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Figurative and Operative Imagery: Essential Aspects of Reflection in the Development of Schemes and Meanings

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Our purpose in this chapter is to clarify the role of imagery in students' mathematical learning and reasoning. In pursuit of this goal, we address interdependencies among imagery, reflection, scheme, and meaning as theoretical constructs and illustrate their interdependence by examples. Our motive for this expanded charge is that while the notion of schemes and scheme development is sometimes discussed in studies of students' mathematical learning, the role of imagery in that process is often neglected, yet it is central to the development of productive mathematical meanings. Often people doing mathematics education research pay insufficient attention to the very nuanced ways in which people understand a situation – what their image of the situation *is*. As a counterpoint, Thompson (1996) spoke of students' images of a solution to an algebraic equation. "Their image of solving equations often is of activity that ends with something like '$x = 2$.' So, when they end with something like '$2 = 2$' or '$x = x$' they conclude without hesitation that they must have done something wrong" (Thompson, 1996, p. 274).

Imagery

We state at the outset that by "imagery" we mean far more than "visualization". Imagining something one has seen, or producing something as if seen, indeed falls within the category of imagery. But recalling something one has said, thought, done, or felt also falls within the category of imagery. In an act of re-presentation[1], a person recalls having an experience. The experience might have been having a thought or feeling, having done something in some context, having interpreted something in a particular way, recalling having recalled something, or other acts of recollection. *In general, by "image" we mean the re-presentation of experience*. We use "imagery" to refer to images collectively.

While we say imagery exceeds visualization, we acknowledge that "visualization" has been used sometimes with broad meaning, including what we call "creative imagery". Creative imagery is the construction of an image one has not experienced but is grounded in experience. One example is Galileo's thought experiment of repeatedly dropping two iron balls of different sizes joined by ever thinner filaments to infer all masses in a vacuum fall identically (Clement, 2018; Miller, 1996). Another is Einstein's thought experiment of a person falling in an elevator to infer gravity is akin to acceleration within an inertial frame of reference (Miller, 1996). Neither Galileo nor Einstein had experiences that were recalled as such. Rather, they assembled images from experience in novel ways.

Experiences are never recalled veridically. Our distinction between creative imagery and recalled experience is therefore muddied by the fact that initial experiences and recalled experiences are constructed using schemes one currently has, which can change in the interim between initial experience and recollection. For example, Piaget (1968a) presented experimental

[1] Glasersfeld (1991) distinguished between "to represent" and "to re-present" this way. To represent an experience is to take one thing as standing for another. To re-present an experience is to bring to mind a record of the experience.

evidence of children's memories of a visual image improving over time. The improvement, Piaget claimed, was due to students having developed more coherent schemes by which they remembered (re-constructed) their original perception. The experiment had first-grade children look for a short period of time at a display of 10 bars aligned vertically in ascending length (Figure 1a). Piaget and colleagues asked children to draw what they remembered seeing one week later and six months later. Children's drawings one week later tended to show local groups of ascension, but not uniform increase in length (Figure 1b). Their drawings six months later, with no mention of the initial episode having been made, were considerably improved—many more children's drawings resembled the figure presented six months earlier. Piaget explained that, in the intervening six months, children had developed schemes for order that included transitivity.[2] In other words, their recollections improved because the schemes by which they re-presented their initial experience of the bars had become more advanced.

Figure 1. (a) The figure as presented to children. (b) Type of drawing common among children one week later.

Another source for our position regarding imagery is our appreciation of action as the foundation of Piaget's genetic epistemology. To Piaget, actions were cognitive activity that might (or not) be expressed in behavior. "One can say that all action—that is to say all movement, all thought, or all emotion—responds to a need" (Piaget, 1968b, p. 6). Actions are the foundation of experience, so the phrase "recall an experience", to be in line with Piaget's genetic epistemology, forces us to include recalled movement, thought, or emotion as imagery.

We also avoid tacitly equating "action" with observable behavior, which happens often when people use the phrase "reflection on activity". Powers (1973a) addressed this when he explained that living organisms cannot organize themselves around how their behavior is perceived by others, but instead according to effects of the organism's actions *as discerned by the organism* (Powers, 1973b, p. 418).

Powers' message for us is that we cannot take students' behavior at face value—it is but an *expression* of their actions—where we use "action" as Piaget intended. This is at the root of Steffe and Thompson's (2000) distinction between students' mathematics (their mathematical realities) and mathematics of students—an observer's understanding of how students might be thinking to behave as they did or might do. Students' mathematics is the dark matter and dark energy of mathematics education. Devising a viable mathematics of students is therefore a core

[2] One way Piaget inferred that children's ordering schemes included transitivity was to have them insert a stick into a series they had constructed, ordered left-to-right ascending by height. If the child found the first stick taller than the one they were to insert and inserted the new stick to the left of the first-taller stick, the child exhibited transitivity. They knew all sticks to the left of that position were necessarily shorter than the one they inserted and all sticks to the right were necessarily taller than the one they inserted (Piaget, 1965).

mission of mathematics education research. Understanding students' imagery while engaged in instruction and while recalling their experiences outside of instruction is central to that mission.

On a related note, it is common for instructors and instructional designers to include visual presentations to supplement prose or have students engage in some form of activity. The thought is that activities or visual presentations help students understand the goal of instruction. Our emphasis on imagery as recalled experience is important for putting these efforts in proper light. By taking seriously the stance that imagery is rooted in recalled experience, we are forced to be mindful that students recall *their* experience of activities or visual presentations; they cannot recall what we understand as having been presented to them. Students' experience of an activity or presentation is conditioned by the schemes through which they understood it and recall it. Since learners are, by definition, new to the ideas being taught, their experience of activities or presentations will be substantially different from the originator's intentions.

Imagery, Schemes, and Meanings

Having spoken of schemes in relation to images repeatedly we feel obliged to say what we mean by a scheme and speak to the role of imagery in scheme development.

Piaget's use of "scheme" was quite utilitarian. It allowed him to speak of mental organizations that supported flexible reasoning across seemingly disparate situations. Montangero and Maurice-Naville (1997, p. 155) presented a compendium of six ways Piaget used "scheme". They ranged from "[Schemes are] organized totalities whose internal elements are mutually implied" (Piaget, 1936, p. 445) to "A scheme is the structure or the organization of actions which is transferred or generalized when this action is repeated in similar or analogous circumstances" (Piaget & Inhelder, 1966, p. 11, footnote not translated in the English version).

Piaget's statement, "organized totalities whose internal elements are mutually implied" derives from his stance that actions are implicative. When someone engages in an action, it creates a new experiential context that could be the trigger for other actions. An action in a context implies other actions. Thus, "… elements are mutually implied" means that a scheme constitutes a locally closed system in which any of its assumed conditions could activate the scheme in its totality. An example is when someone has a mature constant speed scheme. They are aware that a time and a speed are involved when they know an object traveled some distance, that a distance and a speed are involved when they know it traveled some time, and that a time and a distance are involved when they know it traveled at some speed. "Mutual implications" of time, distance, and speed in a person's constant speed scheme is evidenced when they understand that there are implied distances in "A car drove 60 mi/hr for 3 hours and then 40 mi/hr for 5 hours. What was the car's average speed?"

Likewise, the statement, "… organization of actions which is transferred when this action is repeated …" was Piaget's way to account for how a scheme (as an organization of actions) can be activated in seemingly different circumstances. An example of this is when someone uses their constant speed scheme (constant rate of change of distance with respect to time) to understand constant rate of change of volume of a fluid in a container with respect to its height in the container.

Cobb and Glasersfeld (1984) and Glasersfeld (1995, 2001) proposed that, to Piaget, a scheme was a three-part mental structure: an internal condition that would trigger a scheme, an action or system of actions, and an anticipation of what the action would produce. Steffe (2010) expanded Cobb and Glasersfeld's definition to include a scheme's goal (Figure 2). Steffe's motivation for including a generated goal in his definition of scheme was

> The Generated Goal can be regarded as the apex of a tetrahedron. The vertices of the base of the tetrahedron constitute the three components of a scheme. The double arrows linking the three components are to be interpreted as meaning that it is possible for any one of them to be in some way compared or related to either of the two others. The dashed arrow is to be interpreted as an expectation of the scheme's result. (Steffe, 2010, p. 23)

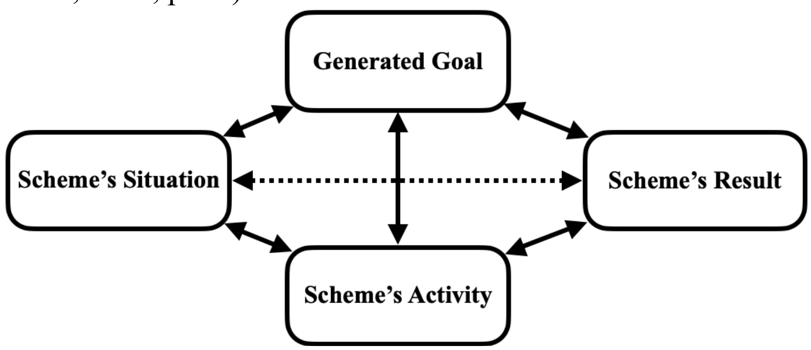


Figure 2. Steffe's (2010, p. 23) characterization of schemes.

A main feature Piaget communicated in his characterizations of scheme is they are cognitive structures that can express themselves in action or behavior. Cobb's, Glasersfeld's, and Steffe's definitions address Piaget's intention well. However, another important aspect of schemes' functioning to Piaget, not captured by those definitions, is that people employ schemes to *comprehend* situations, to *give meaning* to situations (Piaget & Garcia, 1991). To us, a definition of scheme must support interpretations of a person's attempt to *understand* situations in terms of meanings they have for constituent elements and relationships among them. Thompson and Saldanha (2003), for example, spoke of a mature fraction scheme as a network of relationships among schemes for measure, multiplication, division, and relative size, each of which entails aspects of proportionality, as a means to *understand* the broad array of situations one can understand as involving fractions.

To this end, Thompson, Carlson, Byerley, and Hatfield (2014) expanded Steffe's, Cobb's, and Glasersfeld's definitions of scheme.

> A scheme as an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity.[3] (Thompson et al., 2014, p. 11)

We point out that, in this definition, "entry points" often are images of contexts and anticipated outcomes are most definitely images. Unlike Steffe's definition, and like Glasersfeld's and Cobb's, Thompson et al.'s (2014) definition of scheme does not include goals. In a larger context, though, it is like Steffe's definition in that a person can generate a goal in the

[3] This definition is recursive, not circular. A scheme might recruit other schemes when activated.

activity of implementing a scheme or might activate a scheme because a generated goal matches a scheme's anticipated outcome.

Thompson et al.'s definition of scheme is at the root of Thompson and Harel's (Harel, 2021; Thompson et al., 2014) attempt to give coherence among meanings for understandings, meanings, and ways of thinking (Figure 3).

Construct	Definition
Understanding (in the moment)	Cognitive state resulting from an assimilation
Meaning (in the moment)	The space of implications existing at the moment of understanding
Understanding (stable)	Cognitive state resulting from an assimilation to a scheme
Meaning (stable)	The space of implications that results from having assimilated to a scheme. The scheme is the meaning. What Harel previously called Way of Understanding
Way of Thinking	Habitual anticipation of specific meanings or ways of thinking in reasoning

Figure 3. Meanings of "understanding", "meaning", and "way of thinking" (Thompson et al., 2014, p. 13)

This system for the use of "understanding", "meaning" and "way of thinking" aligns with Harel's and Thompson's quest to decouple "understand" and "understand correctly". They do this by resting their system on the idea of assimilation. They rely on Piaget's characterization of assimilation as, in effect, giving meaning.

> Assimilating an object to a scheme involves giving one or several meanings to this object, and it is this conferring of meanings that implies a more or less complete system of inferences, even when it is simply a question of verifying a fact. In short, we could say that an assimilation is an association accompanied by inference. (Johnckheere et al., 1958, p. 59) as translated by (Montangero & Maurice-Naville, 1997, p. 72)

Figure 3's first entry (*Understanding in the moment*) describes a person who has an understanding of something said, written, or done in the moment of understanding it. Technically, all understandings are understandings-in-the-moment. Some understandings might be a state that the person has struggled to attain at that moment through functional accommodations to existing schemes (Steffe, 1991), and is easily lost once the person's attention moves on. This type of understanding is typical when a person is making sense of an idea for the first time.

The meaning of an understanding is the space of implications that the current understanding mobilizes—actions or schemes that the current understanding implies, that the current understanding brings to mind with little effort. An understanding is *stable* if it is the result of an assimilation to a scheme. A scheme, being stable, then constitutes the space of implications resulting from the person's assimilation of anything to it. The scheme is the meaning of the understanding that the person constructs in the moment. As an aside, schemes provide the "way" in Harel's "way of understanding". Finally, Harel and Thompson characterize "way of thinking" as when a person has developed a pattern for utilizing specific meanings or ways of thinking in reasoning about particular ideas or situations.

Images and Schemes

Images enter our definition of scheme, and therefore meaning, in three ways: Images can be contexts that activate a scheme, they can be waypoints in a scheme's activity, and they can be anticipations of a scheme's result within the context of the scheme's activation.

Thompson (1994a, 1996) explained the ways in which the notion of image is intertwined with Piaget's concept of scheme. He pointed out three levels of imagery in Piaget's work. The first level of imagery is when a child engages in deferred imitation. Deferred imitation is when a child enacts the imitated behavior to assimilate (understand) it. The second level of imagery (figurative) is an image of an initial state and actions that are associated with it, but the actions and image are tied tightly—such as a student accustomed to drawing altitudes in a triangle with all angles less than 90 degrees being confused when asked to draw all altitudes in a triangle with one angle greater than 90 degrees.

The third level of imagery is what Piaget called "operative".

> [This is an image] that is dynamic and mobile in character … entirely concerned with the transformations of the object. … [The image] is no longer a necessary aid to thought, for the actions which it represents are henceforth independent of their physical realization and consist only of transformations grouped in free, transitive and reversible combination … In short, the image is now no more than a symbol of an operation, an imitative symbol like its precursors, but one which is constantly outpaced by the dynamics of the transformations. Its sole function is now to express certain momentary states occurring in the course of such transformations by way of references or symbolic allusions. (Piaget, 1967, p. 296)

The three levels of imagery do not differ in type. They are all re-presented experiences. Instead, the levels are differentiated by the ways images are integrated into individuals' reasoning and the types of reasoning into which they are integrated. We unpack the three levels in the following paragraphs.

First-level imagery (deferred imitation)

Piaget's examples of deferred imitation are often about infants or toddlers mimicking their experience of others (e.g., opening their mouth to mimic their mother) or engaging in play to mimic social interactions. But deferred imitation is a broader phenomenon. In psychotherapy it is called re-experiencing (Joseph & Williams, 2005)—the replaying of a traumatic event to assimilate (understand) it by either adjusting one's understanding of a world in which such a thing could occur or adjusting one's understanding of one's place in the world that makes the event sensible. The role of deferred imitation in mathematics learning is unclear to us. This is not to claim it is unimportant. We just say we are unclear as to its role.

Second-level (figurative) imagery

Second-level (figurative) imagery aligns with what Davis, Jockush, and McKnight (1978) called "visually-moderated sequences"—activity sequences triggered by a current visual or cognitive state (e.g., seeing an equation and thinking to add something to both sides) that end in a new visual or cognitive state (e.g., an expression with no constant terms) that triggers another activity sequence (e.g., dividing both sides by the same number). Each activity sequence results in a new state, but upon arriving at a new state it is simply the end of the activity sequence. It is not a goal toward which the student strove and thus the end state is not an anticipated result of the activity, and the actual result does not, to the student, entail an image of the activity leading to it.

Frank (2017) provided an excellent example of a student whose activity ends with a result that, for her, did not entail an image of the activity that led to it. The student (Ali) constructed a graph to represent two quantities' values as they varied simultaneously in an animation of the quantities and their magnitudes. Ali ended with what Frank considered an appropriate graph. But the graph, to Ali, did not entail the covariational reasoning in which she engaged while making it. Ali spoke of the graph which she had just made as if it was a static shape, as if a piece of wire.

> At the outset of this study, I thought that if Ali made a graph by simultaneously tracking two magnitudes, then she engaged in emergent shape thinking. I had not considered that Ali's meaning for her sketched graph might not reflect the thinking she engaged in to make the graph. (Frank, 2017, p. 193)

Research by Lobato, Stump, and Moore provide additional examples of students operating with figurative imagery. Lobato and Thanheiser (2002) reported children thinking the slope of a ramp leading to a platform is affected by the width of the platform. They included the platform as part of the "over" image in their "up and over" slope scheme. Stump (2001) reported a student who thought a slope of $^{-}5/6$ is different than a slope of $5/^{-}6$ because they entail different images of "up and over". Moore and colleagues (Moore et al., 2014, 2019) reported students becoming confused about the slope of a line when *x*- and *y*-axes were switched. They wanted the line to have the same slope because, to them, the line still pointed in the same direction. Their slope scheme depended on an image of a lines' direction and a slope value, to them, was an index of directionality.

Anyone operating with figurative imagery can lose track of their reasoning easily. Byerley and Thompson (2017) report several instances of teachers moving from one meaning of a situation to a contrary meaning within seconds as they employed schemes which used images figuratively. Figure 4 presents an item from the *Mathematical Meanings for Teaching secondary mathematics* (MMTsm) inventory (Thompson, 2016). Its design was motivated by the ambiguity with which teachers in their samples used the word "over" when speaking of rates of change. Thompson and his team could not tell whether teachers used "over" to convey an interval during which an event unfolded or to convey a spatial arrangement of numerator and denominator with respect to a vinculum.

A college science textbook contains this statement about a function *f* that gives a bacterial culture's mass at moments in time.

> The change in the culture's mass over the time period Δx is 4 grams.

Part A. What does the word "over" mean in this statement?

Part B. Express the textbook's statement symbolically.

Figure 4. Meaning of "over" (Byerley & Thompson, 2017). © 2014 Arizona Board of Regents. Used with permission.

For this item, the team took "during" to be a high-level answer to Part A and $f(x_0 + \Delta x) - f(x_0) = 4$ grams, $f(x + \Delta x) - f(x) = 4$ grams, or even $\Delta m = 4$ to be a high-level answer to Part B. Byerley and Thompson reported 113 (45%) of 251 U.S. teachers said "over" meant during or something equivalent in response to Part A, and 71 (28%) of 251 teachers said "over" meant the same as divide in response to Part A. In response to Part B, only 18 (16%) of the 113 teachers who said "over" meant "during" for Part A represented the statement as a difference or change in mass. Forty-one percent (41%) of the 113 teachers who said "over" meant during for Part A responded to Part B by representing the statement as a quotient involving mass and time, writing something like (*change in mass*)/(*change in time*) = 4 grams. In other words, when reading the statement as a plain-language description of a situation, the word "over" for these 113 teachers suggested an image of something happening in time. But upon reading the same statement as something to be represented symbolically, the word "over" suggested an image of numerator and denominator separated by a vinculum.

An interview with James, who had a B.Sc. in mathematics education and was an experienced teacher of algebra, geometry, and precalculus, illustrates how figurative imagery leads to conflicts between schemes.

James: [Over means] during or duration. You could also think of it as a ratio, so change in mass over, yeah so during or duration, so in your math class when they say, "something over something", they always mean a divide sign so a ratio.

Int: Do you think they are both saying the same thing?

James: Well, yeah, I think that. Well yeah, they are saying. I think the during or duration is more saying conceptually what is going on, and the divided by or over I see the reason behind that, I think I'm more pointing out mathematically what we mean when we say over with no explanations as to why, it is just the way it is.

Int: So is the mass, the change in mass divided by the change in time, is that how you write the idea of duration?

James: Can you repeat the question?

Int: Is the "delta mass divided by delta x" a mathematical way of saying duration?

James: I want to say the change in x is the way of saying duration. I want to say the change in x is representing duration. But maybe we could include the division sign. So no, I would not say that "delta mass over delta x" is a way of saying duration. So this is funny. (Byerley & Thompson, 2017, pp. 188-189)

James never reconciled his conflict between "over" suggesting Δx as a representation of elapsed time and "over" suggesting the quotient $\Delta m/\Delta x$. We explain his conflict by appealing to the imagery he apparently evoked in relation to his different purposes for reading the statement. In Part A, his purpose was to read the statement as a plain-language description of a phenomenon, for which "over" suggested an image of something happening as time elapsed. In Part B, James' purpose was to represent mathematically a situation described in plain language, which suggested an image of two numbers or expressions separated by a vinculum. What was new for James is that the interviewer asked him to compare competing implications of his two assimilations of the same word. The images James evoked were figurative—they were tied tightly to the schemes he evoked in the contexts of his different purposes.

We present a second example from Byerley and Thompson (2017) of figurative imagery leading to competing assimilations of "the same" context. Figure 5 shows another item from the MMTsm. Its purpose was to tease out whether teachers interpreted graphs as showing amounts of a quantity despite it being stated explicitly in two ways that the graph showed rates of change of one quantity with respect to another. Parts 2 and 3 to this item (not shown here) allowed teachers to reveal their level of commitment to their initial interpretations of the graph. Thirty-six percent (36%) of 239 high school mathematics teachers appropriately chose (c) for Part 1 of this item; 49% chose (a).

The values of function f give the rate of change (in grams/hr) of a bacterial culture's mass t hours after measurements began.
Over what intervals within the first 8 hours is the culture's mass increasing? Explain.

a) $0 < t \leq 1.4$ and $5.5 < t \leq 8$
b) $0 < t < 8$
c) $0 < t < 3$ and $7 < t \leq 8$
d) None of the above. My answer is ____________________
e) I don't know

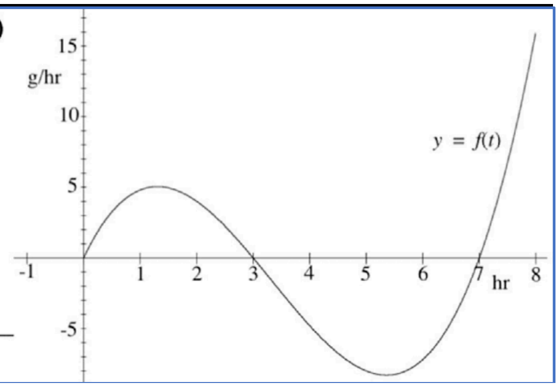


Figure 5. Part 1 of 3-part item "Increasing or decreasing from rate" (Byerley & Thompson, 2017). © 2014 Arizona Board of Regents. Used with permission.

The following excerpt shows a teacher who slipped from one interpretation of the item to a contrary interpretation while explaining her answer to the question.

Annie: [Reads problem aloud, emphasizes "grams per hour".] We interpret increasing. umm...let's see *the function gives the rate of change in grams per hour*... and so umm...what we are going to look at I would look at the rate of change being positive or negative, if we have a positive rate of change the grams per hour the mass is increasing per hour, is getting larger

Annie: So I look at where I have a positive rate of change, and I try to identify where I have no rate of change [highlights maximum at (1.25, 5) where the rate of change is approximately positive 5, but the acceleration is zero], this is telling me where the mass is staying the same, and then I have a negative slope so mass is getting small down to a zero rate of change so I'm not getting any smaller or larger... [Annie chooses (a)]

We do not have direct evidence of where Annie looked as she spoke, but it seems plausible she gazed at the text during her first utterances and gazed at the graph during her second utterances. It appears that in the first utterances Annie had in mind values of the function f as values of the rate of change of the culture's mass with respect to elapsed time, whereas in the second utterances Annie had in mind slopes of tangents to the graph as values of the rate of change of the culture's mass with respect to elapsed time. In other words, Annie slipped from one scheme (values of the function give the rate of change) to a different scheme (slope of tangents to the graph gave the rate of change), and the "slip" was prompted by her different imagistic contexts (text vs. graph). For Annie, with rate of change functions, values of the function give rates of change; with graphs, slopes of tangents give rates of change. We note in passing that Annie felt no conflict between her two schemes because she did not think of the function having positive values in places where its graph had negative slopes.

Third-level (operative) imagery

At the third level of imagery, students' schemes are not dependent upon specific images. Instead, images serve as arbitrary "momentary states" in a scheme's implementation. Thompson and Dreyfus (1988) reported two sixth graders' (Kim's and Lucy's) advancing from second-level imagery, thinking of an integer such as $^-5$ as a location on a number line and later thinking of $^-5$ as a displacement from any starting point. Their imagery moved from Piaget's second level to his third level. The children's schemes no longer needed a definite starting point. They knew they could start anywhere to enact $^-5$. They could then think of $^+3 + {}^-5$ as a composition of two displacements that produced a net displacement of $^-2$, where the second displacement started wherever the first ended. Though they enacted the displacement of $^+3$ from a specific place on a number line, they did not feel required to use a specific place from which they *must* enact it. Their image of an actual starting place was not more general. Rather, it was their scheme that became more general. It did not *require* a specific starting place, thus specific locations on the number line served as "momentary states" in the activity of their integer composition schemes as they conceptualized the net displacement (sum) of several displacements.

While Kim and Lucy developed operative imagery regarding composition of displacements, only Kim thought of *number* in $^-$*number* as itself being a net displacement. Understanding expressions like $^-(^-90+30)$ was unproblematic for Kim because the expression $(^-90+30)$ fit within her image of things constituting *number*—it was a net displacement. For Lucy, only whole numbers fit her image of things constituting *number.* Evaluating the negation of complex expressions was often effortful for Kim, but she nevertheless knew what she was supposed to end with—the negation of a net displacement. In sum, Kim and Lucy had operative imagery with respect to composition of displacements while only Kim had developed operative imagery for negation. Our main point here is that you categorize the level of students' images as second-level (figurative) or third-level (operative) according to your judgment of how necessary those specific images are in students' schemes as they employ them.

Summary

The examples above show our use of "image-level" relatively. Deferred imitation can happen when anyone re-plays an event or collection of actions which they did not fully assimilate—they did not fully understand. Young children go home from school and play school as an attempt to assimilate their new experience of a teacher who attempts to control their thinking. Graduate students in mathematics replay specific aspects of a lecture in their attempt to assimilate them—to develop an understanding. Movement of an hour hand on a circular clock is often offered as a foundational image for understanding cyclical groups. For persons who *must* think of a clock to do arithmetic in a cyclical group, their clock image is figurative because their scheme for addition in a cyclical group requires it. A person who uses a clock whose hour hand varies from 0 to 2π hours to think of equivalent angle measures on a number line, but can also see the repetition as hops on the number line or as the number line collapsing into equivalence classes by the mapping $\mathbb{R} \to \mathbb{R}/2\pi$,[4] is employing images operatively. Any image they employ in their reasoning about arguments to a trigonometric function is a matter of convenience because it fits their purpose in the moment.

Piaget's notion of image is useful because, in developing a scheme, a student must reflect on her reasoning. To reflect on her reasoning, she must create, as best she can, images of having reasoned in the way she did. This means she must develop recollections of "momentary states" in having reasoned. To construct a scheme, students must repetitively engage in variations of the

[4] The mapping $\mathbb{R} \to \mathbb{R}/2\pi$ maps every real number *x* to the non-negative remainder of *x* divided by 2π, or $x \to x - 2\pi\,\mathrm{floor}(x/2\pi)$. Think of all numbers on the number line falling simultaneously, like molecules of water vapor, onto their equivalent positions in the interval [0,2π).

reasoning that will become solidified in that scheme and re-present it as best they can to reflect upon it. Sometimes reflection occurs during moments of confusion, sometimes after having engaged in a chain of interpretations, inferences, and decisions. Nevertheless, in the process of constructing a scheme, *images of having reasoned become students' objects of reflection* (Cooper, 1991; Harel, 2008a, 2008b, 2013).

It is worth noting that the number of schemes students develop is immense. Any word that is meaningful to them is meaningful because hearing or seeing it activates a scheme. Any symbol or symbolic expression that is meaningful to them is meaningful because seeing it or thinking of it activates a scheme. Any diagram or picture that is meaningful to them is meaningful because they assimilate it to one or more schemes. Any time you assess students' thinking or interview students they are interpreting both the setting and your actions through the activation of schemes. Moreover, any combination of the above that proves meaningful to a student is meaningful because of minor or major accommodations in their schemes in the moments of activating them. Any time someone puzzles about the meaning of a word or phrase and resolves their puzzlement has engaged in some form of reflection that engendered an accommodation to their schemes. When you create a scheme as a model of student thinking and impute that scheme to students, you must be cognizant that you have most certainly omitted a vast number of schemes that were at play in students' thinking that might turn out to be important for understanding their thinking. The art of using scheme and image as explanatory constructs is to find the appropriate level of analysis that produces tractable explanations of students' successes and difficulties.

The prior paragraph points to a methodological aspect of scheme as a theoretical construct. On one hand, we say schemes are organizations of a person's mental activity that express themselves in what an observer sees as behavior. From this perspective, schemes reside in individuals. On another hand, we say scheme is a theoretical construct that researchers impute to individuals to explain their behavior. They are a researcher's construct. This is much like stances taken by natural scientists. They realize anything they say is based on models built from theory-laden observations, but in *doing* their science they act as if their models describe reality—until observations force them to step back and question their assumptions and their models. Likewise, we infer schemes from students' behavior in response to carefully defined probes. We impute schemes to students to form explanations of their behavior and to design supports we think will advance their thinking. We step back and question ourselves when our explanations become inconsistent or inadequate, or our designed supports do not have their intended effects.

Imagery, Schemes, and Reflective Abstraction

In this section we expand our earlier discussion of imagery, schemes, and meanings to address what we mean by reflection and the role it plays in a person's construction of schemes.

John Dewey placed reflection at the center of his understanding of thinking, and placed thinking at the center of the development of a critically informed democracy. Dewey defined reflective thought as, "Active, persistent, and careful consideration of any belief or supposed form of knowledge in the light of grounds that support it, and the further conclusions to which it tends" (Dewey, 1910, p. 6). Dewey also anticipated *coherence* as a characteristic of reflective thought that Piaget came to see as central to his genetic epistemology. Reflection on one's thinking leads to

> the organization of facts and conditions which, just as they stand, are isolated, fragmentary, and discrepant, the organization being effected through the introduction of connecting links, or middle terms" (Dewey, 1910, p. 79).

Dewey was also in line with Piaget as to one's motive for reflection.

> Demand for the solution of a perplexity is the steadying and guiding factor in the entire process of reflection – i.e., reflection serves a regulatory function (Dewey, 1910, p. 11).

The key aspect of Dewey's account of reflection is that "to reflect" means to think about thinking. This is in line with our prior discussions of imagery with respect to schemes when one considers that people construct schemes by creating images of having reasoned and take those images as their objects of thought.

A vast difference between Dewey's and Piaget's accounts of reflection is that Dewey thought of reflection as a conscious activity whereas Piaget thought of conscious reflection as the tip of an iceberg. He posited processes of unconscious reflection that must precede anything resembling Dewey's characterization. Piaget took the stance that one can be aware only of images one operates upon. You cannot be aware of the operations you use to operate on an image—to an extent. You can, however, project your operations of thought to a level where they become images upon which you operate. But that is different from being aware of the operations of thought you employ while using them.

As explained by Ellis et al. (this volume) and Tallman and O'Bryan (this volume), the idea of reflection, or thinking about one's thinking, has been on philosophers' minds at least since the time of Aristotle. They also explain that Piaget was the first to break the notion of reflection down into constituent cognitive processes. We will not add to their extensive discussions. Instead, we will highlight essential aspects of reflective abstraction to complete our picture of how imagery and schemes (and therefore meanings) develop and interact in students' thinking across their mathematical development.

Piaget posited five types of abstractive processes: empirical, pseudo-empirical, reflecting, meta, and thematizing

Empirical abstraction is the process of extracting common properties of sensory experience. To empirically abstract a property the person comes to distinguish between objects having and not having the property. It is important to understand that "the property" is the person's construction, not a "real" property. A child abstracting the property of having four legs to apply the word "dog" might at first might also apply "dog" to what we call cats.

Pseudo-empirical abstraction is the process of taking the results of one's activity as objects of empirical abstraction. For example, a child constructed the sequence of arrays in Figure 6. Her actions were to make a vertical line of dots one more than was already there and then complete the figure by making the same number of horizontal dots as in the current array.

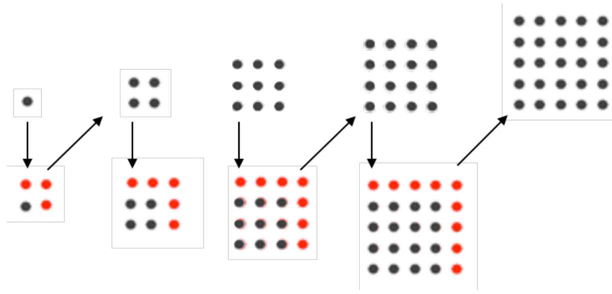


Figure 6. A child's construction of a sequence of arrays.

When the child was asked about the number of dots in each array, she noticed that each array was a square, so she said "1, 4, 9, 16, 25". When asked how many would be in the next array, she squared six to say "thirty-six". In other words, when asked about the number of dots in each array, she took the arrays as if given to her. Here reasoning did not reflect that, by her construction method, the sixth square would have $5^2 + 6 + 5$ dots, or that the $(n+1)^{st}$ array would have $n^2 + (n + 1) + n$ dots, which she could then connect to her observation that each array is a square to conclude that $(n + 1)^2 = n^2 + (n + 1) + n$. This is not to diminish the child's accomplishment. Instead, it is to point out the difference between abstracting one's reasoning from the activity of producing the sequence and abstracting an empirical pattern from the products of one's reasoning.

Reflecting abstraction was already exemplified in the example of pseudo-empirical abstraction. A person engages in reflecting abstraction when she brings to mind (re-presents), as best she can, the reasoning in which she engaged in a prior occasion. A successful process of reflecting abstraction produces a new action, but one that does not need the specific contexts accompanying the original. Successful reflecting abstractions produce *reflected abstractions.*

Metareflection and Thematization are constructs Piaget introduced to account for the ever-increasing level of persons' abstraction of logical, mathematical, and scientific structures. He posited that reflecting abstraction produces reflected abstractions which then can themselves become objects of reflection. He used "metareflection" to capture a person reflecting on reflected abstractions (Piaget, 2001, pp. 82–84).

A person thematizes a scheme via metareflection by developing an image of its major elements, how they work together, and how they might unfold in the context of actual situations. This is not unlike the way storytellers thematize a story. They come to think of the story's major elements, how they work together, and how each element might be unpacked into its details. As Harel (2008c, 2013, 2021) explains, thematization of a scheme evolves over repeated occasions of employing the scheme and reflecting on the activity of employing it. Depending on the scheme's complexity, this could happen over many years. It is worth noticing that when a scheme is well-formed at a reflected level, one can have the experience of having implemented that scheme. Thus, the processes of metareflection and thematization can operate on images of schemes as well as on images of schemes' constituent elements.

Dawkins and Norton (2022) draw upon the construct of metareflection (reflecting on reflected abstractions) to account for students' development of universally quantified conditional statements as logical structures. "We propose populating, inferring, expanding, and negating as four mental actions that, upon becoming reversible and composable, can give rise to the logic of universally quantified conditional statements. We adopt the view that logic is a metacognitive activity in which people abstract content-general relationships by reflecting across their content-specific reasoning activity." (Dawkins & Norton, 2022, p. 1).

We (the authors) had the experience of a calculus student sharing with us her thematization of an approximate accumulation function in DIRACC calculus (Thompson et al., 2019). We were in a staff meeting; the conference room door was open and the statements in Figure 7 were written on a whiteboard visible from the hallway.

$$r_f(x) = 2.1 - e^{\cos(x)}$$
$$r(x) = r_f(\text{left}(x))$$
$$\text{left}(x) = x - \Delta x \left\lfloor \frac{x-a}{\Delta x} \right\rfloor$$
$$a = 2$$
$$\Delta x = 0.01$$
$$A(x) = \left(\sum_{k=1}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} r(a + (k-1)\Delta x)\Delta x \right) + r(x)(x - \text{left}(x))$$

Figure 7. Statements defining an approximate accumulation function in DIRACC calculus.

The student stepped into the room, pointed at the board, and proudly announced,

> I can tell you what every line on that board means and how it all works together! You have a function that gives the rate of change of accumulation for every value of *x*. You want to approximate the accumulation from *a* to *x*, so you cut up the *x*-axis into parts all of length Δ*x* starting from *a* and define a function so its value for everything in a Δ*x* interval is the accumulation's rate of change at the beginning of that interval. Then you assume the accumulation happens at that constant rate over the whole Δ*x* interval. You do that for all complete Δ*x* intervals from *a* to *x*, and then you add the accumulation through the partial interval from left(*x*) to *x*. That's how a value of *A*(*x*) gives an approximate accumulation from *a* to *x*.

While the student omitted some details, such as how the definition of left(*x*) works the way she said and how the summation in the last line gives the accumulation she claimed, she accomplished essentially what she set out to do. She explained each element in her approximate accumulation scheme and how they worked together to produce an approximate accumulation from *a* to *x* for every value of *x* when all one knows is the accumulation's rate of change at every moment.

It is our experience that metareflection and thematization are the least emphasized of all the forms of reflection in Piaget's genetic epistemology—in both instruction and in research. In instruction, we do not envision a teacher commanding his students to metareflect or to thematize. Rather, a teacher *could* promote metareflection and thematization by emphasizing schemes' stories. While the teacher's students must thematize their own schemes, the teacher emphasizing the story of a scheme can open students to the possibility that there is a story to understand. In research, it is inherently difficult to investigate metareflection and thematization as explanatory constructs or as phenomena to investigate, for two reasons. First, these are processes that, even if a student engages in them, occur largely outside of instruction, perhaps even in their sleep, during what Hadamard (1954) called periods of "incubation" and during periods of what Steffe

(1991) called “metamorphic accommodation”—accommodations to schemes that persist over situations and time. Second, metareflection and thematization are difficult to investigate methodologically. Tallman and O’Bryan (this volume) suggest an approach to engendering metareflection in which researchers engage students in what appear to the students as very different situations, but which nevertheless can be understood as similar at a reflected level. Thompson (1994b) used essentially this approach with some success to investigate a fifth grader’s construction of a reflected constant rate of change scheme. Research on metareflection and thematization will be a fruitful area for future advances in theory and practice.

Case Studies

We provide two case studies to illustrate the interconnections among imagery, schemes, meanings, and reflective abstraction. The first case is a 7th-grader learning a mathematical game. It will illustrate a youngster's construction of schemes regarding the game and the crucial role his imagery played in developing them. The second case is of an adult who reasons initially at a figural level about a mathematical task and who projects the situation to a level of existing reflected abstractions upon becoming confused by an unexpected outcome.

Imagery in the Construction of a Nim Scheme

According to Jorgensen (2009), Nim is one of the oldest games in the world. It is a game for two players. In one version, they start with the target number 21 and current total of zero. Players take turns adding a number from one to three to the current total. The player ending with 21 wins.

Diego was a 12-year-old rising seventh grader. We asked him and his parents to let us teach him Nim and allow us to record our Zoom sessions. They agreed. Diego played against a computer program which embedded a general winning strategy that enabled it to win whenever the opportunity arose. Diego played against two versions of the program: (1) A fixed target of 21 and selecting numbers from 1 to 3. He could choose which player goes first. (2) An arbitrary target and arbitrary range of numbers from which to choose. Diego could choose the numbers and which player goes first. Sessions were conducted using Zoom. We met for two sessions—June 17 and 28, 2020. The long break was when Diego attended surf camp.

We report these sessions to highlight the central role of imagery in Diego’s construction of a general Nim scheme by way of the gamut of reflective processes. Diego’s early imagery was figurative, based on re-presenting states of his play. He focused on moves he might make based on the game’s current state. He refined his strategy eventually by taking his reasoning as his object of thought as opposed to states of the game as his objects of thought. His re-presentations of prior reasoning became the images upon which he operated

Session 1: June 17, 2020

21 and 3

Pat explained the game’s rules to Diego and explained that Diego would be playing against a computer program. Pat shared his screen to show the program, which itself explained the game and gave an example (Figure 8).

```
======= RESTART: /Users/patrickt/Dropbox (Personal)/Python Code/Nim-21.py ======
This is the game of NIM. We take turns adding a number from 1 to 3 to our current
total. The player hitting 21 wins.

Enter 1 for you to go first or 2 for me to go first: 2

I choose 1
The current total is 1

Enter a number from 1 to 3: 2

The current total is 3

I choose 2
The current total is 5

Enter a number from 1 to 3: |
```

Figure 8. The computer screen in a game of Nim.

Diego said he would go first. He chose numbers from one to three at random. When the computer played to reach a total of 17, Diego paused. After a few seconds he said, "I lost. No matter what I choose, the computer will get to 21."

In the second game Diego went first, again choosing numbers at random. Diego paused when the computer played to reach 13. "Darn, I'm going to lose again." He remembered that the computer reaching 17 led to it winning, and he couldn't stop the computer from reaching 17.

In the third game, Diego developed the strategy of "goal numbers".

Excerpt 1.

Diego: To get to 21 I have to get to 17. To get to 17 I have to get to 13. To get to 13 I have to get to … nine. To get to nine I have to get to … five. To get to five I have to get to one. To get to one I have to go first.

Diego's image of his participation in the first two games was simply to select numbers from one to three and add his selection to the current total. We say he operated with figurative imagery because each of his activities (adding a new number) resulted in a new state (a new total number) but the new total number was simply the end of an activity sequence (his turn). He was not striving for a particular goal and when he considered a new total number his thinking did not entail an image of the reasoning he used to get to it. He reasoned about specific numbers in specific contexts but not yet re-enacting his reasoning to reflect on it.

Diego's scheme for Nim-21 developed as he reflected on being "blocked". In the first game, he experienced being "blocked" from reaching a desired number (21) because of the total the computer gave him (17). He concluded, by trying all possibilities, he could not reach 21; the computer would win regardless of his choice of number when he is given 17. In the second game, Diego had a similar experience of being "blocked" when the computer presented him with 13—he saw that the computer would reach 17 regardless of his choice and it would therefore win for the same reason he experienced before. In the third game Diego employed his image of being "blocked" to devise a strategy in which he could block the computer from reaching a desired goal number. Diego employed his "blocking" image to block the computer from reaching 21 by him reaching 17, then blocking the computer from reaching 17 by him reaching 13, and so on. Diego's images of being blocked were at first dependent on thinking about the computer reaching 17 and the possible moves he could make from 17 to 21. As Diego repeatedly reasoned about being blocked and additional blocking numbers his image became less dependent on specific game states. As Diego's blocking number scheme became more stable and less dependent on specific states of the game his imagery of being blocked moved from figurative to operative. It became less necessary for Diego to consider a specific game state when he reflected on being blocked. We say that Diego developed a *blocking number scheme* to determine goal numbers and used his goal numbers to ensure he won. He coordinated his blocking number scheme and list of goal numbers in playing the game. Diego had developed a Nim-21 scheme.

38 and 8

Pat then ran a program that played Nim with arbitrary target and range numbers, suggesting they change target and range. Diego chose 38 as target and 1-8 as range. Diego applied his blocking scheme to determine goal numbers of 38, 29, 20, 11, and 2. He chose to go first, selected 2, and won the game.

Diego's behavior suggested he used more than a Nim-21 blocking scheme to determine his goal numbers. He generalized his Nim-21 scheme of "subtract four" to "subtract one more than the largest range number". This had the effect of generalizing his Nim-21 scheme to a Nim scheme.

In using his Nim scheme, Diego made a play to reach a goal number, then awaited the total given him by the computer to determine his next choice. He paid no attention to the number the computer chose. He attended only to the current total given him. The computer's choice did not play into his thinking. His underlying image was to await the computer's total, then use it to reach the next goal number.

Pat raised the matter of Diego's number in relation to the computer's number in the context of reaching the next goal number.

Excerpt 2 (after winning 38 and 8)

Pat: Do you notice a relationship between what the computer chooses and what you choose to get to the next goal?

Diego: No, not really.

Pat: What happens when you are at a goal number so that you get to the next goal number?

Diego: I add a number.

Pat: What about the computer?

Diego: It adds a number, too.

Pat: What about those two numbers? What has to be true about them so you get to the next goal?

Diego: (28 second pause.) They have to add up to 9!

Pat: Why is that?

Diego: Because if I'm at a goal number … if I'm at a goal number the computer will choose a number and then I'll choose … I'll choose … I'll choose another number to get to the next goal number … and the next goal number is 9 away from the [goal] number I have.

Diego's initial responses to Pat's question confirmed our hypothesis about his underlying image of a play. It was not an image that combined his and the computer's moves into one move that satisfies the requirement of reaching the next goal number. Instead, his image of a play was to take the number presented him and to figure the number needed to reach the next goal he'd already determined. At the beginning of Excerpt 2 Diego's image of goal numbers was operative because it was not tied to a specific state of the game—he was able to discuss goal numbers generally. Diego's scheme for goal numbers made it possible for him to use images of goal numbers decoupled from specific game states. His image underlying his decision on the next play was figurative because it was tied to a specific state of the game. He did not have a combined play scheme that would allow him to decouple his image of his next move from the current game state.

Pat's question, "What about those two numbers? What has to be true about them so you get to the next goal?" was instrumental in providing Diego an occasion to reflect on the relationships among current goal, computer's play, his play, and next goal. We suspect that he organized, at least temporarily, the states of current and next goal as being connected by the combination of the computer's and his plays.

We interpret Diego's 28-second pause as him re-playing the computer's and his move in succession. This gave him an occasion to consider the two moves together, as one. We cannot know whether Diego would have thought of this himself, but the fact that he assimilated the question and resolved it suggests that, in re-playing the game, he modified his image of a "play", at least in that moment, to include both players' moves that together would move the total from one goal number to the next. Diego's modification of his image of a "play" in this way was a step towards him developing an operative image of a "play." The development of a stable combined play scheme went along with the development of an operative image of a play that was not tied to a specific game state.

33 and 7

Pat suggested one last game. Diego chose 33 and 1-7 as target and range. Diego went through his working-back strategy to determine goal numbers 33, 25, 17, 9, and 1. He concluded that to reach one he had to go first. Despite the insight Diego stated after the previous game, he still focused on the number presented him and what he had to add to reach the next goal.

Excerpt 3 (in the midst of 33 and 7)
Pat: How are you deciding your number?
Diego: I'm looking for the number to add to get the next goal number.

After Diego won, Pat asked again if there was a relationship between the computer's number and his number when reaching the next goal. Diego quickly noted that the sum of his and the computer's plays needed to be 8—for the same reason he stated earlier. However, it is important to note that, with 33 and 7, Diego did not use the insight he'd stated earlier (regarding sum of his and computer's moves) in deciding his play. Instead of *Number Computer Plays + My Number* = 8, Diego's image of play continued to be *Current Total Given Me + My Number = Next Goal*.

Pat ended the session by thanking Diego for participating and suggested he play the game with his family before the next meeting. Pat texted Diego late the next day to ask if he'd played the game with his family. Diego replied, "No, I'm still trying to get my head around it." We presumed by "it" Diego meant "strategy".

Session 2: July 28, 2020

Diego's attendance at surf camp led to an 11-day break between sessions. He had not played the game with anyone, but he had thought about the game. Pat asked Diego what he remembered.

Excerpt 4

Pat: Do you remember the game we played?
Diego: Yeah. Nim. Yeah.
Pat: Do you remember how it is played?
Diego: Yeah.
Pat: How is it played.
Diego: So … first person to get to 21 wins. If you choose one you go first, if you choose two you go second. And the first person to get to 21 wins.
Pat: How do you get to 21?
Diego: So, you gotta go first, and you choose … two … no. You have to choose//
Pat: //What numbers are you choosing from?
Diego: Oh. One, two, and three.
Pat: I think you were trying to remember the strategy you used?
Diego: Yeah.
Pat: What was that strategy?
Diego: (4 second pause.) His number and your number have to add up to four.
Pat: Why is that?
Diego: So then that, so then (yawns) so you keep getting to the places where you know you're able to get to 21.
Pat: Okay.
Diego: I know … I know that sounds confusing.
Pat: Well, instead of just telling me about it let's play a game.
Diego: Okay.

We were struck by Diego's recollection of his strategy (in bold). What had been a transitory observation in his 7/17/20 session, an observation he never employed, had become a defining feature of his strategy on 7/28/20, despite no interaction in the interim regarding the game with us or between Diego and his family. Moreover, his strategy of $C + D = 4$ entailed a reason for it—this strategy blocked the computer from reaching any goal numbers. This suggests to us that on 7/17 Diego engaged in what Steffe (1991) called *functional accommodation*—the modification of a scheme in the context of using it—and that in the interim he engaged in what Steffe (1991) called *metamorphic accommodation*, which we understand as largely equivalent to Piaget's meaning of projecting images and actions to a reflected level. In Excerpt 4 Diego had an operative image of combined play that was not tied to a specific game state. The development of this operative image was possible due to his reflection upon combined plays and the development of a stable combined play scheme. Diego's combined play image was part of his combined play scheme—the scheme also included the entry points that trigger action and anticipations of action. As Diego's combined play scheme became more stable, his images of combined play became more mobile and less dependent on specific aspects of the game. We say more about this in the discussion.

21 and 3

In playing the first game with 21 and 3, Diego employed his strategy of working backward to determine the first goal number. However, he miscalculated 21 minus 4, saying "sixteen", getting goal numbers of 21, 16, 12, 8, 4, and 0. He chose the computer to go first and selected his number according to his new rule that the sum of the computer's and his number must be four. Diego realized he would lose when the computer presented him with 17.

Excerpt 5

Diego: (Total is 17; D pauses for 21 seconds) Ooohhh. (Pause)
Pat: Why the long pause?
Diego: Um … cause seventeen … if I put three it's twenty and he wins, if I put two its nineteen and he wins, and one he could just put three and he wins.
Pat: What do you suppose the problem is?
Diego: (Pauses for 26 seconds; whistles while thinking.) Oh, I went too far back. I was supposed to get to 17 first. So I lost. So just put two [just to finish the game].
Pat: So, you were supposed to get to 17, but you said 16?
Diego: Yeah.

We interpret Diego's 21-second pause as his imagining all the moves he could make in combination with the computer's subsequent move and his 26-second pause as re-presenting (re-playing) his original reasoning to get his list of goal numbers. He recalled thinking "$21 - 4 = 16$" and realized he should have said "seventeen". We also note that this episode confirms Diego's confidence that applying the condition $C + D = 4$ for each pair of moves was sufficient for him to win.

In the next game (21 and 3, again) Diego insisted on enacting his working-back strategy "just to make sure", ending at one and deciding he should go first. He played appropriately, getting to each of his desired goal numbers. When the computer presented Diego with a total of 10, Diego chose 3 to reach 13. Pat asked about how he was deciding on his choice of numbers.

Excerpt 6

Diego: (Computer presents a total of 10) Three.
Pat: How are you figuring out what number to pick?
Diego: Whatever number he chooses, I just need to pick the number that adds up to four.

Diego "$C + D = 4$" scheme was now at a reflected level. He knew applying it would necessarily land him at the next goal number without having to think of what the next goal number was.

36 and 5

Diego again employed his working-back strategy, but this time just to determine the starting number. He counted 36, 30, 24, 18, 12 6, 0 then said, "But we cannot get to 0, so the computer needs to go first." He did not mention a goal number while playing the game. Instead, as the computer played its number, Diego played a number so the sum of the computer's and his numbers was six.

Excerpt 7

Comp: (Plays 1)
Diego: Okay, five (total is six; computer chooses 5, total is 11). One
Pat: How are you deciding what number to put in?
Diego: Same thing. Whatever number they choose I just have to choose another number that adds up to six.

Pat: Why is six special?
Diego: It's one more than the number we can pick … so that they can't go over the number we have to get to but we also can't … err … and also we … it's too … and also it's not too much for us to get to.

Diego went on to win the game. He again confirmed that his strategy no longer relied on the total given him or the next goal number. It relied only on him determining the first number to play, using this number to decide whether he or computer should go first (computer first if first goal is 0; otherwise him first), and knowing what the sum of his and computer's play must be. Diego felt confident that attending to these conditions would produce a winning strategy. He had developed a Nim scheme.

General Nim
Pat suggested one last game—77 as target and 1-10 as selection range. Diego again worked backward from the target by 11's, getting 0 as the first goal number. Diego chose the computer to go first and won the game. Afterward, Pat asked about his general strategy.

Excerpt 8
Pat: Let me ask you a question.
Diego: Yeah.
Pat: It seems … and correct me if I'm wrong. It seems you start with the target number, and then go back to find the next smallest target number//
Diego: //Yeah. That's what happened.
Pat: And you keep going back [Diego: Yeah] until you find the first target number [Diego:Yeah] and that tells you whether or not you go first? [Diego: Yep] Is that right?
Diego: Yep. If it gets to one, then you have to go first, but if it gets to zero then they have to go first.
Pat: What if you get to two?
Diego: You can still go first … unless … the boundary … the number … unless the number limit you have to choose from is less than two … which would be kinda boring.

Diego's last statement shows he not only reasoned generally about his strategy, he saw implications of alternative conditions ("unless the number limit is less than two …"). We take this as evidence that Diego's goal number images and combined play images were now operative. Because of the generality of Diego's summary, Pat decided to raise the issue of efficiency.

Excerpt 9
Pat: That's very good! Now … could you think of a way to figure out what the first number should be without having to go back step by step?
Diego: I guess that (pause) if … oh, hold on, I'm just noticing. When it was 21, they had to go first. Oh no. We had to go first. So I feel like, if it's 21 … if the number limit is divisible by the number limit you're allowed to choose, then you have to go first.
Pat: So 21 is divisible by 3//
Diego: //Oh, no. But 36 isn't divisible by 5. And do you remember … can you go back and see if they had to go first on 36? Scroll up to see the previous game.
Pat: (Scrolls back to the game of 36 and 5.) You had the computer go first and it chose one.

Diego: Yeah, they went first. If the target number is divisible by the highest number you choose from, then you have to go first.
Pat: But 36 is not divisible by five.
Diego: And the computer went first! (Pause) So if it's not divisible by the highest number you're allowed to choose from, then the other person has to go first!
Pat: Why do you suppose it works that way?
Diego: (Looking into the air) Because that … it's always gonna leave … … like … it's always gonna leave … a higher number than … it's alw … ahh … it's always gonna end up like … it's never gonna end up … hmm, hold on. (Pause) Yeah. That makes no sense. (Pause.) Here. Let's do … so … so … from what we've seen right now it's um an odd number has to make the computer go first and an even number has to make … wait no, not even number. A number divisible by that means I have to go first. So let's just see … make … do … do like 30 … do like 32 and then do 4 … 32 is divisible by 4, yeah.

Diego's response to Pat's question, "Could you think of a way to figure out what the first number should be without having to go back step by step?" has earmarks of what Piaget called pseudo-empirical abstraction. By this we mean that Diego looked for a pattern that related game conditions (target number and range) and the decisions he had made in light of them. He reflected on the products of his reasoning, not the reasoning in which he engaged to create those products. This is not a criticism. Rather, it is an observation.

Diego tested his hypothesis on 32 as target number and 1-4 as range—and lost. Pat suggested he try his working back strategy again. Diego did this—32, 27, 22, 17, 12, 7, and 2—noting he had to go first and start with two. Pat determined it would be a long struggle for Diego to refine his strategy further, so he used an intervention common in exploratory teaching interviews (Castillo-Garsow, 2010; Moore, 2010; Steffe & Thompson, 2000). This was to offer a suggestion to see how Diego might understand it and how he might subsequently use it. An intervention move within an exploratory teaching interview can unveil the nature of and boundaries of the interviewee's schemes.

Excerpt 10
Pat: So going back … I'm going to remind you of something. Each time you went back, you subtracted five, right? [Diego: Yeah] You subtracted one more than the upper limit. [Diego: Hmm hmm] Repeated subtraction is like division. [Diego: Hmm hmm] So you went back some number of fives and you got to two. [Diego: Yeah] So, if you divide 32 by 5, what remainder do you get? [Diego: Two] (5 second pause) And what's special about 2?
Diego: You could go first and get to 2, and then you could get to 7 first, you just … you kinda win.
Pat: So let's try that. This time I'm going to try 45 and 6. Now, without using target numbers, see if you can find the first target number.
Diego: Seven divided by 45 is … (40 second pause; D looks in the air). The remainder would be three. Because the closest number to that number that is divisible by seven is 42, and 42 divided by seven is six. And then the remainder would be three. So the first target number is three.
Pat: So who goes//
Diego: //So I would have to go first.
Pat: And when you say 42 divided by seven is six, what does that six mean?

Diego: (19 second pause) It means I would go back by seven six times.

It appeared Diego easily understood that his going back strategy entailed repeated subtraction, and he appeared to understand Pat's statement, "repeated subtraction is like division." He also inferred that the remainder after dividing 32 by 5 would be the first target number. Diego then applied a "division and remainder" strategy to a game with target 45 and range 1-6, concluding he had to start with three and the sum of plays had to be seven. Diego also understood the relationship between dividing 45 by seven and his working-back strategy—dividing 45 by seven would give the number of times he would go back by seven to get the starting number and the remainder would be the starting number. Afterward, Diego celebrated.

Excerpt 11

Diego: So now I know how to do it without having to go back step by step!

Pat: Isn't that cool that you can figure out [where to start and] what number to add without knowing any of the target numbers except the last one?

Diego: Yeah.

Pat: So, just … okay, we can finish this now. But if you could, tell me what your general strategy would be no matter what numbers I pick.

Diego: Umm. Get to the remainder first.

Pat: Remainder of what?

Diego: Get to the remainder of the number … get to the remainder of the goal number divided by the highest choice number plus one.

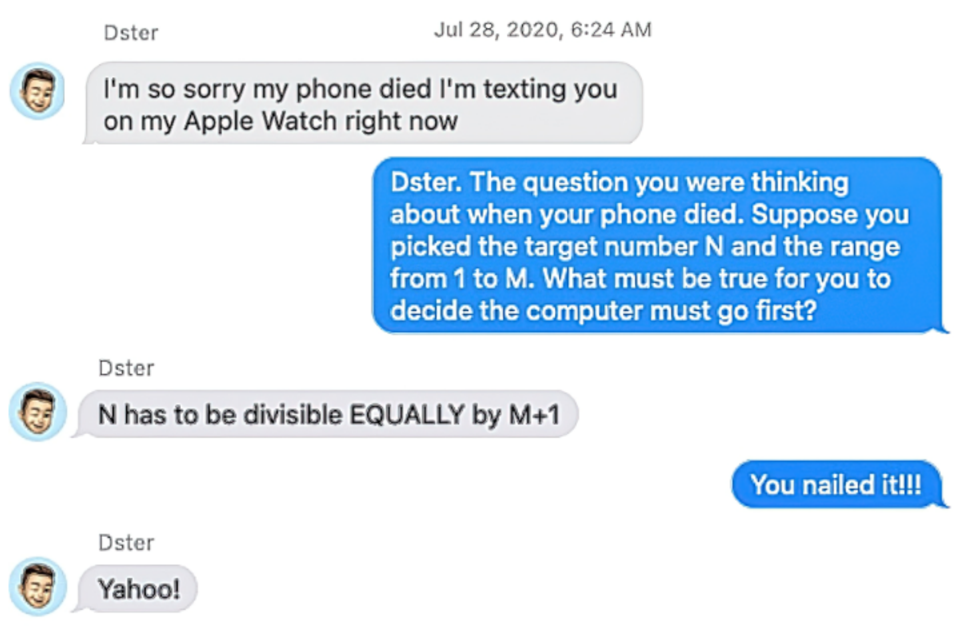
Pat: And that remainder tells you what?

Diego: And then … and then … that remainder tells you if you should go first or if they should go first and then you always have to get … you have to see what their number is and add another number to get the number … to get … to get to the highest number plus one.

Pat: When should you have the computer go first? When should you decide to have the computer go first? (Zoom connection fails) Diego? Diego?

The broken Zoom connection was because Diego's phone battery died. Excerpt 12 shows their exchange of text messages following the Zoom failure.

Excerpt 12



Excerpt 11 and 12 provide evidence that Diego made metamorphic accommodations to his Nim scheme. His final Nim strategy was much more efficient than his first winning strategy. His final strategy was more efficient and qualitatively different than his first winning strategy because he integrated his division scheme to avoid needing to repeatedly subtract to find each goal number. Diego reflected on images of goal numbers and combined play to accommodate his Nim scheme to be more general.

Discussion
It is important to note that Diego did not write anything down, nor was there a visible record of his thinking. There was only what the computer presented after each player played. Diego had no visual record of his reasoning to aid his reflections. He only had what he could recall of his reasoning and its results as images to reflect upon.[5]

Diego's imagery for the game rested in his re-presentations of his reasoning, and sometimes re-presentations of his conclusions. The role of Diego's images evolved as his schemes evolved. His initial imagery consisted of his reasoning about specific numbers in specific contexts. Diego re-enacted his reasoning in order to think about it. This is an instance of deferred imitation. Early on, Diego's remembered decisions did not entail the reasoning leading to them, consistent with his scheme at that time employing imagery figuratively. His decisions were simply the last step in his chain of actions. Later, Diego differentiated between his decisions and the reasoning leading to them, which allowed him to begin projecting his reasoning to a reflected level. Finally, Diego generalized his images from specific game states to arbitrary game states, consistent with his imagery operating at a third level because he had created his Nim scheme at a reflected level. We provide a detailed summary of Diego's development of his Nim and General Nim schemes below.

[5] It is plausible that had Diego created a written record, his writing might have prompted him to engage more frequently in pseudo-reflective abstraction.

Nim Scheme

- An image of being blocked. Diego experienced "I was blocked" twice. He developed the image, "The computer gave me a number that kept me from reaching a goal number. There are numbers that 'block' a player from winning."
- A "Blocking" scheme—find all the blocking numbers. A blocking number keeps the computer from reaching the next goal number and ensures he can reach the next goal number. Diego's blocking scheme relied on his image of a blocking number.
- A "First number" scheme—if the first goal number is 0, the computer goes first. If first goal number is not zero, go first and select that number. This scheme relied on Diego having an image of having executed his blocking scheme.
- A "Combined play" image—a combined play takes the game from one goal number to the next.. In Excerpts 2 and 3 Diego reasoned that the sum of his and computer's play had to be a certain number (9 in one instance, 8 in the other), but he seemed not to have had a "combined play" image. He did not recall these conclusions upon playing the next game. It was after the 11-day break that Diego seemed to have an image of the computer's play and his play as one play.
- A "Combined play" scheme—decide what to play based solely upon the computer's play and the selection range.
- A Nim scheme—we saw in Excerpt 7 that Diego coordinated his First Number scheme with his Combined Play scheme to form a Nim scheme. He was confident his Nim scheme would produce a winning strategy regardless of the target number and selection range.

Diego had developed a scheme for Nim. He developed it over time by coordinating his blocking scheme, first number scheme, and combined play scheme to make a strategy for winning the game regardless of the target number and selection range. His coordination of schemes was enabled by having projected each to a reflected level. We are comfortable saying this because he could articulate each in general terms not reliant on any specific game. In each case, the projection happened as a result of reflecting on images of his reasoning that he gained from playing the game. Diego's images became more mobile and flexible. Figurative images became operative as Diego's schemes developed through reflection and repeated reasoning.

One of Piaget's defining characteristics of operating at a reflected level is that the person is aware of their schemes and uses them to explain their reasoning. That Diego *coordinated* his blocking, first number, and combined play schemes, and explained how they worked together confirms to us he had indeed projected them to a reflected level.

General Nim Scheme

Diego went beyond his initial Nim scheme. He assimilated Pat's suggestion to think of repeated subtraction as division and used that idea to develop more than a winning strategy. He employed already-developed schemes for division (as repeated subtraction) and remainder to understand Pat's suggestion and saw how it related to his blocking scheme (repeated subtraction) in determining the first goal number. He modified his Nim scheme by incorporating the scheme "$First\ Number = mod(N, M)$".[6] Had Diego not had a well-developed division scheme, he would not have seen the relevance of Pat's suggestion.

[6] This is our description, not Diego's.

We note in closing that Diego's speech over time gives an indication of how he generalized his imagery. His earlier statements were about specific numbers. Later statements were about "computer's number", "my number", "goal number", and "one more than the biggest number we can choose". We take Diego's use of literal names for states as enabling him to differentiate his reasoning from the specific contexts in which his reasoning occurred and from the specific conclusions he drew. This also supported Diego's projection of his reasoning to a reflected level. His use of literal names for objects upon which he acted supported his focus on the actions he took to get from one state to another.

We interviewed Diego again after a lapse of 15 months. He did not recall Nim nor how it is played. We reminded him of the rules and played a game of 21. After a short pause, he recalled his Nim scheme (finding goal numbers and first number) and used it to win. We then played 45 and 1-8 and he again employed his Nim scheme to have the computer go first, and he won. In the third game (67 and 7) he recalled his generalized Nim scheme, found the remainder of 67 ÷ 8, chose to go first, gave three as his first number, and won. This all happened in less than 20 minutes. The rapidity with which Diego reconstructed his generalized Nim scheme suggests to us we are correct to have called it a scheme.

Implications for Math Education

We shared Diego's case study specifically to highlight that *students' images of having reasoned* are the primary fodder for productive reflection and hence for productive scheme formation. Students' images of their reasoning become transformed into operations of thought they can apply outside specific contexts in which their reasoning occurred. This fact has implications for mathematics teaching and mathematics education research.

<u>Implications for mathematics teaching</u>

To help students form images of having reasoned so they may reflect upon them is not the same as asking "Why did you do that?" or "How do you know that?" Those questions often sound to students like they are being policed. A teacher highlights reasoning instructionally by engendering reflective discourse (Cobb et al., 1997; Stein et al., 2008) with and among students, and by creating didactic objects to support reflective discourse (Thompson, 2002). Designing instruction to bring students' imagery into the open and to support reflective discourse means to orient students to discuss ways they are understanding situations ("What do you see going on in this situation?" "Share with us what you imagined about this situation when you said that?"), meanings and reasoning they are trying to convey ("Help us understand what you meant by that?"), ways they are understanding diagrams and animations ("What do you see happening in this animation?" "What do you see this diagram depicting?"), and their reasoning in the context of solving a problem ("Please share your strategy, if you can." "What stood out to you when you decided to divide?"). Of course, those cannot be idle questions. A teacher conveys genuine interest by incorporating students' answers into the classroom conversation.

Fostering reflective discourse also entails having students attempt to understand others' reasoning and reflect on meanings they might intend. Reflective discourse takes students' imagery, meanings and reasoning as objects of class discussion. Instructors fostering reflective discourse continually demonstrate that they care about and value students' understandings—instructors convey to students that they are *interested* in students' images and meanings-in-the-moment (Hackenberg, 2010). Cobb et al. (1997) and Clark, Moore and Carlson (2008) reported teachers' consistent support of reflective discourse can positively affect students' attitudes and classroom atmospheres. Under a teacher's guidance, sharing the foundations of one's thinking, and expecting the same of others, becomes a classroom mathematical practice (Yackel & Cobb, 1996).

Although Pat's interactions with Diego were in an interview setting, Pat's questions to Diego did have an instructional effect. For example, Pat's questions "How are you deciding what number to select?" and "Do you see a relationship between the computer's number and your number?" appear to have caused Diego to reflect on his reasoning when he might not have done so otherwise. Also, Pat's questions prompted Diego to formulate *responses* to those questions. As noted by Inhelder and Piaget (1964), the attempt to express one's reasoning to oneself, or to communicate one's reasoning to another person, is a primary stimulant for reflection. Questions like these, offered at timely moments, can be incorporated into instruction.

Implications for mathematics education research

Diego's case exemplifies a methodological focus on students re-presenting their reasoning to themselves as necessary for reflection. This focus entails timely probes about what the situation under discussion is *to the student*. It also exemplifies a focus on asking questions that prompt students to explain their decisions. But probes into students' decision-making processes must be crafted carefully. They must not appear to the student as demands for justification. Instead, you want probes to convey to students that you are genuinely interested in how they are thinking. "Help me understand how you thought about this?" exemplifies a genre of questions that can be useful in gaining insight into students' imagery and reasoning.

Diego's case also highlights our stance that it is students' images of having reasoned that provide the fodder for reflection. To live this stance in your research requires that you take students' verbal statements and symbolic work as a clouded window into their thinking—that you must probe to gain insight into what they meant when they said or wrote what they did and how they imagined their actions being relevant to the situation as they conceived it.

Imagery in the Projection from Figurative to Reflected Thought

Piaget spoke of two kinds of reflecting abstraction. The first is to construct schemes at a reflected level while the second is to reason at first with schemes at a figural level and then move one's reasoning to counterpart schemes that have been created at a reflected level.[7] This case is a study of the latter.

Michael and Robert are mathematics educators. Michael shared with Robert a task for students asking them to predict the graph of $y = \sin(3x + 1)$ as a transformation of the graph of $y = \sin(x)$. Robert first thought about the problem in terms of slots—$(\square + b)$, largely as a figural generalization of $(x + b)$. He envisioned the graph being transformed according to $y = \sin(\square)$ and then translated according to the value of b. Robert knew $y = \sin(3x)$ would "compress" the graph of $y = \sin(x)$ by moving each value of $\sin(x)$ from above the value of x to above the value of $x/3$. He knew "+1" would translate the graph of $y = \sin(3x)$ one to the left by putting each value of $\sin(3x)$ above the value of $3x - 1$. Robert concluded that the graph of $y = \sin(3x + 1)$ is the graph of $y = \sin(3x)$ shifted by -1. Michael said, "That's not correct."

Robert was immediately puzzled—he was sure his reasoning was correct. He first wondered how to test his claim. He knew, according to his theory of the situation, that all points on the graph of $y = \sin(3x)$ would be shifted to the left by 1. He decided to produce both graphs

[7] We have seen this distinction translated in different ways by different translators and we do not know which terms to use for them. REVIEWERS—HELP!!

on the same axes and examine one point on the graph of $y = \sin(3x)$ shifted like he thought it should. Robert's investigation confirmed that the graph of $y = \sin(3x + 1)$ is *not* the graph of $y = \sin(3x)$ shifted by 1 (Figure 9).

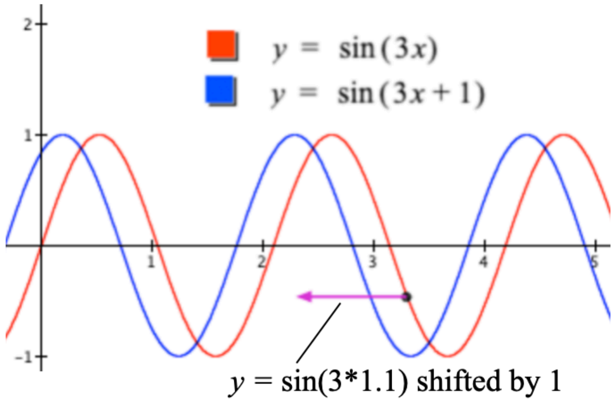


Figure 9. Robert's test of his initial claim.

Robert puzzled about why the graph of $y = \sin(\square + 1)$ is not necessarily the graph of $y = \sin(\square)$ shifted by 1. He used the mouse to highlight points on the graph and noticed the graph seemed shifted to the left by 1/3 instead of 1. He wondered, "How is it possible for three as a coefficient of x to affect the effect of adding 1?"

Robert eventually asked himself, "What am I doing when I shift a graph by changing its argument?" Focusing on the idea of argument opened him to think of $\sin(3x + 1)$ as a composite function. He then considered $\sin(3x + 1)$ as $\sin(ax + b)$ and $\sin(ax + b)$ as the composite function $h(k(x))$. Robert employed similar imagery as initially to understand how a graph is shifted when a function's argument is itself a function: Start with a value $x=c$ (Figure 10a), move the value $x=c$ on the x-axis by the function k to arrive at $x = k(c)$ (Figure 10b), "pick up" the value of $h(k(c))$ (Figure 10c), move from $x = k(c)$ back to $x=c$ by $k^{-1}(k(c))$ (Figure 10d), then plot the value of $h(k(c))$ above $x=c$ (Figure 10e). He concluded that the graph of $y = h(k(x))$ will appear to be the graph of $y = h(x)$ but with each value $h(x_0)$ plotted above or below $x = k^{-1}(x_0)$, provided $k^{-1}(x_0)$ exists.

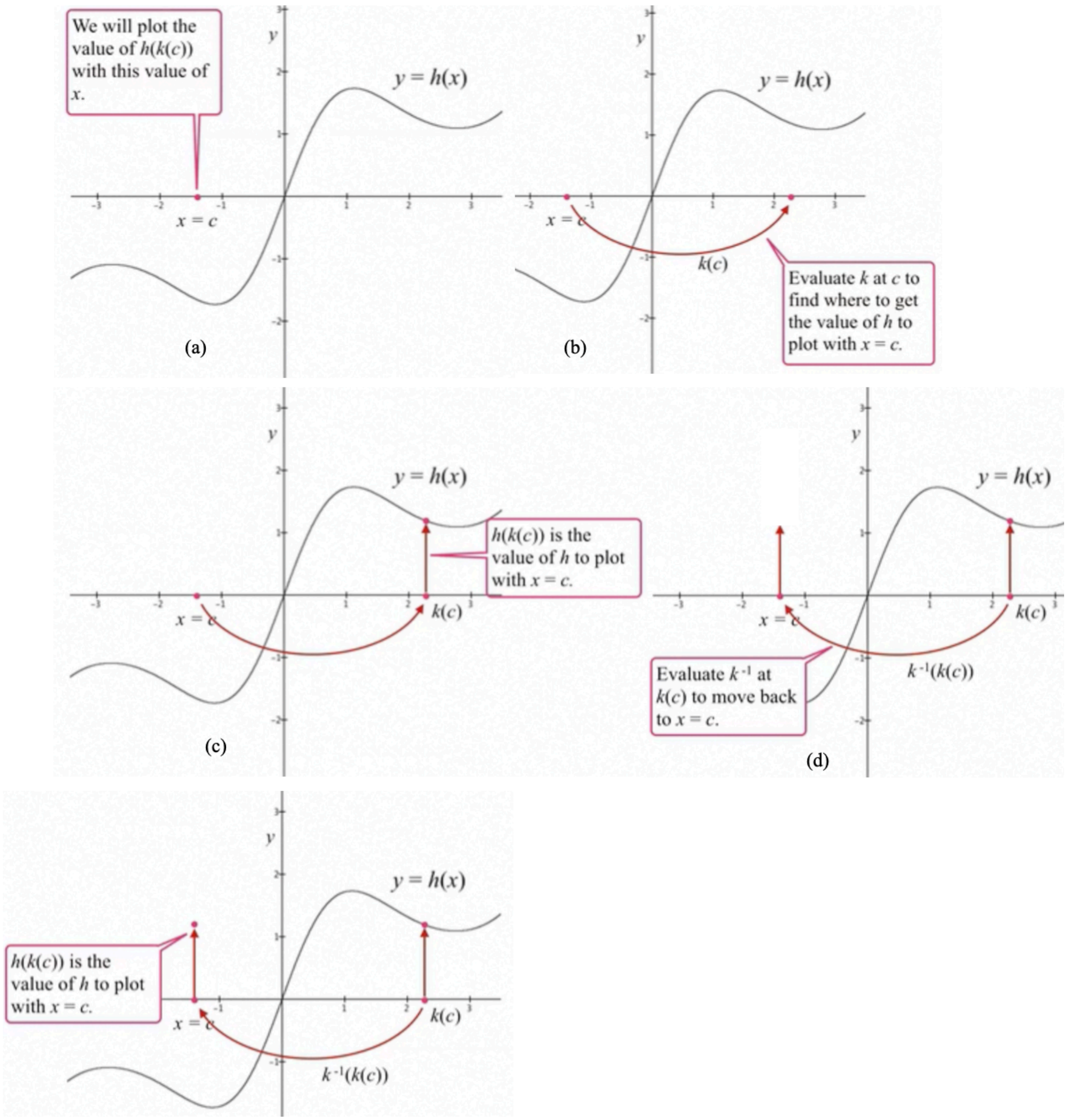


Figure 10. Our rendition of Robert's drawings and his reasoning about them.

Robert then applied this result to the graph of $y=\sin(k(x))$ where $k(x)=ax+b$ to see that the graph of $y=\sin(x)$ is "shifted" by the function k^{-1} so that each value of $\sin(x)$ appears above or below $k^{-1}(x) = (x - b)/a$. In this particular case, the graph of $y = \sin(3x + 1)$ is the graph of $y = \sin(x)$ with each value of $\sin(x)$ plotted above or below $(x - 1)/3$ He explained his new approach as,

> You want to anticipate the graph of $y = h(k(x))$ given the graph of $y = h(x)$. Imagine standing on the graph of $y = h(x)$ at, say, $x = x_0$. Where did this value x_0

> come from? It came from a value $x = c$ so that $x_0 = k(c)$. Where will the value of $h(x_0)$ appear on the graph of $y = h(k(x))$? It will appear above or below $c = k^{-1}(x_0)$. In the case of $y = \sin(x)$ and $y = \sin(3x + 1)$, any value $\sin(x_0)$ will appear at a value $x = c$ so that $x_0 = 3c + 1$, or $c = (x_0 - 1)/3$. This tells me that to get the graph of $y = \sin(3x + 1)$ start with the graph of $y = \sin(x)$, shift it to the left by 1, then compress that graph by 1/3. But this will be true of any function *f*. The graph of $y = f(3x + 1)$ will be the graph of $y = f(x)$ but with each value $f(x)$ appearing above or below $(x - 1)/3$.

Robert shared his conclusion with Michael, who agreed that the graph of $y = \sin(3x + 1)$ is the graph of $y = \sin(x)$ shifted by 1 then compressed by a factor of 1/3. Michael, however, had not considered the general case of composite functions that Robert used to arrive at this specific result.

How do we possess a record of Robert's inner thoughts? Because Robert was Pat Thompson and Michael was Alan O'Bryan. Pat, realizing this was a potentially important event, wrote a log of his thoughts as he puzzled through his confusion. Figure 10 is our rendition of his unorganized drawings.

The central point of this example is that Pat first employed imagery at a figural level—the level of action regarding an existing scheme. He initially assimilated the question to a well-developed scheme for transforming graphs which allowed him to think primarily in terms of imagining specific actions on a graph in relation to the specific symbolic form $y = \sin(\square + 1)$. He thought of $y = \sin(\square + 1)$ as $y = \sin(x + 1)$, concluding that the graph of $y = \sin(\square + 1)$ would be the graph of $y = \sin(\square)$ shifted by 1 to the left. He moved to a level of reflection only upon being faced with the invalidity of his reasoning.

Even at a reflected level, Pat employed images like what he conjured at a figural level. He envisioned how the original and new graphs are related by picking an arbitrary value on the *x*-axis where a value of the composite function would be plotted and moving to a location on the *x*-axis (by evaluating a function's argument) where the original function would be evaluated . These images initially were figural with respect to Pat's "translate a graph" scheme that employed them. They became operative when he moved to a reflected level to think of $y = \sin(3x + 1)$ as $y = h(k(x))$. At a reflected level, he thought of a function and its graph, but not a specific function or a specific graph. Any graph would support his thinking of movements on the *x*-axis by an arbitrary function *k* and its inverse. He also thought of an arbitrary argument to the composite function. His images were arbitrary while still providing a context for the transformations he employed—evaluating the composite function's argument to get a location for evaluating the original function, then using the argument's inverse to move that value of the function back to where it would be plotted as a value of the composite function. He then thought of this transformation as being applied to every value in the domain of the argument. Pat resolved his initial confusion about the graph of $y = \sin(3x + 1)$ in relation to the graph of $y = \sin(x)$ by answering a general question about the graph of a composite function $y = f(g(x))$ in relation to the graph of $y = f(x)$ for any functions *f* and *g*. He also understood his reasoning applied only where *g* has an inverse function.

It is important to understand that this is not a story about Pat *constructing* a higher-level scheme, like Diego, through the abstractive phases of empirical, pseudo-empirical, and reflecting abstraction. He already possessed the schemes for functions, function notation, function inverse, and function graph he eventually employed. Instead, this is a story of someone projecting their figurative reasoning to an already-present reflected level—a level of already-existing operative schemes. Pat understood all along he was solving the original problem, but with the additional understanding that he was solving a general version of that specific problem. Pat's attention was focused initially on the graph of $y = \sin(3x + 1)$ in relation to the graph of $y = \sin(x)$. He re-imagined the problem to be about the graph of $y = h(k(x))$ in relation to the graph of $y = h(x)$ for arbitrary functions h and k.

It is also important to note that the role of Pat's images changed from his initial to final thoughts. Initially, Pat's imagery provided a template for his assimilation of the problem, assimilating $y = \sin(3x + 1)$ as $y = \sin(\square + 1)$, where $\square + 1$ was a figurative generalization of $x + 1$. Even though his imagery for $y = \sin(\square + 1)$ involved movement, it was figural regarding the scheme to which he assimilated it—a scheme in which the added constant specified the direction and distance the function's graph would be translated. Pat's images at a reflected level served as arbitrary states of transformations—moving an arbitrary displacement on a number line with respect to an arbitrary graph. Pat understood movements along an axis being "caused" by evaluating functions and understood locations on the number line as related by being values of an argument or its inverse. This role of imagery is in line with Piaget's third form, describing the role of images in relation to operative thought (see quotation on page __). They served as "momentary states" in Pat's reasoning about transformations from original graph to desired graph. The specific images Pat employed were not necessary to invoke the transformations which related them.

Although Pat did not construct a higher-level scheme through elaborate abstractive processes of differentiation, integration, etc., he did construct a new scheme—and therefore a new meaning—for "transform a graph". He connected (assembled) existing schemes in, for him, a novel way. Initially, his meaning for "transform a graph" was to think of the form of a function's argument and what it implied about how a function's graph changes. His meaning at the end was crystalized as how the original function's values are "re-positioned" on the independent axis by the inverse of the original function's argument when the argument is viewed as a function.

Implications for Mathematics Education

We shared Robert's (Pat's) case study specifically to highlight again that *images of having reasoned* are the primary fodder for productive reflection. This time, unlike Diego's case, "productive reflection" meant to project the situation as Pat originally conceived it to a reflected level of already-existing schemes. That imagery of having reasoned is important even when moving to reflected mathematical thought has implications for mathematics teaching and mathematics education research.

<u>Implications for mathematics teaching and mathematics education research</u>

Solving a specific problem by solving a generalized version of it is a standard move in higher mathematics. At the same time, it is a rare move in school mathematics and a move made without students' noticing it in undergraduate mathematics. We are unaware of research into this phenomenon from a genetic epistemology foundation. The closest we know is research on problem posing (see Cai et al., 2015), especially the early work by Brown and Walter (1983, 1993). Problem posing, as originally crafted by Brown and Walters, is to provide facts about a situation and ask students to craft problems from these facts. Cai et al. (2015) surveyed research on problem posing as an instructional technique, concluding that employing it in instruction has a positive impact on students' problem-solving abilities.

We suspect students in problem-posing studies engaged in various forms of reflection to create their problems. Reports that "more able" students pose more and more complex problems than "less-able" students (Cai et al., 2015; Marsh & Yeung, 1998) suggests to us that "more able" students were engaging in projection to reflected thought. However, we are unaware of problem-posing studies that examined students' behaviors from a perspective of images they formed and their reflections on those images. One technique employed by Brown and Walters is ripe for research on students' imagery and reflection. It is to have students revisit a problem repeatedly, to relax constraints in each iteration so they produce ever more general versions of the original problem. Generating generalized problems, and discussing their generalization processes, could provide opportunities for reflection. Researching their solving activities for the problems they generate could then provide occasions to explore the connections they actually make between their underlying imagery and reflective processes.

Discussion

Our thesis throughout this chapter has been that images of having reasoned are the foundation for reflection and scheme development. We stressed that imagery includes visualization but includes far more than visualization. We recapped Piaget's levels of imagery and expanded their meaning to make them useful for modeling mathematical scheme formation at any level of sophistication. We included the case study involving composite functions specifically to show Piaget's constructs can be used to model higher level mathematical thinking. We illustrated the interplay among imagery, reflection, and scheme formation through two case studies and explained the implications of each for mathematics teaching and research.

In this discussion we will highlight an aspect of Diego's case study that was central to the work with Diego but remained tacit in our accounts. It is our preparation for the interviews—task selection and design together with conceptual analysis of the game.

We settled on Nim as the context of our interviews for two reasons. First, we wanted to avoid as much as possible Diego's need to create written records of his work. We are not suggesting that symbolizing is unimportant—far from it. Our past research and teaching, however, convinced us that one effect of students' mathematical schooling is they often engage in premature symbolization. We say "premature" because students often create inscriptions which then turn into the objects of their attention. Their focus on past inscriptions then diverts their attention from the reasoning that led to them. We also considered that readers might think a case study of learning the game of Nim is unrelated to learning school mathematics. This would be true if we have in mind standard school mathematics. But reflective discourse is not standard in school mathematics, and we wished to highlight that it is students' reflection on their images of prior reasoning that is central to their advancement. We argue that the case of a student constructing a fairly complex scheme without recourse to pseudo-empirical abstractions from written work is highly relevant to ways students *could* learn mathematics in school. As the last interchange with Diego showed, he did symbolize his thinking, but he did so only after his thinking advanced to a state where the symbols retained their meaning within his reasoning.

Glasersfeld defined conceptual analysis by a question, "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (Glasersfeld, 1995, p. 78). Thompson expanded Glasersfeld's meaning of conceptual analysis to include four uses:

1. in building models of what students actually know at some specific time and what they comprehend in specific situations,

2. in describing ways of knowing that might be propitious for students 'mathematical learning,
3. in describing ways of knowing that might be deleterious to students 'understanding of important ideas and in describing ways of knowing that might be problematic in specific situations,
4. in analyzing the coherence, or fit, of various ways of understanding a body of ideas. Each is described in terms of their meanings, and their meanings can then be inspected in regard to their mutual compatibility and mutual support. *(Thompson, 2008, p. 59)

We employed conceptual analysis according to #1 in our analysis of Diego's interviews. We employed it according to #2 in our preparations for the interview—we analyzed the game of Nim according to what someone must understand to play it at the highest level and what schemes might be necessary to get there. We also drew on our own experience playing the game and from watching others play it.

For any game with target *N* and range 1-*M*, we considered $My\ First\ Number = mod(N, M)$ coordinated with $Computer's\ Play + Human's\ Play = M + 1$ to be the most sophisticated strategy. We also anticipated that the first scheme would be Blocking and the second would be Goal Numbers. What we had to consider was how Diego might fill in the gaps between first schemes and final scheme.

We anticipated that the $Computer's\ Play + Human's\ Play = M + 1$ scheme was crucial for Diego to advance to using division, for without that scheme he would not see that successive goal numbers are generated by repeatedly subtracting his and the computer's combined play. This was why Pat repeatedly asked Diego, "How are you deciding your number?" and later asked, "Do you see a relationship between the computer's number and your number?"

We also anticipated it would be crucial that Diego see a connection between repeatedly subtracting the combined play to determine first number and dividing target by one more than the range to find a remainder. When Pat determined this insight would be long in coming, he pointed out to Diego that he was using repeated subtraction and suggested the connection between repeated subtraction and division to see what Diego would make of it. Finally, we decided not to suggest Diego write anything down for the reasons we explained earlier.

The case of Robert (Pat) illustrates a second form of reflection that turns figurative imagery into operative imagery—the projection of a context assimilated figurally to schemes that employ images operatively. The trigger for this projection was Pat thinking of the "input" to the sine function as an argument to the sine function, which then led him to think of the original context more generally as a composite function. The case of Robert is different from Diego's in that Pat already possessed the schemes to which he projected the context. We argued that Pat did not *construct* a scheme that employed images operatively. He already possessed those schemes. Instead, Pat constructed a new *meaning* for "transform a graph".

In closing, we say again that our main purpose was to highlight three arguments. The first is that imagery, as re-presentations of experience, include far more than visualization. The second is that a main function of imagery in students' mathematical learning is that they form images of having reasoned. This includes the kind of reasoning in which Diego engaged, it includes reasoning students use to interpret diagrams or animations, and it includes reasoning they engage in to comprehend a problem situation along with reasoning they engage in to solve it. The third is that imagery, as a construct, does not stand alone. Imagery as a construct is useful only to the extent that it allows one to focus on the contexts of students' schemes and meanings and to employ reflective abstraction as a construct for explaining and investigating students' mathematical learning.

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