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SALIENT ASPECTS OF EXPERIENCE WITH CONCRETE MANIPULATIVES†

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A current hypothesis among many mathematics educators is that it is helpful, and perhaps necessary, for students to be able to represent mathematical ideas in several equivalent ways. This hypothesis is embedded in Dienes' Multiple Embodiment Principle (Bruner, 1963; Dienes, 1960) and is thematic in recent expositions of multiple, linked representational systems (Kaput, 1986). Dienes' rationale was that to develop an abstract concept, one needs more than one example which embodies it and from which the concept is to be abstracted.

From a constructivist viewpoint, the role of "embodiments," including concrete manipulatives, is that they provide situational constraints on students' activities, and hence provide occasions for students to make real for themselves the mathematical constraints that constitute the boundaries and glue of a concept. In encountering a constraint, students are blocked from doing something they want to do. Such disequilibria may foster reflection and abstraction of the mathematical constraints intended by the designer of the materials. For example, the concept of addition is elementary: it is to combine two quantities. The difficulty occurs in *naming* the resulting quantity's value. The task of naming the value of a quantitative operation's result is made even more difficult when all aspects of a naming process are constrained to occur within a specific representational system, such as decimal numeration. Dienes' base-ten blocks make the constraints of decimal numeration explicit. If students are asked to solve, say, a subtraction problem with Dienes' blocks they have occasions to reflect upon, and interiorize, the impact that constraints of decimal numeration have upon methods to name a difference.

The hypothesis of this study was this: The more pronounced in students' experience is the constraining nature of a notational system in relation to what is denoted, the more likely they are to conceive of notational algorithms as deriving from adaptations to the system's constraints. To study this hypothesis, the first author developed a computer microworld that incorporates multiple, linked notations for decimal numeration and compared students' use of it with students' use of Dienes base-ten blocks.

MICROWORLD

The computer program used in the study was a mathematical microworld for decimal numeration. The genre of mathematical microworlds is described elsewhere (P. Thompson, 1985, 1987). This particular microworld, called BLOCKS, runs on a Macintosh.

BLOCKS presents students with a supply of Dienes blocks (Figure 1) and regions where blocks may be stored. Two notational systems corresponding to displayed blocks are presented: expanded notation and traditional numeral. Boxed words (called "Buttons") in the display's top right corner will be explained later.

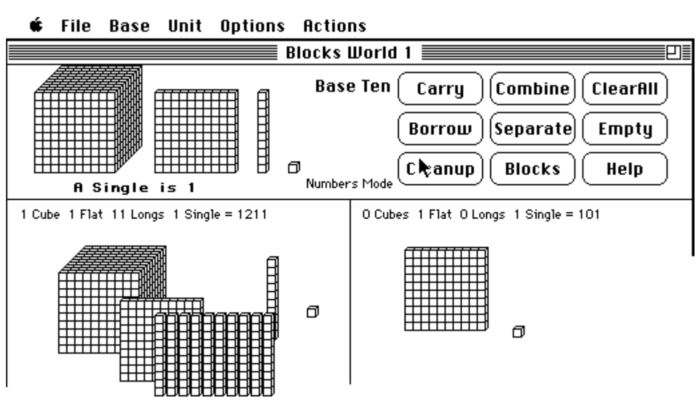

Figure 1

Interaction

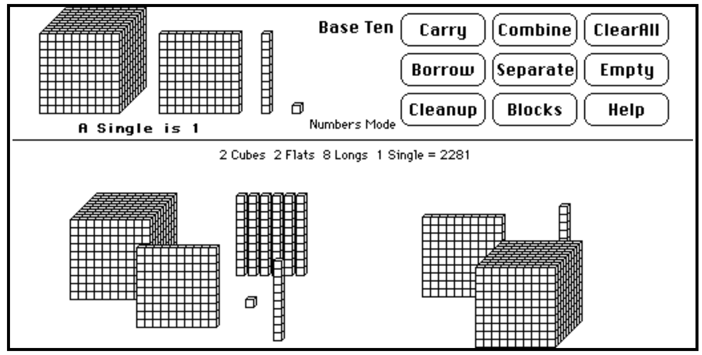

Figure 2

Students create collections of blocks by using a mouse to drag copies of blocks from the source region to the region in which the blocks are to be stored. Subtraction is done by dragging blocks from one storage region to another. Addition is done by creating collections in the two storage regions and by clicking the **Combine** button (Figure 2). Clicking **Combine** causes BLOCKS to remove the vertical dividing line and consider all blocks as one collection. Clicking **Separate** causes the vertical line to be redrawn, splitting the single storage area in two.

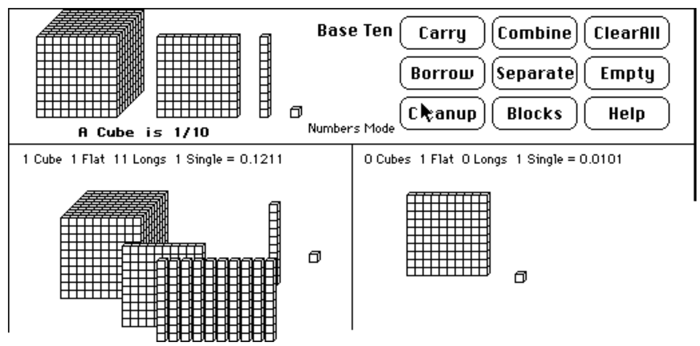

Figure 3

The buttons **Borrow** and **Carry** effect transformations of the digits in a numeral and thereby effect transformations of the blocks in a collection. Transformations of digits are effected by clicking on a digit in the numeral's expansion, then clicking **Borrow** or **Carry**. **Borrow** causes one block of the kind corresponding to the clicked digit to be unglued into 10 of the next smaller kind (with the exception of a single). **Carry** causes 10 blocks of the kind corresponding to the clicked digit (if there are at least 10) to be glued into one block of the next larger size (with the exception of a cube). The transformation is enacted with blocks and the result is reflected in the numeral's expansion.

The ***Unit*** menu contains options for what stands for one. The options range in sequential powers of ten from "A single is 1000" to "A cube is 1/1000". The numeral display reflects the user's choice of unit. Figure 3 repeats the display presented in Figure 1, except a cube denotes 1/10.

METHOD

Subjects

Twenty fourth-grade students enrolled in a midwestern university laboratory school were subjects of the study. Ten students were male; 10 were female. The laboratory school's enrollment is chosen to represent the geographic region's population academically and socio-economically. Average percentile ranks for subjects' Iowa Test of Basic Skills scores were: Concepts-70, Problem Solving-73, Computation-60, and Total Math-71.

Procedures

Students were assigned to two treatments: microworld instruction and wooden-block instruction.

<u>Assignment to treatments</u>

Students were matched according to their scores on a 19-item whole number computation, place value, fractions and decimal fraction pretest (test-retest correlation = .83). Item scores were entered into a stepwise multiple-regression analysis with total test score as the dependent variable. The analysis ended with six items ("six best items") being included in the regression equation. Sums were computed on those six items to give a pretest subscore; these "six best items" subscores were ranked in descending order. Pairs were formed by taking adjacently ranked sums. Members of pairs were assigned at random to experimental conditions through the use of a random number generator. Pairs were further grouped by pretest score: low (3 pairs), medium (5 pairs), and high (2 pairs).

Two procedures were used to test the validity of the rankings. First, students' "six best items" subscores were correlated with their total pretest scores (Pearson's $r = .92$). Second, item scores were analyzed by factor analysis (orthotran-varimax). Two factors emerged: Representations and procedures. Factor scores were computed for each student and correlated with their "six best items" subscores (Pearson's $r = .91$).

Posttest

The posttest was in two parts: the pretest (as given before treatments) together with items on ordering decimals, decimal representations, appropriateness of method, and decimal computation. Items were scored for correctness of result and validity of method. Following the posttest, eight students were interviewed: the two pairs scoring highest on the pretest and the two pairs scoring lowest on the pretest. All interviews were videotaped and transcribed.

Instructors

The students' regular 4th-grade teacher taught the microworld group. A research assistant taught the wooden-blocks group. The regular 4th-grade teacher had never used this instructional approach before, nor had she used a microcomputer in instruction. The research assistant was an experienced teacher who was thoroughly familiar with the aims of instruction and with the computer program being used by the microworld group. We assigned the research assistant to the wooden-blocks group so that any "teacher expertise" bias would favor wooden-blocks instruction.

Instruction

Each instructor worked from a transcript written for the microworld treatment. The wooden-blocks instructor modified segments appropriate only for the microworld so that they were appropriate for wooden blocks. In-class activity sheets and homework sheets were identical for both groups. All microworld instruction was videotaped; field notes were taken during blocks instruction.[1]

Instruction was in three segments: Whole number addition and subtraction, decimal numeration, and decimal addition and subtraction (see below). Instruction on whole number addition and subtraction emphasized place-value numeration, transformations of numerals, the creation of methods for solving addition and subtraction problems, and the recording of actions done while applying a method. An emphasis was placed on students' freedom to create schemes for operating on blocks to solve addition and subtraction problems, with the provision that they had to represent in notation each and every action in their scheme, whether it be a change of representation or an arithmetical operation. How students denoted their actions was in large part left up to them. Alternative action schemes for solving problems were discussed frequently, as were alternative notational schemes for any given action scheme.

RESULTS

Testing

Students' performance on the pretest before and after instruction was stable. Each group's average score increased somewhat after the instructional period, with the exception of the High Blocks group, whose performance deteriorated (Figure 4). The deterioration in the High Blocks group was focused predominantly in the items having to do with concepts of decimal fractions. The increase in both Medium groups was across all items, the greatest increase occurring on items having to do with concepts of decimal fractions.

Students' performance on the posttest's decimal concept and skill items is shown in Figure 5. The disordinal interaction between pretest performance levels (Low, Medium, and High) and treatments (Microworld and Blocks) is obvious. The Low Blocks group outperformed the Medium and High Blocks groups on each of the areas of decimal representations, decimal computations, and decimal ordering; The Low Blocks group outperformed all groups on decimal computation (though all but Low Microworld scored about the same). The Medium Microworld group outperformed all Blocks groups on decimal representations, decimal ordering, and appropriateness of method. The High Microworld group outperformed all groups on each of the concept tasks.

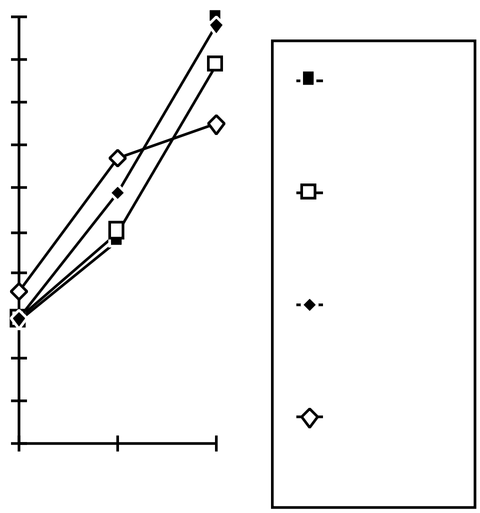

Figure 4

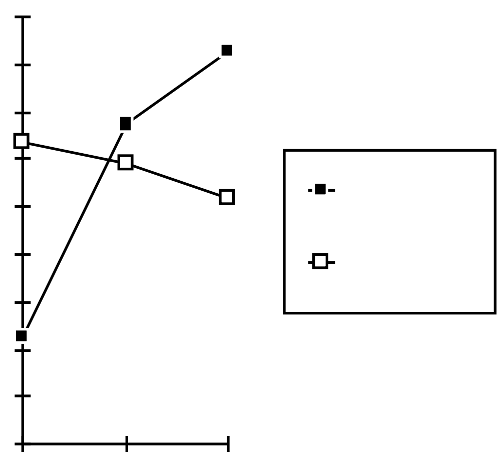

Figure 5

It is noteworthy that all Blocks students used either standard computational algorithms, or used addition and subtraction algorithms based on a method introduced early in instruction as just one example of unconventional approaches to calculation sums and differences. The Microworld group showed a variety of methods, although no student used more than one.

On one item showing the record of a method not used by any students during instruction, all but one Blocks student agreed with the statement that "the answer is correct, but it's not the right way to do it." Among the Microworld group, 3 of the 4 Low students agreed with the statement, while 4 of 5 Medium students and 2 of 2 High students disagreed with the statement.

Interviews

The four students in the High groups and the lowest four in the Low groups were interviewed after instruction and posttest. The interviews focused on students reasoning as expressed on the posttest.

Low, Blocks Group

Despite their performance on the decimal concepts and skills portion of the posttest, the Low Blocks group showed little evidence of having interiorized blocks and numeration as linked representational systems. Students in this group gave explanations that were largely procedural, without reference to actions on blocks or to constraints imposed by the decimal numeration system. The following excerpt illustrates the sorts of explanations given.

AT: I was just wondering why you didn't do it this way [*shown to right]*, why you did it that way. Why did you line up the seven and the...

$$\begin{array}{r} 7.31 \\ \underline{-\ 6.4} \end{array}$$

T: Well, I guess because, because you can't, well cause you need to line up the decimals, because you can't subtract. Well, its four minus, four minus one and decimal. And three minus six. But there's not another decimal, so you can't even do that. The decimals have to be lined up.

AT: You mean the decimal points?

T: Yeah.

AT: Ah. And you don't think it's correct if I say ,"Okay, one minus four, I can't do that so I can borrow from here, and then make this an eleven. Eleven minus four is seven and then let's see. Twelve minus six is six. Then put a decimal here. And then say, six." That wouldn't be right?

T: Well, no, because you'd still have this decimal hanging there, and you wouldn't, you couldn't put that there because it wouldn't, then it would be seven point six point … six point seven [*i.e., 7.6.7*].

Low Microworld Group

Two students in the Low, Microworld group missed two days of instruction, so this may have contributed to their low posttest scores. Regardless of absences, students in this group showed little understanding of decimal numeration, and had lost even the modest facility they previously had with written computation.

AT: Which one (7.9 or 7.89) is bigger than which?

B: This one's bigger than that one [*pointing to 7.89*].

PT: Could you read them?

AT: Seven point eight nine is bigger than seven point nine.

B: Yeah

AT: Why?

B: Because you have an eight, nine and you just have a nine there.

AT: And?

B: And eight, nine is bigger than nine.

AT: Eight, nine is bigger than nine. What place value is this? Where is this eight?

B: Tenths.

AT: In the tenths? And this one here?

B: Is nine ones.

High Blocks Group

The two students in the High Blocks group appeared to have ignored instruction on decimal numeration and concentrated exclusively on “answer getting” procedures for processing numerals. The following excerpt is illustrative of this. It is interesting to note that “J” had the second highest pretest score.

PT: Remember the block stuff? What'd you think of it?

J: Um, I thought it was pretty easy because we had done it in our math books before.

PT: You did it in your math book? Did you do decimals in your math book?

J: No.

PT: Did the blocks help you think about decimals and fractions with tenths and hundredths?

J: No, not really.

PT: Not really, huh. So how was it that you learned about decimals? You know, to add and subtract them.

J: I don't know.

PT: You don't know. What were you thinking? Did it make sense to you?

J: Well, I just thought about it as doing it without the decimal and then I just added the decimal at the end.

PT: Even when things weren't lined up?

J: What do you mean, they weren't lined up?

PT: Well, we'll get to that , okay. So you just ignored the decimal and then treated it...

J: Yeah.

PT: ...like you always did before.

J: Sort of like dollars and cents.

PT: Ah, I see.

J: So you don't have to worry about it.

High Microworld Group

The High Microworld group showed complete facility with decimal numeration and with the source of conventions.

PT: Some people have said that they think that this problem should be written like this. So when you write it up and down, seven point three one take away six point four, it looks like this. They say you always have to line up the right hand side.

$$\begin{array}{r} 7.31 \\ \underline{-\ 6.4} \end{array}$$

K: Well, not exactly. If you have a decimal point on this side, you got to, um, you got to match the decimal point with the, this and this with the ones and this is the tens. So you're actually measuring up by the decimal point.

PT: Well, it sounds like we have two rules that are in conflict. One is to always write, line things up on the right hand side, and the other is to always line up the decimal point. Are those different rules?

K: Yeah, Um, when you line them on up at the right hand side, you usually have two whole numbers which are ones and other, other numbers in front of it. And so when you're adding then or subtracting to the other, they all fit together. But, when you have a decimal point and there's this number has a, a one, a number that is a hundredth and this one is only a tenth, you scoot it over to match the decimal point.

PT: Oh, I see. So lining up the decimal point is what keeps you from...

K: Messing up.

………

PT: Did you feel like you knew what you were doing?

K: Um-hum.

PT: When you were doing these problems, did you ever think about the, the blocks?

K: Yeah, sometimes. When I get stumped or stopped.

PT: Um-hum.

K: Well, I think about the blocks and how, how they would match together and go together.

DISCUSSION

Evidently, the experiences had by the two treatment groups were very different. A review of field notes taken during wooden-blocks instruction and of videotapes recorded during microworld instruction suggests the nature of the difference.

Both groups began instruction expecting to be told “how to do it.” Students in the Blocks group showed great resistance throughout instruction to entertaining alternative methods of solving addition and subtraction problems. On one occasion a student asked, “Can we just do it the old way?” The Microworld group also resisted discussing alternative methods, but only initially. One Microworld student, when asked after class one day how he liked what the class was doing, replied: “I really like it, this way of doing it if it makes sense to you. But I’m afraid to do it, really.” His fear was that “Next year the teacher might mark it wrong.”

On several occasions it was apparent that wooden blocks were of little value in constraining students’ actions and thinking relative to mathematical concepts. For example, one in-class activity said “select a block to stand for one, then put blocks out to represent 3.41.” One student selected a cube to stand for one, then looked back at her paper, reading “three hundred forty-one.” She put out 3 flats, 4 longs, and 1 single, looked at the paper and back at the blocks, then went to the next task. A student in the Microworld group started similarly, selecting "A Cube is One" from the **Unit** menu, and then putting out 3 flats to make "three hundred forty-one." After putting out 3 flats, the Microworld student looked at the screen and said, "Point three? That's not what I want. ... Oh! A cube is one!"

Though the wooden-blocks teacher frequently oriented students toward correspondences between what they did with blocks and what they might write on paper, the Blocks students showed little evidence of *feeling* constrained to write something that actually represented what they did with blocks. Instead, they appeared to look at the two (actions on blocks and writing on paper) as separate activities, related only tangentially by the fact that the written symbols could have reference in the world of wooden blocks.

Except for the Low students, students in the Microworld group repeatedly made references to actions on symbols as *referring* to actions on blocks. One reason for this might be that their attention was always oriented toward the symbols, *even when their intention was to operate on blocks*. That is, the Microworld group acted on blocks by acting on their symbolic representations. Thus, the relationships between notation and manipulatives was always prominent in their experience.

Wearne and Hiebert (1988) outlined a local theory of competence with written symbols. In their theory manipulatives serve as referents for symbols, and actions on manipulatives serve as referents for actions on symbols. The idea of manipulatives-as-referents is not in contrast to the issue of interiorization of constraints. Rather, it highlights one aspect of interiorizing concrete materials as embodiments of a mathematical system. The contribution of interiorized constraints to a student's thinking is that they provide the principles by which the system works.

Constraints are what make situations problematic, and it is overcoming constraints that constitutes problem solving. Students must conceive of notation (literal or manipulative) as *representing* something. Notations themselves cannot be the object of study. Also, students must construct an equivalence between notational systems. Multiple, linked representational systems do not make these achievements easy. Rather, they can have the effect of orienting students' attention to the issues of representational equivalence.

FOOTNOTES

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[1] Blocks instruction could not be videotaped. Two special education students were present during instruction, but not part of the study. State and university policy does not allow videotaping or photography of special education students without parents' permission.

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