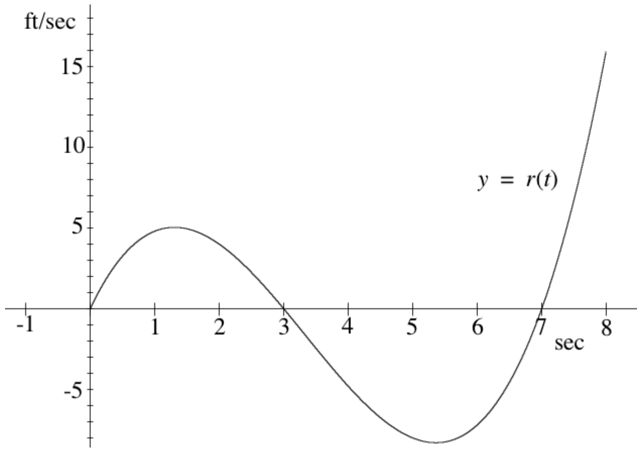
It is reasonable that this city uses less electricity on weekends than on weekdays.

a) Define a function that, in your estimation, reasonably models the rate of electricity usage on weekends. Explain your definition.
b) Redefine the overall rate function so it models rate of change of energy usage for any (real) number of days since the beginning of some Monday.
c) Give new answers to the original questions.

Appendix 2

Pretest/Posttest for Comparing DIRACC and Traditional Calculus 1

1. The function r is the *rate of change function* with respect to time for a particle's displacement from its initial position while it moves in a straight line. The graph of $y = r(t)$ is given to the right.
   The function s is the particle's *displacement function.* Its values (measured in feet) give the particle's displacement from its initial position t seconds after starting. At what time, approximately, during the first 7.5 seconds does $s(t)$ have its smallest value?



A. 1.2 sec
B. 3 sec
C. 5.4 sec
D. 7 sec
E. None of the above

2. A company produces different sized smartphones with rectangular screens. The screen's dimensions are w and h, where the height (h) is half the width (w) for all sizes of smartphones. Which of the following functions represents any screen's diagonal length as a function of its width?

A. $L(w) = \dfrac{\sqrt{5}w}{2}$

B. $L(w) = wh$

C. $L(w) = \dfrac{1}{2}w^2$

D. $L(w) = \sqrt{w^2 + h^2}$

E. None of the above

3. The Trans-Port Company manufactures containers of various dimensions, with heights x up to 4.5 yards. The volume of their containers of height x is given by the function g, where $g(x) = 4x^3 - 50x^2 + 144x$ is measured in cubic yards. If the height of the container is increased from 1.5 yards to 2 yards, what is the corresponding change in the container's volume, in cubic yards?

A. $(2 - 1.5)^3$ B. $g(2 - 1.5)$ C. $2^3 - 1.5^3$

D. $g(2) - g(1.5)$ E. $g(1.5) - g(2)$

4. When a rocket is launched, its speed increases continually until its booster engine separates from the second stage. During the time it is continually speeding up, the rocket is <u>*never*</u> moving at a constant speed. What, then, would it mean physically to say that at *precisely* 2.15823 seconds after launch the rocket is traveling at *precisely* 183.8964 miles per hour?

   A. 183.8964 is the limit of a difference quotient as time approaches 2.15823 seconds.

   B. If you were to freeze time at 2.15823 seconds after launch, the rocket's speedometer would point at 183.8964 miles per hour.

   C. The rocket traveled at the speed of 183.8964 miles per hour for the first 2.15823 seconds of its flight.

   D. The rocket's speed over the time interval of 2.15822 seconds to 2.15824 seconds after launch is essentially 183.8964 miles per hour.

   E. None of the above is an acceptable meaning for the statement that the rocket was going precisely 183.8964 miles per hour 2.15823 seconds after launch.

5. The table below gives information about functions f and g. Let h be defined as $h(x) = f(g(x))$. What is the rate of change of h at $x = 4$?

	x			
	1	2	3	4
$f(x)$	20	23	18	14
Rate of change of f at x	7	-3	-7	-2
$g(x)$	-10	-11	-4	2
Rate of change of g at x	-0.5	2	4	3

A. -2 B. -3 C. -6 D. -7 E. -9

6. What is the primary focus of calculus?

   A. Properties of graphs, mainly slopes and areas

   B. Finding values of derivatives and integrals

   C. Modeling and analyzing how quantities vary together

   D. Learning complex operations with symbols and numbers to improve cognition

   E. Finding limits

For questions 7 – 10: Let $F(x)=\int_a^x f(t)\,dt$

7. What does f represent?

   A. A distance function with respect to time
   B. A small change in a quantity
   C. A rate of change function for some quantity
   D. A total amount of some quantity
   E. None of the above

8. What does $f(t)dt$ represent?

   A. A distance function with respect to time
   B. A small change in a quantity
   C. A rate of change function for some quantity
   D. A total amount of some quantity
   E. None of the above

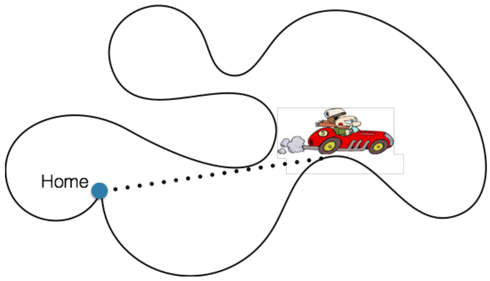
9. What does F represent?

   A. A distance function with respect to time
   B. A small change in a quantity
   C. A rate of change function for some quantity
   D. A total amount of some quantity
   E. None of the above

10. What does t represent in the expression $f(t)$?

   A. Time
   B. The value half way between a and x
   C. A variable that varies from a to x
   D. Nothing, it is a dummy variable
   E. None of the above

11. Bob traveled in his car at a constant speed along a complicated loop, beginning and ending at his home. What must be true about the rate of change of the car's straight-line distance from home with respect to time at the moment it is farthest from home?



A. The rate of change will be largest at the moment the car is farthest from home.

B. The rate of change will change from negative to positive at the moment the car is farthest from home.

C. The rate of change will be zero at the moment the car is farthest from home.

D. The rate of change will be smallest at the moment the car is farthest from home.

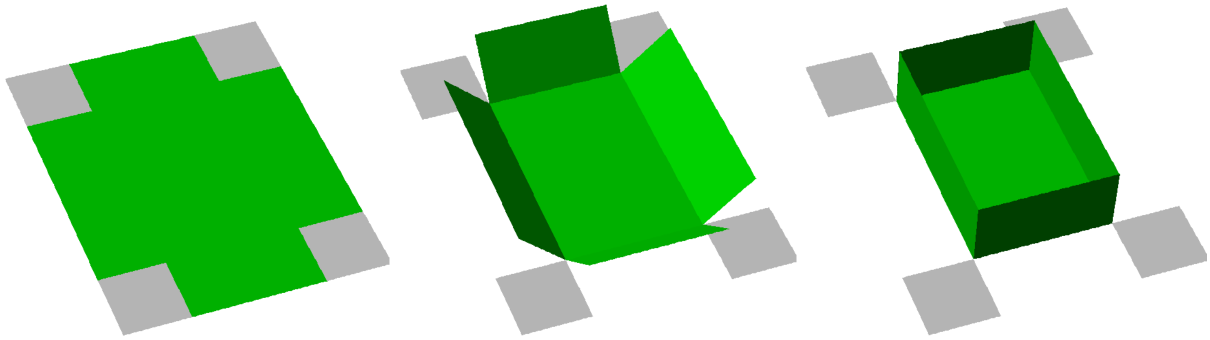
E. None of the above.

Appendix 3

Sample C1CI Items (all © 2018 Arizona Board of Regents)

Variation and Covariation

1. You have an x cm by y cm rectangular sheet of cardboard. You can fold the sheet into a box by first cutting squares with side lengths a cm from each of the four corners. Which of a, x, and y have values that vary when you think of finding the box with the largest possible volume?



a) a

b) x

c) y

d) x and y

e) a, x, and y

Function

2. A function f converts weight in pounds (at a particular location on earth) to the equivalent mass in kilograms. Another function g determines the volume of a certain liquid in liters as a function of the total mass of the liquid in kilograms.

   Given a certain volume x of this liquid in liters, which of the following is the weight of the liquid in pounds?

a) $f^{-1}(g(x))$

b) $f(g^{-1}(x))$

c) $g^{-1}(f(x))$

d) $g(f(x))$

e) $f^{-1}(g^{-1}(x))$

Modeling/Quantitative Reasoning

3. The Trans-Port Company manufactures containers of various dimensions, the tallest being 3.5 yards tall. The volume of a container depends on its height; $g(x) = 4x^3 - 50x^2+144x$ is the volume (in cubic yards) of a container with height x yards. If the height of the container is increased from 1.5 yards to 2 yards, what is the corresponding change in the container's volume, in cubic yards?

   a) $(2 - 1.5)^3$

   b) $g(2 - 1.5)$

   c) $2^3 - 1.5^3$

   d) $g(2) - g(1.5)$

   e) $g(1.5) - g(2)$

Structure Sense

4. For the following function, which differentiation rule applies to the expression as a whole?

$$((f(s)/g(s)+k(s))^3\sqrt{h(s)}$$

   a) Chain Rule

   b) Power Rule

   c) Product Rule

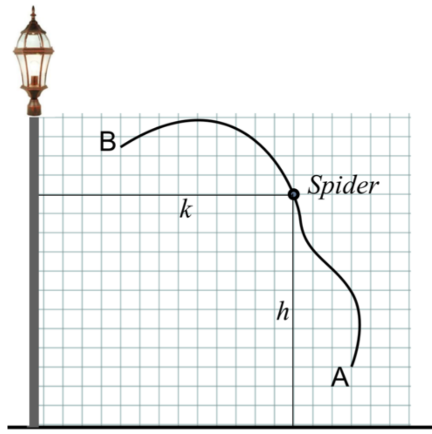
   d) Quotient Rule

   e) Sum Rule

Rate of Change

5. At one end of a brick wall is a vertical light pole. A spider walks on the wall from point A to B along the path shown. The number of feet the spider is above the ground (h) and the number of feet the spider is to the right of the light pole (k) both vary as the spider walks the path.

   Estimate the rate of change of k with respect to h at the moment shown in the illustration.



   a) 1

   b) 2

   c) -2

   d) -1/2

   e) Not enough information to make an estimate

Accumulation

6. On Mars, an astronaut dropped his watch from a cliff. Its speed at every moment was $w(t)$ meters per second, where *t* is the number of seconds after the watch was released. Which expression gives the *best* estimate for the distance the watch fell from 8 to 8.04 seconds after being released?

a) $w'(t)\cdot(8.04-8)$ b) $w(8)(8.04-8)$ c) $\dfrac{w(8.04)-w(8)}{8.04-8}$ d) $w(t)+w(t+0.02)(0.02)$

e) $w(8)(0.02)+w(8.02)(0.02)$

Fundamental Theorem of Calculus

7. Given a differentiable function *k*, for what values of *a* is $\int_a^x k'(t)\,dt = k(x)$ for all values of *x*?

All values of *a* such that…

a) $k'(t) = k(a)$

b) $x = a$

c) $k(a) = 0$

d) $k'(t) = k(x)$

e) $a = 0$