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Quantitative Concepts as a Foundation for Algebraic Reasoning: Sufficiency, Necessity, and Cognitive Obstacles[1,2]

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This premise was investigated: Quantitative reasoning provides a genetic foundation for algebraic reasoning in applied situations. A model-as-cognitive-objective was developed and tested with 25 average-ability eighth graders over a 6-month period. The premise received indirect confirmation, in that students who were severely deficient in quantitative reasoning as characterized by the model were also poor at arithmetic and algebraic problem solving.

A primary concern in mathematics education, at least within the United States, is to incorporate aspects of algebraic reasoning into the elementary curriculum (NCTM, 1987; Kieran & Wagner, 1988). That is to say, students' study of arithmetic is supposed to prepare them for algebra and currently is not doing so.

How transitions between the study of arithmetic and the study of algebra might be made is an important area currently being researched by many in the PME (see Kieran, 1988, for an excellent review). Difficulties caused by that transition is another important research area (Booth, 1981; Küchemann, 1980; Herscovics & Kieran, 1980; Vergnaud & Cortes, 1986).

The premise of this research is: *Competence in algebra is founded genetically upon competent quantitative reasoning.* This premise is similar to a view of algebra as "generalized arithmetic," but is different in key aspects, for here it is also argued that *arithmetic* is founded on quantitative reasoning.

The claim that quantitative reasoning is genetically foundational to formal algebra

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[2] Software reported in this paper and associated materials can be obtained from the author by sending a 3.5" double-sided, double-density disk and a cheque for $5.00.

does not preclude the importance of "non-quantitative" algebra (e.g., the study of group, ring, and field properties of polynomials, or the study of algebras of real- and complex-valued functions). Rather, it provides a link between "non-quantitative" algebra and the algebra of applications.[3]

QUANTITATIVE REASONING

A quantity is a measurable quality of something. A magnitude of a quantity is the quantity's measure in some unit. Quantities are categorized into two kinds: extensive and intensive quantities (Freudenthal, 1973; Schwartz, 1987). An extensive quantity is one which may be measured in absolute units. An intensive quantity is one which may be measured only by units that are constructed comparatively. Lengths, areas, volumes, cardinalities, and ordinalities are examples of extensive quantities. Speeds, densities, temperatures, pressures, and pitches are examples of intensive quantities, as is a multiplicative comparison (ratio) of two quantities.

To reason quantitatively is to reason about quantities, their magnitudes, and their relationships with other quantities. Quantitative mental operations include these:

Operation	*Example*
• Combine quantities additively	*Unite two sets; consider two regions as one.*
• Compare quantities additively	*"How much more of this is there than that?"*
• Combine quantities multiplicatively	*Combine distance and force to get torque; combine linear dimensions to get regions*
• Compare quantities multiplicatively	*"How many times as large is this than that?"*

Quantitative reasoning is to reason about situations in terms of quantities and quantitative operations. This characterization does not identify quantitative reasoning with arithmetic; in fact, it is independent of arithmetic. A quantity within a structure of quantitative operations may require any arithmetic operation to determine its value.

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[3] I have maintained elsewhere that even the more structural features of elementary algebra can be taught with an explicit foundation in arithmetic. See Thompson and Thompson, 1987.

The appropriate operation to compute a quantity's value can be inferred only within the context of a quantitative structure together with assumptions about what values are known.

Complex quantitative reasoning entails relating groups of quantitative mental operations, such as in forming a multiplicative comparison of an additive comparison and an additive combination (i.e., "How many times as large is this difference than is this combination?"). Quantitative reasoning also entails reasoning relationally about quantitative structures, entails the constitutive mental operations for comprehending a quantity situationally, and entails the constitutive mental operations which allow one to recognize a quantity as one whose value varies or can vary.

To investigate the sufficiency of quantitative reasoning as a foundation for arithmetic and algebraic reasoning, the author constructed a computer program (called *Word Problem Assistant*, or *WPA*)which makes numerical or algebraic inferences solely on the basis of quantitative structure (Thompson, 1988). Once given a representation of a situation in terms of quantities, units of measurement, initial numerical information, and pairwise relationships among quantities, the program will generate arithmetic or algebraic expressions for computing unknown values within the situation, including the setting up of equations where appropriate. For example, here is a situation with a question about it.

> MEA Export is bidding to supply an oil valve to Costa Rica. They sell this valve for $5000. Freight charges to Costa Rica are $100. The insurance cost is 1.25% of the total price to Costa Rica. The total price to Costa Rica includes the cost of the valve, the cost of freight, and the cost of insurance.
>
> What is the total price to Costa Rica?

Figure 1 shows a representation of this situation made within WPA, and also shows the inferences WPA made. The boxes in Figure 1 (called notecards) represent quantities.[4] The bottom cell in a notecard contains the name of the unit in which the quantity is measured. The *Value Cell* in a notecard holds the quantity's measure, if one is known. The *Description Cell* of a notecard is a place that holds an expression which tells how to compute the quantity's value.

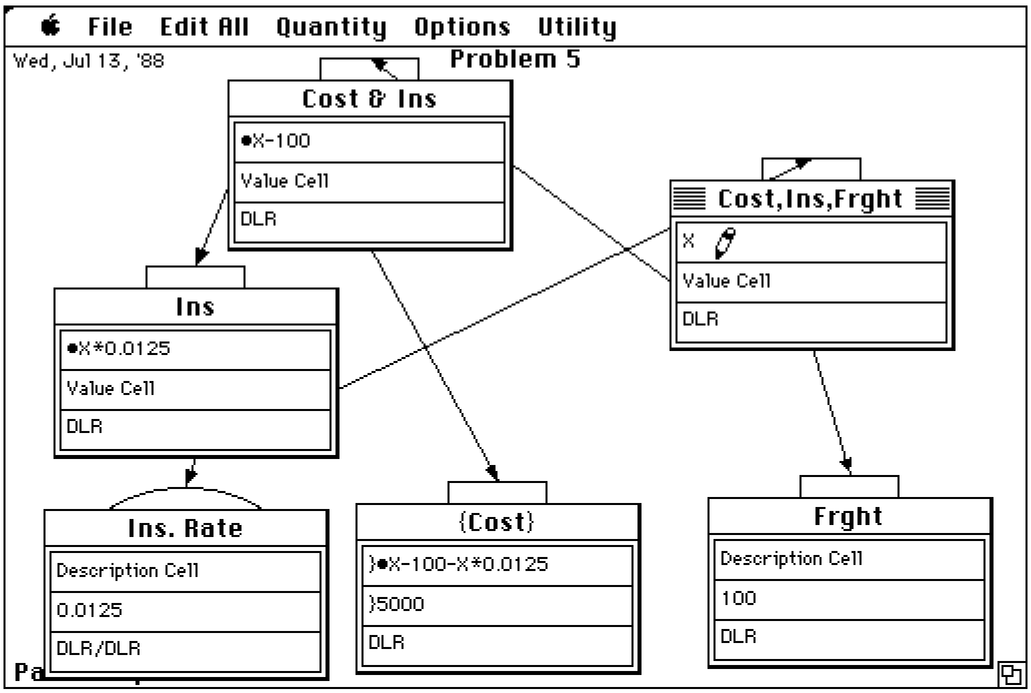

Figure 1. A representation in WPA of an algebra story problem, along with the inferences it made based on the quantitative information presented about the situation.

Entries with bullets (•) are the results of inferences made by WPA. Any entry without a bullet was entered by hand. The arrows show quantitative relationships. For example, the arrows coming out of **Cost,Ins,Frght** show that it is made of **Cost & Ins** together with **Frght**. Since the units are all in dollars (DLR), WPA inferred that this is an additive relationship, and hence that if **Cost,Ins,Frght** is *x*, and if **Frght** is 100, then **Cost & Ins** must be *x*-100. Other inferences were made accordingly, resulting in the inference that *x*-100-*x**0.0125 is a formula for computing the value of **Cost**. Since

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[4] The shape of the notecards and the convention of using arrows to show quantitative relationship was developed by Valerie Shalin (Greeno et al, 1985; Shalin, 1988).

Cost already had a known value of 5000, WPA decided to ignore **Cost**'s value (and indicated as much by inserting "}" before its value) and to propagate the formula—resulting in an equation (i.e., two descriptions of a single quantity's value) whose solution will determine numerical values of all quantities in the representation.

AN EXPLORATORY STUDY OF QUANTITATIVE REASONING

WPA constitutes a cognitive objective—a model which can be taken as depicting knowledge students are to build as a result of instruction (Greeno, 1980). As a cognitive objective, it can be used as a guide for instruction and as a guide for assessing students' knowledge.

WPA was used on an ongoing basis for 6 months with 25 average-ability eighth graders at the ISU laboratory school. Students were given pretests on problem structure (similar to Silver, 1981), rate and ratio, concepts of expression and variable (similar to Küchemann, 1980), and arithmetic computation. Only on the computation pretest did the class average exceed 50%. Other pretest averages were 14% (Structure), 30% (Expressions and Variable), and 41% (Rate and Ratio).

The classroom was equipped with 11 Macintosh Plus computers. Students worked in groups of two or three, whether in activities away from the computers or in problem solving at the computers. Whole-class instruction was done using the blackboard, overhead transparencies, or with a Macintosh connected to a projector. When students worked in groups, one group was selected for videotaping. If the videotaped group used a computer, their computer was connected to a projector so that both they and their computer screen could be videotaped simultaneously.

The user-interface to WPA was made so that, to use it in solving word problems, students needed to make decisions only about quantities involved in the described situation. Decisions to be made about identified quantities were: a quantity's name, its type ("number of things," difference, ratio, or rate), the unit in which it is measured, and quantities that it is made of (by some quantitative operation, but they did not have to specify the operation—WPA inferred one). The following are mistakes that students made on such a frequent basis that they are indicative of cognitive obstacles to quantitative reasoning, and hence are perchance indicative of obstacles to algebraic reasoning.

<u>Naming Quantities</u>

A predominant tendency was for students to name quantities by their numerical value, e.g. “250 dollars” instead of “Sam’s Savings.” Their failure to distinguish between a quantity and its measure hindered their ability to explicate relationships. This was for two reasons. First, students could identify only “quantities” that had given numerical values. It was difficult to identify quantities in a situation that had an unknown value, and especially difficult to recognize the need for including “intermediate,” but unmentioned quantities. Second, when students had only numerical values in mind they had no rational basis for deciding what operation to perform to find an unknown value. When one has in mind, for example, only “17.32 dollars” and “12.87 dollars” without relational information about the quantities having these values, one cannot decide, on this information alone, what operation would be appropriate to compute the value of a third quantity.

<u>Identifying Kinds of Quantities/Assigning Units of Measure</u>

Students reliably identified and assigned appropriate units only to extensive quantities of a linear dimension. Multiplicative quantities of any sort (products, ratios, rates) were commonly mis-identified or given an inappropriate unit.

The most common mis-identification was of rates. The typical mistake was to give the “numerator” of a rate or ratio as the unit of the identified quantity (with the exception of speed in miles per hour). Pay rates per unit time were given “dollar” as the unit. Prices were given “dollar” or “cent” as the unit. Ratios between children and buses were given “child” as the unit. And so on.

WPA required students to pick the kind of quantity a notecard was to represent and to supply the unit in which the quantity was to be measured. When a mismatch occurred (e.g., saying that a rate had “meter” as a unit), it would say that the unit and kind of quantity were not consistent, and would not let students proceed until the two were made consistent.

<u>Showing Relationships</u>

WPA checked compatibility of units and kinds of quantities when students attempted to show relationships among quantities. If they attempted to show, by drawing arrows, that an extensive quantity measured in “meter” was made by combining a rate measured in “ft/sec” and a time measured in “sec,” WPA would explain that the quantities could not be related in this way, and would not establish a relationship (i.e., would not draw the arrows). The constraint of compatibility was one which students bumped against commonly.

Students’ consistent violation of the two constraints imposed by WPA proved to be so severe that direct instruction on quantity, units of measure, and relationship was necessary. This instruction, however, uncovered affective and sociological norms which severely obstructed students’ abilities to assimilate instruction on quantitative reasoning and its generalization to algebra. Another publication will address issues of instruction, affect, and effects of the “social contract” of schooling.

CONCLUSION

The premise that quantitative reasoning provides a genetic foundation for arithmetic and algebraic reasoning remains to be tested directly. The current study provides, at best, indirect confirmation of the premise. This is that students who are not good arithmetic or algebra problem solvers have substantial deficiencies in quantitative reasoning. The next phase of this project will be a longitudinal study of quantitative reasoning and its relationship to algebra. We will follow a class of fifth-graders for four years, instructing them in quantity, quantitative reasoning, and the representation of quantitative structure. It is hoped that a longitudinal approach will illuminate issues regarding sufficiency of quantitative reasoning for algebra.

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