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School students' preparation for calculus in the United States

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Abstract
Researchers have been interested in students' transition to calculus since the early 1900s. One line of inquiry highlights students' understandings of high school mathematics as impeding or supporting their successful transition to university mathematics. This paper addresses an underlying question in this line of inquiry: does school mathematics provide opportunities for students to develop productive meanings for calculus? This article reports on U.S. calculus students' responses to items that assessed students' variational reasoning, meanings for average rate of change, and representational use of notation—ideas ostensibly addressed in school mathematics. To make sense of students' difficulty on these items we sought to understand the opportunities students had to reason with these ideas prior to calculus. We use two data sources to understand the likelihood that students have opportunities to construct productive meanings for function notation, variation, and average rate of change in their secondary mathematics education: meanings for these ideas supported by precalculus textbooks and meanings secondary teachers demonstrated. Our analysis revealed a disconnect between meanings productive for learning calculus and the meanings conveyed by textbooks and held by U.S. high school teachers. We include a comparison of meanings held by U.S. and Korean teachers to highlight that these meanings are culturally embedded in the U.S. educational system.

Keywords Transition to calculus · Mathematical meanings for teaching · International comparisons · Precalculus textbook analysis

1 Introduction

Mathematics educators have attended to students' transitions from school to university mathematics since the early 1900s; Felix Klein diagnosed the problem of transitions from high school to university and from university back to the classroom as resulting from a double discontinuity in high school teachers' mathematical education. The first discontinuity was an abrupt change in mathematical rigor and content from school to university. The second was teachers' lack of connection between their university mathematics education and the content of secondary mathematics they were to teach. Klein's solution to the double discontinuity was to bring students' school mathematics and their university mathematics into closer alignment by improving teachers' understandings of the deep mathematical roots of the secondary curriculum (Kilpatrick, 2019; Klein, 1932).

Others, since Klein, examined students' difficulties in transitioning from school to university. Some approached the problem generally as a matter of affect or identity (Cassidy & Trew, 2001; Parker et al., 2004). Others looked at students' mathematical preparation in light of broad cultural movements (Hoyles, Newman and Noss, 2001). A third line of inquiry highlights students' understandings of high school mathematics as impeding or supporting their successful transition to university mathematics. Stewart and Reeder (2017) argue that algebraic skills developed in high school impede students' ability to participate fully in calculus and higher courses in university.

In this article, instead of examining the broad issue of transitioning from high school to university mathematics, we focus on transition from pre-calculus mathematics to calculus wherever it happens in students' mathematical education. In the U.S. students take a course called "precalculus" the year before taking calculus. While precalculus is a capstone in U.S. students' preparation for calculus, our position is that students' readiness to develop conceptual understandings of key ideas in calculus depends significantly on their development of important meanings in their earlier schooling.

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Thus, investigating students' calculus experiences should also include investigation into their prior schooling.

In this article we illustrate the need to investigate students' prior schooling in order to understand their difficulties in calculus. We provide three sources of data to illustrate our concern about students' opportunities to construct productive meanings throughout their prior schooling. We use U.S. calculus students' responses to a Calculus 1 Concept Inventory to shed light on university calculus students' meanings and examine those for indications of problematic meanings these students developed in their prior schooling. We focus on students' variational reasoning, meanings for average rate of change, and representational use of function notation. We examined secondary math teachers' meanings for these same ideas because ways teachers understand a mathematical idea contribute to the mathematical understandings students actually construct (Copur-Gencturk, 2015; Thompson, 2013). Thus, understanding teachers' meanings provides insights into the opportunities their students have to construct productive meanings. Finally, we examined the treatment of these ideas in precalculus textbooks under the assumption that the meanings conveyed in curricular materials inform students' meanings both through students' interactions with the text as well as through a teacher's use of the text to inform their instructional decisions. Through this data we illustrate what seem to be limited opportunities for students to construct meanings necessary to be conceptually prepared for calculus and argue that this is a problem requiring further examination.

2 Students' opportunities to be conceptually prepared for calculus

We follow Thompson in our broad meaning of "meaning" (see Thompson, 2013, 2016 for details). An understanding is a cognitive state resulting from an assimilation. A meaning is the space of implications of an understanding. Even ritualized performance can be a student's meaning for what is commonly characterized as "acting with meaningless symbols" in the moment of enacting it.

The meanings a student constructs throughout their prior schooling have the potential to provide a coherent platform that supports the student in developing conceptual understandings of key ideas in calculus—we classify such meanings as productive. This implies that the targeted understanding informs what constitutes a productive meaning from one's prior schooling. As an example, the idea of quotient as a measure of relative size is a productive meaning that supports a conceptual understanding of difference quotients in calculus (Byerley, 2019; Mkhastwha, 2020). Alternatively, a meaning students construct throughout their schooling, such as part-whole meanings for fractions or quotient as just the result of division, can be unproductive and inhibit their understanding of key ideas in subsequent courses. Our examination of students' meanings for key ideas in calculus revealed that students' difficulties were often not specific to the calculus concept but instead rooted in meanings they likely developed in their prior schooling.

We illustrate these problematic meanings with data from the Calculus 1 Concept Inventory (C1CI).[1] This instrument was administered to 356 calculus students[2] at a large U.S. public university. We examined the meanings students demonstrated on the instrument for indications of problematic meanings and ways of thinking these students developed in prior schooling. In this paper we illustrate the problematic nature of students' variational reasoning, students' meanings for average rate of change, and students' usage of function notation.

To understand the root cause of these problematic meanings and ways of thinking we considered the mathematics instruction students experience in high school. We define mathematics instruction as the conveyance of mathematical meaning where the student constructs meaning by trying to understand what a textbook said or in trying to understand what their teacher said or did (Thompson, 2013; 2016). Conveyed meanings are therefore foundational for what students learn from classroom instruction. We recognize that a thorough investigation of the root cause of students' meanings requires a longitudinal study to trace the development of students' meanings and how these meanings are constructed in light of their educational experiences. This paper is intended to illustrate the need for such a study by examining how mathematical meanings conveyed in classroom instruction, either by a teacher or curricular materials, might impact students' opportunities to construct meanings productive for classroom learning.

2.1 Data on teachers' meanings

While we recognize the variety of sources that influence students' meanings such as their interactions with peers, their existing meanings, and online resources, we highlight teachers' meanings because we suspect teachers' meanings

[1] The C1CI was developed within Project DIRACC (Developing and Investigating a Rigorous Approach to Conceptual Calculus, NSF Grant DUE-1625678, http://patthompson.net/ThoompsonCalc.) This instrument consists of 43 multiple choice items designed to reveal students' meanings for function, function notation, rate of change, accumulation, and the fundamental theorem of calculus.

[2] Students were volunteers and recruited from students currently enrolled in calculus 1 or calculus 2 at a large U.S. public university. There were two iterations of this instrument. Questions that were on both versions were seen by 356 students while questions that were modified between versions were seen by 224 students in the second iteration.

have a large impact on students' meanings. We theorize that a student will adjust what they say and do based on how they assimilate a teacher's words and actions, and these words and actions are informed by the teacher's own mathematical meanings. We recognize that a teacher's instructional decisions may not always reflect their most robust mathematical meanings, but it is unlikely students will construct meanings more productive than meanings their teacher holds.

To understand potential meanings teachers convey we used data from the *Mathematical Meanings for Teaching secondary mathematics* instrument (MMTsm; Thompson, 2016) to characterize teachers' mathematical meanings. This instrument was designed to reveal aspects of teachers' mathematical meanings whether productive or unproductive for student learning. The instrument consists of both open ended and multiple-choice questions; clinical interviews were conducted to validate that a teacher's response provided insights into their mathematical meanings.[3] The project team conducted an international comparison between U.S. and South Korean teachers to see if U.S. and Korean teachers demonstrate different meanings. If they do, it suggests teachers' meanings reflect obstacles or affordances within the different educational systems for teachers to have developed productive mathematical meanings for ideas they encountered in their schooling. In selecting a country for the comparison the project team was constrained to languages for which they had native speakers and prioritized a country that performed highly on international math comparisons (PISA or TIMSS). Thus, they selected South Korea. The MMTsm was administered to 253 U.S. high school teachers participating in NSF-funded professional development projects in the Midwest and Southwest U.S. and 366 South Korean teachers (102 middle and 264 high school) participating in a qualification training program required for teachers to earn their "1st class" teaching certificate.

2.2 Textbook analysis

Textbooks can contribute to a student's opportunities to construct mathematical meanings either through the student's own interactions with the text or as their teacher uses the text as a resource to inform their instructional decisions. Textbooks have the potential to be a critical resource for teachers and students by reinforcing their existing meanings, providing new interpretations they were not aware of, or conveying meanings that cause the teacher/student to experience cognitive conflict and perhaps make a modification to their existing meanings.

We considered ten textbooks sorted into four categories: commercially published textbooks marketed for the high school market, commercially published textbooks for the college market, textbooks whose development was funded by the National Science Foundation, and open-source textbooks. When initially selecting texts we considered the market share of secondary textbooks[4] and then prioritized texts that we thought were most likely to support students in constructing meanings necessary to develop a conceptual understanding of calculus. We eliminated all texts that treated precalculus as a miscellaneous topics course including topics such as complex numbers, vectors, matrices, and parametric functions. Finally, we selected texts whose authors also wrote Calculus texts with the expectation that these authors would be more likely to convey meanings that would support students in being successful in calculus. The selected textbooks are *Glencoe Precalculus* (Carter, Cuevas, Day & Malloy, 2011), *Precalculus* (Sullivan & Sullivan, 2013), *Functions Modeling Change* (Connally, Hughes-Hallet & Gleason, 2019), and *OpenStax Precalculus* (OpenStax College, 2017).

We used textbooks indexes to identify sections that focused on the topics relevant to our analyses (e.g., average rate of change). We analyzed the textbooks' definitions, explanations, examples, and homework problems. We also examined the previous and subsequent sections to see how the authors built a foundation for an idea or built upon an idea. Finally, we analyzed homework solutions in the teacher's edition of the text to understand the authors' expectations for students' reasoning.

3 Variational Reasoning

Variational reasoning is an essential way of thinking in order to construct and reason about rate of change functions and accumulation functions—the building blocks of calculus. Variational reasoning is not unique to calculus; productive meanings for constant rate of change are built upon images of constrained variation and covariation (Thompson, 1994). For example, to construct an image of a car traveling at a constant speed requires one first construct two varying quantities: distance traveled and time elapsed. Each quantity

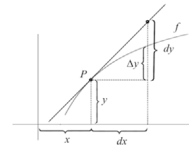
[3] Thompson (2016) elaborates on instrument development and validation.

[4] Houghton Mifflin Harcourt, Pearson, and McGraw-Hill account for a combined 83% of secondary math market share Fulkerson, W. O., Campbell, K., & Hudson, S. B. (2013). *2012 National survey of science and mathematics education: Compendium of tables.*. These three publishers have seven programs available in the secondary market: Saxon Math, AGA, Integrated Math, Envision, CME Project, Illustrative Math, and Glencoe. Out of these seven series only one, Glencoe Math by McGraw-Hill, offers a precalculus textbook. All other series offer a three course sequence: Algebra I, Geometry, and Algebra II.

Fig. 1 Linear Approximation Task from Calculus 1 Concept Inventory

The graph depicts the linear approximation of *f* near the given point *P*. Which of the following will vary as the value of *dx* varies in this context?

a. $dy, \Delta y$
b. $x, y, dy/dx$
c. $x, y, dy, \Delta y$
d. $dy, \Delta y, dy/dx$
e. $x, y, dy, dy/dx$



n=222[5] students	a	b	c	d	e
count	77	23	35	66	21
percent	34.7	10.4	15.8	29.7	9.5

varies by accumulating bits of change to an already accumulated amount of distance (or time). We do not need to attend to the already accumulated amount beyond assuming that there is one. After constructing such images of two varying quantities one can then construct the relative size of the increments of accumulation—the bits of change—and anticipate that as the quantities continue to vary the relative size of the increments remains constant.

In this section we focus on constant rate of change as a context to analyze opportunities for students to develop variational reasoning in school mathematics. We should expect to see evidence that teachers spontaneously employ continuous variational reasoning when reasoning about constant rate of change and we should expect precalculus texts to attend to quantities' continuously varying values, increments of change, and constrained variation in their discussion of constant rate of change. As we illustrate in this section, U.S. teachers experienced difficulty differentiating between a quantity's value and changes in the quantity's value and they demonstrated thinking of discrete changes instead of thinking of changes happening continuously. Also, precalculus texts were no more likely than teachers to support students' variational reasoning—they too discussed variation as happening in completed chunks.

3.1 Calculus students' variational reasoning

When we consider students' variational reasoning we are interested both in ways they imagine quantities varying as well as their ability to construct an invariant relationship between changes-in-progress. One's variational reasoning can be classified by whether they are engaged in chunky thinking which involves reasoning about quantities varying in discrete amounts or imagining smooth change which involves reasoning about change in progress (Castillo-Garsow, 2010; 2012).

To illustrate students' difficulty maintaining an invariant relationship in a dynamic situation consider U.S. calculus students responses to the task shown in Fig. 1. To successfully complete the task students needed to reason that a constant rate of change implies $\frac{dy}{dx}$ remains constant as the values of dx and dy vary. Approximately 35% of students saw the diagram as implying varying values of x and y (answer choices b, c, and e) and nearly 50% of students saw the diagram as implying a varying value of $\frac{dy}{dx}$ (answer choices b, d, and e). Only 34.7% understood the diagram as implying fixed values of x and y and varying values of dx, dy, and Δy (answer choice a). These results suggest students were unprepared to interpret $\frac{dy}{dx}$ as expressing the invariant relative size of increments of change. This suggests to us these calculus students did not have sufficient opportunities in their prior schooling to differentiate and coordinate fixed and varying quantities in the context of linear approximation.

3.2 Teachers' variational reasoning

We highlight high school teachers' variational reasoning because of its importance to students' understanding of variation and rate of change in calculus. How teachers understand a mathematical idea shapes the ways they talk about and enact it, which in turn contributes to the mathematical understandings students construct (Copur-Gencturk, 2015; Thompson, 2013). One difficulty students experience with variation is to conflate change with amount. The following item suggests many U.S. high school teachers make the same conflation. It is unlikely these teachers design instruction with the intent of supporting their students in reasoning about change in progress and constrained variation—ideas foundational to productive meanings for constant rate of change.

3.2.1 Teachers' constructions of increments of change

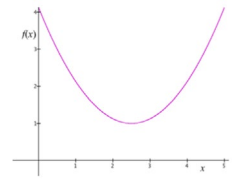
Reasoning productively about a quantity's variation requires one to construct two quantities: the quantity that is varying and the change in that quantity. Figure 2 shows an MMTsm item[5] that asked teachers to use the graph of $y=f(x)$ to describe the behavior of the *changes* in the dependent variable over an interval of the domain. 45.6% of U.S. high school teachers correctly identified that the values of the changes in the dependent variable were negative and only 15.2% of U.S high school teachers correctly identified that these negative changes were increasing.

Over 50% of U.S. high school teachers selected option (b), describing the *behavior* of *the function's value* (positive and decreasing) as opposed to describing the behavior

[5] This item was not scored for Korean teachers because of a mistranslation from English to Korean.

Fig. 2 MMTsm item that probes teachers' meanings for changes in quantities' values

The graph below is of a function *f* over the interval [0,5]



For small equal increases of the value of *x* starting at *x* = 1 and ending at *x* = 2, the corresponding changes in the value of *f* are…

a. positive and increasing
b. positive and decreasing
c. negative and increasing
d. negative and deceasing
e. I cannot tell

	a	b	c	d	e	total
US <calc	4	92	21	59	0	176
	2.3%	52.3%	11.9%	33.5%	0%	100%
US ≥calc	2	37	17	17	1	74
	2.7%	50.0%	23.0%	23.0%	1.4%	
Total US	6	129	38	76	1	250
	2.4%	51.6%	15.2%	30.4%	0.4%	

Fig. 3 MMTsm item "Mrs. Samber" probes teachers' meanings of a computed slope

Part I:

Mrs. Samber taught an introductory lesson on slope. In the lesson she divided 8.2 by 2.7 to calculate the slope of a line, getting 3.04.
Convey to Mrs. Samber's students what 3.04 means.

Part II (on the page following Part I):

A student explained the meaning of 3.04 by saying, "It means that every time *x* changes by 1, *y* changes by 3.04." Mrs. Samber asked, "What would 3.04 mean if *x* changes by something other than 1?"
What would be a good answer to Mrs. Samber's question?

of *changes in the function's value* (negative and increasing). Surprisingly, U.S. calculus teachers focused on the behavior of the dependent variable at essentially the same rate as U.S. pre-calculus teachers. Teachers' focus on the behavior of the function's values, as opposed to the behavior of *changes* in the function's values, suggests they are unlikely to provide opportunities for students to reason about constant rate of change as an invariant proportional relationship between *changes in quantities' values*.

		Chunky Thinking				
	Continuous Relationship	*Discrete Changes*	*Over and Up*	*Average Rate*	*Not categorized*	*total*
Korea	7	119	31	195	10	362
	1.9%	32.8%	8.6%	53.9%	2.8%	
US	3	41	78	28	4	154[7]
	1.9%	26.6%	50.6 %	18.2%	2.6%	

Fig. 4 Teachers' responses to Part I of "Mrs. Samber"

3.2.2 Teachers' variational reasoning in the context of slope

The task in Fig. 3 was designed to probe meanings of slope teachers intend their students to develop. Because textbooks commonly portray slope as a change in *y* for an integer change in *x*, we were interested in the extent to which teachers would display this as their default meaning (Part I). Part II, on a separate page, probed implications, to teachers, of their default meanings of slope by explicitly raising the possibility of a non-unit change in *x*, opening them to think about continuous changes in *x*. We include Korean data on this item to show that the meaning of measured slope among teachers in the U.S. is specific to the U.S., suggesting it is not necessary for teachers to think of slope in the ways common in the U.S.

Korean high school and middle school teachers responded similarly to both Parts I and II as did U.S. pre-calculus and calculus teachers. We therefore aggregated results by country to highlight relationships between teachers' responses by country. Teachers' responses to Part I were coded based on what the response suggested the teacher intended students understand about 3.04:

- 3.04 as implying a continuous relationship between Δy and Δx
- 3.04 as coordinating discrete changes in *x* and *y*: as *x* changes by 1 (or 2.7) *y* changes by 3.04 (or 8.2)—a chunky way of thinking
- 3.04 means "over (by 1) and up (by 3.04)"—a chunky way of thinking
- 3.04 is the "average rate" without saying what they meant by average rate
- The response could not be categorized such as "it is the value of *x* and *y*"

Figure 4 gives the level of teachers' responses organized by country.

In analyzing results from Part II, we were interested in whether the teacher could form an image of continuous change even though their default way of speaking was in terms of coordinating discrete changes. Of the 119 U.S. teachers who demonstrated a chunky way of thinking in

Part I (77% of all U.S. teachers) 60 conveyed a continuous relationship between Δy and Δx in Part II. In contrast 120 of the 150 Korean teachers who demonstrated a chunky way of thinking in Part I conveyed a continuous relationship between Δy and Δx in Part II. Moreover, of the 195 Korean teachers who, in Part I, said "average rate", 151 (77% of 195) responded to Part II in a way that suggested continuous variation.

In Part I both U.S. and Korean teachers expressed default ways of speaking about the meaning of a calculated slope that were based in images of discrete change (either chunky thinking or average rates of change). However, their responses in Part II suggest that U.S. and Korean teachers had different meanings behind their default ways of speaking. Most U.S. teachers (53%) really meant what they said, expressing discrete images in both Part I and Part II. Korean teachers had a way of speaking which suggests discrete changes, but for 78% of these teachers this discrete way of speaking was grounded in images of a proportional relationship between continuous changes.

3.3 Textbook's discussions of variation within constant rate of change

In this section we document the meanings four precalculus texts convey about constant rate of change and theorize how these conveyed meanings might support students' and teachers' development of continuous variational reasoning.

3.3.1 Textbook fails to promote images of variation

Carter et al. (2011) assume precalculus students are proficient with constant rate of change, saying: "In algebra, you learned that the slope between any two points on the graph of a linear function represents a *constant* rate of change" (p. 38; italics in original). The authors repeat this statement two other times (pp. 82, 757) prior to connecting average rate of change to instantaneous rate of change via limits (p. 82) and prior to connecting the slope of the tangent line to the instantaneous rate of change (p. 757). In all three instances the authors focus on the numerical computation of slope and do not provide contextual interpretations for slope.

In no instance do authors convey an image of variation. It seems they assume implicitly that students have compatible meanings for slope and constant rate of change. While an expert can conceptualize slope as a ratio that measures the constant rate of change of one quantity with respect to another, many students conceptualize slope as an index of steepness of a line and thus experience difficulty interpreting slope as a constant rate of change. For example, Lobato and Thanheiser (2001) report on an Algebra I student who "conceives of slope only as a number, not as a measure of the dripping rate of the faucet" (p. 162). Zaslavsky, Sela and Leron (2002) introduce the constructs of 'visual slope' and 'analytic slope' to explain students' difficulty coordinating their meanings for slope and constant rate of change. They explain that the 'visual slope' is a geometric entity and property of the line "but this [visual slope] has no relation whatsoever (except perhaps in sign) to the 'analytic slope', that is, the rate of change (or 'slope') of the function represented by this line" (p. 137). This suggests students who conceptualize slope as a 'visual slope'—the steepness of the line—will experience difficulty coordinating their understanding of slope and constant rate of change. This body of work suggests that using constant rate of change as a synonym for slope does not support students in constructing productive meanings for constant rate of change.

3.3.2 Textbook conveys discrete images of change

Sullivan and Sullivan (2013) and Connally et al. (2019) also define constant rate of change as slope and provide interpretations for the slope which convey discrete images of change. For example, Connally et al. (2013) provide the following interpretation of a slope of 0.05: "For each minute the phone is used, it costs an additional \$0.05" (Sect. 1.3). This explanation does not challenge students to account for partial minutes and thus supports students' natural inclinations to think of rate of change as involving quantities that change in discrete chunks. This chunky reasoning was supported by the examples and homework in this section: these problems asked students to compute the constant rate of change over an integer change in the independent variable.

Sullivan and Sullivan consistently convey discrete images of change when interpreting slope. For example, they say:

"The slope m of the line containing the points (1,2) and (5, − 3) may be computed as $m = \frac{-3-2}{5-1} = \frac{-5}{4} = -\frac{5}{4}$ or as $m = \frac{2-(-3)}{1-5} = \frac{5}{-4} = -\frac{5}{4}$. For every 4-unit change in x, y will change by – 5 units. That is, if x increases by 4 units, then y will decrease by 5 units" (Sullivan & Sullivan, 2013, p. 30).

Even if the computation results in a quotient that can be reduced and expressed in lowest terms (e.g., 2/6 = 1/3), a student with this interpretation will understand slope as a comparison between an integer change in the independent variable and an integer change in the dependent variable. A student with this meaning is not prepared to reason about slope when the independent variable changes by an amount different from the denominator (e.g., $m = -5/4$ and $\Delta x = 3$). Nor could students envision a non-integer change, such as $\Delta x = 0.36$, or continuous changes in the independent variable. In other words, the interpretation provided by Sullivan and Sullivan supports students in thinking of -5/4 as meaning "over 4 and down 5"; it does not support students in thinking about an invariant proportional relationship between *any* change in the independent variable and the corresponding change in the dependent variable.

3.3.3 Textbook conveys incoherent meaning for constant rate of change

An exhaustive search of OpenStax College (2014) for the phase "constant rate of change" produced four instances conveying a meaning for constant rate of change:

- "A constant rate of change, such as the growth cycle of this bamboo plant, is a linear function" (p. 125).
- "Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value" (p. 126).
- "When exploring linear growth, we observed a constant rate of change—a constant number by which the output increased for each unit increase in input" (p. 328).
- "The common difference (of the arithmetic sequence) is the constant rate of change, or the slope of the linear function" (p. 955).

From our perspective, these four statements convey incoherent meanings for constant rate of change. In particular, for a pre-calculus student without a robust meaning for limits, the second and third statements convey incompatible meanings for constant rate of change. More specifically, the second and fourth statements convey that the constant rate of change is associated with individual values of a function's independent variable. The third statement, however, conveys the constant rate of change is associated with a unit increase in the independent variable. While ideas of instantaneous rates of change based on understandings of limits allow mathematicians to see these statements as synonymous, without an understanding of limits students have no basis to interpret a rate of change without an interval over which two quantities change. As we illustrated above, chunky thinking can be problematic for students reasoning about invariant relationships in dynamic situations. Taken together, we conjecture that students who attempt to understand the OpenStax text will develop fragmented meanings for constant rate of change and little opportunity to reason about continuous variation.

The analysis above reveals few opportunities for students to engage in continuous variational reasoning in the context of reasoning about constant rate of change. The textbooks did not support students in anticipating that, as the quantities continue to accumulate, the relative size of the increments remains constant. Instead, the texts seem to support students in understanding constant rate of change as a visual property of the graph—an understanding that many researchers have documented to be problematic. Relative to variational reasoning, both the U.S. texts and U.S. teachers conveyed discrete ways of thinking about slope by comparing the change in the dependent variable for unit or integer changes in the independent variable.

4 Average rate of change

In thinking about students' preparation to understand derivatives, we identified three essential components to calculus students' meaning for average rate of change. Suppose we are considering the average rate of change of the function f on the interval from x_1 to x_2.

- The function's average rate of change over the interval $[x_1, x_2]$ is the constant rate of change that produces the same net change in $f(x)$ as x varies from x_1 to x_2. (Thompson, 1994)
- A differentiable function's behavior over a sufficiently small interval resembles a function with a constant rate of change over that interval. The constant rate of change is the function's average rate of change over that interval.
- The average rate of change over a sufficiently small interval quantifies a relationship between respective *changes* in values of two quantities.

A calculus student's ability to construct this meaning is largely dependent on the meaning for average rate of change they construct prior to calculus. At the precalculus level, a productive meaning for average rate of change would entail interpreting the average rate of change as the constant rate of change needed to produce the same net change in the dependent variable for a specified change in the independent variable. We would also expect to see average rates of change computed and interpreted over small intervals in order to describe the rate of change of $f(x)$ with respect to x. However, we question whether students have opportunities to construct these precalculus level understandings. Data from the MMTsm suggests many teachers did not have these meanings and the four texts we analyzed did not attend to these ideas. In this section we characterize U.S. calculus students' problematic meanings for average rate of change. We also characterize U.S. high school mathematics' teachers meanings for average rate of change to reveal similarities between teachers' and students' meanings. Finally, we document the meanings the four texts potentially convey about average rate of change and illustrate why this is problematic for the teaching and learning of calculus.

4.1 U.S. calculus students' meanings for average rate of change

Few calculus students responding to the C1CI demonstrated a productive meaning for average rate of change. Consider, for example, the rate of change item in Fig. 5.

Fig. 5 Calculus 1 Concept Inventory Item "San Diego to New York"

A car left from San Diego heading to New York. The average speed for the first 4 hours of the trip was 52 mph. In the next 0.003 hours, the car had an average speed of 71 mph. Which is the best estimate of how fast the car's distance from San Diego was changing at 4 hours after leaving San Diego?

a. 52 mph
b. 52.014 mph
c. 61.5 mph
d. 71 mph
e. Cannot be determined.

n=356 students	a	b	c	d	e
Count	92	66	92	51	55
percent	25.8	18.5	25.8	14.3	15.4

Fig. 6 MMTsm item "Meaning of Average" that probes teachers' meanings for average rate of change

Task: A car went from Phoenix to Tucson. The car's average speed was 62 mi/hr. What would you like your students to mean by the phrase, "the car's average speed was 62 mi/hr"?

	Constant Rate of Change	$\frac{distance}{time}$	*Distance traveled in one hour*	*Arithmetic mean or circular reasoning*	*Not categorized*	*total*
Korea	8	77	82	169	30	366
	2.19%	21.0%	22.4%	46.2%	8.2%	
US	0	23	36	43	19	121
	0.0%	19.0%	29.8%	35.5%	15.7%	
total	8	100	118	212	49	487
	1.6%	20.5%	24.2%	43.5%	10.1%	

To select answer choice (d) one must consider the car's number of miles from San Diego as a function of elapsed time since leaving—as say $f(x)$. The value of 71 is then computed by $(f(4.003) - f(4))/0.003$. Out of the 356 U.S. calculus students who completed this item only 14.3% chose 71 mph suggesting these students understood that the rate of change of one quantity with respect to another is a property of respective *changes* in quantities' values. Students' choices (a), (b), or (c) (25.8%, 18.5%, 25.8%, respectively) suggests students included 52 mph, the car's average speed over the first 4 h in their thinking. They seem to have believed that to measure the rate of change from San Diego they needed to account for all time since they left San Diego and thus did not conceptualize rate of change as relating corresponding increments in distance from San Diego and elapsed time from start. In other words, they did not express a way of thinking about the rate of change 4 h after leaving San Diego that did not entail the entire 4-h interval.

One obstacle students encounter in constructing productive meanings for average rate of change is their meaning of *average*. Dorko and Weber (2013) explain: "While average rate of change has a specific mathematical meaning in calculus, the word average may have lexical ambiguity because of its use in statistics and everyday language" (p. 386). While these students could correctly identify the algebraic computation for computing an average rate of change, their interpretations of the computed value were unproductive for constructing a quantitative interpretation of average rate of change as an estimate for the rate of change of $f(x)$ with respect to x. Although there is little research on students' meanings for average rate of change, existing research (Ärlebäck et al., 2013) does suggest U.S. students' meanings for average rate of change are incompatible with the idea of derivative as a linear approximation of a function's behavior over an interval.

4.2 U.S. high school teachers' meanings for average rate of change

Like calculus students, many U.S. high school teachers' meanings for average rate of change are based on their meanings for average as an arithmetic mean. Figure 6 shows an MMTsm item that asked teachers to articulate the meaning they want students to hold for "the car's average speed was 62 mi/hr". This item was administered to 487 teachers (366 Korean, 121 U.S.).

Teacher responses were categorized into five levels:

- The teacher conveyed that average rate of change is the constant speed a car would travel in order to travel the same distance in the same amount of time.
- The teacher conveyed an algebraic equivalent of distance/time.
- The teacher conveyed that the average speed tells the distance traveled in one hour.
- The teacher conveyed that average speed is an arithmetic mean, or provided circular reasoning ("The car *averaged* 62 miles for each hour.")
- The response could not be categorized (e.g. "the speed as a rate involving mi/hr").

A meaning consistent with arithmetic mean was the most common response amongst all teachers (43.5%) and also the most common within each country (35.5% U.S. teachers and 46.2% of Korean teachers). This response is consistent with many calculus student meanings for average rate of change (Dorko & Weber, 2013). The second most common response was the chunky way of thinking that the average speed gives the distance traveled in one hour. In fact only 1.6% of all teachers expressed a meaning for average rate of change that aligns with the productive meaning we identified for learning calculus.

Since a teacher's conveyed meanings in the classroom are likely to be more calculational and less conceptual than the meanings the teacher conveys independent of their instruction (Nagle & Moore-Russo, 2013), we conjecture it is unlikely that U.S. teachers convey meanings for average rate of change that support students in coordinating their meanings for average rate of change and constant rate of change.

Other data suggests we must interpret the Korean data on Meaning of Average cautiously. One question involved weighted averages: A car traveled one way with an average speed of 40 mi/hr and returned by the same route. Its overall average speed was 60 mi/hr. What was its average speed back? Forty-seven percent (47%) of U.S. teachers answered 80 mi/hr, the solution to $(40 + x)/2 = 60$. Only 7% of Korean high school teachers solved an arithmetic mean and 89.4% understood the problem as traveling the same distance twice at constant speeds and twice the one-way distance at a constant speed.

4.3 Textbook analysis of potential conveyed meanings for average rate of change

We analyzed four precalculus texts' potential conveyed meanings for average rate of change. In this section we document the meanings conveyed by each text and discuss potential implications for both teachers' instruction and students' learning.

4.3.1 Text conveys geometric meaning for average rate of change

Sullivan & Sullivan (2013) and Carter et al. (2011) convey geometric meanings for average rate of change. Sullivan and Sullivan (2013) explain:

> "The average rate of change has an important geometric interpretation. The average rate of change of a function *f* from *a* to *b* equals the slope of the secant line containing the two points $(a, f(a))$ and $(b, f(b))$ on its graph" (pp. 87–88).

Carter et al. (2011) convey a similar meaning: "The average rate of change between any two points on the graph of *f* is the slope of the line through those points" (p. 38). They provide no alternate meaning for rate of change nor do they support students in connecting their meaning for slope with their meaning of average rate of change of one quantity with respect to another.

4.3.1.1 Implications for students' learning and teachers' instruction of average rate of change If learners understand average rate of change as stated in these textbooks they will have understood average rate of change as a property of a line connecting two points on the function's graph. Numerous researchers illustrated students' difficulty coordinating meanings for slope and rate of change (Lobato & Thanheiser, 2001; Teuscher et al., 2010). This suggests that the meaning conveyed by these texts does not support students in constructing a coherent meaning for average rate of change.

While these texts define average rate of change as slope, the worked examples and homework problems focus on computing the average rate of change with an algebraic rule. We found no problems asking students to provide a contextual interpretation of average rate of change. Additionally, textbook problems consistently ask students to compute the average rate of change over intervals of integer width (e.g., $\Delta x = 3$). This type of task can be easily assimilated into teachers' and students' chunky variation schemes.

In summary, these two texts convey a meaning for average rate of change that focuses on geometric properties of a line connecting two points on a curve and a procedure for computing a number called "average rate". This meaning does not support students in interpreting average rate of change as the constant rate of change needed to produce the same net change in the dependent variable over a specified interval in the independent variable. Thus, we claim that these texts do not provide opportunities for teachers, and thus students, to construct meanings for average rate of change that will be productive for understanding rate of change in calculus.

4.3.2 Text conveys average rate of change is an arithmetic mean or "smoothed out" change

Both Connally et al. (2019) and OpenStax College (2017) convey meanings for average rate of change that leverage a reader's colloquial meaning for "average". Connally et al. (2019) state:

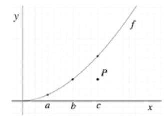
> "The average rate of change of the function $Q = f(t)$ over an interval tells us how much *Q* changes, on average, for each unit change in *t* within that interval. On some parts of the interval, *Q* may be changing rapidly, while on other parts *Q* may be changing slowly. The average rate of change evens out these variations." (Sect. 1.2).

Here, the authors do not say what they mean by "average". Instead, they state that the "average rate of change evens out these variations". This language supports students in thinking about a leveling out perspective consistent with the arithmetic mean.

Fig. 7 Calculus 1 Concept Inventory item that assesses students' meaning for function notation

Consider the graph displaying the function *f* and point *P* in the *xy*-plane.
Express the coordinates of point *P*.

a. (c, P)
b. $(c, f(b))$
c. $(c, f(c))$
d. $(P, f(b))$
e. $(P, f(c))$



n=224 students	a	b	c	d	e
count	43	115	58	2	6
percent	19.2	51.3	25.9	0.89	2.68

OpenStax College (2017) defines average rate of change computationally, stating "the average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values" (p. 38). They provide examples of this computation:

"Over 7 years, the average rate of change was. $\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196$ dollars per year.
On average, the price of gas increased by about 19.6 cents each year." (OpenStax College, p. 37).

This interpretation leverages a colloquial use of the word average as meaning "smoothed", and since no meaning is given for the phrase "on average", this interpretation of 0.196 provides a circular meaning of average rate of change.

4.3.2.1 Implications on students' learning and teachers' instruction of average rate of change The notion of average rate of change as an arithmetic mean or a smoothed-out change is consistent with the meanings demonstrated in earlier examples by both U.S. calculus students and U.S. high school mathematics teachers. However, these meanings are problematic for understanding calculus. These texts support interpretations for average rate of change that are not quantitative. Additionally, their emphasis on large intervals does not support thinking about average rate of change over very small intervals, which gives meaning to the rate of change of $f(x)$ with respect to x at a value of x.

5 Representational Use of Function Notation

Function notation is ubiquitous in calculus because of its representational power to define a relationship in a concise way. We identified two essential components to students' meanings for function notation:

- The notation $f(x)$ is shorthand way to simultaneously describe the relationship between a value in the domain, x, and the corresponding value in the range, $f(x)$, as well as the name of the invariant relationship between the domain and range.
- "$f(x)$" can be used representationally—it can be used to describe relationships without specifying or repeating a rule of association.

In this section we characterize U.S. calculus students' meanings for function notation in order to highlight students' lack of preparation to use function notation representationally. We then characterize U.S. high school mathematics' teachers' meanings for function notation to conjecture implications for the meanings they convey in their classroom instruction. Finally, we examine the ways in which four pre-calculus texts support students in constructing this two part meaning for function notation and illustrate how students and teachers likely internalize the textbooks' presentations in ways problematic for teaching and learning calculus.

5.1 Students' meanings for function notation

To illustrate calculus students' difficulty with function notation consider students' responses to the C1CI item in Fig. 7. While 51.3% of calculus students successfully completed the item, students experienced difficulty in both using function notation representationally and seeing function notation as more than a four-character name for a computational rule. More specifically, 25.9% of students selected answer choice (c) where the x-coordinate matches the independent variable to the function. This suggests these students think of function notation as an idiom (the letter inside the parentheses is part of a function name). On the other hand, 19.2% of students selected the answer choice (a). In addition to conflating the name of the point and the y-coordinate of the point, students who selected answer choice (a) rejected all answers that involved function notation. These students seem to experience difficulty using function notation representationally and we conjecture these students might see function notation as a command to evaluate as opposed to representing a value of a dependent variable for a specified value of the independent variable.

5.2 Teachers' meanings for function notation

We see teachers' meanings as an indicator of the meanings they convey in their instruction and thus the opportunities they create for students to construct productive meanings for function notation. In this section we examine U.S. teachers'

Fig. 8 MMTsm item to probe teachers' meanings for function notation

Here are two function definitions:

$w(t) = \sin(t-1)$ if $t \geq 1$

$q(s) = \sqrt{s^2 - s^3}$ if $0 \leq s < 1$

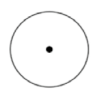
Here is a third function c, defined in two parts, whose definition refers to w and q. Place the correct letter in each blank so that the function c is properly defined.

$$c(v) = \begin{cases} q(\_\_\_\_) \text{ if } 0 \leq \_\_\_\_ < 1 \\ w(\_\_\_\_) \text{ if } \_\_\_\_ \geq 1 \end{cases}$$

	v through-out	*Mix of s,t,v*	*s,t*	*Not categorized*[8]	*Total*
Korea MS	65	0	6	31	102
	63.7%	0.00%	5.88%	30.4%	
Korea HS	203	1	14	46	264
	76.9%	0.38%	5.30%	17.4%	
US<calc	53	7	74	45	179
	29.6%	3.91%	41.30%	25.1%	
US≥calc	32	5	25	12	74
	43.2%	6.76%	33.8%	16.2%	
total	353	13	119	134	619
	57.0%	2.10%	19.2%	21.6%	

Fig. 9 MMTsm item that probes teacher's understanding of the representational power of function notation

Hari dropped a rock into a pond creating a circular ripple that spread outward. The ripple's radius increases at a non-constant speed with the number of seconds since Hari dropped the rock. Use function notation to express the area inside the ripple as a function of elapsed time.



	Function notation represent-ationally	*Function notation left hand side only*	*Inconsistent use of variables*	*Not categorized*	*Total*
Korea MS	39	16	5	42	102
	38.2%	15.7%	4.90%	41.2%	
Korea HS	163	20	10	71	264
	61.7%	7.58%	3.79%	26.89%	
US<calc	31	58	24	54	167
	18.6%	34.7%	14.4%	32.3%	
US≥calc	25	27	7	15	74
	33.8%	36.5%	9.46%	20.3%	
total	258	121	46	182	607
	42.5%	19.9%	7.58%	30.0%	

meanings for function notation. We present data from two MMTsm items to illustrate teachers' difficulty both using function notation representationally and understanding function notation as more than an idiom.

5.2.1 Teachers' understanding of function notation idiomatically

The item shown in Fig. 8 was designed to reveal the extent to which teachers understand function notation as a four-character idiom, where the letter inside the parentheses is part of the function name. This item was administered to 619 teachers (253 U.S. and 366 Korean). Teacher responses were categorized based on the variables used to complete the definition of c. A teacher who uses the variables t and s as arguments of q and w in the definition of c likely sees function notation idiomatically—as a four-character name for the rule on the right side. We include Korean data on this item to show that idiomatic meanings for function notation expressed by U.S. teachers are not shared with Korean teachers.

While over 73% of Korean teachers used v throughout the function definition for c, only 33% of U.S. teachers did. U.S. high school teachers who had never taught calculus were most likely to demonstrate an idiomatic understanding of function notation. These teachers saw the function names as "$w(t)$" and "$q(s)$" as opposed to "w" and "q". We emphasize that while 39% of all U.S. teachers expressly showed an idiomatic understanding of function notation, only 5% of Korean teachers did. This suggests thinking of function notation idiomatically is a way of thinking that is culturally embedded in the U.S. educational system; it is not necessary for teachers to think about function notation in this way.

5.2.2 Teachers' representational use of function notation

The item shown in Fig. 9 was designed to reveal the extent to which teachers use function notation representationally—meaning a teacher would take "$f(x)$", by itself, as naming a relationship ("f") between values of one quantity (labeled "x") and values of another quantity (labeled "$f(x)$") without the need to write a defining formula.

We considered teachers as having used function notation representationally only if they used it to represent the radius length as a function of time—if they used function notation on the right side (as in $A(t) = \pi(r(t))^2$).We did not count using function notation only on the left side as evidence of using function notation representationally since this response does not differentiate between a teacher who understands $f(t)$ as the name of a rule as opposed to representing a relationship between values of two quantities. Teachers who did not use or attempt to use function notation, such as writing $A = \pi(r^2)t$, were classified as "not categorized".

Responses from 607 teachers (366 Korean and 241 U.S.) are shown in Fig. 9. The percentage of Korean teachers who used function notation representationally was nearly twice as large as the percentage of U.S. teachers. U.S. teachers used function notation on the left side only at three times the rate of Korean teachers. This suggests that many U.S. teachers interpreted the prompt "use function notation" to mean "write $f(x)$ instead of y". The difference in U.S. and Korean teachers' responses suggests there are aspects of the U.S. educational system that likely contribute to teachers' limited use of function notation representationally. U.S. teachers' responses to the items in Fig. 8 and Fig. 9 suggest it is unlikely that they support students in using function notation representationally.

5.3 Textbook analysis of potential conveyed meanings for function notation

Since teachers do not plan their instruction in isolation, it is possible that the curricular materials could support teachers in conveying a meaning other than the one they demonstrated in these items. In this section we document the meanings for function notation conveyed by four precalculus texts and discuss potential implications for both teachers' instruction and students' learning.

5.3.1 Text conveys function notation as a command to operate

Carter et al. (2011) conveyed a meaning consistent with that of U.S. high school teachers: function notation is an idiom—a four character symbol to name the function. The text states: "In function notation, the symbol $f(x)$ is read f of x and interpreted as the value of the function f at x. Because $f(x)$ corresponds to the y-value of f for a given x-value, you can write $y = f(x)$" (p. 7). This excerpt supports the reader in understanding $f(x)$ as a single symbol as opposed to a relationship between three mathematical entities. To understand the extent to which this text supports representational use of function notation we examined the homework set associated with the function notation lesson and found 27 problems that asked students to "find each function value" given the algebraic presentation of the function. We found no instances of the text using function notation in the absence of a defining rule. There was no evidence that the text supports students in thinking about function notation as something other than an alternative to y that goes on the left side of the function definition.

Sullivan and Sullivan (2013) also focus on function notation as a command to operate. The authors do not attempt to provide a conceptual meaning for function notation and instead introduce function notation (p. 61) under the title "find the value of a function" (p.61) and the corresponding homework set includes 45 problems that ask students to evaluate an algebraic rule at specific values of the independent variable.

5.3.1.1 Implications for students' learning and teachers' instruction of function notation To illustrate the problematic nature of understanding function notation as a command to operate, consider the symbolic presentation of the difference quotient: $\frac{f(x+h)-f(x)}{h}$. For a student who understands function notation as stating a relationship between mathematical entities, this expression represents the relative size of a change in the dependent variable with respect to a change in the independent variable. However, for a student who understands function notation as a command to operate, this notation means to compute three values in a specific order: $f(x+h)$, $f(x)$, and then: $\frac{f(x+h)-f(x)}{h}$. In fact, Carter et al. (2011) explicitly supports this computational understanding of the difference quotient in their homework exercises by having students evaluate $f(a)$, $f(a+h)$ and $\frac{f(a+h)-f(a)}{h}$ for 12 function definitions (p. 11).

We see it as essential for students' learning of calculus that students understand function notation as more than a command to operate. If students conceptualize the difference quotient as an expression of relative size then they have an opportunity to coordinate their meaning for the difference quotient with their meanings for constant and average rate of change. This coordination of meanings would support the student in understanding the derivative as resulting from a linear approximation of the function's behavior over an interval.

Since teachers also engage with the text, perhaps more so than students, we also focused on how teachers might implement their meaning for function notation in instruction. Since the texts focus on function notation as meaning a computation, we anticipate that these texts do not provide opportunities for teachers, and thus students, to construct more robust and productive meanings for function notation that will be productive for understanding many ideas in calculus.

5.3.2 Text conveys relational meaning for function notation

Both OpenStax College (2017) and Connally et al. (2019) differentiate between the function name, independent variable ("input"), and dependent variable ("output") when introducing function notation. For example, OpenStax College (2017) states:

> "To represent 'height is a function of age,' we start by identifying the descriptive variables h for height and a for age.

h is *f* of *a* We name the function *f*; height is a function of age.

$h=f(a)$ We use parentheses to indicate the function input.

$f(a)$ We name the function *f*; the expression is read as '*f* of *a*'" (p. 4).

However, after introducing this meaning for function notation the authors focus on the skill of interpreting function notation as a command to evaluate. The homework section associated with this lesson includes 27 problems on function notation and 23 of these problems ask students to evaluate $f(x)$ for a specified value of x given a function presented graphically or algebraically. Only four problems ask students to interpret the meaning of, for example $f(2)=300$, in a contextual situation.

Similarly, Connally et al. (2019) states:

"To indicate that a quantity Q is a function of a quantity t, we abbreviate using function notation and write $Q=f(t)$. Thus, applying the rule f to the input value, t, gives the output value, $f(t)$, which is a value of Q" (Sect. 1.1).

While the authors include some homework items that focus on interpreting function notation in context over 70% of the function notation problems focus on the skill of evaluating a function given a graphical or algebraic presentation of the function.

5.3.2.1 Implications for students' learning and teachers' instruction of function notation Given that a large percentage of textbooks' function notation exercises focus on using a function's rule to calculate a value, we anticipate teachers and students who already see mathematics as about calculations are reinforced by the textbooks' heavy focus on calculating "outputs" from "inputs". Thus, it seems unlikely that these texts provide sufficient motivation for teachers to highlight representational power of function notation in ways that will be productive for students in calculus.

6 Conclusion

The analyses presented here suggest that U.S. students have limited opportunities to construct mathematical meanings productive for understanding calculus. We argue there is evidence of a large disconnect between meanings conveyed by textbooks and held by teachers and meanings that would be productive for students' understanding of major ideas in calculus.

We documented similarities among U.S. calculus students' meanings, U.S. high school teachers' meanings, and textbooks' presentations. These similarities highlight an ecology of unproductive meanings in U.S. mathematics education. The data suggests U.S. teachers and students share many meanings for slope, average rate of change, and function notation and these meanings are unproductive for understanding calculus. As stressed by Thompson and colleagues (Thompson, 2013, 2016; Thompson & Milner, 2019), it seems U.S. teachers' university mathematics and professional training have little influence on the mathematical meanings they developed in high school. A majority of U.S. teachers in our sample appear to have experienced a continuity of meanings as they progressed from their high school education through their university experiences and back into the high school classroom as teachers.

That teachers experience this continuity of meanings suggests that attempting to support students in reconstructing their meanings in calculus is too late. Students are able to assimilate much of calculus with their existing meanings – meanings that worked in school because they were consistent with textbook authors' and teachers' meanings. Thus, we conjecture that if we limit our conception of calculus reform to the teaching and learning of calculus we are too late. Instead, as we consider calculus reform we must give serious attention to middle school and high school curricula as well as professional development opportunities for middle school and high school teachers.

We are optimistic that such reforms are possible in the U.S. since U.S. and Korean teachers demonstrated significantly different meanings for slope, average rate of change, and function notation. We take this as evidence that the meanings prevalent among U.S. high school teachers are culturally embedded in the U.S. educational system; teachers do not experience occasions where they are prompted to rethink their meanings for slope, function notation, and average rate of change. This suggests that calculus reform efforts in the U.S. must support both school teachers and school students to overcome what Artigue (1992) called *obstacles of a didactic nature*: "obstacles linked to the choices and characteristics of the educational system" (p. 110). In other words, the system of meanings embedded in U.S. school mathematics is culturally embedded and these meanings are deeply rooted as teachers carry their meanings with them through university mathematics. Thus, it will take carefully designed experiences to support students and teachers in reconstructing their mathematical meanings.

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