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**Experience, Problem Solving,  
and Learning Mathematics:  
Considerations in Developing  
Mathematics Curricula**

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Recent research in cognitive psychology and artificial intelligence (collectively called cognitive science) has made impressive progress in revealing the varieties and intricacies of mathematical problem solving. However, the connections between studies of mathematical problem solving and the practice of teaching mathematics are not always clear. That is to be expected. Due to the very nature of scientific inquiry, studies of problem solving must focus on questions of limited scope with simplified hypotheses, if only to provide a filter with which to separate signal from noise.

Also, cognitive scientists tend to focus on what *is* in the world of mathematical problem solving, which is essential. But mathematics educators have an equal charge to consider what *ought to be* and how it might be achieved. With this in mind, it is clear that research in cognitive science on mathematical problem solving can inform mathematics educators of the current state of affairs and can even suggest constructs that promise powerful ways of thinking about teaching problem solving, but it cannot dictate mathematics curricula or methods of teaching. To improve mathematics teaching and learning, mathematics educators must consider the students' passage through an entire curriculum. Thus, those involved in curriculum development will of necessity always be traveling untrodden terrain, always working beyond established databases.

My purpose in this paper is to discuss an attempt at developing mathematics curricula that draws from research on problem solving and mathematical cognition, but goes beyond it in addressing issues unique to mathematics education—primary among them, mathematics education's concern with the learner throughout a mathematical program. By itself, a collection of models of problem solving on relatively restricted problem sets is insufficient as a basis for designing a mathematics curriculum. A curriculum developer must augment it with some sort of model of a learner passing through the curriculum.

As I have noted, cognitive science has developed a wealth of constructs that have powerful implications for mathematics curriculum development. I wish to make clear from the start, however, that I am writing from the perspective of a practicing mathematics educator. The examples that will be given come largely from my experience as a teacher of mathematics, albeit one who is fairly well informed of the methods and constructs of cognitive science.

Perhaps another caveat is necessary. Ideas are significant only within the context of an overriding universe of discourse. The significance for mathematics education of much of recent research on mathematical problem solving can be appreciated only when we consider the aims of mathematics education. If one thinks of learning mathematics as tantamount to memorizing mathematical “facts,” or an accretionary building of elaborate sets of behaviors, then much of what follows will be irrelevant. If, however, one accepts that the aim of mathematics education is to promote mathematical thinking, then this paper will be of interest.

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A predominant theme of this paper, as one would guess from the title, is that learning mathematics is a constructive process. This is an idea on which Dewey and Piaget each based over 50 years of work. I make no pretense here of extending their ideas. Rather, I will show that much of current research on mathematical problem solving is consistent with Dewey's and Piaget's ideas, and will suggest ways that the three might be joined in mathematics curricula.

Another theme of the paper is that learning mathematics (and solving mathematical problems) is by and large a reflective activity. I will discuss the theoretical side of this claim later. For now, let me give an example. A student in my content course for prospective elementary school teachers was having difficulty understanding the idea of the composition of two "flips" in the coordinate plane. I had constructed several examples, drawing diagrams with specific values for the parameters of the problem, and was at the point where I was hoping that the student would generalize to arbitrary values. After much discussion about method, the student paused and then remarked, "Oh, I see! You don't care what the numbers are; you only care about where you get them and what you do to them!" Though I must have said essentially the same thing (though not as lucidly) at least five times in class, that was not sufficient for this student; she had to do her own reflecting.

A third theme is that to learn mathematics is to learn mathematical problem solving. This is in contrast to the common view that one learns a set of mathematical skills and then learns to apply them to solve problems (e.g., Gagné, 1983). However, the essential feature of constructing mathematical knowledge is the creation of relationships, and creating relationships is the hallmark of mathematical problem solving. I will return to this point in the next section and later in case studies.

The remainder of the paper is divided into three major sections. The first addresses issues that are preliminary to developing problem-based mathematics curricula. The second shows two curricula constructed according to the principles developed in the first section. The third gives a summary and concluding remarks.

## **PRELIMINARIES**

This section is in four parts. The first addresses the issue of what a mathematics curriculum can be. This is especially important when we take the position that the development of mathematical knowledge is based on problem solving. The second part discusses the idea of cognitive objectives of instruction as a basis for curriculum development. The third investigates the role of problem solving in learning mathematics, and its place in mathematics curricula. The fourth part examines the influence of environments in learning mathematics.

### **Mathematics Curricula**

It is common to find references in the literature to the notion that "problem structure" is a determinant factor of students' problem-solving activities (Carpenter et al., 1981; Newell & Simon, 1972). There are two ways to view this idea, depending upon one's ontology of problems. The first is to think of problems existing independently of any solver. Looked at this way, it would be possible to design a problem-based curriculum independently of the cognitive characteristics of the intended audience. It would be the teacher's (pedagogical) problem to "connect" his or her students with the problems. What it means to "connect" students with problems is not clear (Thompson, 1982a).

The second way to think of problems as determinant factors in students' problem solving activities is to locate the problems inside the students' heads. But if we locate problems inside students' heads, then it becomes uncertain what the problem is that a student is solving at any given time. As a result, it would appear that we have no control over what we have taken to be the foundation of the curriculum—the problems.

The issue I am raising is the conflict between realism and constructivism (Lakatos, 1962, 1976; Thompson, 1982a; von Glasersfeld, 1978). The reason for speaking of epistemologies here is that I will be characterizing a curriculum as a collection of activities from which students may construct the mathematical knowledge that we want them to have. There are issues central to curriculum development that must be addressed when “curriculum” is used this way, but if the reader is not first made aware of the special sense given here to curriculum, and if the underlying constructivist epistemology is not made explicit, he or she may wonder about the points of the discussions.

Let me say it again in a slightly different way. By a mathematics curriculum I mean a selected sequence of activities, situations, contexts, and so on, from which students will, it is hoped, construct a particular way of thinking. This characterization differs considerably from most that I have seen. Notice that there are no concepts in a curriculum, nor are there topics. Concepts are the structures of thought that a curriculum is aimed at promoting. Topics are our constructions—resulting either from differentiating among concepts or classifying types of activities. Also note that a curriculum never exists in its entirety. There are the planned curriculum, the past curriculum, and the current activities of the students.

To characterize mathematics curricula as I have in the preceding discussion would seem to open any suggestions that I might give about curriculum development to the same criticism made of Dewey's progressive education—that one loses the subject matter when grounding a curriculum in students' experiences. At the time, that criticism was valid (Eisner, 1982). However, as Dewey later noted, poor implementations of sound ideas do not affect the quality of the ideas. Subject matter *had* been neglected.

One consideration stands out clearly when education is conceived in terms of experience. Anything which can be called a study, whether arithmetic, history, geography, or one of the natural sciences, must be derived from materials which at the outset fall within the scope of ordinary life-experience. . . . But finding the material for learning within experience is only the first step. The next step is the progressive development of what is already experienced into a fuller and richer and also more organized form, a form that gradually approximates that in which subject-matter is presented to the skilled, mature adult. (Dewey, 1945, pp. 73ff)

Dewey (1945) also noted that not just any set of experiences would suffice as a curriculum. Special care had to be taken in the selection of experiences for their educative potential—what could be made of them later in the curriculum.

It is also essential that the new objects and events be related intellectually to those of earlier experiences, and this means that there be some advance made in conscious articulation of facts and ideas. It thus becomes the office of the educator to select those

things within the range of existing experience that have promise and potentiality of presenting new problems which by stimulating new ways of observation and judgment will expand the area of further experience. He must constantly regard what is already won not as a fixed possession but as an agency and instrumentality for opening new fields which make new demands upon existing powers of observation and of intelligent use of memory. (p.75)

The principal difficulty that Dewey had in implementing his theory, as he admitted, was that he lacked a philosophy of experience (p. 91). By this I mean that he had neither a metaphor nor a technology through which to make specific his ideas about how people can create something like mathematical knowledge from experience. He did not have a genetic epistemology. Dewey was confident that problem solving was the key to the growth of knowledge, as evidenced by his repeated references to it, but he never explained how problem solving provided the key to intellectual growth, nor how one might actively promote it.

By characterizing a mathematics curriculum as a sequence of activities that (it is hoped) leads to students constructing mathematical knowledge of a particular kind, we are forced to face the same difficulty as Dewey. We must make explicit the nature of the knowledge that we hope is constructed and make a case that the chosen activities will promote its construction. How one characterizes the knowledge to be constructed, and how the selected activities might lead to its construction, will greatly influence the choice of activities.

The general stance taken here is that anything we might wish to call mathematical knowledge is a structure of thinking—the structure is a structure of processes. Also, it is assumed that mathematical structures arise from abstracting the invariant features of one's thinking in problematic situations. From a constructivist perspective, any curriculum aimed at promoting mathematical thinking *must*, by the very nature of the phenomenon, be problem based. The task of the curriculum developer is to select problematic situations that provide occasions for students to think in ways that have a generative power in regard to the objectives of instruction. These points will become clear from the examples given in the case studies.

The advantage that we have today over progressive educationists of mid-century in regard to developing problem-based curricula is one of both theory and technique. Cognitive science has developed a technique (and a metaphor) for explicating processes of which concepts are made. Piaget's genetic epistemology gives us a theory of structure in cognitive processes and its development. The two are supportive of one another in many areas, and there is tension between them in others. Yet, overall they both speak to difficulties in designing problem-based mathematics curricula.

### **Cognitive Objectives of Instruction**

The objectives of a curriculum are the concepts that the curriculum is aimed at promoting. One might ask, what of skills? If one takes a schematic view of concepts, where a schema is one of mental processes, and if one also supposes that skilled behavior is merely an expression of highly structured thinking, then skills as they are colloquially thought of are subsumed by concepts. That is to say, the assessment of concept formation is made by examining the degree of skill shown in problematic situations. From this perspective, students cannot fail to be skilled if, by the above criterion, they demonstrate that they have formed the aimed-for concepts.

The objectives of a curriculum must be expressed in cognitive terms if we are to take them as goals of instruction. This idea did not originate with me. Resnick (1975, 1981) and Greeno (1980) have made a strong enough case that I need not cover that ground here. What I will argue is that cognitive objectives of instruction must have a developmental side as well. Resnick (1981) lamented that there is no cognitive theory of learning upon which to base instruction. I disagree. Piaget, though not normally thought of as a learning theorist, provided a strong framework for addressing knowledge development. His equilibration theory, which says that the epigenesis of knowledge comes from a tension between accommodation and assimilation, says essentially that one learns by solving problems.

Problem solving as a basis for learning mathematics was explicitly promoted by Gestalt psychologists (Duncker, 1945; Wertheimer, 1945). Wertheimer focused more on the idea of mathematical structure than Duncker, who concentrated along the lines of Polya (1973), stressing general strategies for solving problems. Neither Duncker, Wertheimer, nor Polya suggested how a curriculum could be based on problem solving. That is, they did not address the issue of how major mathematical concepts, such as whole number, rational number, and integer, might result from a problem-solving program. To do this they would have had to explicate the composition of the aimed-for concepts and how the problem situations they might set could lead to them.

Piaget and Wertheimer shared the view that equilibrium was a natural state toward which cognition tended, but their views of what constituted cognitive equilibrium, and of avenues to it, differed significantly. Wertheimer (1945), and Gestalt psychologists of his time, saw an isomorphism between the formation of perceptual and conceptual constancies. Piaget (Inhelder & Piaget, 1969) rejected this position on the ground that it could not account for a special character of mathematical concepts—that whatever is thought of as being done can be thought of as being undone.

Gestaltism, as we know, interprets all intelligence as an extension, to wider and wider areas, of the “forms” initially governing the world of perceptions. What we have just said in the preceding pages, however, contradicts such an interpretation. Moreover, as far as the operations [of thought] are concerned, there are the following considerations. The perceptual structures are essentially irreversible. ...But the operations, although they constitute integrated structures, are essentially reversible:  $+n$  is completely cancelled out by  $-n$ . ...It seems obvious, therefore, that operations, or intelligence in general, do not derive from perceptual structures. (pp. 49–50)

In all his work, Piaget clearly distinguished between two modes of thought, which he alluded to in the preceding quotation. The distinction is between figurative and operative thought. In one of Piaget's (1970) more classic characterizations, he described the distinction between figurative and operative thought as one between actions of thought directed towards objects and actions of thought directed towards transformations of objects.

I shall begin by making a distinction between two aspects of thinking that are different, although complementary. One is the figurative aspect, and the other I call the operative aspect. The figurative aspect is an imitation of states taken as momentary and

static. In the cognitive area the figurative functions are, above all, perception, imitation, and mental imagery, which is in fact intereorized imitation. The operative aspect of thought deals not with states but transformations from one state to another. For instance, it includes actions themselves, which transform objects or states, and it also includes the intellectual operations, which are essentially systems of transformation. They are actions that are comparable to other actions but are reversible, that is they can be carried out in both directions...and are capable of being intereorized: they can be carried out through representation and not through actually being acted out. (p. 14)

Piaget's distinction between figurative and operative thought is the most significant that I know of for mathematics education, and especially for the development of problem-based mathematics curricula. A modern translation of Piaget might substitute 'control structures' for 'systems of transformation' (Lawler, 1981; Thompson, 1982b, 1982c). That is to say, operative thought in a particular domain allows students to make propitious decisions about what to do next, and allows them to see what they might do next in relation to what has already taken place.

Recent developments in research on problem solving suggest a parallel with Piaget's ideas regarding levels of thought, lending a specificity that was otherwise lacking. Prominent among them are recent studies in artificial intelligence on planning and understanding. Sacerdoti (1977) implemented a program, called NOAH, that operated at a number of levels in the process of comprehending a problem statement and devising a plan of action. NOAH understood problems by assimilating them to a network of high-level actions, each of which served as a controlling mechanism for conglomerates of lower-level, context-specific actions. A solution to a problem in NOAH took the form of an arrangement of high-level actions with associated constraints that, were they implemented in lower-level actions, would (in NOAH's anticipation) produce a solution to the problem. In other words, the bulk of NOAH's high-level processing was devoted to making decisions about what to do, as opposed to actually doing it.

Hayes-Roth (1980) extended Sacerdoti's ideas by introducing the notion of "opportunities"—situations that, in their model's evaluation, hold some promise for achieving the current goal. The difference between Hayes-Roth's and Sacerdoti's models of planning is that Sacerdoti's emphasizes top-down analysis, while Hayes-Roth's allows both top-down and bottom-up processing. That is, Hayes-Roth's model allows information generated at the more data-specific levels of processing to have both proactive and retroactive implications for assumptions made at higher levels of processing. Also, Hayes-Roth's model is more easily extended so that it learns from experience. It does this by looking back to lower-level actions as a source of unexpected outcomes (Hayes-Roth et al., 1981).

Other artificial intelligence programs that reflect Piaget's division between levels of thought are DEDALUS, a program to construct computer programs to solve problems (Manna & Waldinger, 1977, 1978), Lenat's (1981) program to create heuristics, and Schank's (Schank & Abelson, 1977; Schank & Riesbeck, 1981) programs for comprehending natural discourse.

Recent studies of problem solving using expert/novice comparisons give another perspective on the idea of levels of thinking in problem solving. Larkin (1980) found that the better problem solvers in her physics study tended to operate at both highly structured/low detailed and highly structured/highly detailed levels of thinking. The thinking of poorer problem solvers tended to

have little structure and tended to be bound to highly detailed levels.

Several studies in cognitive psychology and mathematics education have also shown the importance of structure in one's thinking in mathematical problem solving. Krutetskii (1976) found in over 12 years of research that students who were able to grasp the structure of a problem showed the greatest flexibility in problem solving. Resnick (1983) and Thompson (1982b) both found that children whose thinking attained a structural character in regard to whole-number numeration showed the best understanding of the subject, as manifested in the flexibility of their problem-solving behavior. Riley et al. (1983) and Briars and Larkin (1982) showed the importance of structure in children's thinking for solving addition and subtraction word problems.

In summary, concepts are not only natural goals of a mathematics curriculum, they are natural goals of problem-solving instruction. What needs to be made clear is the connection between concepts and curriculum, and between problem solving and concepts.

### **Problem Solving and Learning Mathematics**

The aim of a constructivist mathematics curriculum is that students going through it develop operative structures of thought in relation to the domain of problems constituting the subject matter of the curriculum. The questions to be addressed in this section are, (1) How does operative thought develop? and (2) How does problem solving fit into our attempts to promote its development?

Before taking up those questions, I believe that it will be profitable to return to Piaget's distinction between figurative and operative thought. Piaget apparently offered his distinction in order to contrast sensorimotor and concrete-operational intelligence. I have found it quite useful to generalize it to any level of thought.

When a person's actions of thought remain predominantly within schemata associated with a given level (of control), his or her thinking can be said to be figurative in relation to that level. When the actions of thought move to the level of controlling schemata, then the thinking can be said to be operative in relation to the level of the figurative schemata. That is to say, the relationship between figurative and operative thought is one of figure to ground. Any set of schemata can be characterized as figurative or operative, depending upon whether one is portraying it as background for its controlling schemata or as foreground for the schemata that it controls. For instance, the thinking of a college mathematics major in an advanced calculus course, which certainly would be classified as being formal operational in Piaget's fixed sequence of cognitive development, could nevertheless be classified as figurative in regard to the kind of thinking required in a graduate course in real analysis. Of course we would have to make apparent to ourselves the possibility that the "objects" of such a student's thinking are things like functions, classes of functions, and associated operations.

By generalizing Piaget's distinction between figurative and operative thought we have a way to capture a phenomenon that I call figuration-boundness. This is the inability of a student to go beyond the elements of a problem to a network of relationships and potential transformations in which the elements exist—a phenomenon that I have seen in students ranging from first-graders studying elementary arithmetic to college mathematics majors studying point-set topology.

#### *Development of Operative Thought*

In Piaget's genetic epistemology, the key to the development of operative thought is what he calls reflective abstraction. Piaget characterized two complementary forms of reflective abstraction. The first was the attainment of more systematic control over one's mental actions through the construction of schemata composed of higher-level operations. The second was the reflection of a state of affairs in schemata at a figurative level to a representation of that state in schemata at an operative level. The first might be said to address learning, while the second addresses operative comprehension.

To understand what Piaget meant by reflective abstraction, it is worthwhile to examine its counterpart at the figurative level, empirical abstraction. Empirical abstraction is abstraction from objects (recalling that objects are mental constructs). It is the separating of the object or object's composition into similarities and differences—what Piaget (1951) also called (schematic) differentiation. Piaget (1970) gave the following example:

A child, for instance, can heft objects in his hands and realize that they have different weights—that usually big things weigh more than little ones, but that sometimes little things weigh more than big ones. All this he finds out experientially, and his knowledge is abstracted from the objects themselves. (p. 16)

Reflective abstraction, on the other hand, is knowledge abstracted from coordinated actions. The emphasis is on the transformations these actions bring about and on that which remains constant when they are performed. To extend the preceding example, it is through reflective abstraction that the child comes to know that whatever the weight of an object, it remains the same under transformations of elongation (if it is malleable) or other deformation (as long as nothing is added or taken away). That is, the child's conservation of weight can only be abstracted as an invariant of his or her actions on objects, and not as a property of objects per se.

As the child establishes systems of operations and coordinates them relationally in terms of inversions, reciprocities, or compositions, he or she comes ever closer to a stable state within that system—a form of equilibrium. Equilibrium in this sense means that the operations, through their system of relationships, are capable of compensating perturbations of the system. This was Piaget's definition of conservation. The system of *relationships* is conserved—any state of the system is attainable from any other state (Inhelder & Piaget, 1969). The closure of any system is only relative, however, in that the child may construct operations *from* the system that result in perturbations that the system itself cannot handle (e.g., a concrete operational child trying to coordinate operations on composite units, such as combining torques).

One might think that Piaget's idea of reflective abstraction is similar to abstraction as operationalized in information-processing models (Newell & Simon, 1972), in which the "thing" abstracted is a condensed version of the original. Piaget (1976) made it clear that that is not the case:

Indeed, it should be well understood that an operation is not the [semantic] representation of a transformation; it is, in itself, an object transformation, but one that can be done symbolically, which is by no means the same thing. (pp. 76ff)

The point to be drawn from the quotation is that an operation is not a datum that represents (in a linguistic sense) a transformation: it is itself a transformation that can be executed in place



of, but with the significance of, its figurative correspondent. For example, abstracting the operative structure of constructing an equilateral triangle with ruler and compass results in operations that can be carried out in thought, in place of the figurative transformations that result in a specific triangle. The difference between the two is that logical deductions can be made from the operative structure (e.g., deducing equiangularity of an equilateral triangle), whereas only figurative analogies can be made from the transformations from which the operative structure was abstracted.

As was noted earlier, once a student has created a structure of operations, he or she may reflect the current state of affairs into that structure, and think in terms of possibilities: What would happen *if I did* (or did not) do this? By working within an operative structure, the student may then consider consequences of actions, since then one is not bound by the present situation. That is, the student may begin to generate plans.

### *Problem Solving and Operative Thought*

The complementary forms of reflective abstraction in Piaget's genetic epistemology highlight the reciprocal relationship between problem solving and the development of operative thinking within the content domain of the problems. As students create mathematical objects, they may relate them through the construction of transformations of one to the other. As they relate them, they may reflect those relationships into subsequent problems, in turn enriching their understanding of the objects.

As an example, consider a student's creation of functions as mathematical objects (cf. Piaget et al., 1977). At first, a function is at most a rule for assigning objects (usually numbers) in a domain to objects in a range. The rule of assignment together with its domain and range is the function, but it is often not an object to students in the same sense as the objects being assigned to one another. There are "too many parts" to a function for it to be one thing, as one student said to me.

As students solve problems relative to properties of functions, such as injectivity, surjectivity, and bijectivity (e.g., What could you do to this function so that it is injective?), the objectivity of functions develops. Functions, as mathematical objects in a student's thinking, culminate at two levels. The first is the level of thinking of functions as objects to be composed and decomposed. The second is the level of thinking of functions as objects in a domain and range of a function. Examples of the latter are problems of morphisms between structures, derivatives as mappings between function spaces, and functions defined as limits of sequences of functions (for example, functions as elements in a metric space).

Hidden behind these examples are systems of transformation that allow students to move from aspect to aspect of their concept of function, and systems of transformation that allow them to move between levels of detail. For example, if  $F$  is thought of as a set of differentiable functions, then  $D_x: F \rightarrow F'$  can be thought of in a number of ways, among which are: (1)  $D_x$  is a linear function that associates functions with functions (leaving it at that); (2) pick an element  $f$  in  $F$  and apply  $D_x$  to it (getting a function as a result); (3) pick an element  $f$  in  $F$ , apply the definition of  $D_x$  to  $f$  (getting a derived rule of assignment), and then evaluate  $D_x(f)$  at an element  $a$  in the domain of  $f$ ; and (4) pick an element  $f$  in  $F$ , pick an element  $a$  in the domain of  $f$ , and then apply the definition of  $D_x$  to  $f(a)$ .

Each way of understanding  $D_x: F \rightarrow F'$  has its advantages under some circumstances, and disadvantages under others. The ideal situation is for students to be able to jump from one form

of understanding to another according to the conditions of the problem at hand. The way to promote this ability, as proposed here, is for students to solve problems in which they must construct, and reconstruct, functions as objects, and to solve problems in which they must establish relationships among their various ways of thinking of functions.

Another example of the relationship between problem solving and the development of operative thought can be found in whole-number numeration. Children learn to construct sequences of number-names both forward and backward in increments of one and ten; they learn how to count-on and count-back; they learn how to construct units of ten; they learn how to (conceptually) add and subtract. To have a full concept of ten as a base of a numeration system, they need to relate these schemata in a systematic, highly structured way—knowing each in isolation is insufficient for solving complex problems involving whole-number numeration (Thompson, 1982b). To establish relationships, they must construct transformations among schemata so that they may adjust to the conditions of a problem—for example, by transforming the adding of ten to a quantity into the linguistic transformation of the number-name signifying that quantity by a rule system for “next-ten.” Again, the pedagogical avenue to establishing such relationships is by posing questions of such a nature that students must create relationships to answer them.

A related view of the development of structures in problem solving is set forth by Anderson and colleagues (Anderson et al., 1981; Neves & Anderson, 1981). They do not have a direct counterpart of Piaget’s notion of reflective abstraction. The closest construct to reflective abstraction in their accounts is what they call generalization (Anderson et al., 1981). Generalization is the abstraction of similarities in solutions to problems (in this case, geometry proofs). In effect, they posit that students create data structures of steps in solutions to problems, and abstract common steps taken in problems with similar conditions and goals. The resulting “rule” is a generalization from previous problem solutions in that, since its conditions for application are less specific than the conditions of any of the problems from which it was abstracted, it may be applied to a larger class of problems than that from which it was abstracted.

Although Anderson et al.’s notion of generalization and Piaget’s notion of reflective abstraction would appear to lead to parallel accounts of the development of proof-generation abilities (the subject of Anderson et al.’s article), it becomes less apparent when we look at geometric thinking in a larger context. To Anderson and colleagues, the objects operated upon in geometric proofs are lines, line segments, angles, and so on, which in turn are nodes in a declarative semantic network (Anderson, 1976; Greeno, 1976; Neves & Anderson, 1981). That is, it appears that they view the items of knowledge toward which proofs are aimed as syntactic structures that are generated by manipulating and relating letters, words, and, possibly, visual images. Theirs seems to be a linguistic theory of geometric thinking.<sup>2</sup>

Piaget (1951), on the other hand, put his emphasis on the operational structure of the objects of reasoning, and stressed that operations upon those objects had to be consistent with their nature. In his account, the word (mental sound-image) “triangle” would be associated with certain operations of thought; for example, the operations of moving, turning, moving again, turning towards the starting position, and then going to it. If we take this view of a triangle as a geometric object, then properties of triangles in an individual’s thinking must be theoretically characterized in terms of properties of their operational structure. They may well be

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<sup>2</sup> This statement is not completely accurate. There was some controversy among the authors as to the nature of the objects of geometric thinking. The discussion of learning by subsumption given by Greeno is less linguistically oriented than the other sections. However, subsumption and reflective abstraction address different issues.

characterized in terms of a “semantic” structure, but we would have to expand our meaning of “semantic” to that usually given to “semiotic.” That is, we would have to allow operations as representations of actions (Piaget, 1968). In terms of Piaget’s theory, the highest form of abstraction allowed in Anderson et al.’s theory is pseudoempirical (Piaget, 1977; von Glasersfeld, in preparation)—abstraction from (linguistically encoded) objects. In Piaget’s theory, there is empirical abstraction, but there is also abstraction that has its roots in the operational structure of both the objects and the subject’s actions upon them. If by “declarative” Anderson et al. mean “figurative” in Piaget’s sense, then there could be a great deal of similarity between the two. I find it difficult, however, to read figurative thought into Anderson et al.’s discussions of semantic networks.

### **Influence of the Environment**

Radical constructivist that I am, I feel a bit awkward about having a section with a title like the above. But there is a pressing problem in math education that has become ever more apparent to me: Why, even under what might appear to be the most ideal circumstances, do people in general find it difficult to construct relationships? Raising this issue here is especially important. Not every calculus student of mine creates the different ways of understanding differentiation that I have outlined; not every child that I have worked with has constructed concepts of base-ten numeration. Why? I am sure there are relatively simple explanations for why any given student does not grasp a relationship for which I have set the stage, but I believe that there is a deeper principle in operation as well.

Papert (1980) raised a similar question. He asked: If the vast majority of people construct concrete operations at a relatively early age, then why is it that it takes so long for them to construct formal operations, if they construct them at all? His answer is that the common environment of humans is impoverished of objects amenable to formal thinking. Though Papert’s answer seems to conflict with his constructivist epistemology, that appearance can be avoided. To understand Papert’s observation in a constructivist framework, and its significance for developing mathematics curricula, we need to return to Piaget’s account of the development of concrete operations.

In interacting with a physical environment, a child isolates certain features and disregards others, but always acts on it. In cybernetic terms, the child attempts to control sensory and motor inputs by affecting the environment through actions (MacKay, 1969; Piaget, 1964; Powers, 1973, 1978; Skemp, 1979; von Glasersfeld, 1978). As a cybernetic system, the child is constantly receiving feedback from which the efficacy of actions relative to goals may be judged, and which can thus serve as a basis to refine actions into systems for achieving goals. The key feature here is the continual feedback that the child generates. The child knows when he or she is right or wrong by evaluating achievement of goals. The biological nature of humans is such that concrete operations are an inevitable outcome, within normal constraints, of interaction with the physical environment.

When we examine the development of formal operations, or mathematical thinking, the situation is quite different. The salient actions are entirely mental, and the objects acted upon are ideas. The environment of ideas *is* impoverished, in the sense that there need be nothing intrinsic to an idea that tells the person holding it that what he or she just did (with that idea) will lead to conflict if the line of thought is continued. That is, feedback need not be an intrinsic part of an action upon an idea, beyond the feedback that the action was carried out.

From this perspective it is entirely understandable why students so frequently rely solely on an answer key to tell them whether or not their answers are right. They apparently do not have the mental means to judge the efficacy of their thinking; so they employ a primitive working-backward heuristic. Similarly, it is understandable why children are quite happy to count, say, 12 Dienes' ten-blocks as "10, 20, ... 90, 100, 101, 102." There is nothing in their thinking at that moment that signals to them that their method is faulty. Without an operative structure, they have no means to expect other than that with which they end.

Feedback analogous in its richness to that at the sensorimotor level becomes intrinsic to mental actions only when the outcome of the actions is compared with an expectation. But an expectation of an action upon an idea is an idea, and ideas can only be compared within systems of transformation (Hayes-Roth et al., 1981; Lunzer, 1969). The implication is that we must design mathematics curricula so that students can generate feedback for themselves as to the efficacy of their mental actions (methods of thinking) vis-à-vis the "environment" provided them by mathematical problems. An ideal curriculum would have the potential of being manipulable by students to the extent that they may generate feedback of whatever degree of subtlety they feel is necessary.

### CASE STUDIES

Two examples of mathematics curricula developed in accord with the considerations we have discussed are outlined in this section. They are based on five guiding principles: A mathematics curriculum should

1. Be problem based.
2. Promote reflective abstraction.
3. Contain (but not necessarily be limited to) questions that focus on relationships.
4. Have as its objective a cognitive structure that allows one to think with the structure of the subject matter.
5. Allow students to generate feedback from which they can judge the efficacy of their methods of thinking.

The curricula were designed for use in a middle school mathematics program; I have used them in a mathematics course for prospective elementary school teachers. The first example is drawn from a curriculum for developing a concept of the ring of integers; only that part of the curriculum dealing with the group of integers will be discussed. The second example is a curriculum for developing a concept of Euclidean transformations in a coordinate plane. Each focuses first on developing the mathematical objects of the subject matter, and then on developing operations upon objects, relationships between objects, and relationships between operations. Both curricula are in their formative stages and are being refined as I use them. They are stable enough in their structure and spirit, however, that I feel comfortable sharing them.

The cognitive objectives of the curricula with respect to the mathematical objects are that the students first construct them as transformations between staves, and then reconstruct them as elements themselves to be transformed. This is shown in schematic form in Figure 1. Put another way, the objectives are to have students first construct transformations as "things to act with," and then to reconstruct them as "things to act on."

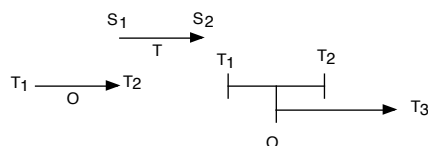


Figure 1. Schematic for levels of cognitive objectives, where “S” stands for a state, “T” for a transformation, and “O” for an operation.

In order to satisfy the principle regarding feedback (5), I have taken an approach consonant with Papert (1980), and Abelson and diSessa (1981), in that I ask my students to use the computer as a medium for exploring mathematics. The implementations that serve as the focus of the curricula are examples of what I call mathematical microworlds. A colleague has characterized them as computerized axiom systems, but I would not put it that way. I prefer to think of them as environments in which students explore their understanding of the subject matter in the same way that scientists test their conjectures about the way the world works.

It is the rare explorer, however, who can begin the journey without a map. The curricula that surround the microworlds are intended, first, to get the students started by enticing them to begin and, second, to refine their knowledge of the terrain by throwing ever more difficult obstacles in their paths. I have yet to answer the question of how, in principle, to handle the interesting side trips that students so often find on their own. That is an issue likely to remain with the classroom teacher.

The two case studies are presented in the following format: a general description of the objectives of the curriculum, then a description of the microworld, and finally examples of activities and problems aimed at achieving the objectives. The activities and problems will be annotated wherever a discussion of their purposes seems warranted.

### Case Study One: Integers

The following discussion of a curriculum for integers owes much to the spirit of Steffe (1976) and Weaver (1982) in taking transformations as a basis for instruction. The work of Vergnaud (1982) was particularly suggestive of the form that the curriculum for integers has taken.

#### *Objects*

An integer as a mathematical object is to be, in the student’s thinking, a composite of a direction and a whole-number magnitude. A direction is not to be universally fixed, such as left/right, but is to be understood as being relative to the currently assumed direction. The genesis of integers is supposed to be (1) double counting-on and counting-back as mental operations, (2) transformations of quantity, (3) directed whole-number magnitudes (Thompson, 1982b).

#### *Operations*

The basic operations upon integers are negation and affirmation. Negation is the operation of deciding to take an integer oppositely from its statement. Affirmation is the operation of deciding to take an integer as it is stated. Negation and affirmation as operations are similar to the ideas of “crossing” and “crossing again” in Brown’s (1972) theory of indication.

### Milieu

The objects and operations are supposed to exist within the general environment of the student's developing ability to compose transformations, construct operations, and to apply operations recursively. This supposition is based on the further assumption that the student's experiences include solving problems in which, to the student, these abilities are demanded.

The epigenesis of addition and subtraction of integers is supposed to be of two sources. The first is compositions of transformations—composing two transformations gives *one* transformation—which have as their source consecutive execution of transformations. The second is an analogy with addition and subtraction of whole numbers.

### Microworld

The integers microworld, called INTEGERS, has a graphics display of a “turtle” (a triangular graphics object) on a number line (Figure 2). At any time, the turtle is facing either left or right; when doing nothing it is, by convention, facing to the right. The turtle thus has a two-property state—a list containing its position on the number line and its current direction.

The number line is divided into three parts: zero, positions to the right of zero, and positions to the left of zero. Positions are named by numbers indicating their distance from zero. Numbers for positions to the right of zero are prefixed by a “+”; numbers for positions to the left of zero are prefixed by a “-”. Note that +5, -70, and so on, are nominal numbers; they are not integers. They merely name positions on a number line by a given convention, no more and no less.

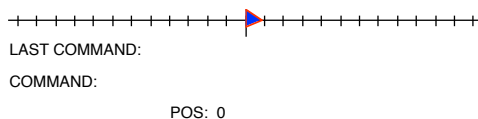


Figure 2: Initial display of INTEGERS. Turtle is at 0; slashes are 10 turtle-steps apart.

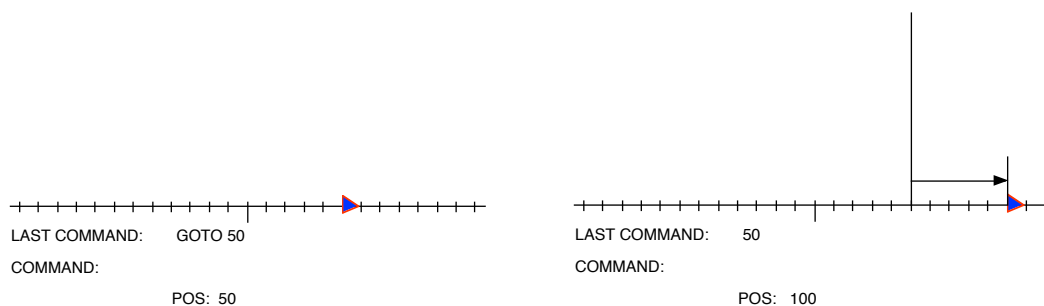


Figure 3: Effect of entering GOTO 50 vs. entering 50; GOTO 50 puts the turtle at the position 50; 50 moves the turtle 50 turtle-steps.

### Curriculum

The curriculum has two levels, corresponding to the levels of cognitive objectives discussed earlier, and illustrated in Figure 1.

*Level I.* Students are introduced to the curriculum with a discussion of points relative to an origin, such as temperature scales on a thermometer, numbering floors in a building that extends above and below ground, and so on. Then they are introduced to the naming conventions of “+”

and “-“ to indicate on which side of the origin the position lies. The students are given a sheet with a cursory description of the microworld, its conventions, an explanation of the display, and a discussion of the idea of entering commands on the keyboard to make the turtle move. The sheet also contains a description of some of the primitive commands that the students have available to them. The commands are given in Table 1.

The students are introduced to INTEGERS by having them locate points on the number line and set the turtle at them by using the GOTO  $x$  command, where  $x$  is a nominal number for a point on the line. Then they are allowed to play with the microworld so that they may begin to develop schemata at an intuitive level for its operations. Finally, they are told that they are going to be asked, eventually, to think of numbers in a new way.

Table 1. Subset of commands in INTEGERS .

Command	Action
<number>	The turtle moves <number> steps in its current direction.
N<number> <sup>a</sup>	The turtle turns around, does <number>, and then turns back around.
GOTO <position>	Places the turtle at <position>.

a. The use of “N” to designate negation is found in the version of INTEGERS for college students. The version for middle school uses “-“.

If a student enters, say, 50 as a command, the turtle will move 50 turtle steps (approximately 5 centimeters) in its current direction (Figure 3) . Fifty, and whole numbers in general, are no longer measures of sets, they are transformations of quantities (Vergnaud, 1982). To highlight this new way of thinking of numbers, the students are asked to solve problems like the following:

- I a.* Enter GOTO 30. What number takes the turtle to 70?
- b.* Someone entered GOTO 25. Then they entered a number. The turtle ended up at 60. What number was entered?
- c.* Develop rules by which you can always predict the answers to the questions like these:
  - (1) The turtle is at  $\blacklozenge$ . What number would you make it end up at #?
  - (2) The turtle began at #. Someone entered a number. The turtle ended up at  $\blacklozenge$ . What number was entered?

At this point in the curriculum, the students are being asked to reflect on connections between successive states of the turtle. That is to say, they are at the entry level of creating integers (transformations) as mathematical objects. The first of the above situations focuses on an integer as a transformation from one state of the turtle to another; the second focuses, reciprocally, on two states of the turtle being related by a transformation. To this point we have involved only what we see as positive integers.

Note that variations on the above situations could be used to promote the idea that two states uniquely determine a transformation, but a transformation determines at most an equivalence class of pairs of states. The idea of transformations as equivalence relations gives a psychological basis for the classical definition of integers as equivalence classes of pairs of whole numbers that are related by a common difference (Herstein, 1975). A pair of states, on the other hand, may be identified with the integer that relates them, but we must be careful not to

equate the two. A pair of states is only representative of those pairs that can be so related.

Students tend to be confused at first when applying the double meaning of numerals in the microworld. There is 50 as the nominal number naming a particular position to the right of zero, and there is 50 as a command that tells the turtle to go 50 steps in its current direction—which doesn't necessarily have anything to do with the position 50. They do, however, quickly separate the two by differentiating the meanings by the conditions of their context. To *enter* "50" is to move; to *name* as "50" is to identify. This situation is not new to the students, and they tend to realize it when it is pointed out that they overcame essentially the same confusion when they were in first and second grade. Then, they double-counted—five (is one), six (is two), seven (is three), and so forth—keeping the two meanings quite separate. However, the students must yet reconstruct this distinction several times over, as we shall see.

There is no reason at this time in the curriculum for students to give any significance to "in its current direction" as a qualifier of the turtle's movement in response to a number being entered. They typically think "to the right" in its place. This thinking will lead them astray in later problems.

The next set of questions and activities brings out the difference between the students' old ways of thinking of numbers and the new ways required of them.

- 2 a. Enter GOTO 70. What number will take the turtle to 50?
- b. Someone entered GOTO -20. Then they entered a number. The turtle ended up at -55. What number did they enter?
- c. When Mary tested her answer to 2a, she intended to enter N 30, but entered 30 instead. Tell Mary what she could enter now to make her turtle end up at 50.

Problems 2a and b are stated as they are purposely. The wording in either could have indicated "something," rather than "number," to be entered. This would have made it easier for students to conclude in b, for example, that N 35 made the turtle move from -20 to -55, because they would not have had to classify "N 35" as representing a number. But the objective of these problems is precisely that the students begin to rethink their idea of number so that N 35 falls under it. By making the problems initially easier to solve, we would merely postpone the time when the students must come to grips with the idea that N x is a number just as much as x is. In fact, by making the problems easier, we might make it more difficult for the students to rethink their idea of number when they must.

There are two ways of thinking of N x that are explicitly addressed within the curriculum. The first is the idea of N as a unary operator upon x; the second is N x as a number. The closest parallel to this distinction that I know of is found in Logo. In Logo, a procedure that takes one or more inputs and produces an output is called an operator. At one level, you can think of an operator as a relationship between three objects: input, procedure, and output. At another level, you can identify the object-output with the procedure and its input. Thus, to think of N x as a number, one must identify the action resulting from N x with the composite object made of the procedure N and its input x. To create N x as a composite object is an extremely difficult task for students. To get somewhat ahead of the discussion, they may do so only when they come to look at x formally, in the sense that it can take on any INTEGERS expression as a value. To appreciate the difficulty of creating integers as composite objects, recall that any INTEGER expression is a procedure. That is, the input to N, N itself, and the output of N are all procedures. The realization that a composition of integer expressions is an integer is similar to students'



creation of derived units, such as specific gravity (Lunzer, 1969).

Problems from non-INTEGERS settings that parallel the structure of those in INTEGERS are also given to the students.

- 3 a. Mr. Quimby had 50 dollars in his pocket. He went into a bank. He came out with 20 dollars.
- (1) Rewrite the statement as if it were given about the turtle in INTEGERS.
  - (2) What happened in the bank?
- b. Gary played a game of marbles. He started with 30 marbles. He ended with 16 marbles.
- (1) Rewrite the statement as if it were given about turtle in INTEGERS.
  - (2) What happened during the game?

The students have already formed a generalization for 1c, which was given in the context of the turtle always moving to its right. After solving a number of problems in which the turtle ends up moving to the left, the students are asked to form a generalization paralleling, and including, 1c, if they haven't done so already.

- 3 c. Develop rules so that you can predict the answers to questions like these:
- (1) The turtle is at  $\bullet$ . What number would you enter to make it end up at  $\#$ ? (Remember,  $\#$  can be either to the right or to the left of  $\bullet$ .)
  - (2) The turtle began at  $\#$ . Someone entered a number. The turtle ended up at  $\bullet$ . What number was entered? (Remember,  $\bullet$  can be either to the right or the left of  $\#$ .)

To this point in the curriculum, students still have not been explicitly asked to think of an integer (transformation) as an object. To do so, they need to abstract the transformation from any particular starting and ending position, and any particular direction, so that it represents a class of possibilities. To draw their attention to this, the students are asked questions such as the following:

- 4 a. Someone entered N 60. What happened?
- b. List as many possible starting and ending positions of the turtle as you can. List them as pairs in the form (start, end). Can you list them all? Why?
- c. What do the pairs of starting and ending positions in b have in common?
- d. Is moving the turtle from 100 to 40 *exactly* the same as entering N 60? That is, will doing one always have the effect of doing the other?
- e. Will  $\langle \text{number} \rangle$  always move the turtle to the right? Why?
- f. Will N  $\langle \text{number} \rangle$  always move the turtle to the left? Why?

To further push the students to thinking of an integer as a formal entity, they are asked to predict the turtle's actions upon their entering such commands as N [N 30], N [N 60], and then to generalize their predictions to N [N  $x$ ]. Their generalization, that N [N  $x$ ] =  $x$ , is explicitly formalized in INTEGERS by the instructor's offering to adopt their generalization as a new convention. That is done by having the students enter the command DEF "P [N [N :X]].

The DEF command is INTEGERS's way of allowing the students to define their own constructs as new operations in INTEGERS. Once P is defined, entering P 50 produces the same

effect as entering N [N 50]. With P defined, N has a counterpart in INTEGERS—the operation of affirmation. To paraphrase one student, using N means that you want to reverse what you had planned to do; using P means that you *really* want to do what you had planned.

*Level II.* The second level of the curriculum, namely, having students think about integers as objects to operate upon, begins with problems like the following:

- 1
  - a. Predict how much and in what direction the turtle will end up moving after you enter N 80 P 70.
  - b. Test your prediction.
  - c. What is the net effect of N 80 P 70?
- 2 John entered GOTO -60 on his computer and Fred entered GOTO 40 on his. They then (independently) predicted what would happen if they entered N 60 P 90, and found that they made the same prediction. Can they both be right? Why?
- 3
  - a. Helen entered P 80 N 90. What is the net effect of Helen's command? What could she enter to make the turtle return to its starting position?
  - b. Bruce entered P 70, and then he entered N 70. Where did the turtle end up?
  - c. Rebecca entered three separate commands. The first two commands disappeared under the turtle's screen after she entered the third, so she couldn't see them anymore. She noticed that the net effect of all three was P 30. She remembered that the second command was N 60; she could see that the third was P 100. Tell Rebecca what her first command was.
  - d. James entered P 60 N 30 P 40. Then he entered N [P 60 N 30 p 40]. Where did the turtle end up?
- 4
  - a. Paul played two games of marbles. At the first game he won 6 marbles. At the second game he lost 4 marbles. What has happened altogether? (Vergnaud, 1982)
  - b. Mary made two transactions at a bank. The second transaction was a deposit of \$40. She left the bank with \$25 less than when she entered. What was the first transaction?

Problems 1 and 3 leave the turtle's initial and ending positions unspecified. The students must think, either initially or in retrospect, in terms of an arbitrary starting position, if they think of one at all. Some students will pick a starting position—usually 0. If they do, and if they answer in terms of a specific ending position (e.g., the turtle ended at 0), they are immediately asked if their answers would be correct were the turtle to start at, say, 10? 20?—40? That is, they are encouraged to rethink their method of answering, as distinct from rethinking their answer. Questions in problem 4 move out of the INTEGERS microworld, but ask students to think with the same structure as within it.

Questions in Level II are also an entrance into compositions of transformations as consecutive execution of two or more integers. The term “net effect” (lc) is used to refer to the elementary integer that could be entered in place of the more complex expression. As an aside, students frequently have trouble deciding the net effect of something like N 60 P 60. It seems that it is not until they think of entering 0 as a way to tell the turtle to stay put that they see 0 as an integer.

Question 3d stresses the formal relationship between  $x$  and  $N x$ . Often students can say that expressions like 30 and N 30 cancel each other, yet cannot see the same relationship in 3d, instead acting out each of the steps in the turtle's itinerary.

Consecutive execution is the first time that it is important for students to understand that the

turtle resets its heading after performing a command. If they don't recall this, the students will predict that  $N \times P \times y$  results in  $N(x + y)$ . I should point out that, in line with much of the work surrounding Logo (Papert, 1980), the students are encouraged to act out their commands with the turtle whenever they either get an unexpected result or cannot predict the effect of a command.

One formalization of composition of integers that the students are asked to make is that of consecutive execution as addition of integers. The context of the activity is that they are asked to generalize from addition of whole numbers in order to define addition of integers. The only guidance that they are given is that they should make *some* definition, named ADD, and test it to see if it is satisfactory—that it adds “whole numbers” in the way we think it should. The definition that is usually given is, in form,

DEF "ADD [P : X P :Y]  
or DEF "ADD [:X :Y].

Either way of defining addition of integers results in the turtle doing the first argument and then doing the second. The second definition may appear strange to the programming-wise reader, since there are no commands within the defining part of it, only variables. However, the values of the variables are integers or integer expressions, which are in fact “commands” to move the turtle—they are procedures. That is, declarative knowledge in INTEGERS is no more than represented procedural knowledge (see Figures 4 and 5).

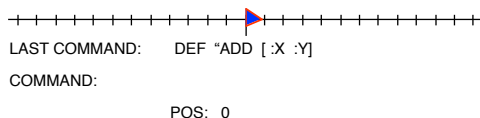


Figure 4. DEF “ADD [:X :Y]. Commands INTEGERS to add ADD to its set of defined commands. The colon stipulates that X and Y are variables whose values will be given by the user at the time of ADD’s execution. If the value entered for either X or Y is not an integer, INTEGERS will complain.

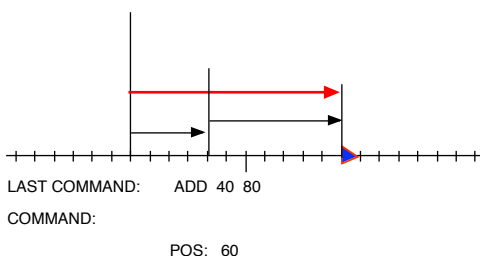


Figure 5. ADD 40 80. Commands the turtle to do 40 and then to do 80. (See Figure 4.)

A goal of the curriculum for integers is that the students develop a method of thinking in regard to definitions. They are encouraged to explore the consequences of a definition. For example, once they define addition of integers in a way that conforms to their intuitions of addition of whole numbers, they are asked to predict the results of the addition of non-whole-number integers, such as ADD 80 [N 70] and ADD [N 40] [N 60].

By defining addition and subtraction of integers, the students begin to solidify integers as mathematical objects—as something to be acted upon. They are asked to further generalize their notion of an integer as an object so that it includes integer expressions. Questions that focus on integer expressions as integers are

- 5 a. Predict the net effect of N [ADD 50 [N 70]] .
- b. Translate problem 4a into an INTEGERS expression that uses ADD.
- c. ADD [N 60] [\_\_\_] = P 30. What integer goes in the blank space?

d.  $N 60 = \text{ADD } [\_] [P 70]$ . What integer goes in the blank space?

With regard to 5a, note that in the list of commands for INTEGERS (Table 1) N is said to operate only on numbers. When this is pointed out to students, it is a real problem for them to reconcile that requirement with their picture of  $\text{ADD } :X :Y$  as a procedure that itself operates on numbers.

Just as students are asked to construct the binary operation of addition of integers, they are asked to construct the binary operation of subtraction. Several definitions (called SUB for uniformity) have been given:

```
DEF "SUB [:X [N :Y]],
DEF "SUB [P :X N :Y]
DEF "SUB [[:X] [N :Y]],
DEF "SUB [ADD [:X] [N :Y]].
```

I will not say anything about the variations in the use of braces in the above definitions, except that the only constraint placed by INTEGERS upon the definer is that an expression that is to be taken as a unit must be enclosed in braces. Braces must also be used to avoid ambiguity, as in  $N 30 50$ , for example.  $N [30 50]$  is unambiguous, being the negation of  $[30 50]$ , but in  $N 30 50$  it is unclear whether  $[N 30] 50$  or  $N [30 50]$  is intended.

Students who define subtraction as  $\text{ADD } [:X] [N :Y]$  have already formed a generalization that others will be asked to make, namely that  $\text{SUB } :X :Y$  is equivalent to  $\text{ADD } :X [N :Y]$ . Further generalizations that they are asked to explore are

```
SUB :X [N :Y] = ADD [ ] [ ],
N [ADD :X :Y] = SUB [ ] [ ],
ADD :X :Y = N [ADD [ ] [ ]],
N [SUB :X :Y] = ADD [ ] [ ],
SUB :X :Y = N [SUB [ ] [ ]], and so on.
```

Structural properties of addition and subtraction of integers are emphasized by having the students develop “convincing arguments” (i.e., arguments that will convince their doubting neighbors) that their generalizations will always work. Structure is also emphasized by having them examine questions such as

- 6 a. Does the net effect of ADD depend upon the order in which you supply its inputs?  
 b. Does the net effect of SUB depend upon the order in which you supply its inputs?  
 c. John played three games of marbles. In one game he won 8 marbles, in another he lost 5, and in another he lost 9. (The actual order in which he played the three games is not the order in which they are listed.)
- (1) Write an INTEGERS expression for John’s marble account.
  - (2) Compare your expression with your neighbors.’ Are they exactly the same? Does it matter? Why?

So that students may “reify” relationships established among operations and structural properties of operations, they are given expressions to simplify and new definitions to create, as in

- 7 a. Predict the net effect of SUB [SUB 50 [N 30]] [N[ADD 40 [N 80]]].  
(Figure 6 shows what would be displayed if this expression was entered.)
- b. Predict the net effect of N [ADD [N [SUB 70 20]] 90].
- c. Define a new operation, called "CANCEL, using a combination of ADD, SUB, and N so that CANCEL 50 30, CANCEL 70 [N 40], and CANCEL [N 20] [N 80] each end up doing nothing. That is, define it so that no matter what two integers you put into CANCEL, the net effect is 0.

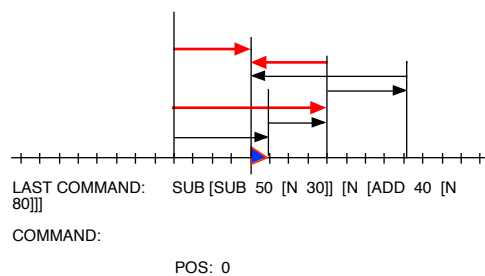


Figure 6. Effect of entering SUB [SUB 50 [N 30]] [N [ADD 40 [N 80]]]. Black arrows correspond to “simple” integers; white arrows correspond to “structured” integers.

Another type of question focuses on the layers of structure that can be put on complex expressions. Students are asked to describe pictures like those in Figure 7 in terms of integer expressions, at three levels of description.

To summarize, the curriculum for integers comprises a collection of activities and questions that are aimed at promoting the students’ construction of integers as mathematical objects, and their construction of transformations and operations upon those objects. All of this takes place within the context of exploring the behavior of a turtle on a number line when it is given commands to perform various activities. The theoretical basis of the curriculum is the position that meaningful, formal mathematical reasoning develops semiotically—a transformation at one level of thought comes to stand for multitudes of possibilities at a lower level. The significance of this curriculum for solving problems involving integers is, as Lawler (1981) put it, that “the control structures of mind embody the genetic path of learning.” As students formalize their actions of thought in regard to integers, the figurations (operations of mentally moving an object) then become models they can drop back to in overcoming obstacles during formal reasoning.

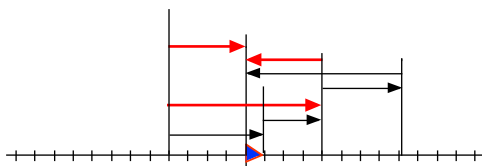


Figure 7. Picture of the effect of entering an integer. Students are to describe this integer (uppermost colored arrow) at three different levels of structure. Lowest level would be [P 50 P 30 P 40 N 80]. Highest level would be ADD [ADD [P 50] [P 30]] [ADD [P 40] [N 80]] or its equivalent (cf. Figure 6).

### Case Study Two: Transformation Geometry

The curriculum for transformation geometry is similar in spirit to that for integers. The students are asked to create the objects of the subject matter, to relate them, and to construct operations upon them. However, the structure of the subject matter in transformation geometry

differs qualitatively from that in integers. The objects are more numerous, and their composition is more complex. Therefore, the relationships among the objects and the operations upon them are more difficult for students to construct than is the case with integers. The curriculum reflects this.

### *Objects*

The objects within transformation geometry are rotations, translations, and reflections (“flips”). Each is to be, in the students’ thinking, a mapping of the plane onto itself by the criterion that the images of points remain the same distance apart as were their preimages. Rotations, translations, and flips are special cases of this definition.

- A rotation is a mapping of the plane that leaves all points equidistant from a given point.
- A translation is a mapping of the plane that makes all segments joining images and preimages parallel.
- A flip is a mapping of the plane that leaves images and preimages equidistant from a given line.

I have found that as characterizations of the intended objects in students’ thinking, the preceding are too facile and too powerful. They are too powerful in that most of transformation geometry can be constructed from them; they have far-reaching implications that are uncovered only after a great deal of elaboration. They are too facile in that students can “record” them as verbalizations, and yet fail to understand them in terms of anything resembling an operational structure.

The cognitive objectives of the curriculum are transformations of figures in a coordinate plane. In terms of the schematic in Figure 1, transformations of a figure are at Level I. The transformations serve to connect “states” of a figure, where a state is determined by some well-defined criterion. A state can be at, or between, two levels in a student’s understanding: it can be a perceptual image, on par with any object in the student’s field of vision, or it can be a list of property values. The properties used in the curriculum are (1) the figure’s position in a rectangular coordinate plane, (2) its heading within a polar coordinate system, and (3) its orientation (each determined by an appropriately well-defined method).

As a connection between visual states of a figure, a transformation is nothing more than the students’ estimate of the effects upon the figure of performing a physical relocation. As a connection between property values of a figure, a transformation is a multivariate mapping. The latter is clearly the more formal of the two, in that once students have the means to predict the effects of a transformation upon a figure’s properties, they have in essence developed a system in which they can consider possibilities regardless of whether or not the figure is in their field of vision, and regardless of whether a figure has even been specified. That is, it is only in the case of the latter that a transformation in the students’ thinking approximates the idea of a transformation being an isometric map of the plane onto itself. The mapping formally preserves perceptual structure.

I should point out that the creation of a system in which my students may predict the effect of a motion upon a figure’s properties has been extremely difficult for them. One reason for this, I believe, is that they have difficulty equating a particular figure in the plane with an instantiation of the variable list [figure (position heading orientation)].

Put another way, students appear to consider a transformation at first as being applied to, and only to, that figure that is currently in their attention, and not to the class of congruent figures of which the current one is only representative. They do not look at a figure as a variable. Thus, for the curriculum for transformation geometry to emphasize transformations as objects, it must first emphasize the states upon which transformations operate as variables. This was not apparent in the curriculum for integers, as the college students in my courses thought naturally of position as a variable value. It is quite possible that middle school students will not.

One could make a case that the ontogeneses of variables and functions are inextricably linked, as did Piaget et al. (1977). If that were the case, the implication would seem to be that instructionally the two are inseparable. From another perspective, however, they are separable—to an extent. To bring the students to a level where, to them, the states upon which a transformation operates are objects in and of themselves, one might ask them to build concrete-operational action schemata for the transformations, indirectly placing the states into the realm of possibilities. At least then it would make sense to them to speak of a transformation independently of any particular state of the figure being transformed.

The genesis of transformations in the plane as mathematical objects can now be supposed to be (1) action schemata of physical displacement, (2) formalization of variable figures (both within and across equivalence classes of figures), (3) transformations as functions between lists of property values of a class of figures, and (4) transformations as isometric mappings of the plane.

### *Operations*

The essential operation upon transformations in the plane is composition. The genesis of composition is supposed to be (1) coexecution—doing one transformation (of any state), doing another (of any state), and differentiating between the two; (2) consecutive execution—doing one transformation and then doing another upon the state resulting from the first; and (3) composition—the result of consecutively executing two transformations being taken as a transformation in and of itself. Note that the genesis of transformations as characterized here assumes the complexity of the genesis of transformations as mathematical objects.

### *Milieu*

As with the integers curriculum, the objects and operations of transformation geometry are supposed to exist within the students' emerging abilities to compose transformations, construct operations, and apply operations recursively. As always, this supposition is made with the assumption that the students' experiences include solving problems in which, to them, these abilities are demanded.

### *Microworld*

The transformation geometry microworld, called MOTIONS, has as its initial display a picture of a Cartesian coordinate system and a flag standing upright on the origin (Figure 8). The flag lies within two frames of reference, the Cartesian coordinate system and the polar coordinate system. It thus has several properties that serve to uniquely identify it—its heading (the direction that the turtle is pointing as it sits waiting for a command from the keyboard), its position (the position occupied by the turtle), and its orientation (the direction that the turtle turns when

constructing the flag).

Other properties for which the program maintains values, but which are redundant with the above three, are the flag's distance from the origin (measured from the turtle); the measure of the angle formed by the turtle, the origin, and the positive  $x$ -axis; and the turtle's  $x$ - and  $y$ -coordinates. Even though these are redundant, it is pedagogically useful to have them maintained.

As a note, the students are not limited to operating upon the figure initially supplied by the program; they may define new figures at will. The pedagogical utility of the capability to define new figures will be discussed in the section Symmetries of Plane Figures.

The transformations initially supplied to the students are translations, rotations about the origin, and flips through lines that pass through the origin. The microworld works as follows: The student enters a command, or sequence of commands, and then the program executes it. For each command, a flag is drawn according to the command's effect upon the flag's properties.

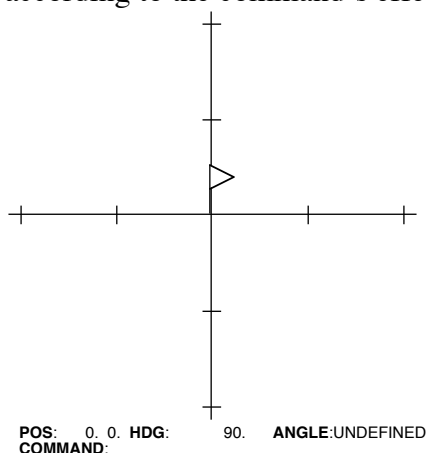


Figure 8. Initial display of MOTIONS. Slashes on axes are in increments of 50 turtle-steps. Information given in information line: the flag's (turtle's) position, heading, and the measure of the angle formed by the turtle, the origin, and the positive  $x$ -axis.

To rotate the flag about the origin, the student enters  $R$  <number>, where <number> is the measure of the angle through which to rotate it (Figure 9). To translate the flag, the student enters  $T$  <number1> <number2>, where <number1> is a distance to move and <number2> is the heading in which to move it (Figure 10). To flip the flag, the student enters  $F$  <number>, where <number> is the heading of the line passing through the origin through which the flip is to be made (Figure 11).

A number of other commands are available to the students for "housekeeping" (e.g., clearing the screen) or for pedagogical purposes. The full glossary of commands, operations, and maintained variables for MOTIONS is given in the Appendix of this chapter.

### *Curriculum*

Each student is given a copy of the MOTIONS program and a copy of a module that contains activities for them to do and questions for them to consider. In its current form, the module is separated into five sections: Getting Acquainted, Investigations, Symmetries of Plane Figures, Systematic Investigations, and General Motions. In the discussion of each section, I will refer to translations, rotations, and flips as motions, since that is the generic term used in the module.



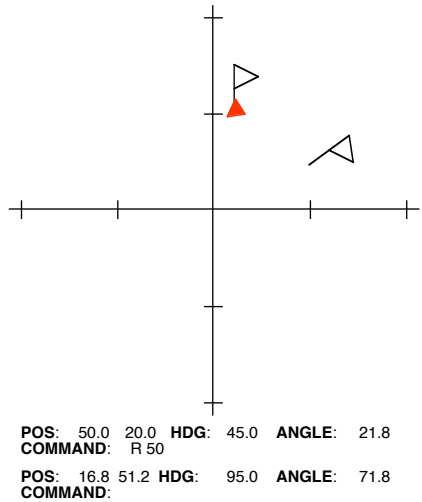


Figure 9. Example of a rotation. Flag started at [50 20] with heading 45 and orientation "RT". Student entered R 50; the flag ended at [16.8 51.2] with heading 95 and orientation "RT".

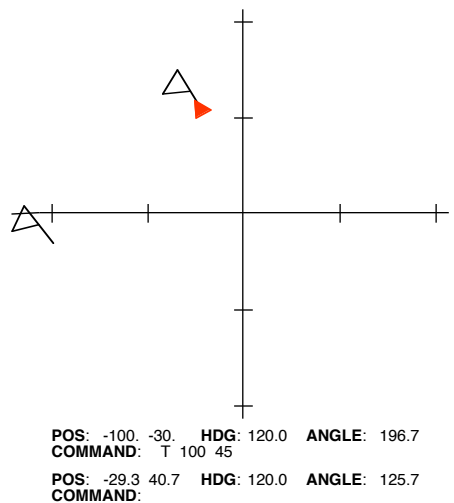


Figure 10. Example of a translation. Flag started at [-100 -30] with the heading 120 and orientation "RT". Student entered T 100 45 (100 turtle-steps in a heading of 45 degrees); the flag ended at [-29.3 40.7] with the heading 120 and the orientation "RT".

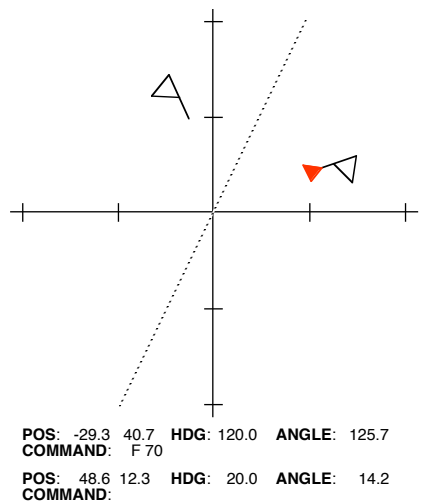


Figure 11. Example of a flip. Flag started at  $[-29.3 \ 40.7]$  with heading 120 and orientation "LT. Student entered F 70 (flip through the line passing through the origin with a heading of 70). Flag ended at  $[48.6 \ 12.3]$  with heading 20 and orientation "RT.

*Getting Acquainted.* In this section, the students are informed of the screen, the information line, and how they should read the screen. They are introduced to the conventions of polar and rectangular coordinate systems, and to the idea of heading (equivalence class of rays). There is usually quite a lot of discussion about heading, as students typically fix upon the origin as the endpoint of any ray. Their fixation upon the origin leads to inordinate difficulties when they come to heading as a parameter of a translation.

The activities in the Getting Acquainted section are designed with two aims in mind—that we have occasions to discuss the idea of any flag being only one of an infinite number of possibilities, and that the students internalize the motions, that is, construct schemata whereby they can give visual estimates of the effect of a motion upon any particular flag. By “estimate” I mean that the students predict what they will see on the screen as a consequence of commanding the flag to move by a particular motion.

The activities that focus on the idea of the flag as a variable have the students use the GOTO command to make the flag appear on the screen as it does in a diagram in the module. The parameters of GOTO are the flag’s  $x$ - and  $y$ -coordinates, its heading, and its orientation. The assumption is that in their examination of the pictures, the students differentiate their “flag” schema so that position, heading, and orientation are variable values in it. That is, the aim is that they see any particular flag as an instantiation of its formal properties. I must point out that these activities by themselves are not sufficient to accomplish that aim, nor are they expected to. It has been my experience that most students do not formalize the flag until they develop schemata for transformations of it. More than a few never formalize it.

The activities directed toward having the students internalize each of the motions are similar to those for formalizing the flag. Students are presented with pictures to create (Figure 12), where the means of creation is restricted to using the three motions. The kind of thinking aimed for is: The flag is here and I want to get it there. Which motion should I use? What values shall I put into it? It is hoped that in deciding upon what needs to be done to a flag in order to generate a picture, the students will separate the motions from one another by the effects they have upon the flag’s properties.

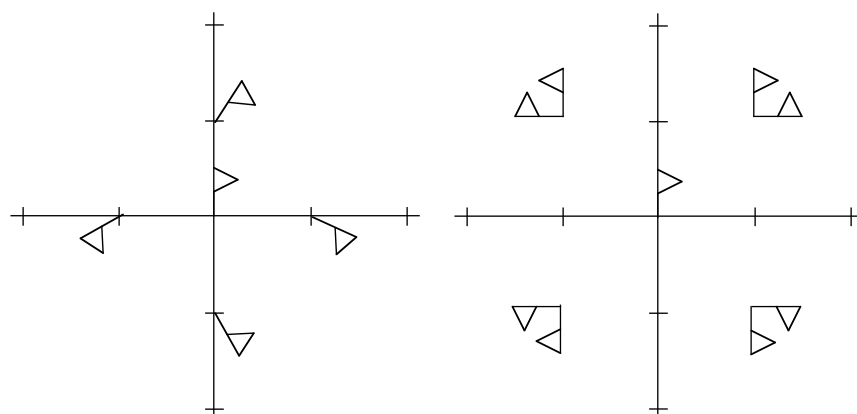


Figure 12. Pictures for the student to create using rotations, translations, and/or flips.

As an aside, I have found that in class discussions my use of the definite article with “flag” often has a very different meaning for me than it has for my students. When I speak of “the” flag, I normally mean the equivalence class of “particular” flags. Through the beginning of the curriculum, when students use the definite article, they are normally referring to a specific visual image. Our differences in significance became most apparent when we were discussing the general effects of a motion upon the properties of the flag. I was speaking of “the” flag, when from their perspective, sometimes there were many flags and sometimes none. It took some time before we sorted this difficulty out. Afterward, I revised the module so that it directly addressed the notion of the flag as a variable.

*Investigations.* This section contains two distinct sets of activities and questions. The first focuses on motions as mappings between properties of the flag—that is, at this point in the curriculum the emphasis is on the initial development of motions in the plane as transformations of states. The second set of activities and questions focuses on composition as an operation upon motions. It is in the second set that the students are asked to create a motion as an object in and of itself.

The following are examples of activities and questions that focus on motion as a mapping between states of the flag.

1. Do each of the following and fill in the table.

a. (GOTO 80 0 90 "LT) CL

b. R 40

R 70

R -180

R 60

R -50

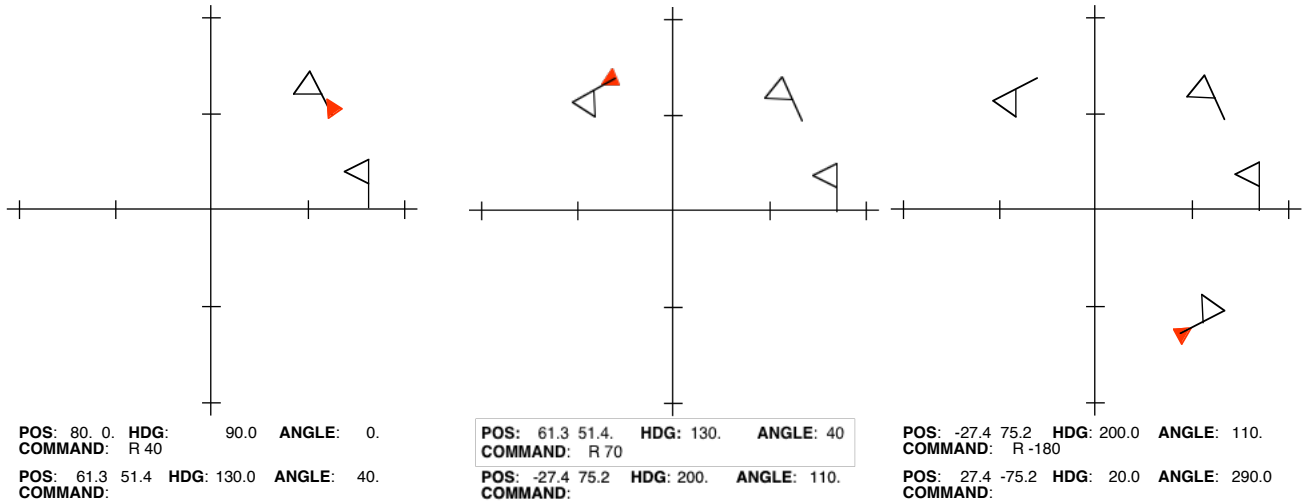
Old heading	New heading
90	130
130	—
—	—
—	—
—	—

(Figure 13 shows the displays that a student would see.)

2. Complete this statement: If the flag’s heading is  $x$ , its heading after doing R  $y$  will be \_\_\_\_.

3. Do CLS R 670. Does the flag’s new heading agree with your generalization in 2?

4. Do CLS F 45. The result looks like a rotation by  $-90$  degrees. Is it?



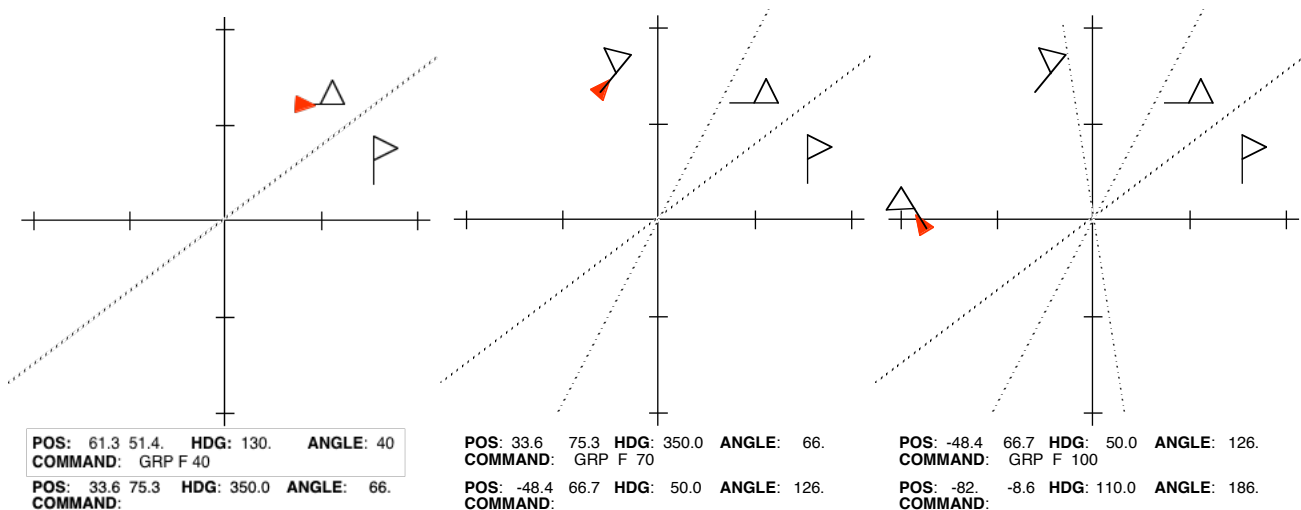
**Figure 13.** The display as a student does successive rotations. First rotation is R 40; second is R 70; third is R -180. Students are to abstract (1) the effect of R  $x$  upon the heading of the flag, and (2) that the flag remains the same distance from the origin.

- Do (GOTO 80 20 90 "RT) CL RP. Now fill in the following table. Precede each flip with GRP.<sup>3</sup>

Old heading		New heading
90	F 40	—
90	F 70	—
90	F 100	—

(Figure 14 shows the displays that student would see.)

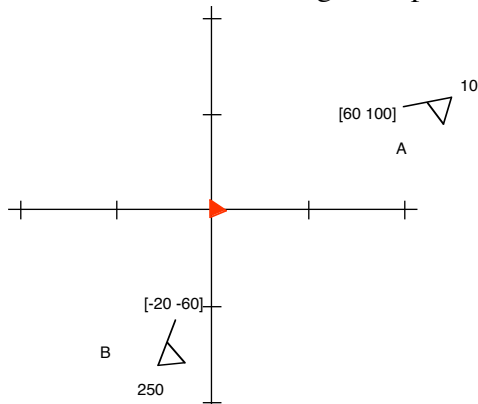
- The initial heading of the flag was 40. Someone entered F \_\_. The flag's heading ended up being 110. What was the flip's heading?



**Figure 14.** The display as a student does successive flips (each flip is performed upon the flag with properties [[80 20] 90 "RT]). The first flip is F 40; the second is F 70; the third is F 100. Students are to abstract (1) the effect of a flip upon the flag's heading and orientation, and (2) that the segment formed by any point and its image is bisected perpendicularly by the line of the flip.

<sup>3</sup> RP tells MOTIONS to record the current figure's position, heading, and orientation. GRP tells MOTIONS to return to the last state recorded by RP.

7. Give a sequence of motions that will take the flag from position *A* to position *B*.



Predictions of the effect of a motion on the flag's position are not stressed, except in very simple situations. To make predictions in general would require a background in trigonometry, which these students typically do not have.

The following problems aim at having the students separate a motion from its operand, and at having them abstract the properties of a motion per se. These problems usually prove difficult for students. This appears to be because they must consolidate the details of a motion into one unit, which requires a great deal of reflection. That is, they must separate the action of mentally carrying out a motion from the experience of applying those actions to a particular flag. Apparently, the schema formed from such abstractions is at a pseudooperative level, as it is not bound to any particular image (visual or mental) or to a flag. However, they must still understand that a figure is being transformed and that it has a position, a heading, and an orientation.

8. Clear the screen.
  - a. Do R 30 R 60. Now do R 90.
  - b. Compare the two results of a. How are they different? How are they the same?
  - c. Do (GOTO -50 -70 120 "LT) CL. Do R 40 R 35. Now do R 75.
  - d. Compare the two results of b. How are they different? How are they the same?
9. What motion is [R 20 R 80]? How is it related to R 20 followed by R 80?
10. Complete this statement: [R x R y] =
11. What translation is
  - a. [T 60 270 T 80 0]?
  - b. [T 30 80 T 40 170]?
  - c. [T 50 0 T 80 90]?
12. Predict the effect of [T 80 45 T 60 0]. Test your prediction. Does the result of your test agree with your expectation? Why?
13. a. Do CLS. Now do T 60 160 T 40 -50. (See Figure 15.) What translation will take the flag back to its starting position?
  - b. Do (GOTO 80 -50 45 "LT) CL. Now do T 60 160 T 40 -50. (See Figure 16.) What translation will take the flag back to its starting position? [*Hint*: Refer to a.]
14. Do DEF "TWO.FLIPS [[X Y] F :X F :Y]. (See Appendix for a description of the DEF command.)
  - a. What does TWO.FLIPS do?
  - b. Test TWO.FLIPS with various inputs. What kind of motion is TWO.FLIPS?
  - c. Complete this statement: TWO.FLIPS x y = \_\_\_\_.

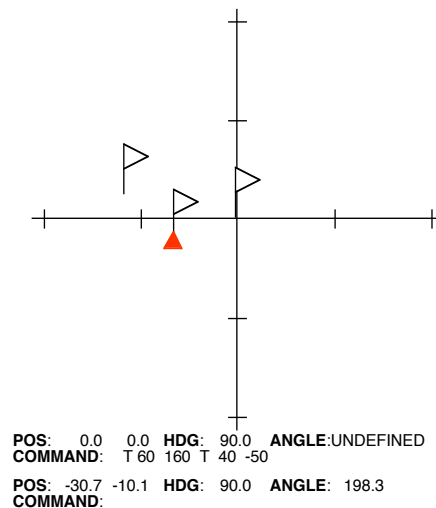
- d. What is the effect of  $[F x \text{ TWO.FLIPS } y z]$ ?
15. The flag was taken from state  $A$  to state  $B$  by  $[F 40 T -100 120 F 30 R 70]$ . Write a sequence of motions that will take the flag from state  $B$  to state  $A$ .

The reader should note the use of brackets in the above questions. Whenever you enter a sequence of commands on a single line, MOTIONS carries them out one at a time, just as it would were they entered separately. In other words, MOTIONS shows the effect of each motion upon the flag that is current at the time of its execution. If the sequence is enclosed in brackets, however, MOTIONS shows only the result of the composition of the sequence. In other words, enclosing a sequence of motions within brackets signifies the composition of the sequence.

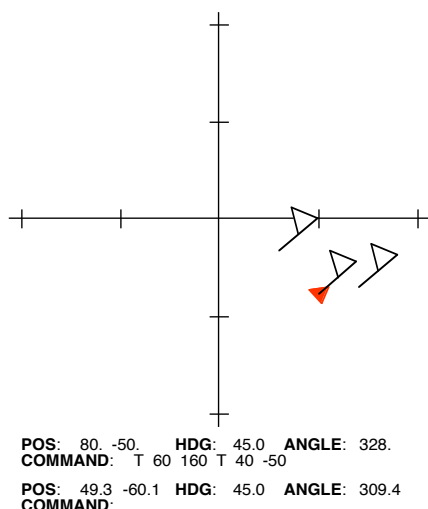
Each of the activities and questions in the preceding group of problems fits with the general aim of promoting students' reflection of the actions of carrying out a motion to a level of operativity. Perhaps an anecdote will be illustrative.

I was working in my office one evening when two students literally banged on my door. They wanted to share a discovery. They had been working problem 14d and quickly decided that since they knew that the net effect of TWO.FLIPS is a rotation, they would focus upon the net effect of  $F x$ . After they entered a flip on the computer, and stared at the screen for a number of minutes, it dawned on them that the net effect of a flip is a flip. This is the discovery that they wanted to share. That a flip is a flip.

I take their discovery (and to them it was a discovery) to mean that up to that point they had looked at finding net effects much in the same way that young children look at finding sums. There was the first motion, the second motion, and the result—three separate and distinct entities. What these students discovered, I believe, is that the net effect (composition) of two motions *is* those two motions, but in combination. That is, I believe that what they discovered had nothing to do with flips. Rather, they created (at least for a time) a composition as an object in and of itself, but one that has an interior.



**Figure 15.** CLS followed by  $T 60 160 T 40 -50$ . What translation will take the flag back to its starting position (in this case, home)?



**Figure 16.** GOTO 80 -50 45 "LT followed by T 60 160 T 40 -50. What translation will take the flag back to its starting position? (Compare with Figure 15.)

*Symmetries of Plane Figures.* There are two reasons for including a study of symmetry in the curriculum. The first is that the investigation of symmetries of plane figures provides a natural occasion to discuss sets of motions that are closed under composition. The second is that it provides an opportunity to pose questions that aim at having the students generalize their concept of a motion in the plane.

16. Enter FIG [PU FD 80 LT 54 PD REPEAT 5 [LT 72 FD 94.02]].  
 This defines a regular pentagon centered at the turtle. (See Figure 17.)
- a. Enter CLS.
    - (1) List the inputs to R that will result in the figure coinciding with itself.
    - (2) List the inputs to F that will result in the figure coinciding with itself.
  - b. Enter (GOTO 0 0 100 "RT) CL.
    - (1) Answer 16a(1).
    - (2) Answer 16a(2).

Students typically answer question 16a without considering that their answers assume the figure to be in its home position, and are surprised when their answers no longer work for the figure in a non-home position. They are equally resurprised when they find that in 16b they end up giving the same answer for rotations as before, but different answers for flips.

- 17 a. Define a motion, called F.SYM, for which the inputs that you listed in 16a(2) will make the figure coincide with itself (assuming that the turtle is on the origin).
- b. Define a motion, called R.SYM, for which the inputs that you listed in 16a(1) will make the figure coincide with itself, regardless of the figure's position and heading.

Most typically, question 17a is answered  
 DEF "F.SYM[[X] F :HDG + :X].

The most typical answers to 17b are

DEF "R.SYM [[X] (T -:DIST :ANGLE) (R :X) (T DIST :ANGLE)],  
 and DEF "R.SYM [[X] (T -:XCOR 0) (T -:YCOR 90) (R :X) (T :XCOR 0) (T :YCOR 90)].

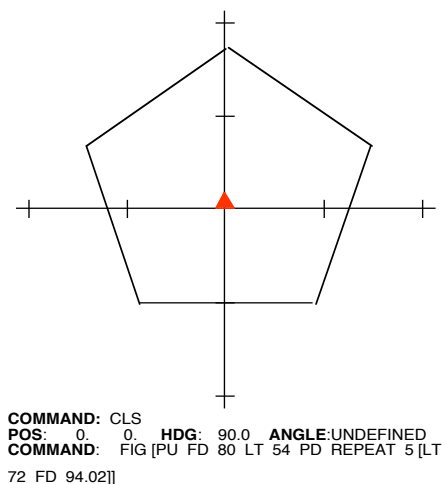
The following questions focus upon aspects of symmetries in a more general context than did Problem 16.

- 18 a. The motions  $M_1$ ,  $M_2$ ,  $M_3$ , and  $M_4$  each make a figure coincide with itself. Will  $[M_1 M_2 M_3 M_4]$  make the figure coincide with itself? Why?
- b. A figure is coincident with itself under each of  $F 40$ ,  $F 80$ , and  $F 120$ . What might the figure look like? What is the minimum number of sides in such a figure?
- c. Suppose that someone tells you of finding a figure for which  $R 180$ ,  $F 45$ ,  $F 90$ , and  $F 135$  are the only motions under which the figure is symmetric (assuming it is at the origin). Do you believe it? Why?

The questions in 18 are the first in the curriculum that begin to focus on the formal properties of motions in the plane. In 18a the students are asked to think strictly in terms of possibilities. Typically, those who have not objectified motions in the plane (as indicated by their performances in other contexts) either will not know how to approach the question, or will answer it in terms of a specific example. For 18b they will try to construct various figures, and then test them against the requirements stated in the question. As far as I can tell, question 18c is accessible only to those who have built an operational structure, for it requires a determination of impossibility. For instance, if  $F 45$  and  $F 90$  are symmetries of the figure, then  $[F 45 F 90] = R 90$  *must* be a symmetry of the figure. Therefore no figure can be symmetric under just those motions.

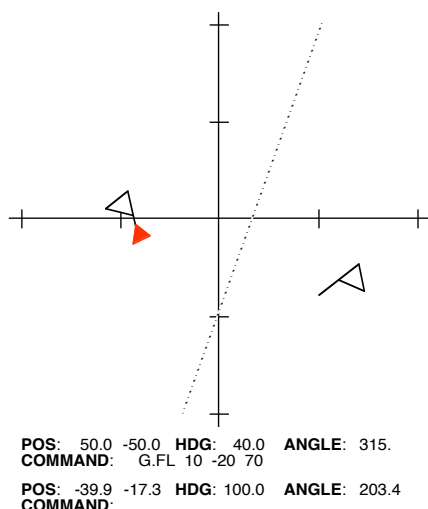
*Systematic Investigations.* The previous sections aimed at having the students create motions as objects in and of themselves, and at having them create composition as a binary operation upon motions. The Systematic Investigations section aims at their coalescence of motions in the plane into sets that have intrinsic algebraic structures. Again, the focus is on reasoning formally about sets of motions—reasoning about possibilities. Problem 19 is an example of questions in this section.

19. Find two sets of motions, each with four elements, that form closed addition tables.



**Figure 17.** New figure created by FIG command. T, R, and F will still operate as before.





**Figure 18.** Example of a student-defined command, called G.FL (glide-flip), that will allow the student to flip through any line in the plane. In this example, G.FL is defined so that its line is represented by a heading and a point that the line passes through. This example commanded MOTIONS to flip the plane through the line passing through [10 -20] with a heading of 70.

Invariably, some students come up with R 0, R 90, R 180, and R 270, as one set, while others come up with R 0, F 0, F 90, and R 180 as another. I count on someone coming up with something like R 0, F 45, F 135, and R 180 so that I can ask if this set is different enough from the second to count it as a different set of motions. Essentially, the subject of the discussions is isomorphism. To emphasize the point, I ask the students to characterize all sets of motions that have four elements, two of which are flips, that form a closed addition table. As an additional challenge, I ask them to try to find two different sets of motions, each with five elements, that form a closed table. There is, in actuality, only one. The real aim of the question is for them to find this out.

*Generalized Motions.* This last section aims at a synthesis of the students' various ways of thinking about motions in the plane—as mappings between properties of a figure, as objects to operate upon, and as elements in algebraic structures. The context of the questions and activities in this section is the problem of how to generalize the motions supplied by the microworld so that it will perform rotations about any point in the plane, and perform flips through any line in the plane (Figure 18). I will not discuss this section or the students' difficulties with it at any length. I will just note that it is only after solving the problems of defining a general rotation and a general flip that they come to think of motions in the plane as point mappings, and that they are truly surprised when all of the generalizations that they have made up to this point continue to hold as special cases for motions in general. Apparently, even when working abstractly with motions, they imagined *a figure* being moved, and not the entire plane—even though it was mentioned many times over that it is the plane that is moved, and not just a figure. This suggests to me that there is much to be uncovered before a complete characterization of operativity in transformation geometry can be made.

## CONCLUSION

I have taken the position that skilled behavior in mathematical problem solving is a

manifestation of highly structured thinking at a number of levels of thought. With that position in mind, I outlined a theoretical basis for design principles of mathematics curricula and offered two examples of curricula designed accordingly.

The examples of curricula that were designed according to these five principles were drawn from integers and transformation geometry. Each was structured so that its aim was first to have the students construct the objects of the subject matter as transformations between states, and then to reconstruct them as objects themselves to be transformed. The case of transformation geometry was the more intricate of the two, as both the states and the transformations are more complex than is the case with integers.

The preservice elementary school teachers in my courses found the curriculum for transformation geometry to be very difficult. Many found the one for integers to be difficult as well.

The case of integers deserves a special comment. It was common at the outset for the students in my courses to attempt to translate complex INTEGERS expressions into  $x$ 's,  $y$ 's,  $+$ 's, and  $-$ 's, especially when all they wanted was an answer. Apparently they tried to assimilate the questions into structures they already possessed, as would anyone. However, one must not assume too much about the nature of those structures. As part of the introductory lecture to integers, I asked the students why it is *reasonable* (as opposed to correct) to rewrite  $-(x - y)$  as  $-x - y$ . There was not a single response that suggested other than an instrumental understanding (Skemp, 1978) of signed expressions. Nor could I elicit a response that referred to quantities. Approximately three weeks after they had been through the INTEGERS curriculum, I gave them the expression  $-(-x + y) = x + -y$ , and asked them *why* removing the parentheses and changing the signs is a reasonable thing to do. They resorted, without my encouragement, to an INTEGERS interpretation, or they said nothing at all. This suggests to me that prior to working with the INTEGERS microworld they had a weak semantic base, if they had one at all, for the rules by which they transformed algebraic expressions. This further suggests that those who do develop mental operations in regard to algebra do so as reflections of transformations of quantities.

### The Missing Link

I hope the reader has not developed the impression that the curricula discussed in the case studies are teacher-independent. On the contrary, they are actually more dependent upon the teacher than are conventional curricula. Moreover, I can easily imagine that in the hands of some teachers we would see very little difference, in terms of what students learn, between them and non-problem-based curricula. All it would take is a teacher whose approach is to demonstrate “the way” to solve problems “like this.”

Underlying the case studies was an assumption that instruction would be in the hands of teachers who are thoroughly intimate with the subject matter, and intimate in two ways. Ideally, teachers should not only possess the cognitive structures that the curriculum aims for in the students, but possess them at a reflected level. They should understand the subject matter both intuitively and formally.

Second, teachers must be—for want of a better word—choreographers. They must have a structure that they can “dance” through as they confront the obstacles that are inevitable in the flow of classroom interactions. To my way of thinking, the only structure that can provide a basis for such flexibility is a personal model of the genesis of the cognitive objectives.

Another issue in the case studies was student affect. In my experience, it was not un-

common, at the outset of a course, for students to go through an activity, record the information that they generated, and then say, “I don’t understand what the answer is supposed to be.” Many students never stopped being “answer” oriented, and they became extremely frustrated. If getting right answers wasn’t the name of the game, then they didn’t know what the game was. They did not understand that “the answer” was most typically a method, or a generalization of a method, and not a number. It was extremely frustrating to me not to be able to influence those students’ perspective on mathematics.

### **The Role of Cognitive Science**

My concluding remarks are on the role that cognitive science can play as an inspiration for problem-based mathematics curriculum development. Reif (1980) and Larkin (1980) each commented that the nature of skilled problem solving and cognition is well enough understood that educators should look to cognitive science for guidance in how to bring them about. My assessment, however, is that at this moment the role of cognitive science in mathematics curriculum development should be minor and indirect.

I have three reasons for such an assessment. The first is that the models of mathematical problem solving one finds in cognitive science tend to do their jobs all too single-mindedly—and all too knowledgeably, in the sense that a model knows what it means to solve a problem. I have yet to see a problem-solving program daydream about an upcoming fishing trip while working a problem, or be distracted by a stray, but related, thought. Nor have I seen a program play a guessing game with the questioner as to what type of answer, among all the worldly categories it possesses, the questioner wants (Thompson, 1982b). Does the questioner want an answer, an historical account of the answer, or an explanation of why an answer has to be correct? The decision is usually made by the programmer.

The second reason is that, as far as I can tell, little attention has been given to the issue of the development of mathematical objects in people’s thinking. Numbers become things to most children, while for quite awhile they were not; operations become things to some high school students; functions, groups, classes, categories, and functors become things to mathematicians. Until cognitive science deals with the nature and development of mathematical objects, I see little that it can offer as guidance for developing problem-based mathematics curricula.

The third, and most important reason for believing that cognitive science is of limited help in developing mathematics curricula is that most models of mathematical cognition fail to take into account the conceptual bases from which cognitions arise. For example, Silver (in press) found that young adults’ thinking about operations on rational numbers is tremendously influenced by the specific models of rational numbers that they bring to mind (e.g., pie charts and measuring cups). Admittedly, mental representations have been of interest in cognitive science for some time now, but the issue that seems to be left aside is that of a mental re-presentation of an action upon a representation. As pointed out in the discussion of integers as mathematical objects, a transformation is not identifiable with the states upon which it acts. It can only be identified with a class of states, and in fact genetically predetermines that class. In my way of thinking, until the notion of action is brought into investigations of mathematical cognition, we will get little guidance from cognitive science regarding the promotion of operative thought in a subject matter area.

I do not mean to give the impression that cognitive science is irrelevant to mathematics education. I find the work of Lawler (1981) and diSessa (1982) to be fascinating and very much

in line with the aims I have expressed here, but theirs seems to be a minority view in cognitive science.

The recent focus in cognitive science on investigations of structures in problem solving has been highly illuminating about the ways we *don't* want students to understand mathematics—for example, understanding mathematics as consisting largely of collections of routine algorithms and associated stereotypical problems (Hinsley et al., 1977). Also, the computer metaphor of cognitive science provides a powerful way to think about mathematical cognition. But we should not reify the central processing unit; we should not forget that the computer metaphor is only a metaphor (cf. Simon, 1972).

Where, then, can mathematics education find guidance for developing problem-based curricula? I find Piaget's genetic epistemology to be most helpful, but only in that it frames the task of influencing others' thinking. It sets design principles for the theory of cognition that will underly the specifications of the cognitive objectives (recalling that cognitive objectives must include a developmental side).

The details of a curriculum must come from an interaction between conceptual analyses of the subject matter (knowledge as held by competent knowers) and investigations of the genesis of that knowledge vis-à-vis instruction. That is, mathematics education must look more to itself than to any other source for guidance in developing problem-based mathematics curricula.

#### REFERENCES

- Abelson, H., & diSessa, A. *Turtle geometry: The computer as a medium for exploring mathematics*. Cambridge, MA: The MIT Press, 1981.
- Anderson, J. *Language, memory, and thought*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1976.
- Anderson, J., Greeno, J., Kline, P., & Neves, D. Acquisition of problem-solving skill. In J. Anderson (Ed.), *Cognitive skills and their acquisition*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1981.
- Briars, D.J., & Larkin, JR. *An integrated model of skill in solving elementary word problems* (Technical Report A.C.P. No. 2). Pittsburgh: Carnegie-Mellon University, Department of Psychology, November 1982.
- Brown, G. *Laws of form*. New York: Julian Press, 1972.
- Carpenter, T.P., Hiebert, J., & Moser, J.M. Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education* 12, 27-39, 1981.
- Dewey, J. *Experience and education*. New York: Collier Books, 1963. (Originally published in 1945.)
- diSessa, A. Unlearning Aristotelian physics: A study of knowledge-based learning. *Cognitive Science* 6, 37-76, 1982.
- Duncker, K. On problem solving. *Psychological Monographs* 58(270), 1-112, 1945.
- Eisner, E. *Cognition and curriculum: On deciding what to teach*. London: Longman, 1982.
- Gagne, R. Some issues in the psychology of mathematics instruction. *Journal for Research in Mathematics Education* 14, 7-8, 1983.
- Greeno, J. Indefinite goals in well-structured problems. *Psychological Review* 83, 479-91, 1976.
- Greeno, J. Some examples of cognitive task analysis with instructional applications. In RE. Snow, PA. Federico, & W.E. Montague (Eds.), *Aptitude, learning, and instruction*. Volume

- 2: *Cognitive process analysis of learning and problem solving*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1980.
- Hayes-Roth, B. *Human planning processes* (Rand Report R-2670-ONR). Santa Monica, CA: Rand Corporation, 1980.
- Hayes-Roth, F., Kahn, P., & Mostow, D. Advice taking and knowledge refinement: An iterative view of skill acquisition. In J. Anderson (Ed.), *Cognitive skills and their acquisition*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1981 .
- Herstein, I. *Topics in algebra* (2nd ed.). Lexington, MA: Xerox College Publishing, 1975.
- Hinsely, D., Hayes, J., & Simon, H. From words to equations: Meaning and representation in algebra word problems. In M. Just & P. Carpenter (Eds.), *Cognitive processes in comprehension*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1977.
- Inhelder, B., & Piaget, J. *The early growth of logic in the child*. New York: W.W. Norton, 1969.
- Krutetskii, V. *The psychology of mathematical abilities in schoolchildren*. (J. Teller, Trans., J. Kilpatrick & I. Wirszup, Eds.) Chicago: University of Chicago Press, 1976.
- Lakatos, I. Infinite regress and the foundations of mathematics. *Aristotelian Society Supplementary Volume* 36, 156-84, 1962.
- Lakatos, I. *Proofs and refutations: The logic of mathematical discovery*. London: Cambridge University Press, 1976.
- Larkin, J. Teaching problem solving in physics: The psychological laboratory and the practical classroom. In D.T. Tuma & F. Reif (Eds.), *Problem solving and education: Issues in teaching and research*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1980.
- Lawler, R. The progressive construction of mind. *Cognitive Science* 5, 1-30, 1981.
- Lenat, D. *The nature of heuristics* (Cognitive and Instructional Science Series CIS-12 [SSL-81-1]). Palo Alto, CA: Xerox PARC, 1981.
- Lunzer, E. Introduction. In B. Inhelder & J. Piaget, *The early growth of logic in the child*. New York: W.W. Norton, 1969.
- MacKay, D. *Information, mechanism, and meaning*. Cambridge, MA: The MIT Press, 1969.
- Manna, Z., & Waldinger, R. *Synthesis: Dreams → Programs* (Report No. STAN-CS-77-630). Stanford, CA: Stanford University, Computer Science Department, 1977.
- Manna, Z., & Waldinger, R. DEDALUS—The Deductive Algorithm Ur-Synthesizer. In *Proceedings of the National Computing Conference*, Anaheim, CA, 1978.
- Neves, D., & Anderson, J. Knowledge compilation: Mechanisms for the automatization of cognitive skills. In J. Anderson (Ed.), *Cognitive skills and their acquisition*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1981 .
- Newell, A., & Simon, HA. *Human problem solving*. Englewood Cliffs, NJ: Prentice-Hall, 1972.
- Papert, S. *Mindstorms. Children, computers, and powerful ideas*. New York: Basic Books, 1980.
- Piaget, J. *Psychology of intelligence*. London: Rourledge & Kegan Paul, 1951.
- Piaget, J. Development and learning. In R. Ripple & U. Rockcastle (Eds.), *Piaget rediscovered*. Ithaca, NY: Cornell University, School of Education, 1964.
- Piaget, J. *Six psychological studies*. New York: Basic Books, 1968.
- Piaget, J. *Genetic epistemology*. New York: W.W. Norton, 1970.
- Piaget, J. *The child and reality*. New York: Penguin Books, 1976.
- Piaget, J. *Recherches sur labstraction reflechissante* (Vol. I). Paris: Presses Universitaires de France, 1977. (Translation provided by E. von Glasersfeld.)
- Piaget, J., Grize, T., Szeminska, A., & Bangh, V. *Epistemology and psychology of functions*. Dordrechr, The Netherlands: D. Reidel, 1977

- Piaget, J., & Inhelder, B. *The psychology of the child*. New York: Basic Books, 1969.
- Polya, G. *How to solve it* (2nd ed.) . Princeton, NJ: Princeton University Press, 1973 .
- Powers, W. *Behavior: The control of perception*. Chicago: Aldine, 1973.
- Powers, W. Quantitative analysis of purposive systems: Some spadework at the foundation of scientific psychology. *Psychological Review* 85, 417-35, 1978.
- Reif, F. Theoretical and educational concerns with problem solving: Bridging the gaps with human cognitive engineering. In D.T. Tuma & F. Reif (Eds.), *Problem solving and education: Issues in teaching and research*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1980.
- Resnick, L. Task analysis in instructional design: Some cases from mathematics. In D. Klahr (Ed.), *Cognition and instruction*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1975 .
- Resnick, L. *The psychology of mathematics for instruction*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1981.
- Resnick, L. A developmental theory of number understanding. In H.P. Ginsburg (Ed.), *The development of mathematical thinking*. New York: Academic Press, 1983.
- Riley, M., Greeno, J., & Heller, J. The development of children's problem-solving ability in arithmetic . In H.P. Ginsburg (Ed.), *The development of mathematical thinking*. New York: Academic Press, 1983.
- Sacerdori, E. *A structure for plans and behavior*. New York: Elsevier, 1977.
- Schank, R., & Abelson, R. *Scripts, goals, plans, and understanding*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1977.
- Schank, R., & Riesbeck, C. *Inside computer understanding*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1981.
- Silver, E. Probing young adults' thinking about rational numbers. In *Focus on Learning Problems in Mathematics*. In press.
- Simon, H. On the development of the processor. In S. Farnham-Diggory (Ed.), *Information processing in children*. New York: Academic Press, 1972.
- Skemp, R. Relational and instrumental understanding. *Arithmetic Teacher* 26, 9-15, 1978.
- Skemp, R. *Intelligence, learning, and action*. London: John Wiley & Sons, 1979
- Steffe, L. On a model for teaching young children mathematics. In A. Osborne (Ed.), *Models for learning mathematics*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education, 1976
- Thompson, P. *Children 's schemata in solving problems involving whole-number numeration*. Paper presented at the annual meeting of the American Educational Research Association, New York, March 1982(a).
- Thompson, P. *A theoretical framework for understanding children's concepts of whole number numeration*. Unpublished doctoral dissertation, University of Georgia, 1982(b).
- Thompson, P. Were lions to speak, we wouldn't understand. *Journal of Mathematical Behavior* 3(2), 147-65, 1982(c).
- Vergnaud, G. A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T.P. Carpenter, J.M. Moser, & T.A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1982.
- von Glasersfeld, E. Cybernetics, experience, and the concept of self. In M. Ozer (Ed.), *Toward the more human use of human beings*. Boulder, CO: Westview Press, 1978.
- von Glasersfeld, E. On reflection and abstraction. In preparation.
- Weaver, F. Interpretations of number operations and symbolic representations of addition and

subtraction. In T.P. Carpenter, J.M. Moser, & T.A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective*. Hillsdale, NJ: Lawrence Erlbaum Associates, 1982.

Wertheimer, M. *Productive thinking*. New York: Harper & Row, 1959. (Originally published in 1945.)

APPENDIX: SUBSET OF COMMANDS  
AVAILABLE IN MOTIONS

<i>Command</i>	<i>Action</i>
<i>CL</i>	Cleans the screen; erases all but the current figure.
<i>CLS</i>	Clears the screen; places the figure at the home position (HDG 90, POS 0 0, and orientation right).
<i>DEF "name</i> <i>[list of commands]</i>	<p>Defines "name as the name of <i>[list of commands]</i>, where the commands in the list are among those allowed by MOTIONS. MOTIONS will respond "<i>name DEFINED</i>" if it accepts the definition.</p> <p>If the commands in <i>[list of commands]</i> take the variable input, then the list must begin with a list of the variable names. A variable name must be preceded by a colon within the definition itself. The colon means "value of the following name."</p> <p>Example: DEF "TWO.FLIPS [[X Y] F :X F :Y] defines a motion named <i>TWO.FLIPS</i> that takes two inputs, and performs two flips. The first flip is preformed about the line defined by the first input; the second flip is preformed about the line defined by the second input.</p>
<i>F&lt;degrees&gt;</i>	Flips the plane through the line that passes through the origin at the angle of measure <i>&lt;degrees&gt;</i> .
<i>GRP</i>	Places the current figure at the position, heading, and orientation last recorded by RP.
<i>ID</i>	Identity motion—leaves plane as it is.
<i>R&lt;degrees&gt;</i>	Rotates the plane about the origin through an angle of measure <i>&lt;degrees&gt;</i> .
<i>RP</i>	Records the current figure's position, heading, and orientation. (See <i>GRP</i> .)
<i>T&lt;distance&gt;&lt;heading&gt;</i>	Translates the plane <i>&lt;distance&gt;</i> turtle steps in a heading of <i>&lt;heading&gt;</i> .

MOTIONS maintains several variable values to which reference may be made within a definition. These are:

<i>Variable</i>	<i>Value</i>
:XCOR	The current figure's <i>x</i> -coordinate.
:YCOR	The current figure's <i>y</i> -coordinate.
:HDG	The current figure's heading.
:ANGLE	The measure of the angle formed by the current figure's position, the origin, and the positive <i>x</i> -axis.

For example, T:XCOR 180 translates the plane so that the current figure has an *x*-coordinate of 0.