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MICROWORLDS AND VAN HIELE LEVELS

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Two sixth graders were observed as they used a computer microworld in which integers are represented as transformations. It is described how the students (re)construct the mathematics as they progress from one van Hiele level to the next.

Freudenthal [1972] has distinguished ready-made mathematics from mathematics as an activity. He made it clear that the active performance of mathematics is essential in the learner's progress from one van Hiele level to the next. The relationship among van Hiele levels is that the organizing activity of any level becomes the object of analysis on the next higher level. The question arises as to what settings would facilitate progress from one level to the next.

It has been argued elsewhere [Dreyfus, 1984] that computer environments can be consonant with Freudenthal's theory. Such an environment, or microworld, should give the student the opportunity to construct mathematical knowledge. Thompson [1985] has given a detailed analysis of the principles according to which a microworld should be built. In the present paper, one such microworld will be analyzed with respect to its incorporation of van Hiele levels. Two students will be followed as they progress through these levels.

The integers microworld, named INTEGERS [Thompson, 1984] is a learning environment for the additive group of integers. It is built in such a way that it can be used by students of elementary arithmetic as well as by students of group theory. This has been achieved by adhering to the following principles:

1. There must be a strong distinction between the curriculum and the environment upon which the students act (and the metaphor within which they reason). The software presents the environment and metaphor, and incorporates the mathematical structure within a model. Instructions and questions for the student are given in print. This allows teachers to adapt the curriculum to the purposes of instruction as well as to the levels of the students. Thence, continued use of this model is apt to contribute to the integration, in students' minds, of the various levels at which the concept can be examined.[3]

2. The representation of a mathematical structure by a model must be such that the model incorporates the structure in detail, yet must be easy to use even at relatively low levels where the full power of the model is not needed. The presentation of mathematical content should make aspects of its structure accessible to students regardless of their level of thinking.

The Environment for Integers

In the case of integers, these principles were expressed by presenting an integer as a transformation of position, and operations upon integers as compositions of transformations. More precisely, an integer is reflected in integers by having a turtle walk left and right on a number line in response to one or more numbers being entered. The response of the turtle to a number being entered is governed by these rules:

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[3] For example, INTEGERS can be used in Grade 1 to teach addition, in Grade 6 to teach negative numbers, in Grade 8 to teach variables and operations, and in Grade 12 to teach the Abelian group structure of integers.

number The turtle walks *number* turtle-steps in its current direction.

-number The turtle turns around, does *number*, and then turns back around.

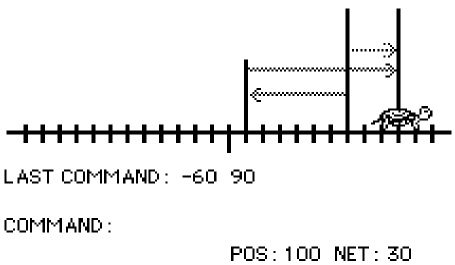
The turtle's grammar for numbers is:

i. A whole number is a number.

ii. The negative of a number is a number.

iii. Two or more numbers entered simultaneously is a number *(the net effect of executing them consecutively).*

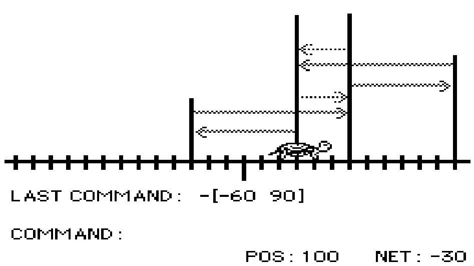
Figures 1 and 2 illustrate the turtle's grammar. The readers should note the grammar's recursive structure. The combination **-60 90** is a number (by iii.), and hence **-[-60 90]** is a number (by ii.). Figure 3 explains the grammatical structure of Figure 2.



The effect of entering **-60 90**. The tall vertical line on the left shows the turtle's beginning position. The tall vertical line on the right shows its ending position. The intermediate (shorter) vertical line shows where the turtle finished executing **-60** and began executing **90**. The turtle did **-60** (lower heavy arrow) and then did **90** (upper heavy arrow), which resulted in a net effect of **30** (top arrow). POS tells the turtle's current position. NET tells the net displacement of the turtle's position caused by the last-entered command.

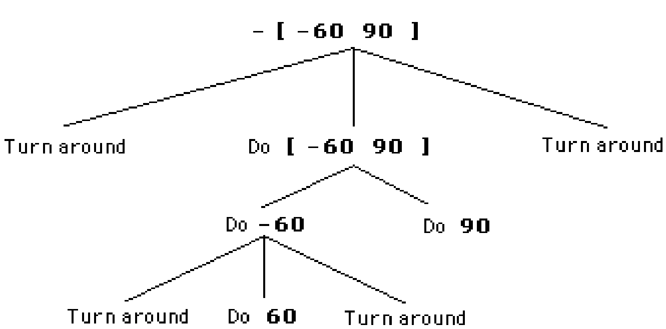
Figure 1.

INTEGERS has been used by a student assistant in tutoring two sixth graders, whom we will call Kim and Lucy. They met for eleven sessions of approximately 40 minutes each. the sessions were audiotaped and transcribed. These data have been analyzed from several points of view: (1) progress through several van Hiele levels, (2) the formation of the concept of composition of integers, and (3) the formation of the concept of negation. In this paper we concentrate on the first two analyses. The formation of concepts of negation will be discussed in a separate paper [Thompson & Dreyfus, In press].



The effect of entering **-[-60 90]** after having entered **-60 90**. The right tall vertical line shows the turtle's position when it began executing **-[-60 90]**. The upper three arrows document the turtle's itinerary in response to **-[-60 90]**.

Figure 2.



The method by which the expression **-[-60 90]** was parsed. Read the parse tree thus: To do **-[-60 90]**, turn around, do **[-60 90]**, and then turn back around. To do **[-60 90]**, do **-60** and then do **90**. To do **-60**, turn around, do **60**, and then turn back around.

Figure 3.

Kim's and Lucy's Progress

Level 0

The lowest van Hiele level (Level 0) is the distinction between position and transformation of position. Typically, a student on this level would be asked, for example, to place the turtle at 30 and to make it walk to any of 90, 10, or -80. Variations on this problem are: (1) give the starting position and the transformation; ask where the turtle will end, and (2) give the transformation and the ending position; ask where the turtle began. The objects of attention at this level are the states (positions) of the turtle. The organizing activity is the determination of relationships among beginning position, ending position, and transformation.

While at Level 0, Kim and Lucy showed a persistent confusion between integer (transformation) and position. Their confusion is illustrated in the following excerpt.

	(Turtle is at position 30.)
INT:	What would happen if you just put in thirty.
LUCY:	Would it just stay there?
INT:	Try it. (Lucy enters 30 and presses RETURN).
LUCY:	Oh yeah, it does that line.
INT:	And what did the turtle do?
LUCY:	It—then it goes thirty.
INT:	Okay, try it again. Put in thirty, but don't press RETURN. (Lucy types 30) Where is the turtle going to end up when you press RETURN?
LUCY:	Over there (points to the position 30).
INT:	Go ahead and press RETURN.
LUCY:	(Presses RETURN) Oh, it'll go another thirty spaces.

It took Kim and Lucy till the third session before they had created a reliable distinction between position and transformation. Even then, it was not uncommon for them to lapse back into confusion when questions got intricate or required a new focus. That is to say, the separation of Level 0 and Level 1 was not pure, as both Lucy and Kim would at times behave as if their thinking was Level 0 while at the same time they were working more or less successfully at Level 1. For this reason we use van Hiele levels more as descriptions of performance than as statements of competence.As statements of competence, one could say only that a child *can* think at a given level, but not that he or she *will* think at that level given the capability to do so.

Level 1

While the objects of thought at Level 1 are transformations, the students' activities are organized by their comparison of different transformations and by problems that emphasize the independence of position and transformation. The extent to which students formalize an integer transformation as an equivalence class of position pairs is open to question. The question of formalization arises at each level of our analysis. It will be treated more fully in the discussion of Level 2.

Kim and Lucy progressed fairly quickly through Level 1 and seemed to have little trouble conceptualizing a turtle movement as a transformation from one state to another. Since a turtle movement is accomplished by entering an integer numeral, we surmised that they had conceptualized integers as transformations. Quite surprisingly, this was true for negative as well as positive integers. They quickly apprehended that a minus sign in front of a number causes two turns of the turtle—one before and one after its walk.

Level 2

At Level 2, integer transformations are the objects of thought. Transformations can be composed (integer addition) and/or inverted (negation). The students' activities were organized by composing and inverting integers and integer expressions. The following excerpt from the fifth meeting is representative of Kim's and Lucy's activities at Level 2. The new level of discourse is suggested by the introduction of the term "net effect," which means the net displacement caused by the composition of two or more integers.

INT: (Enters **20 -30**; the turtle walks accordingly) What is the net effect? What is that little white dotted arrow?
LUCY: Ten.
KIM: No, see, it's fifty, because it ...
INT: I wanted to know ...
KIM: Because like, it would be going ...
INT: If I was sitting right here at the end of this arrow and I wanted to move the same amount as that arrow, what number could I put in?
KIM: Fifty, no ...
LUCY: Yeah, negative fifty.
INT: Negative fifty?
KIM: I doubt it.
LUCY: Well, if I was facing ...
INT: Which way is the turtle facing? How do we make it turn around?
KIM: Negative.
INT: Negative. Right. Okay, I want to make an arrow just like this (net effect arrow).
LUCY: Oh!! Negative ten.
INT: Negative ten. Try it.
LUCY: (Enters **-10**; the turtle walks accordingly)
INT: Are those two arrows the same?
LUCY: Yeah.
INT: So, what is the net effect of negative thirty twenty?
KIM: Fifty.
LUCY: Fifty.

It turned out, in this instance, that by "net" the students meant "total," as in "total number of turtle-steps walked." We contend that the driving force behind the attribution of "total steps" to "net" was Kim's and Lucy's conception of number. Minus signs were not part of numerals; rather, they qualified numerals. Similarly, the direction of a transformation was not part of the transformation; rather, it qualified the transformation. To add two of *their* transformations, one would add the number of turtle steps in each. A direct analogy is college students' common mistake of adding two vectors' magnitudes to determine the magnitude of their sum.

The above dialogue is typical of the kinds of discussions that are elicited by the didactical situation of two or three students, with or without a tutor, engaged in solving a problem and having the opportunity to make inquiries by means of a microworld. As illustrated, inquiries do not guarantee insight, but they provide opportunities for insight.

Kim and Lucy proceeded to understand what happens in concrete "turtle actions" in response to the entry of two integers. Over the next four sessions they became more proficient in constructing these descriptions, and they became able to handle a wide variety of cases, such as **-70 -40** and **-90 50**. But this observation is of little interest in regard to our focus on their understanding of the structural properties of integer addition (composition of transformations). Did they, in fact, grasp that two transformations, applied one after the other, are equivalent to one appropriately chosen transformation? Did they formalize the notion of net effect by understanding that "net effect" denotes a *resulting* transformation? The answer was *no* to both questions. The following excerpt illustrates why we say this.

INT: I'm going to put in twenty ten (enters **20 10**). Now, what is going to happen?
LUCY: It'll go thirty.
INT: It'll go thirty.
LUCY: Thirty, without having to turn around and stuff.
INT: It'll go ...
KIM: Twenty and then it'll go ten.
INT: The net effect is thirty?
KIM: Thirty. Yeah.
INT: (Presses RETURN; turtle moves accordingly) Right. So he's moved thirty.

	Now, if I put in a negative twenty ten (types **-[20 10]**), what is he going to do?
KIM:	Face the other way, and then go twenty, and then go on and go ten, and then turn around.
INT:	So, what's the net effect going to be?
KIM:	Negative ... (pause) ten.
INT:	Negative ten?
KIM:	Mmm-hmmm (yes).
INT:	Okay, let's look at this one more time. Here we have twenty ten (points to arrow diagram), and what did we say the net effect of that was?
LUCY:	Thirty.
KIM:	Thirty.
INT:	Okay. Now here we have the *negative* of twenty ten.
LUCY:	Negative ... (pause) negative thirty?
INT:	Press RETURN and see what happens.

Kim and Lucy afterward expressed their feeling that they had understood the point of the question, so the interviewer gave them **-[40 -20]** and asked them to predict its net effect. There answers were:

LUCY:	Twenty, ... no, negative twenty, ... no ...
KIM:	Negative sixty?

Clearly, they were still uncertain. It must be concluded that they had not yet formalized the net effect of two integers as being a single integer. It was only at the end of the eighth session that a question about the net effect of **-[-50 30]** immediately brought the answer "negative negative twenty." This may be taken as an indication that by then an abstract formalization of the composition of integer transformations had taken place.

Level 3

At Level 3, composition and inversion of integers and integer expressions become the objects of attention, and their structural properties are emphasized. Kim and Lucy did not reach Level 3.

We argue elsewhere [Thompson & Dreyfus, In press] that Level 3 in regard to integers may be Level 0 in regard to algebra. We cannot address that issue here.

Conclusion

It may be hypothesized that abstractions made in the context of a powerful and consistent model and through explorations of that model are more stable than rules for operating upon integers that have been learned by rote. At this time, no data exists to support that hypothesis.

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