A Theoretical Model of Quantity-Based Reasoning in Arithmetic and Algebra *

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Running Head: Quantitative Reasoning

ROUGH DRAFT

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Mathematics is done within a tradition and with customary modes of reasoning. In the United States, it is customary to approach complex mathematical problems with the idea that we will use the tools of algebra as an aid to manage such complexity. One byproduct of this custom is that we are predisposed to move away from the situation itself, with all its complex relationships, and move toward the formalities of algebraic and numerical manipulations. This is completely natural and quite acceptable—except in the case of persons for whom "formalities" are magical and devoid of situational reference. Another byproduct of our predisposition to think that complexity is a domain for formal algebra is that we are undisposed to reason in other, more concrete ways and we are undisposed to hold such reasoning as a type for which teachers should aim. Our predispositions also have the effect of directing us away from examining kinds of concrete, intuitive reasoning that might foster the development of the algebraic reasoning which we value so highly.

Here is an example of the distinction I have in mind. Two approaches to solving a problem are depicted: customary algebra and what I call reasoning with quantities and relationships.

> I walk from home to school in 30 minutes, and my brother takes 40 minutes. My brother left 6 minutes before I did. In how many minutes will I overtake him? (Krutetskii, 1976, p. 160)

An algebraic approach would be to set up an equation, such as $(6+t)\frac{d}{40} = t\frac{d}{30}$, and then solve for t. This is a customary approach, and today any problem like this would be placed in an algebra textbook.

On the other hand, this problem can be approached in a much more "situation-sensitive" manner. Here is an example of one such chain of reasoning:

- Imagine myself and brother walking: What matters is the distance between us and how long it takes for that distance to become zero.

- The distance between us shrinks at a rate that is the difference of our walking speeds.
- I take 3/4 as long as brother to walk the same distance, so I walk 4/3 as fast as brother.
- Since I walk 4/3 as fast as brother, the difference of our speeds is 1/3 of brother's speed.
- The distance between us shrinks at 1/3 of brother's speed, so the time required for it to become zero is 3 times what brother took to walk it.

∴ I will overtake brother in 18 minutes.

If we take this example as illustrative of a *type* of reasoning—what I call quantitative reasoning—and if we take quantitative reasoning as an objective of arithmetical instruction, then problems like the one discussed above can be included in the middle school and junior high curriculum.[1] To take quantitative reasoning as an objective of instruction, however, it will help to have a clear, detailed picture of the mental operations and conceptual structures that enable quantitative reasoning to happen.

Before continuing, some issues must be brought out. First, "students' quantitative reasoning" is almost an oxymoron. For the most part students do not reason quantitatively within school mathematics. Textbooks and curricula do not promote quantitative reasoning (Stigler, Fuson, Ham, & Kim, 1985; Fuson & Stigler, 1989; Porter, 1989). It would be surprising to find many teachers teaching for quantitative reasoning since a large portion do not reason quantitatively themselves (Post, Harel, Lesh, & Behr, in press). So, from the beginning I am in the somewhat awkward position of speaking of a model of a particular style of cognition that is rare, but which should be a primary aim of instruction in the elementary, middle, and junior-high grades. One purpose of this paper is to explicate a meaning of quantitative reasoning as it pertains to reasoning arithmetically and algebraically. An explication will enable us to speak with precision about cognitive objectives for which we might aim and about what might be wrong when students perform poorly.

[1] Krutetskii (1976) considered this problem to be an arithmetic problem.

Second, the theoretical model of quantitative reasoning presented here is in its formative stages. Its inspiration comes from many sources, most prominent of which are research on additive and multiplicative conceptual fields (e.g., Vergnaud, 1982, 1983, 1988; Kaput, 1985; Schwartz, 1988; Post, Lesh, & Behr, 1988) and interviews of research mathematicians, mathematics and science graduate students, middle school and junior high students, middle school and junior high teachers, and elementary/middle school education majors.

Many sources suggest that students rarely reason in terms of quantitative operations. Rather, their thinking is dominated by numerical operations, which (as will be shown later) are orthogonal to quantitative operations. The fact that students may be correct or incorrect in solving a problem is explained, in terms of accounting for observed behavior, more often by appealing to models of text-processing schemata than by models of quantitative reasoning (e.g., Mayer, 1983, 1987; Kintsch & Greeno, 1985).

Third, the model reflects my bias against local, decontextualized hypotheses. The model is of a *development* of quantity-based algebraic reasoning. In addressing the issue of development of a concept field such as quantity-based algebraic reasoning, one is constrained severely by the supposition that people are not born as ninth-graders. They were also eight-graders, and before that they were seventh-graders. That is to say, quantity-based algebraic reasoning must be evidently constructible from the kind of knowledge expressed in competent quantity-based arithmetical reasoning.

What This Model is About

The model proposed in this paper is about a stratum of reasoning that lies beneath both applied arithmetic and applied algebra. It is about people using “rigorously qualitative” reasoning, where rigor derives from the intention to attend to the quantification of a situation’s qualities.

It is also about the potential that learning to reason quantitatively has for the kinds of algebra children could be expected to learn. That is, it looks for the boundaries of quantitative reasoning in the world of algebraic reasoning.

The model presented here is about the cognitive processes and conceptual structures that *enable* the kind of reasoning illustrated in the previous section. These processes and structures are at the heart of what I call quantitative reasoning and its expression in applied arithmetic and applied algebra.

Definitions

I will use the following as heuristic, working definitions of quantity, quantitative operations, and quantitative reasoning. I ask you to keep in mind that these definitions have a purpose: They are meant to support the *operationalization* of notions of quantitative and algebraic reasoning. That is, they should be examined as constructs in a system. The next section (Definitions in Action) will show the operationalizations that they enable.

Quantity

A quantity is a quality of something that one has conceived as admitting some measurement process. Part of conceiving a quality as a quantity is to explicitly or implicitly conceive of an appropriate unit.

This definition hides a lot of cognition. It assumes that someone has isolated an object or has objectified a phenomenon having qualities that, to them, may be measured. To comprehend a quantity, children's conceptions of the "something" must be elaborated and analyzed (by them) to the point that they see characteristics of the object which are admissible of processes of gross or numerical quantification.

By defining the construct of quantity in this way, it is possible to describe pre- and non-numerical comprehensions of what, to us, are quantitative situations. It is quite possible that earlyestablished conceptions of gross quantity provide cognitive “cues” for expert problem solvers of applied algebra problems as to productive directions for elaborating their comprehensions of a situation in terms of quantities and relationships.

Quantification

Quantification is a process by which one assigns numerical values to qualities. That is, quantification is a process of direct or indirect measurement.

This definition does not imply that a quantification process result in a class of magnitudes (Freudenthal, 1972.) I am not speaking of quantification processes as producing mathematical systems, although to a sophisticated quantifier they could. Young children’s counting can be a quantification process (to measure numerousity) just as can a nutritionist's attempts to quantify the energy required to dispose of complex sugars.

It is in in the process of quantifying a quality that it becomes truly analyzed. What is a quantity from our perspective will become a quantity from a child’s perspective only in the context of the child’s development of quantification processes for it. Quantity and quantification are a dialectic.

Value

A quantity’s value is the numerical result of a quantification process applied to it.

Extensive Quantity (“Number of things”)

An extensive quantity is a quantity that may be measured directly or is a combination of directly measurable quantities. Put another way, extensive quantities are quantities whose measures admit normal arithmetic (Cohen & Nagel, 1939).

Intensive Quantity

Intensive quantities are quantities whose measures are non-additive as in normal arithmetic (Cohen & Nagel, 1939). Speeds are not additive in general, although velocities are additive as vectors. Temperatures, densities, and frequencies are not additive in general.

These definitions of extensive and intensive quantities differ somewhat from others given in the literature (e.g., Kaput et al., 1986; Ohlsson, 1988; Schwartz, 1988). According to these authors, an intensive quantity is one that can be measured by a ratio. I use Cohen and Nagel's terminology largely because the central issue concerning intensive quantities has to do with the distinction between ratio and rate (discussed below) and that the distinction between intensive and extensive quantities must not interfere with this more central distinction.

Difference

A difference of two quantities is the quantity by which one quantity exceeds or falls short of another.

It seems reasonable to allow a weaker definition: A difference of two quantities is the amount by which one quantity's *value* exceeds or falls short of another's. The reason for allowing the weaker definition is that, conceptually, it seems reasonable to make comparisons like "How much greater is my height in centimeters than your age in years?" The stricter definition allows comparisons only within a dimension.[2]

There can be no direction implied in the definition of difference. If the compared quantities are measured in the same unit, then the direction of the comparison must be decided upon by considerations external to the comparison itself. For example, "the" difference between a boy's height measured in centimeters and his dog's height measured in centimeters is not well defined until we make explicit our assumptions about the direction of the comparison. Are we asking how

[2] The weaker definition reflects our ability to turn any measure into a counting measure. It says, in effect, that we can ask "How many more of these units than those units are there?"

much taller the boy is than the dog or how much taller the dog is than the boy? In “competent” quantitative reasoning, this decision is made independently of the quantities’ values.[3]

On the other hand, if the two quantities being compared are measured in different units, then the unit of the difference determines the direction of the comparison. If one is comparing a quantity of apples with a quantity of oranges, and conceives of the difference as a number of oranges, then the comparison is “How many more oranges than apples are there?” If there are more apples than oranges, the difference will have a negative value.

Ratio

A ratio is a multiplicative comparison between two quantities.

The goal of forming a ratio is to conceive “how many times as large” is one quantity than another, or to conceive “how many of these is in that.” If the quantities being compared are measured in the same unit, then the comparison can be conceived as being a direct comparison of qualities. If the quantities are measured in different units, then it is segmentations (measures) of their qualities that are being compared.

This definition of ratio emphasizes the quantitative-relationship aspect of ratio. It allows us to speak of ratios without implying an inextricable link between a ratio and an arithmetic operation to calculate its value. Of course, in the canonical situation of knowing the values of a ratio’s “numerator” and “denominator” quantities, division is the operation that determines the ratio’s value. However any operation may be used to calculate a ratio’s value, and that operation is determined by the relationship of the ratio to other quantities in a specific situation. This point will revisited later in the discussion of quantitative operations.

[3] A comparison’s direction must be fixed independently of the quantities’ values. Many students will assume that the direction is always from the greater measure to the smaller—to ensure a positive value for the difference between two quantities. This gets them into trouble when variable quantities are compared; the difference may have positive or negative values as the values of the compared quantities vary.

Rate

A rate is a quantity that may be analyzed into a multiplicative comparison between two other quantities—where one quantity's value is conceived as varying in constant ratio with variations in the value of the other.

There is a strong relationship between concepts of rate and ratio. The principal distinction that I draw is this: When one conceives of two quantities in a multiplicative comparison, and conceives of the compared quantities as being compared in their, *independent*, *static* states, one has formed a ratio. As soon as one re-conceives the situation as one in which the value of one quantity varies *in constant ratio* with variations of the other, one has conceived a rate.[4]

Note the wording "as soon as one re-conceives …". It is possible, indeed likely, that people first conceive a multiplicative comparison in terms of a ratio and re-conceive that ratio as a rate. A stereotypical occasion for this to happen in schools is when students are asked to find an "average speed" when all that is known is a distance traversed and the time of traversal. However, occasions for re-conceptualizing a ratio as a rate are rife in applied problem solving. We see the re-conceptualization of a ratio as a rate as being a critical, fundamental conceptual ability that supports much of successful performance in proportional reasoning tasks (Lesh et al., 1988)[5], and it may be a foundation for the concept of derivative in the calculus.

This definition of rate is different from those given by Lesh et al. (1988), Kaput et al. (1984), Ohlsson (1988), Post et al. (1988), and Schwartz (1988), where the principal distinction between ratio and rate has to do with the natures (measure spaces, or dimensions) of the quantities being compared. According to these authors, a comparison is a ratio if the quantities

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[4] I feel very sorry for school students who are told by texts that, on the one hand, "a ratio is used to compare parts of one thing to parts of another," while, on the other hand, "a rate is a ratio comparing different units." Any student who attempts to understand this distinction can justifiably conclude that mathematics is not meant to be sensible.

[5] A proportion in the form $\frac{a}{b} = \frac{c}{x}$ can represent, in principle, two situations: a statement of equivalent ratios or a statement of two instantiations of one rate. It is clear that the latter involves proportional reasoning; whether or not the former involves proportional reasoning is a current issue.

being compared are within the same measure space, and a comparison is a rate if the quantities being compared are of two different measure spaces.[6]

Some researchers distinguish between ratio and rate by the convention that ratios are multiplicative comparisons and rates are "unit ratios" (e.g., Kaput, ___). In that distinction, "3 oranges per 2 apples" expresses a ratio, while "3/2 oranges per apple" expresses a rate. In the present model, this is a false distinction. If one expresses a relationship as "3 oranges per 2 apples" and has in mind two collections—one having 3 oranges and the other having 2 apples, then the underlying conception is of a ratio. On the other hand, if one expresses a relationship as "3 oranges per 2 apples" and has in mind that as the value of a quantity of oranges changes, the value of the quantity of apples also changes, and they *always* retain the relationship "3 oranges per 2 apples," then the underlying conception is of a rate. One cannot tell by forms of expression alone whether a speaker is conceiving of a ratio or of a rate.

Another difference between conceptions of a rate and conceptions of a ratio is that, typically, a ratio describes a situation that is phenomenologically bounded whereas a rate does not. To conceive of a speed requires neither a conception of a distance travelled nor an amount of time spent travelling. To conceive of an average speed requires the conception of a "trip-so-far"—a completed or partially completed trip.[7]

In the definition of rate given here, the natures of the compared quantities and the manners in which relationships are expressed are irrelevant. Instead, the essential characteristic of a rate is that it is conceived of as constituting a functional relationship. This is consistent with

[6] Schwartz (1988) makes a slightly different distinction. He calls a ratio a rate if it describes a "real" quality of an object, e.g. an object's density. There is no disagreement here with Schwartz's position, except that the definition of rate I have given includes the possibility of generalizing a ratio to a rate, which can produce rates that are not qualities of any "real" thing (e.g., generalizing a ratio between students in a school and planets in the solar system to the rate "students per planet"). In this instance, the quantity is a *constructed relationship* between students and planets, yet it does not refer to a "real" quality of some object.

[7] One student, responding to the question "Can you measure the speed of a car in miles per century?", said "You can't measure the speed of a car in miles per century because the car would rust away or the driver would die before a century." This student was not conceiving of speed as a rate. "Miles per century" meant that one had to drive for a century.

Karplus' definition of rate as a linear function (Karplus et al., 1983), and with conceptions of a single-variable derivative evaluated at a point (i.e., an instantaneous rate of change) as being the slope of a tangent line. It is different from Karplus et al.'s notion of rate as a linear function in that, here, a rate is a *quality* of something that is measured through a functional relationship; it is not a function. "Linear function-ness" is a numerical relationship between the *values* of two quantities.

Quantitative Operation

A quantitative operation is the conception of two quantities being taken to produce a new quantity.

The model includes eight quantitative operations. These are sufficient for generating viable comprehensions of the great majority of quantitative situations commonly depicted in arithmetic and algebra. This list is influenced by Greeno et al. (1985) and by Shalin (1988), which in turn are based on the work of Judah Schwartz and Jim Kaput.

<u>Operation</u>	<u>Example</u>
• Combine quantities additively	*Unite two sets; consider two regions as one.*
• Compare quantities additively	*"How much more (less) of this is there than that?"*
• Combine quantities multiplicatively	*Combine distance and force to get torque; combine linear dimensions to get regions; combine force applied and distance travelled to get work.*
• Compare quantities multiplicatively	*"How many times as large is this than that?" "This is (multiplicatively) what part of that?" "How many of these in those?"*
• Generalize a ratio	*"Suppose this comparison applies generally (i.e., suppose it were to continue <u>at the same rate</u>)."*
• Instantiate a rate	*"Travel 50 miles per hour for 3 hours." "Travel 5 hours per*

	mile for 6 miles."
• Compose ratios	*"Jim has 3 times as many marbles as Sally; Sally has 4 times as many marbles as Fred. Jim has so many times more marbles than Fred."*
• Compose rates	*"A German mark is 75.53 Japanese yen. A US dollar is 1.88 marks. A dollar is some number of yen."*

It is important to note that quantitative operations are not the same as arithmetic operations. Arithmetic operations are numerical operations that are used to calculate a quantity's value; there is no direct correspondence, except in a canonical sense, between quantitative operations and the arithmetic operations actually used to calculate a quantity's value in a given situation. Here is an example:

> Jim is 15 cm taller than Sarah. This difference is five times greater than the difference between Abe and Sam's heights. What is the difference between Abe and Sam's heights?
>
> • *Difference between Sam's and Abe's heights: (15÷5) cm.*

In this situation division is used to calculate the value of a difference—even though, canonically, "difference" means "subtract."[8]

It is important to note that a quantitative operation is, in a sense, a description of how a quantity comes to exist. A quantitative operation is a conception of *one* quantity, but a conception which includes its relationship with the quantities operated upon to make it.[9]

[8] It is clear why students *do* confound quantitative terms like difference and ratio with the arithmetic operations of subtraction and division. In one text series, the word "difference" is used exclusively to name the answer in a subtraction calculation, and the word "ratio" is used exclusively in conjunction with the writing of a fraction or the computation of a quotient.

[9] This is analogous to a conception of a sum (a+b) as one number, but made in a particular way.

Quantitative Relationship

A quantitative relationship is the conception of three quantities, two of which determine the third by a quantitative *operation.* [10]

The key phrase is that two quantities determine a third by a *quantitative* operation. In an arithmetic relationship, any two of three related numbers can be thought of as determining the third. This is not the case of quantities in a quantitative relationship. If a quantity is conceived as resulting from a quantitative operation on two other quantities, then the relationship cannot be changed without re-conceiving at least one of the quantities.

For example, suppose an average speed is conceived by a multiplicative comparison of a distance and an interval of time. That average speed is a ratio. To conceive of the distance traveled as being made by moving at an average speed for an interval of time, we must re-conceive the average speed as a rate of change of distance with respect to time. In this re-conception, “average speed” is no longer a ratio; it is no longer a multiplicative comparison of two quantities.

Quantitative Structure

A quantitative structure is a network of quantitative relationships.

The idea of a quantitative structure will be illustrated in the next section.

Quantitative Reasoning

Quantitative reasoning is the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships.

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[10] This is analogous to thinking of a sum in two ways: as ($a+b$) and as a number independently of the fact that it is a sum — $z = a+b$.

Complex quantitative reasoning entails relating groups of quantitative relationships, such as in forming a multiplicative comparison of an additive comparison and an additive combination ("This situation is about how many times as large is this difference than is that combination.").

Formulas

A formula is an expression that describes a numerical method to calculate a quantity's value.

Quantity-based Arithmetic

Quantity-based arithmetic is:

1) *Quantitative reasoning*
2) *Determination of appropriate operations to calculate quantities' values*
3) *The propagation of calculations*

Later sections will amplify this characterization of quantity-based arithmetic. For now, the important feature to notice is that arithmetic operations are inferred according to relationships among quantities.

Quantity-based Algebra

Quantity-based algebra is the same as quantity-based arithmetic, except:

1) *Representations of situations are under-constrained in terms of quantities' values (there is not enough numerical information to propagate calculations)*
2) *Some value or values is represented symbolically*
3) *Formulas are propagated instead of values being propagated*

Inferences drawn must be propagated in "open" form because of the presence of one or more indeterminate values in resulting expressions. Thus, every expression formed in the process of solving a problem quantitatively is a formula for calculating a quantity's value.

Equations

An equation is (1) a formula for a quantity's value together with the value that the formula must yield, or (2) two formulas for a single quantity's value.

Quantity-based algebra includes issues not present in quantity-based arithmetic. These are: (1) propagating expressions for the explicit purpose of *describing* how to calculate a quantity's value; (2) explicitly holding the possibility of propagating an algebraic expression to describe how to calculate a quantity's value when that value is already known; and (3) propagating two non-equivalent algebraic expressions, where both expressions describe how to calculate the same quantity's value. In cases (2) and (3), propagation produces equations.

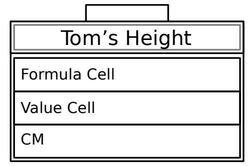
Definitions in Action

The definitions given in the previous section are the essential stuff of a model of quantity-based arithmetic and algebra. They have been incorporated into a computer program, named WPA, that is designed to have an observable feature that corresponds directly with an issue or distinction addressed by the model. The program also contains an "engine" that makes inferences about what arithmetic operations are appropriate for calculating a quantity's value, and which propagates inferences through the structure.

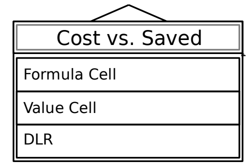
The structure of this section parallels that of the previous. It will use the same subheadings, but the content of a subsection will describe how the design of the program (as an implementation of the model) incorporates a definition or addresses an issue brought out earlier. Examples are given to show implications of each design feature (i.e., assumption) for the behavior of the model as a whole. The program's user-interface was inspired by the work of Valerie Shalin and Nancy Bee (Greeno, 1985, 1987; Shalin & Bee, 1985; Shalin, 1988).

Quantity

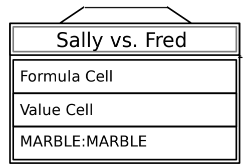
The correspondent in the program of someone thinking of a quantity is for the program to display a "notecard."[11] The program contains four types of notecards (shown at the right), one for each of four types of quantities: Number of things, difference, ratio, and rate. Any notecard has places for four items of information:



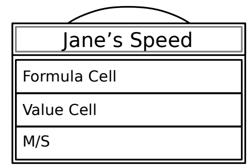
Number of things



Difference



Ratio



Rate

- A name for the quantity (the title-bar of the notecard),
- a place for a numeric or algebraic expression that describes how to calculate the quantity's value (the *Formula Cell*),[12]
- a place for the quantity's value (the *Value Cell*), and
- a place for the unit in which the quantity is measured.

Every notecard must have a title. This reflects the assumption that competent problem solvers isolate quantities by their features, and that in isolating a feature one constructs a linguistic or imagistic description of that feature.

The presence of a formula cell in a notecard corresponds to an *awareness of the possibility* that one may not be able to get direct access to a quantity's value, but may have to infer it by its relationships with other quantities.[13]

[11] A notecards does not appear magically. It appears because whoever is using the program directs it to display one. The only things the program does automatically are to check the consistency of the structure, infer information about what operations to perform, and propagate inferred information through the structure as it exists at the moment of inference.

[12] Actually, there are five places. One may set the program so that notecards have two formula cells. The eventual need for two formula cells is described later.

[13] Another name for a formula cell might be "Calculation Cell." However, that would not be entirely accurate, as the cell itself does no calculations. It contains an expression that describes what calculations need to be performed.

Quantification

The presence of a unit cell in a notecard corresponds to the assumption that a feature is well-conceived as a quantity only after one has settled the issue of quantifying it, and a quantification process necessarily involves a unit. If counting is the quantification operation appropriate for a situation, then one must decide what is to be counted and what is not. In counting, a unit is a situationally-defined countable item.

Also, the process of forming a unit gives insight into the operational composition of a quantity. A standard textbook example is to introduce Cartesian-product concept of multiplication by way of a problem like this:

> Bob has four school shirts and five school trousers. How many school outfits can he wear?

Unless a student conceives of *one* outfit as a shirt-trouser combination, then that student will be unable to conceive of a *set* of such combinations, and hence will be unable to understand that set as something to be quantified by counting.

Quantitative Operation

A quantitative operation is the conception of making a new quantity from two others. The computer implementation of the model does not make a clean distinction between quantitative operation and quantitative relationship. This, in fact may, or may not be a shortcoming. Discussion of this issue is postponed to follow the discussion of quantitative relationship.

Here it is appropriate to point out a common confounding of issues. We need to distinguish between quantitative operations that are correct by their logic and quantitative operations that are conventionally prescribed. For example: Many middle-school teachers have seen students compute the area of, say, a 3 cm. x 4 in. rectangle by multiplying 3 and 4. On the one hand, this could be interpreted as inattention to the quantities involved in the situation—one length measured in centimeters and another in inches—and the need for converting one measurement unit into another. On the other hand, the student could have, quite correctly, conceived of the rectangle as being measured in the unit *cm-in*, the unit being a rectangle having one side of length 1 cm. and the other side of length 1 in. That is, we might not know the appropriateness, or even the potential reasonableness, of a student's calculations until we know the unit he or she assumed for the result.[14]

Quantitative Relationship

As already mentioned, the computer implementation of the model does not make a clean distinction between quantitative operation and quantitative relationship. I am unclear as to whether this constitutes a bug in the program or a feature of quantitative reasoning.

Ratio's and differences, by definition, are comparisons. So by representing a ratio or a difference as a quantity in and of itself, one *implies* a quantitative operation, but without actually specifying one.

On the other hand, one has not performed a quantitative operation until one has a result from operating. To operate means that the operand quantities are known. To have a result demands that the result be known—the result's type and unit cannot be in question, for otherwise the quantity is under-specified and one does not *have* a result. In short, it appears that one cannot have *completed* a quantitative operation without having constructed a quantitative relationship, and that one may not have constructed a quantitative relationship without having conceived of a quantitative operation.[15] The difference between the two appears to be a matter of focus. When

[14] Evidently, it would be rare, today, for us to be wrong when we assume the student failed to attend to the quantities. However, were students to *come* to attend to quantities, and were they to feel free to do whatever might make sense relative to the situation, we would need to pay attention to *their* assumed units for results.

[15] Alba Thompson has suggested a nice distinction. It addresses the issue of relationship-without-operation. One can know that three quantities are somehow related (e.g., distance, speed, and time) but not know how they are related, nor know what quantitative operations are at the heart of the relationship. She suggests that this knowledge be called an "association." One can know (have the feeling) that three quantities are somehow associated, but without knowing how they are related.

focusing primarily on the result of operating, we tend to think of the object of attention as an operation. When focusing primarily on the result *in relation to its operands*, we tend to think of the object of our attention as a relationship.

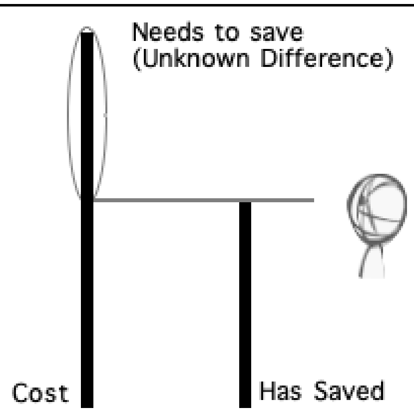
An example:

> John wants to purchase a bicycle. The bicycle costs $143.95. John has a total of $83.48 available to him in cash and savings. How much more money does John need?

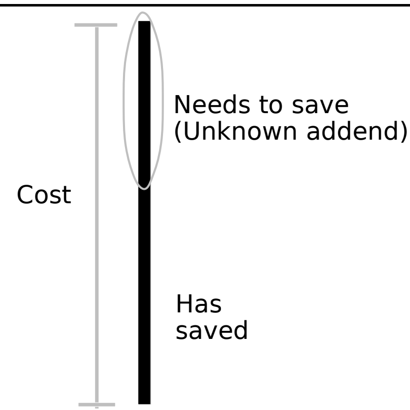
There are, in principle, two ways to conceive of the situation described in the above statement. These are shown in Figure 1.

(1) *We can think of the amount that John needs to save as the amount by which the bike's cost exceeds what John has saved, and*

(2) *We can think of the bike's cost as being made up of what John has saved together with what John needs to save.*



Conception 1



Conception 2

Figure 1

Quantitative operations and relationships are reflected in the program by drawing arrows among the notecards. *Conception 1* is modeled by showing a difference (Needs to Save) that comes from an additive comparison of Bicycle's Cost and Has Saved (Figure 2a). The triangular handle on Needs to Save reflects the conception of it is a difference. The arrows from Needs to

Save to Bicycle's Cost and Has Saved, together with the knowledge that Needs to Save is a difference, means that the relationship is made by an additive comparison.

Conception 2 is modeled by showing Bicycle's Cost as a number of dollars that results from an additive combination of Has Saved and Needs to Save (Figure 2b). Needs to Save is no longer a difference, as in Figure 2a, since it is not made by an additive comparison. It is simply a number of dollars.

In either case—*Conception 1* or *Conception 2*—the resulting arithmetic is the same: one subtracts the amount one has saved from the cost of the bicycle. Yet, the underlying conceptions of the situation are quite different.

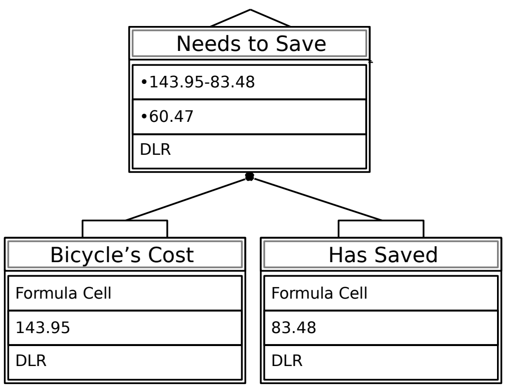


Figure 2.a

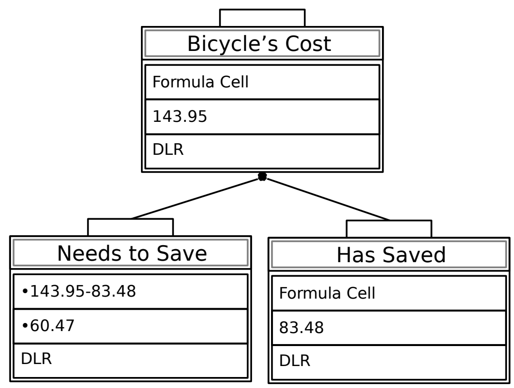


Figure 2.b

Another example of conceptual opacity can be found in textbook proportion problems. The following is typical (including the typical necessity of assuming "at the same rate"):

> Tom ran 3 miles in 17 minutes. How long would it take him to run 5 miles?

The prescribed representation (in schools) of this situation is in the form of a proportion, such as = . However, this proportion does not necessarily reflect any particular underlying conception of quantities and relationships. We have, in principle, three conceptions of the situation, two of which correspond to this proportion. The two conceptions that correspond to this proportion are: (1) that the proportion is a statement of equivalence between two ratios, and (2) that the proportion is a statement of generalizing a ratio to a rate and instantiating that rate with a value. The third conception is unrelated to the idea of proportion: that (3) John always runs at the same rate, so the two trips are just instantiations of his “running rate” over two periods of time.

Figure 3 shows the conception of three quantities — Distance Traveled, Time Traveling, and Travel Rate. Travel Rate is given as a ratio to reflect the anticipation that it will be the result of a multiplicative comparison.[16] Figure 4 shows the establishment of a relationship among these three quantities. The arrows show that Travel Rate is the *result* of an operation on the other two quantities. Travel Rate is a ratio, and it is made by comparing Distance Traveled and Time Traveling.[17]

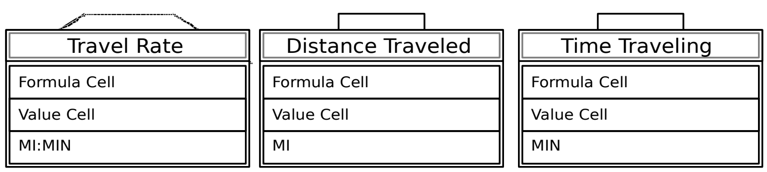


Figure 3

[16] The program reflects the primacy of isolating a quantity before determining its value by insisting that any notecard be given a name and a unit before it is placed with other notecards on the screen.

[17] It should be pointed out that the program demands that units be consistent. Had Distance Traveled been measured in kilometers, the program would not have allowed the structure to be completed, giving the explanation that there is no way to get a ratio in MI:MIN by comparing a number of things measured in KM with a number of things measured in MIN. The program’s demand for consistency among units before allowing a structure to be completed makes it a model of competent quantity-based reasoning, but disallows it from being a model of many kinds of incompetent reasoning.

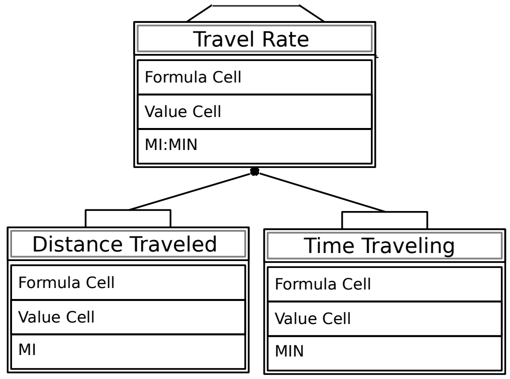


Figure 4

We have two ways to complete the representation of the situation. The first is to represent it as two ratios having the same value. Figure 5 shows this. The two ratio structures reflect a conception of two different runs, each with its own average rate. The heavy line between the two ratio notecards reflects an imposed constraint: the two ratios must have the same value.

Quantity-based Arithmetic

Quantity-based arithmetic is quantitative reasoning together with the determination of appropriate arithmetic operations to calculate a quantity's value, and the propagation of those calculations throughout the quantitative structure. Figure 6 continues the example of the *Tom's*

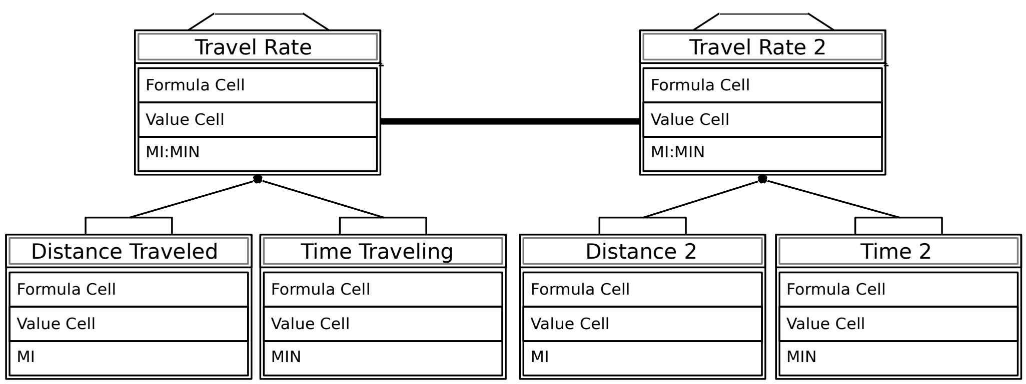


Figure 5

Run problem of the previous section. It shows the results of inferences made by the program and the program's propagation of inferences through the structure. Values were entered in the order

Distance 2 *[5]*, Time Traveling *[17]*, and Distance Traveled *[3]*. Upon entering the value for Distance Traveled, the program had enough information to begin drawing inferences and propagating inferences through the structure. The program indicates that a cell was *inferred* (as opposed to entered by a user) by putting a bullet (•) before its entry.

Note: *v*( ) stands for "value of (whatever is in parentheses)"

- Travel Rate is a multiplicative comparison between Distance Traveled and Time Traveling, so $v(\text{Travel Rate}) = \frac{v(\text{Distance Traveled})}{v(\text{Time Traveling})}$.

- Travel Rate 2 is constrained to have the same value as does Travel Rate, so a calculation to compute its value is also 3/17.

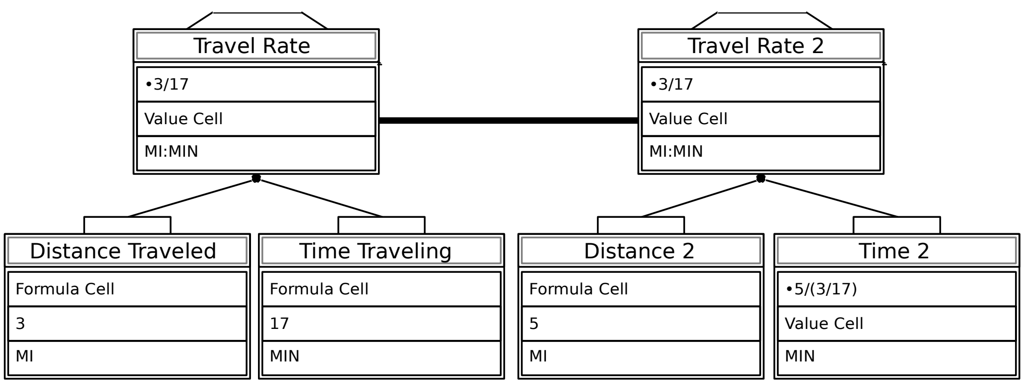


Figure 6

- Since $v(\text{Travel Rate 2}) = \frac{v(\text{Distance 2})}{v(\text{Time 2})}$, $v(\text{Time 2}) = \frac{v(\text{Distance 2})}{v(\text{Travel Rate 2})}$.

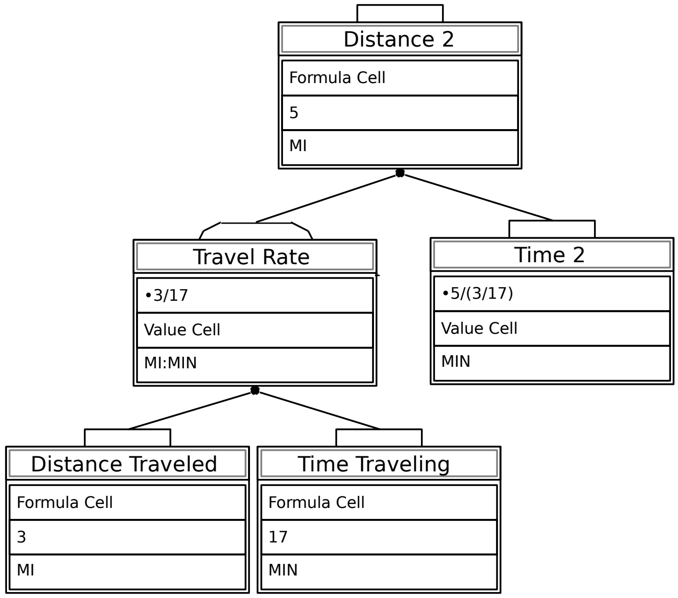


Figure 7

Figure 7 illustrates a conception of the situation as a Travel Rate being a ratio between Distance Traveled and Time Traveling, and then that ratio being used *as if it were a rate* together with Time 2 in order to determine Distance 2. As soon as an arrow was drawn from Distance 2 to Travel Rate, the program reflected the rate-like use of the ratio by "rounding" the corners of its handle to show that even though it is a ratio, it is being used as if it were a rate. This reflects the mental operation of generalizing a ratio to a rate.

The inferences made by the program with respect to calculations to perform were:

- Travel Rate is a ratio comparison between Distance Traveled and Time Traveling, so

  $$v(\text{Travel Rate}) = \frac{v(\text{Distance Traveled})}{v(\text{Time Traveling})}.$$

- Distance 2 is a number of things made by instantiating a rate in MI/MIN with a number of things measured in MIN, so $v(\text{Distance 2}) = v(\text{Travel Rate}) * v(\text{Time 2})$. Hence

  $$v(\text{Time 2}) = \frac{v(\text{Distance 2})}{v(\text{Travel Rate})}.$$

A third conception of the *Tom's Run* is expressed in Figure 7b.[18] In this conception, Tom has a constant running rate (i.e., whenever he runs, he runs at that rate). Thus, both distances are made by running at that travel rate over an interval of time. The order in which the program drew inferences was Travel Rate [•3/17] and then Time 2 [•5/(3/17)].

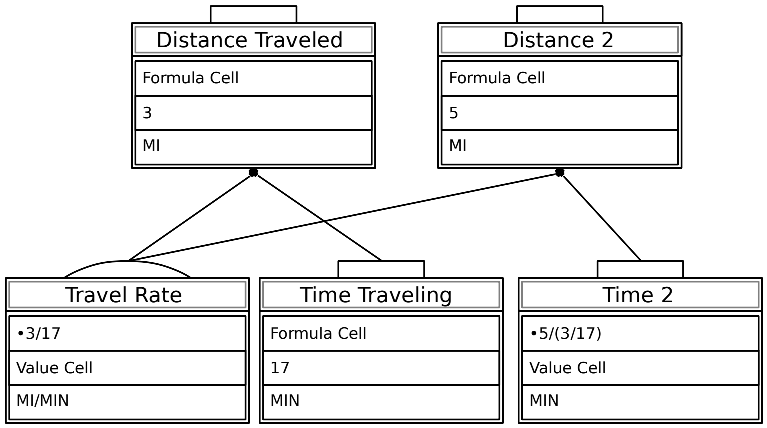


Figure 7b

Each of the three conceptions of the initial problem—the equality of ratios (Figure 6), the generalization of a ratio to a rate (Figure 7), and the notion of "constant running rate" (Figure 7b)—results in the same calculation being done to compute the value of Time 2. However, as conceptions of the described situation in terms of quantities and their relationships, the three conceptions are quite different from each other.

Relationships between Quantity and Arithmetic

I have said repeatedly that there is no necessary correspondence between a quantity's type and the numerical operation actually used to calculate its value. To illustrate this explicitly, we shall look at a variation of the problem used in the previous examples:

> Tom went for two runs. On the first, he ran 3 miles. On the second, he ran 5 miles in 28 1/3 minutes. He ran 1/30 mi/min faster on his first run

[18] This conception was suggested by David Tall and Carlos Vasco.

than on his second. How long did it take him to complete his first run?

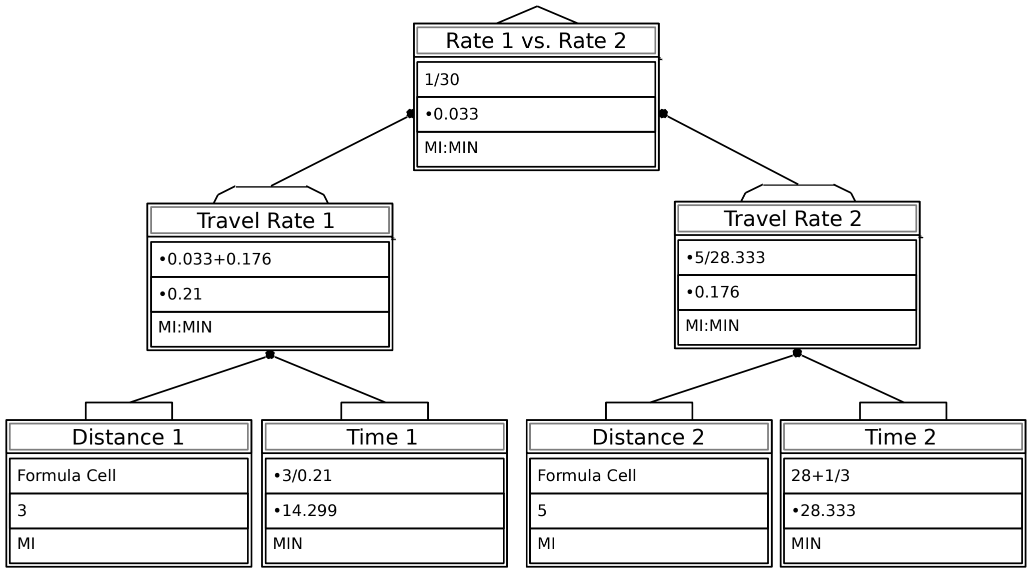


Figure 8

Figure 8 depicts a conception of this situation as a comparison of ratios-as-rates (velocities). The top notecard, Rate 1 vs. Rate 2, is a difference, which means it is the result of an additive comparison between Travel Rate 1 and Travel Rate 2. Since the difference between Travel Rate 1 and Travel Rate 2 has a value of 0.033, and since Travel Rate 2 has a value of 0.176, the calculation giving Travel Rate 1's value is 0.033+0.176. Thus, we have used addition to calculate the value of Travel Rate 1 *even though Travel Rate 1 is a ratio.*

Inferences and Propagation

Given a quantitative relationship, one can determine the appropriate arithmetical operation to evaluate any quantity in the relationship. There are two cases: (1) The resultant quantity is evaluated based on values of the operand quantities, or (2) An operand quantity is evaluated based on the values of the resultant and the (other) operand quantity, and on the canonical operation for the relationship.

*1) Canonical Arithmetic for Quantitative Relationships: Arithmetic operation to evaluate a qu*antity that results from a quantitative operation.

Structure	*Arithmetic Operation to Evaluate the Resultant Quantity*
- A quantity is the result of an additive combination of two quantities[19]	Addition
- A quantity is the result of an additive comparison of two quantities	Subtraction
- A quantity is the result of a multiplicative combination of two quantities	Multiplication
- A quantity is the result of a multiplicative comparison of two quantities	Division
- A quantity is the result of an instantiation of a rate	Multiplication
- A quantity is the result of a composition of ratios	Multiplication
- A quantity is the result of a composition of rates	Multiplication

2) *Non-canonical Arithmetic for Subordinate Quantities in a Quantitative Relationship: Arithmetic operation to evaluate a quantity that is an operand of a quantitative operation.* Inference of operation is based on knowledge of canonical operation.

- If $a = b * c$, then $c = a \div b$ and $b = a \div c$.
- If $a = b \div c$, then $c = b \div a$ and $b = a * c$.
- If $a = b + c$, then $c = a - b$ and $b = a - c$.
- If $a = b - c$, then $c = b - a$ and $b = a + c$.

3) Any time a quantity is evaluated or has a formula constructed for it, every structure of which that quantity is part is examined to see if anything new can be inferred about it. If new information can be inferred, then an inference is made and propagation continues.

[19] "Additive combination" includes disjunctive combinations, such as "Put 3 apples with 4 oranges." The unit for this particular combination is APPLE|ORANGE—any item in this combination is an apple or an orange. Paying attention to the unit avoids the asymmetry of Schwartz's (1988) scheme of calling the resulting unit a "fruit." That is, if the unit of the combination is FRUIT, then from "Sally has 7 [FRUIT], four of which are [ORANGE]" we cannot conclude that she has 3 apples. However, from "Sally has 7 [APPLE|ORANGE], four of which are [ORANGE]" we can, indeed, conclude that she has 3 apples.

Figure 9 repeats Figure 8, but with the program set to accumulate expressions instead of being set to evaluate expressions whenever it can do so (this setting also was used in generating

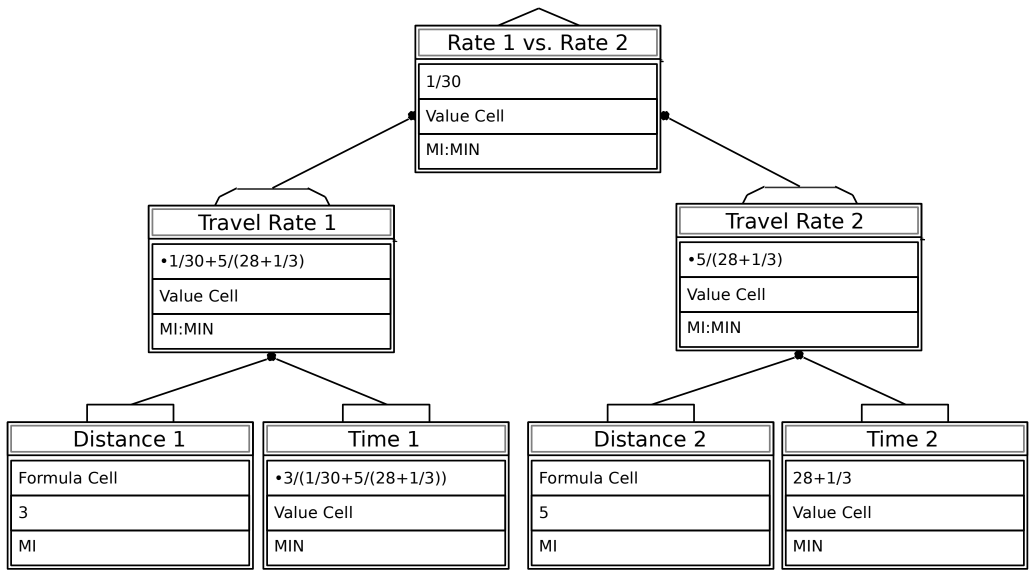


Figure 9

Figures 6, and 7).

The program's setting to accumulate expressions instead of evaluating them immediately reflects the model's hypotheses about key shifts in students' goals and key shifts in students' cognitive processes that constitute a foundation in quantity-based arithmetical reasoning for the development of quantity-based algebraic reasoning. The shift in goals is from the goal of "getting an answer" to the goal of "laying out a pattern of reasoning." The shift in cognitive processes is from immediate evaluation of expressions (to make propagation simpler in terms of subsequent calculations) to the *postponement* of calculations so that one constructs numeric expressions that capture a "history" of a quantity's value. These shifts in goals and processes amount to students developing the intention of constructing "numeric formulas" for calculating quantities' values.

Quantity-based Algebra

Formulas

A formula is an expression that describes a numerical method to calculate a quantity's value. As noted in the section *Transitions to Algebra*, the model proposes that students' transitions from quantity-based arithmetical reasoning to quantity-based algebraic reasoning begin with the onset of their intentions to generate "numerical formulas" for quantities' values.[20] The transition is continued by their developing the intention to create *actual* formulas for quantities' values.

The development of the intention to create formulas requires students to be cognizant of a particular social setting for doing mathematics. It is that there are other persons (including oneself) who might be interested in the solution of this *type* of problem (type being defined by structural similarity), and who are not interested in deriving, or are not capable of deriving, a solution from first principles. The fact that generating formulas has important cognitive benefits is of interest to us (as pedagogues), but generating formulas, as such, cannot be important to students unless they see formulas as being useful constructions which serve definite purposes.[21]

[20] I am not saying that this will necessarily begin spontaneously in children. Rather, I am saying that whether as a result of instruction, reflection, or a combination of the two, the transition begins when children come to *have as a goal* the generation of numerical formulas as a facilitator for reflecting on their patterns of reasoning.

[21] One eighth-grade girl, after constructing a formula for the solution of a problem, asked "Now what?" I said, "Now give this formula to your little sister, explain to her what each letter stands for, and tell her to use a calculator to do the next 10 problems for you." She paused, then said "Oh!!! So that's what a formula is for!" Her sister was in the fifth grade.

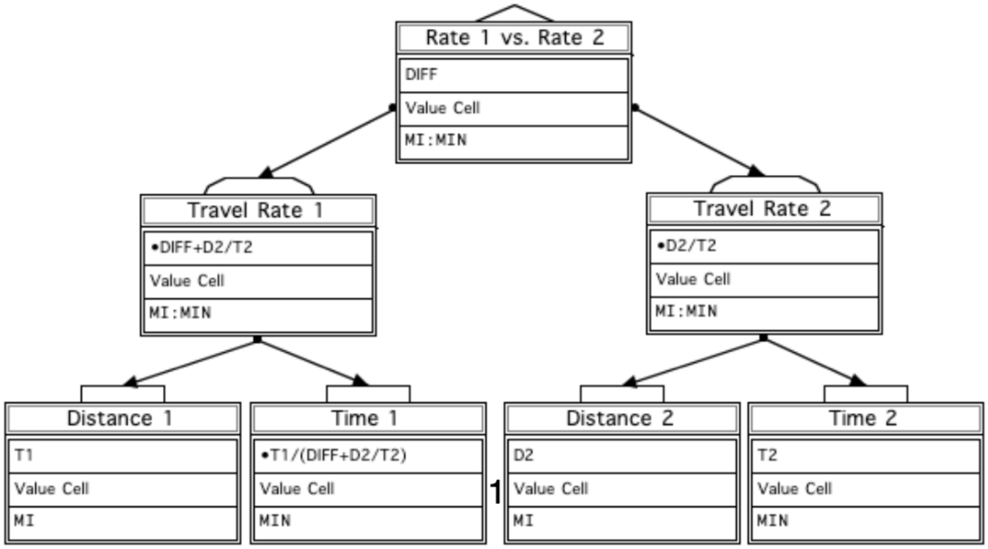


Figure 10

Figure 10 repeats Figure 9, but with the use of letters to represent to-be-known values.[22] The inferences drawn and the propagation of inferences through the structure are *identical* to what was done in the arithmetical setting of postponed evaluation of expressions, with the exception that now the inferences are expressed symbolically instead of numerically. The cognitive context in which those inferences are made, however, is quite different from the arithmetical setting of calculating answers. When a student's goal is to perform calculations, it would be unnatural to introduce letters. The use of letters would be completely contrary to the goal of calculating an answer.

Equations

A formula describes calculations to be done to compute a quantity's value. An equation is (1) a formula for a quantity's value together with the value that the formula must yield, or (2) two formulas for a single quantity's value.

[22] Notice: Thinking of letters as representing "to-be-known" values is quite different from thinking of letters as representing "unknown" values. The former is a cognitive root (Tall, 1989) of *problem parameters.* The latter is an artifact of Algebra I, where the "real" solving is portrayed as starting with an equation, instead of starting with derivation of formulas for calculating quantities' values.

Cognitive Prerequisites

The present model postulates that for a student to construct an equation when reasoning quantitatively about a situation, he or she must have (at least) the following intentions and propensities:

1) The intention to generate formulas for quantities,

2) The “willingness” to construct a formula for any quantity that already has a known value (i.e., the willingness to ignore, for the moment, that he or she already knows a value for a quantity), and

3) The “willingness” to construct more than one formula for any quantity.

Equations Involving a Formula and a Value

The following examples illustrate the effects, as postulated by the present model, of having the ability to reason quantitatively and having intention (1), while having or not having propensities (2) and (3).

> Two trains leave the same station at the same time. They travel in opposite directions. One train travels 60 km/hr and the other 100 km/hr. In how many hours will they be 880 km apart? (Hall, in press).

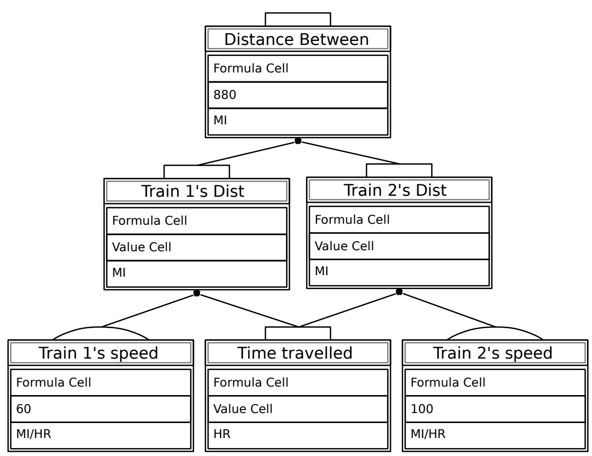


Figure 11

Figure 11 reflects a conception of the problem in terms of each train's travel distance resulting from going at its own speed for a common amount of time, and the distance between them being a combination of the train's distances (see Thompson, 1988 for three other conceptions.)

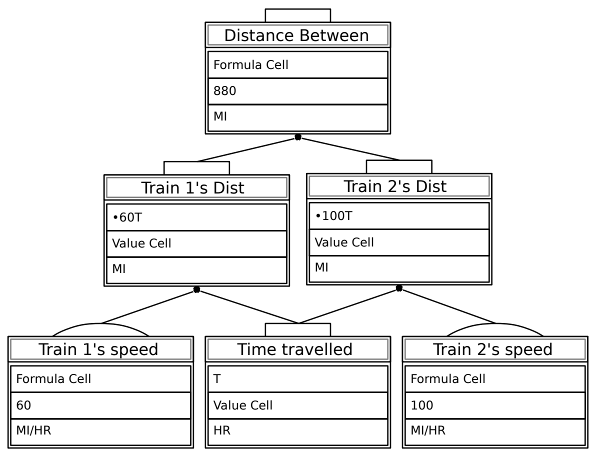


Figure 12

Figure 12 shows the effect of propagating inferences without the propensity to construct a formula for a quantity which has a known value. Upon entering *T* for Time Traveled, we end up with formulas for Train 1's Dist and Train 2's Dist, but nothing more.

Figure 13 shows the effect of having the propensity to ignore already known information when generating formulas. Upon (my) entering *T* to stand for the value of Time Traveled, the program calculated formulas for Train 1's distance and Train 2's distance. It then noticed that it could also generate a formula for Distance Between. But, it also noticed that Distance Between already had a known value, and it went ahead to generate a formula. To signal that it noticed that Distance Between already had a value, but that it went ahead as if that information were unknown, the program placed the name of the quantity in braces (i.e., {Distance Between}) and preceded information about the quantity by a right brace (}).

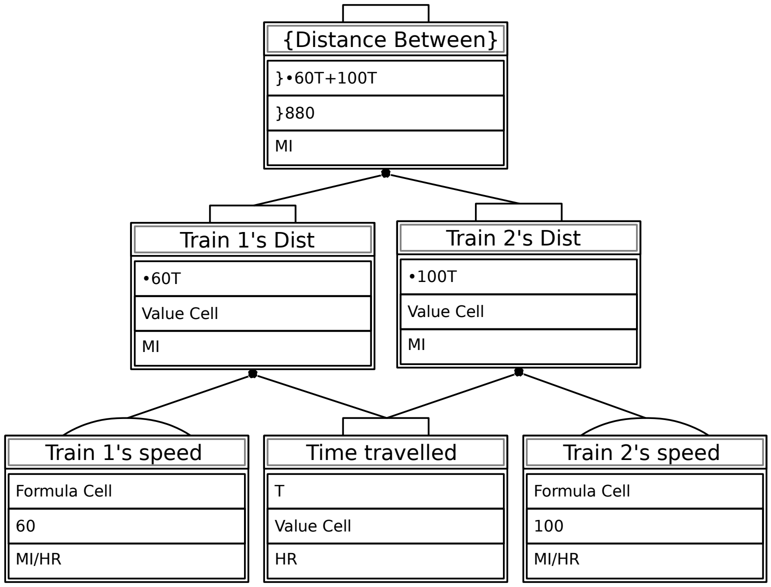


Figure 13

It is worthwhile to note that this model makes apparent the relative arbitrariness of equations. The equation depicted in Figure 13, viz. $60T + 100T = 880$, came about because of supplying the letter T to stand for the value of Time Traveled. I could just as well have supplied a letter to stand for the value of Train 1's Dist or for the value of Train 2's Dist. Figure 14 shows the inferences and resulting equation relative to entering D to stand for the value of Train 2's Dist. One would answer the stated question ("In how many hours …") by solving for D and then substituting for D in the formula for Time Traveled (viz., $D/100$).

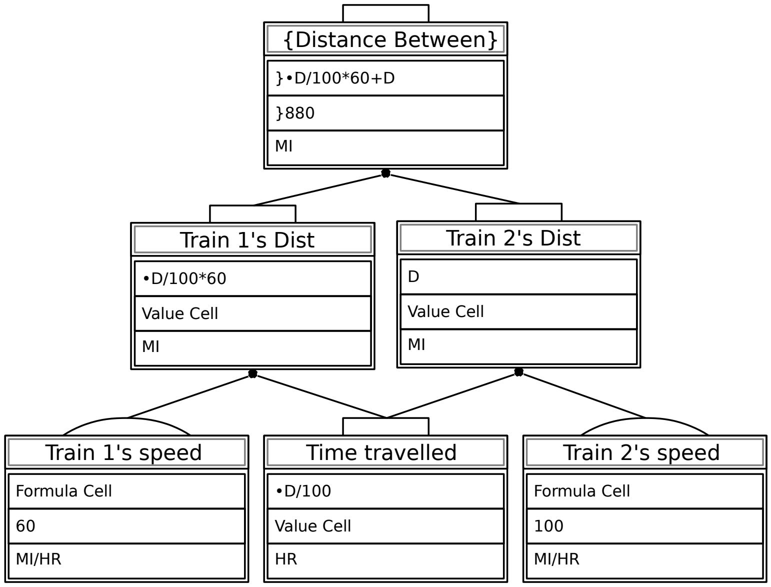


Figure 14

There is still more to the notion of relative arbitrariness of equations. One could also begin a solution with the intention of deriving an equation *for a particular quantity.* That is, to continue the present example, one could start out with the intention of deriving an equation involving Train 2's Speed. Figure 15 illustrates this strategy. I directed the program to ignore the fact that Train 2's Speed had a value *before* I entered *D* for the value of Train 2's Dist. Upon my entering *D* to stand for the value of Train 2's Dist, the program made the following inferences to arrive at the equation

$$\frac{D}{\left[\frac{(880 - D)}{60}\right]} = 100$$

- Distance Between is an additive combination of Train 1's Dist and Train 2's Dist, so v(Distance Between) = v(Train 1's Dist) + v(Train 2's Dist), and therefore v(Train 1's Dist) = 880 - D.[23]

[23] The program could not, at this moment, infer a formula for Time Traveled, for after having been directed to ignore the fact that Train 2's Speed has a value of 100, there was not sufficient information to infer a formula for Time Traveled.

- Train 1's Dist is an instantiation of Train 1's Speed at the value of Time Traveled, so v(Train 1's Dist) = v(Train 1's Speed) * v(Time Traveled). Therefore v(Time Traveled) =

$$\frac{(880 - D)}{60}.$$

- Train 2's Dist is an instantiation of Train 2's Speed at the value of Time Traveled, so v(Train 2's Dist) = v(Train 2's Speed) * v(Time Traveled). Therefore v(Train 2's Speed) =

$$\frac{D}{\left[\frac{(880 - D)}{60}\right]}.$$

Since v(Train 2's Speed) is also known to be 100, the program arrived at the equation

$$\frac{D}{\left[\frac{(880 - D)}{60}\right]} = 100.$$

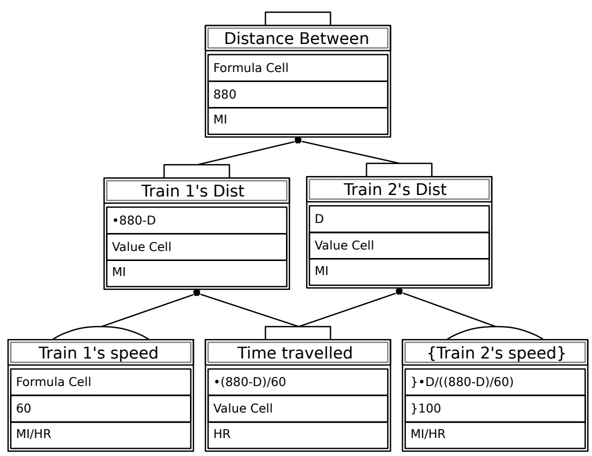


Figure 15

Equations Involving Two Formulas

Propensity (3), the willingness to construct two formulas for one quantity, is reflected in the program by its having a "Tall Notecards" setting. Tall Notecards are notecards with two formula cells. Notecards having two formula cells reflect an “openness” to the possibility that they might both be filled, thus reflecting a willingness to have two formulas for computing a quantity’s value.

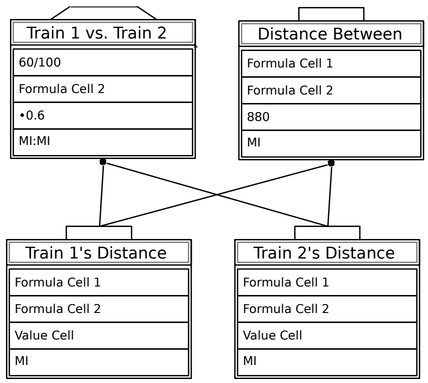


Figure 16

Figure 16 shows how Fay, a high-school teacher, conceived Hall’s train problem (of the previous example). Fay’s thinking was that there are two relationships between Train 1’s Distance and Train 2’s Distance—a constant ratio between their distances and the combination of their distances.

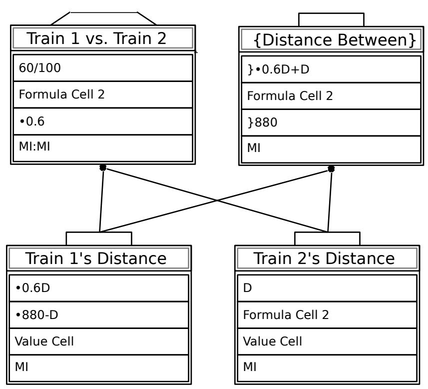


Figure 17

Figure 17 shows that this structure leads to two formulas for calculating the value of Train 1’s Distance (it also leads to an equation involving the value of Distance Between, which the program derived, but which Fay did *not* derive). The inferences made by Fay, and captured by the program, were:

- The ratio of Train 1's Distance to Train 2's distance is 60/100, so if Train 1's distance is *D*, then Train 2's distance is $\frac{60}{100}D$.
- The combined distances traveled by the two trains is 880 miles. So if Train 2 travels *D* miles, Train 1 travels 880 - *D* miles.
- The equation to solve is $\frac{60}{100}D = 880 - D$.

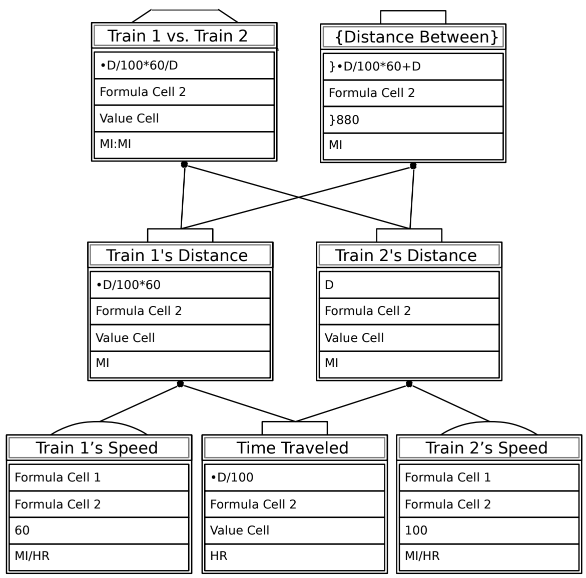


Figure 18

Fay did a "quick calculation" to conclude that the ratio of the train's distances is constant, and that its value is 60/100. The program fails to capture that aspect of Fay's reasoning. Figure 18 shows the incorporation of the trains' speeds and distances. Upon my entering *D* to stand for the value of Train 2's Distance, the program propagated appropriately, and propagated an expression for the ratio that reduces to 60/100, but then derived an equation involving the value of Distance Between and stopped there. The reason it stopped is that the formula for Train 1 vs. Train 2 and the equation involving Distance Between were both propagated from Train 1's Distance—they were not propagated from Train 2's Distance, as was the case in Figure 17. The program is designed so that no notecard will propagate to itself, either directly or indirectly. In principle this constraint precludes propagating equations that are identities. However, this constraint also causes the program to not construct some correct equations.

On the other hand, if we incorporate a temporal dimension into our model of Fay's solution, then the program behaves as did Fay. Figure 19 reflects the computation of a formula for the ratio between Train 1's Distance and Train 2's Distance. Relationships involving Distance Between are ignored.

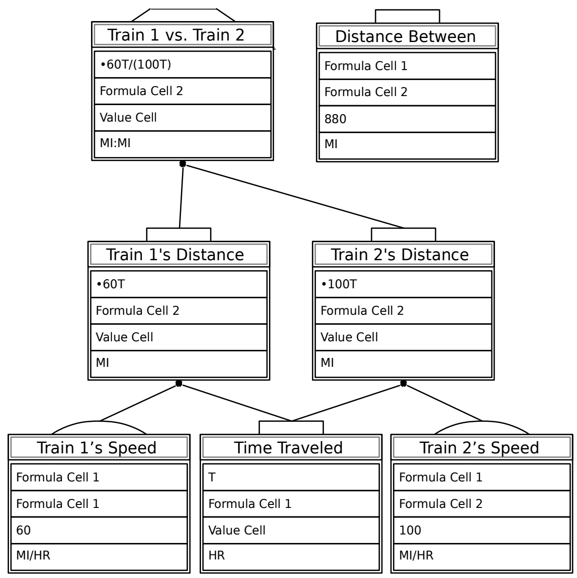


Figure 19

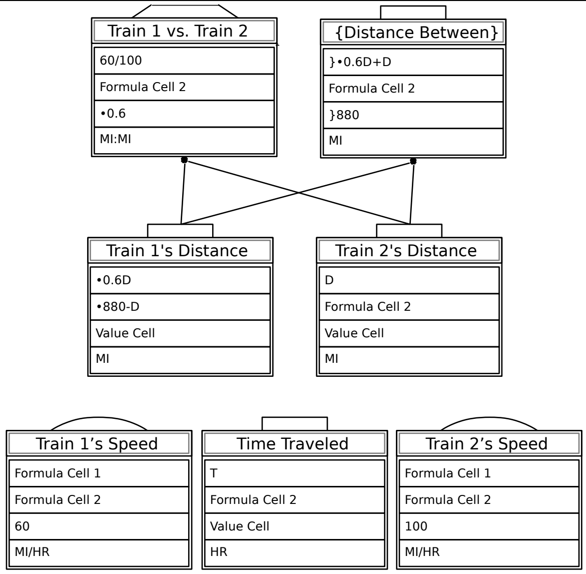


Figure 20

Figure 20 reflects the subsequent decision to ignore relationships involving speeds and times, since they are irrelevant to the ratio (as it is constant). Figure 20 also reflects the decision to attend to relationships involving Distance Between. By incorporating these two decisions we produce a model of Fay’s quantitative reasoning in relation to her construction of a solution.

One aspect of Fay’s solution that remains unmodeled is her *predisposition* (as distinct from willingness) to construct two formulas for one quantity. The program is designed so that it is predisposed to look for a formula and a value, although it is “willing” to generate two formula’s for one quantity.

It was evident in Fay’s comments that she did not *think* she was trying merely to get two formulas for one quantity. Instead, what she *thought* she was doing was trying to construct as many relationships as she could, because she knew that relationships constrain possible solutions.

Equations that Model Recursive Relationships

Students find it extremely difficult to construct equations for situations that involve mutual, recursive relationships among quantities. It is frequently the case that the difficulty is more an artifact of instruction that teaches transliteration of text to equations than a difficulty inherent in the problem. The next example clarifies this claim.

> MEA Export is to deliver an oil valve to Costa Rica. The valve's price is $5000. Freight charges to Costa Rica are $100. Insurance is 1.25% of Costa Rica's total cost. Costa Rica's total cost includes the costs of the valve, insurance, and freight. What is Costa Rica's total cost? [24]

The confusion most typically encountered among students has to do with the mutual, recursive relationship involving Costa Rica's total cost and the cost of insurance. The total cost includes the cost of insurance, yet the cost of insurance is based on the total cost. When students think that, in writing a formula, they are "pretending" that they know some quantity's or quantities' values, then they are quite naturally confused—pretensions are time dependent. By this frame of mind, even in pretense one must know Costa Rica's total cost in order to compute the cost of insurance, while at the same time one must know the cost of insurance to compute Costa Rica's total cost. In this particular case, a frame of mind that suggests one is "pretending" to know values when constructing formulas in fact becomes an obstacle to constructing formulas (and hence equations). That pretension leads one into a "calculating" frame of mind that results in a never-ending series of calculations.

[24] This problem is courtesy of Daniel Gonzalez. According to Mr. Gonzalez, it is standard practice for shipping and insurance to be paid by the customer. Thus, total cost to Costa Rica must include insurance. However, it is the exporter who writes the check for the insurance, and if anything were to go wrong he or she needs to recover that cost as well as all others. Thus, it is the total purchase cost, including the cost of insurance, that must be insured.

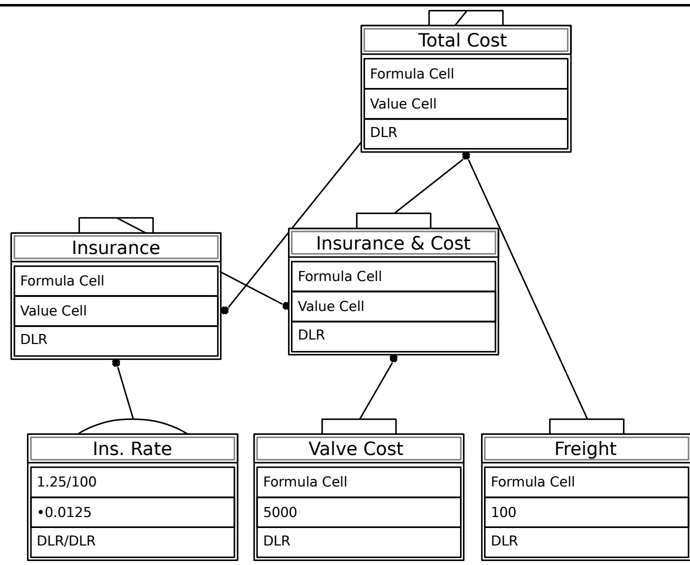


Figure 21

Figure 21 shows one model of the situation in terms of quantities and relationships. The diagram shows the mutual reliance between Total Cost and Insurance.

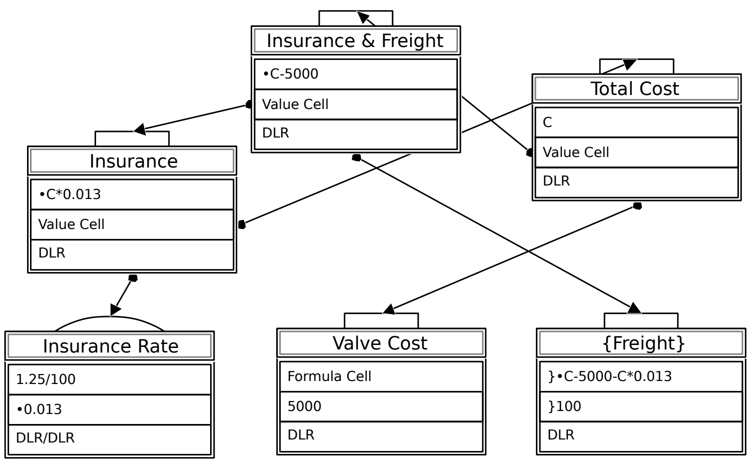


Figure 22

Figure 22 shows the effect of using a parameter to represent Total Cost's value. The program propagated normally, ending with the equation c-100-.0125c = 5000.

A point raised by this example is this: While one must have the intention of constructing formulas for calculating quantities' values in order to construct equations, the way one conceives of a formula's parameters has a direct influence on one's ability to *enter* the process of constructing formulas. If the use of letters is understood as "pretending" that you know a value, then difficulties can arise. If the use of letters is understood as *representing to-be-known* values, then no difficulty arises. The fact that a value is considered “to-be-known” implies nothing about a process of coming to know it. In some cases it can be chosen arbitrarily; in other cases it must be chosen in order to satisfy a set of constraints, as illustrated in this example.

Other Issues

The concept of unary operations on quantities and the concept of scalars are issues of quantitative reasoning that do not come under the heading of Quantity-based Arithmetic or Quantity-based Algebra, but which are nevertheless important. An examination of relationships between the mathematics of scalars and the mathematics of quantity helps to clarify several issues, particularly the concept of percent as a quantity.

The principal unary operation discussed here is inversion of ratios and rates. An examination of situations requiring inversion to comprehend them quantitatively highlights potential sources of students’ difficulties with them.

Percents and “Scalar” Quantities

Rates and Ratios as Scalar Quantities

To clarify a quantitative meaning of percent, I will first discuss the general issue of scalars. A scalar is typically characterized as a “dimensionless” quantity—a ratio between two quantities in the same unit. This, however, is more an artifact of an arithmetic of units than of conceptual analysis. There are many situations where it is not only sensible, but *desirable* to retain units in a rate involving quantities having the same unit. Here is one example.

> Jane is a golfer with a keen eye but a weak swing. She can accurately estimate distances, but consistently hits the ball 4/5 as far as she needs to hit it. If Jane is 150 yards from where she wants the ball to go, what estimate should she use in order to hit the ball far enough?

Figure 23 shows a conception of Jane's situation as one where she has a constant "hit rate". The value of that rate is 4/5. The unit of that rate is "yd/yd." Together these mean that there are 4/5 yards of "hit" per each yard of distance to the hole. To consider4/5 as a scalar in this situation would be unnatural. A unit of "yd/yd" follows naturally from conceiving the situation quantitatively.

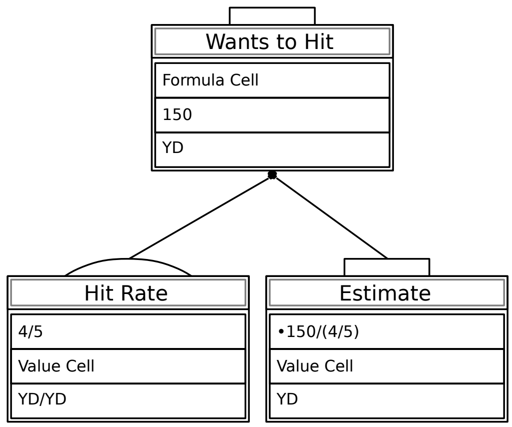


Figure 23

The interpretation of Jane's hit rate as having a unit of *yd/yd* is not necessary, and perhaps seems unnatural to the sophisticated reader. The statement "… hits the ball 4/5 as far as she needs to hit it" does not, itself, imply a unit for Jane's hit rate. Rather, it is the unit *yd* for Wants to Hit and Estimate that suggested the unit *yd/yd* for Jane's hit rate.

The crux of the issue of scalars resides in the question: Can one quantitatively interpret the statement "… hits the ball 4/5 as far as she needs to hit it" independently of specific context (i.e., units of other quantities)? This question is answered affirmatively, in the present model, by allowing the possibility of someone conceiving of a situation in terms of *parametric* units. The idea of parametric units is the same as that of parametric values. You know that a unit will be associated with a quantity, but it need not be given specifically to know something about the quantity.

The notion of parametric units is implemented in the program by having one use a question mark as the first character in a unit-name. The question mark indicates that the unit is a parameter; the name itself is used as a matching variable so that, once bound by the matching process in checking the consistency of a quantitative structure, the same name implies the same unit within the formal structure of a quantity's unit.[25]

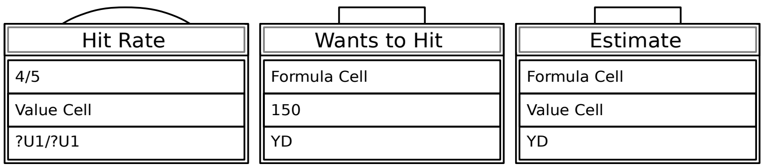


Figure 24

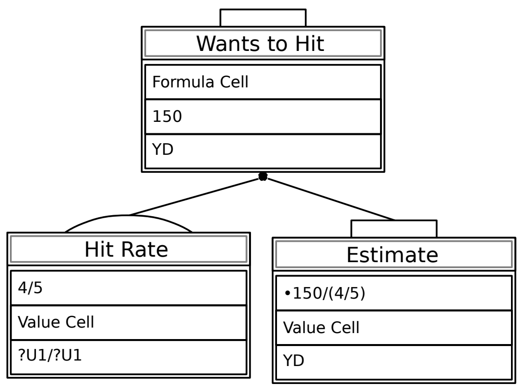


Figure 25

Figure 24 reflects the temporal dimension of a (fictitious) person's reading of Jane's golfing problem. This person conceived of Jane's hit rate, at the time of reading the passage describing it, as a rate with unspecified, and hence parametric, units. Figure 25 shows this person's conception of the situation after having had related the hit rate and the two distances.

---

[25] Whether the same parameter-name needs to imply the same binding throughout a structure is unclear. If a non-parametric unit name were to be thought of as binding to itself, then the answer would have to be that the same parameter-name in different quantity's units would have to represent the same binding. However, it seems entirely reasonable that a person could be thinking of parametric units for different quantities, use the same "mental symbol" in constructing the quantity's units, but intend them to be different. In essence, the issue is whether bindings are local to a quantity-as-conceived or global to the person-as-conceiver. I imagine that either could be the case for any given individual.

The unit *?U1/?U1* for Hit Rate says that this person thought of Hit Rate as postulating a linear relationship between values of two quantities, and expresses this person's conclusion that the units of the other quantities really don't matter as long as they are the same.

Percents as Scalar Rates and Scalar Ratios

The quantitative concept of percent occurs as a special case within the general concept of quantitative scalars. Quantitative scalars are ratios or rates involving parametric units, where the units of numerator and denominator quantities are anticipated to be the same.[26] Within the present model, a percent is a scalar rate or a scalar ratio—with the additional restriction that its value is expressed in terms of multiples of 1/100.[27]

Unary Operations on Quantities

Algebra teachers will attest to students' difficulty with problems like the following.

> There are 640 acres in 1 square mile. A mile is 5280 feet. A city lot is 50 ft × 100 ft. What part of an acre is one city lot?[28]

[26] Even though it is a matter of definition, the restriction that to conceptual a quantitative scalar one must anticipate numerator and denominator quantities having the same unit is unnecessary. "Scalar-like" thinking can arise naturally whenever one anticipates the *structure* of a quantity's unit and anticipates that quantity's relationships with other quantities.

[27] It is interesting that in this formulation "" itself is used as a scalar rate, meaning th of whatever it is that you are measuring.

[28] In central Illinois, where farmland (measured in acres) is routinely converted to residential area (measured in lots), this is a real problem.

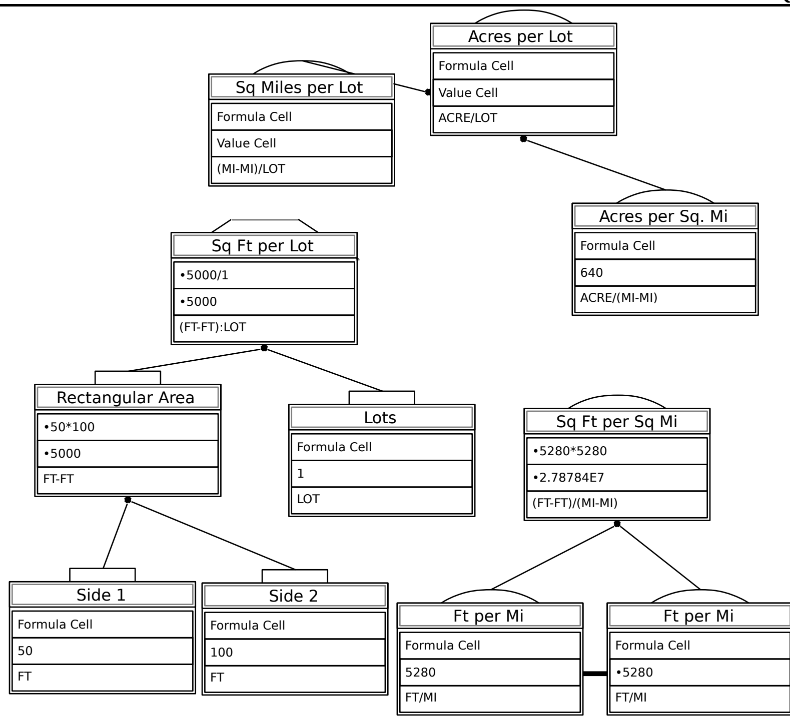


Figure 26

Figure 26 shows one reasonable way to conceive the relationships described in the text.[29] Acres per Lot is a rate made by composing the rates Acres per Sq Mile and Sq Miles per Lot. The quantity Sq ft per Sq Mi is made by multiplicatively combining two Ft per Mi rates (the sides of a square which is growing).[30] The area of a lot is made by comparing the area of a 50×100 rectangle with the number of lots with which it is associated.[31]

After this we see an obstacle: To make Sq Miles per Lot, we need to combine Sq Ft per Lot with the *inverse* of Sq Ft per Sq Mi. We could invert the originally given rates of feet per mile as

29 By “reasonable” I mean that three college mathematics majors initiated their solutions to this problem in a way consistent with this particular setup.

30 Note: (1) We might object that “Ft per Mi” is one quantity, and hence should not have to be repeated in Figure 25. However, what is being conceived here is a square whose sides are growing at the same rate. Each side *could* grow at its own rate; the sides’ growth rates must be constrained to be equal in order that the initial square remain a square as its sides grow. (2) The unit for Sq Ft per Sq Mi, *(ft-ft)/(mi-mi)*, is equivalent to the unit *(ft/mi)-(ft/mi)*. The latter was entered as the unit for Sq Ft per Sq Mi; the program converted that unit to the former, it being a more standard form.

31 Forming the ratio Sq Ft per Lot is more like an act of definition than like a multiplicative comparison.

miles per feet, or we can invert Sq Ft per Sq Mi and use it as a derived quantity. Figure 27 shows the effect of apply the “invert rate” operation to Sq Ft per Sq Mi.[32]

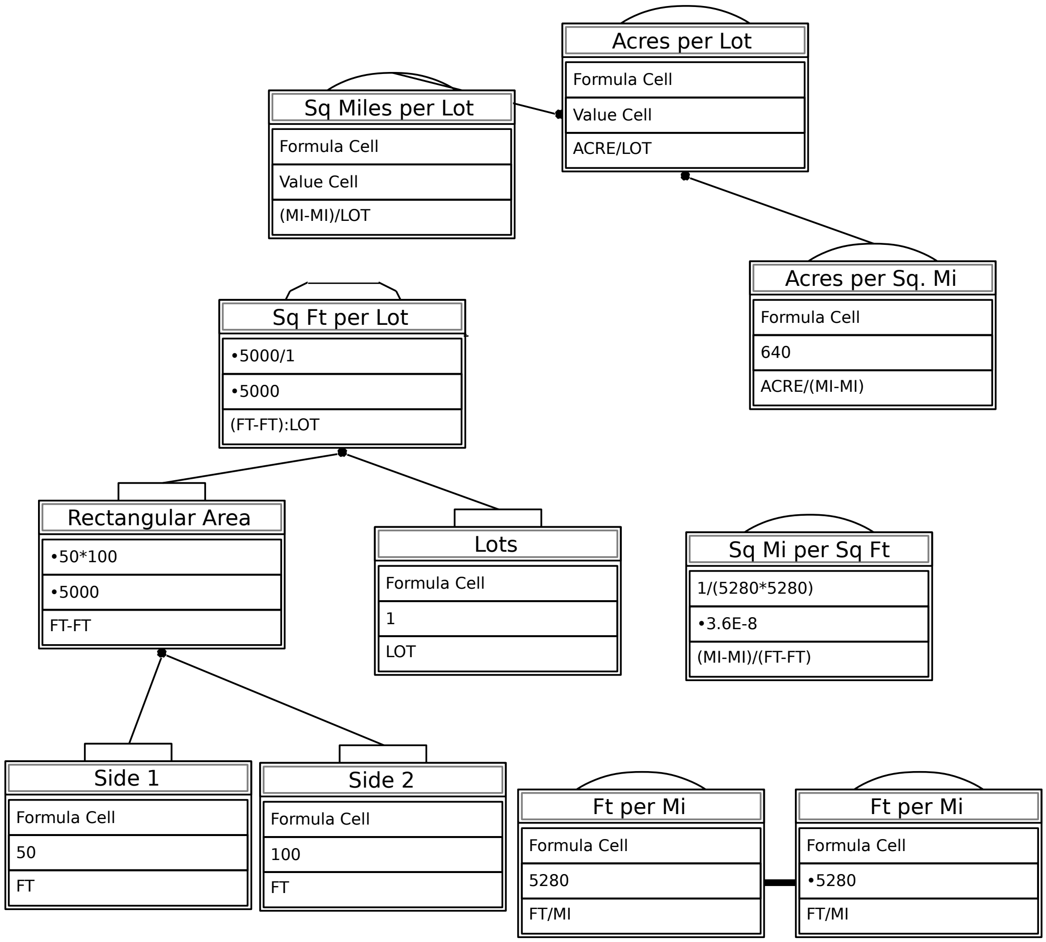


Figure 27

Upon inverting Sq Ft per Sq Mi, the quantity was divorced from the structure originally making it. It retained its (inverted) formula and associated value, but it could not remain connected to the two copies of Ft per Mi. Figure 28 shows the completion of this problem’s solution.

[32] The program inverted the quantity, but it left the name alone. The program is ignorant of naming conventions, so it does nothing with names. It is up to the user to change the name so that it reflects the inverted quantity. As such, it was I who changed the name, it was not the program.

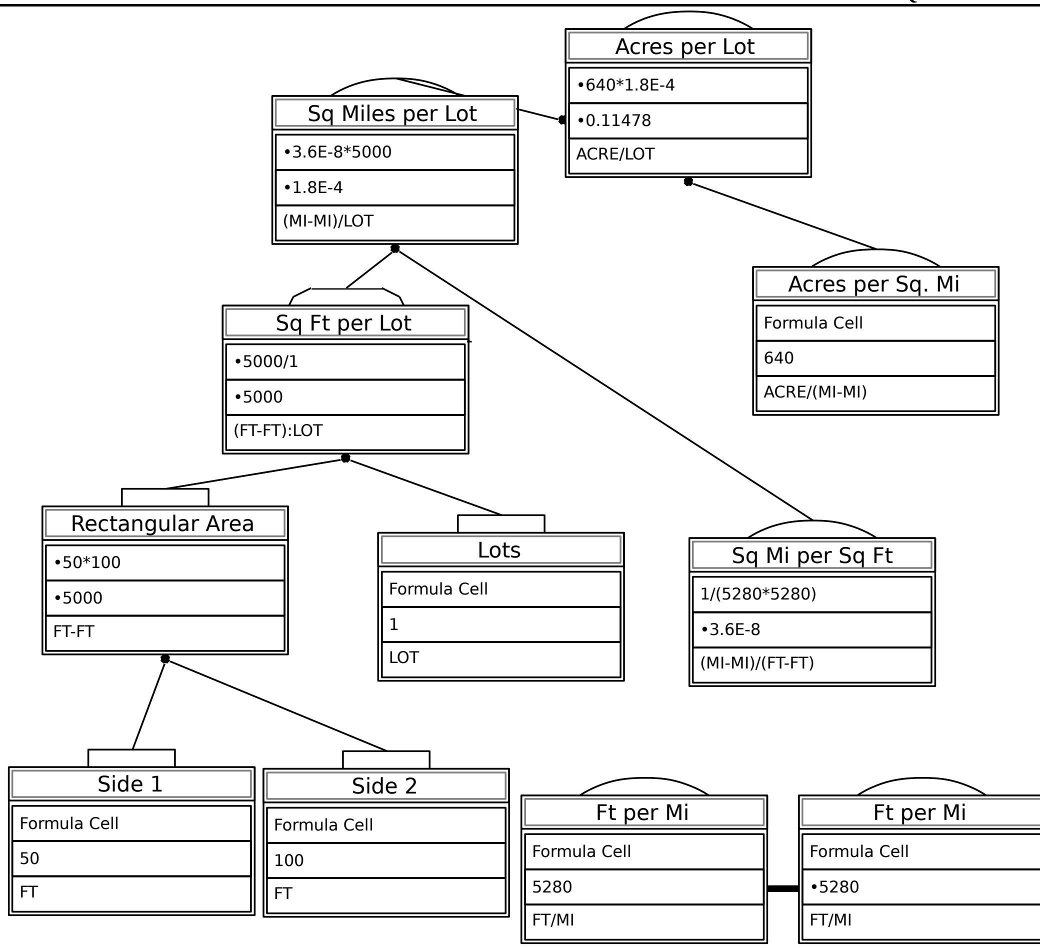


Figure 28

The need to invert a quantity midway through conceptualizing a problem is a major obstacle for many students. Another source of students' difficulties with this particular problem is that they frequently mis-conceive the "target" quantity (the answer to the question) as being a number of things measured either in ACRE or in LOT, *while at the same time conceiving of "5280 feet in a mile" and "640 acres in a square mile" as denoting rates*.[33] These particular conceptions are incompatible.

A third, clear source of difficulty in this problem is that, though its statement is short, the number of quantities involved is relatively large. Students who anticipate that they will need only

[33] This was true of 18 out of 24 middle-school mathematics majors. Sixteen of these 18 got nowhere when trying to solve the problem.

the information given, and therefore are biased against conceiving of "intermediate result" quantities, will make little progress on problems like this one.

Simultaneous Equations: Concept or Technique?

Algebra textbooks traditionally devote a chapter to systems of simultaneous equations. This suggests that simultaneous equations, and situations modeled by them, are conceptually different from single equations and situations modeled by them. I will argue that a more appropriate view is that systems of equations are a *technique* to handle complexity of information and relationship, and that the technique is an artifact of paper-and-pencil technology. There is no intrinsic *conceptual* difference between situations modeled by single equations and situations modeled by systems of equations.

The following problem is from Eggan and Van Den Eynden (1979). It is typical of problems normally offered as practice in setting up simultaneous equations in one variable.

> A factory makes toy trucks and toy trains. Each toy needs to be assembled and painted. Toy trucks each require 5 minutes assembly time and 7 minutes painting time. Toy trains each require 6 minutes assembly time and 10 minutes painting time.
>
> The factory rents time on the machines needed for painting and for assembling these toys. They have reserved 300 minutes assembly time per day and 600 minutes painting time per day. Is there some combination of trucks and trains that will enable the factory to use all the available time on both machines?

Here is a traditional approach: If we let x be the number of trains manufactured, and y be the number of trucks manufactured, then a system of equations for this situation is:

$$10x + 7y = 600 \quad (1)$$

$$6x + 5y = 300 \quad (2)$$

Solving for y in equation (2) and making a substitution in (1) gives the equation

$10x + 7\left(\frac{300 - 6x}{5}\right) = 600$. If solving for x produces positive, integral values for both x and y, then the factory may use all its rented time.

Figure 29 shows a quantitative model of the described situation. One can see that equations (1) and (2) get their "simultaneity" from the shared structures hanging from Assembly Time and Ptg. Time.

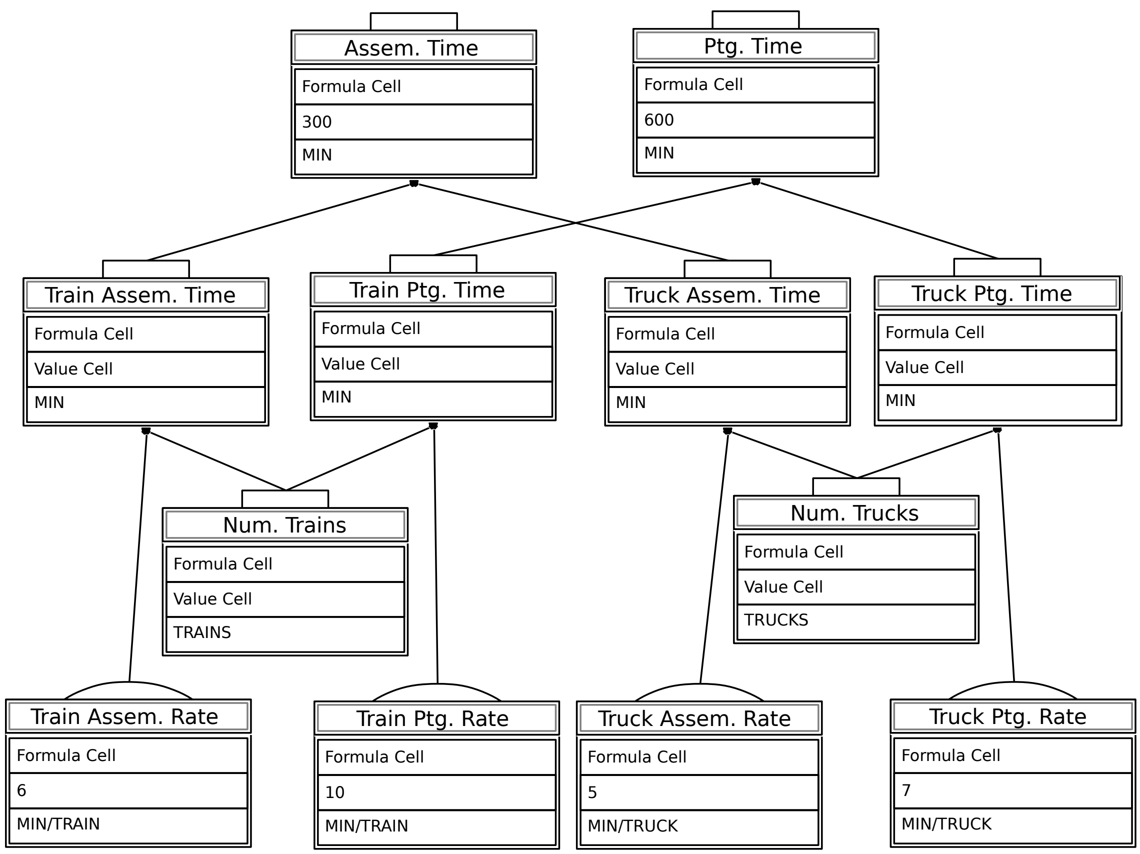


Figure 29

Figure 30 shows the effect of entering x as a representation of the number of trains. The propagation scheme produces the equation shown in Ptg. Time—which is the very same equation we ended up with when using the method of substitution with equations (1) and (2), above.

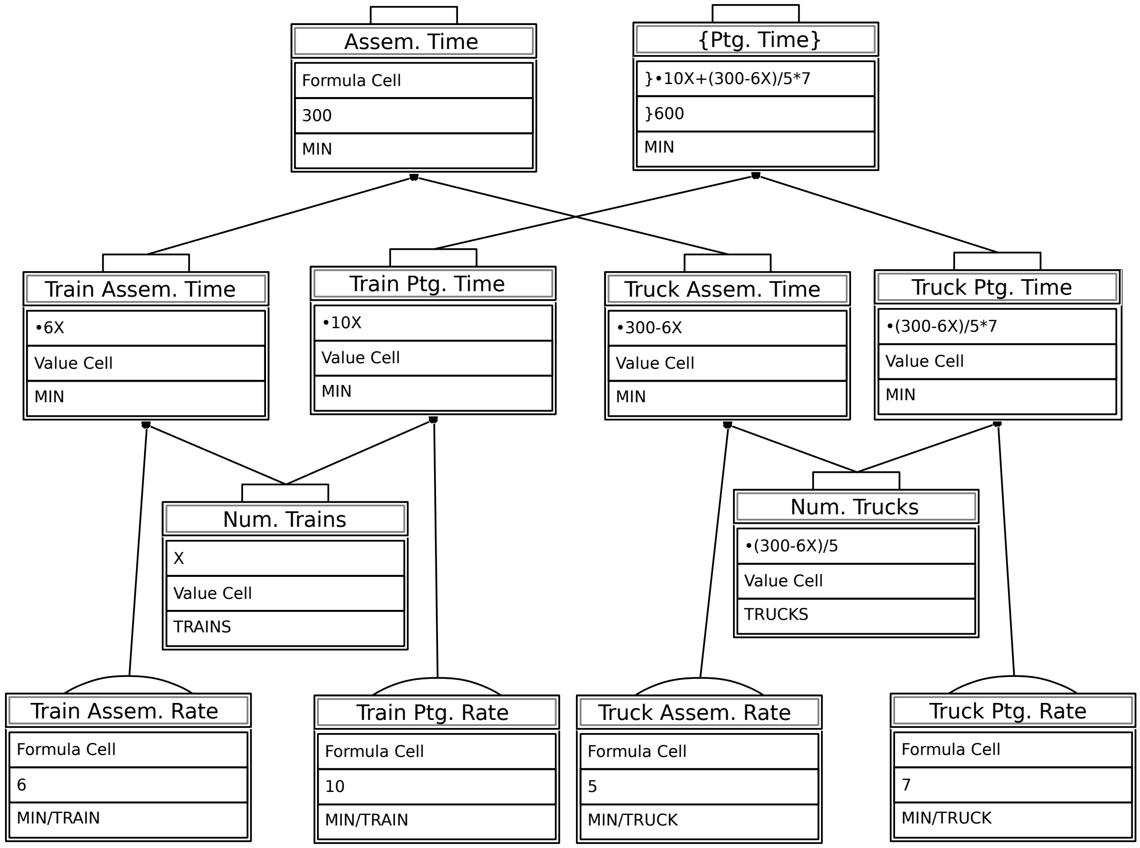


Figure 30

The propagation scheme that produced the equation for Ptg. Time is the same scheme that propagates formulas. However, the propagation of formulas is tantamount to substitution. The only issue is, "When does one substitute?" If one follows the heuristic of letting different letters stand for the values of different quantities, then one is forced to use substitution at the end of "setting up" the problem.[34] On the other hand, if one attends to quantitative relationship *while* "setting up" the problem, then substitution will happen incrementally, and the number of independent variables introduced will necessarily be minimal. In short, simultaneous equations are an artifact of an implicit decision to postpone substitutions.

[34] A common difficulty among algebra students when following this heuristic is how to decide what relationships to ignore and what relationships to follow. For example, we could let *x* stand for the number of trains, *y* stand for the number of trucks, *z* stand for the painting time for trains, and so on. But then we end up with "a bunch of letters and no equations."

As an aside, the model shown in Figures 29 and 30 does not address the question “Can the factory manufacture some combination …” with precision. The unit for assembly time and painting time is MINUTES when it should be MINUTES/DAY. Figure 31 shows a model of the situation in an appropriate unit. All quantities in this representation are rates. However, the same equation as before shows up in Ptg. Time. Yet, the arithmetic done to get that equation is based completely on combinations and compositions of rates whereas, previously, the arithmetic was based on combinations of numbers of things and instantiations of rates. The same equation is gotten as before, but for completely different reasons.

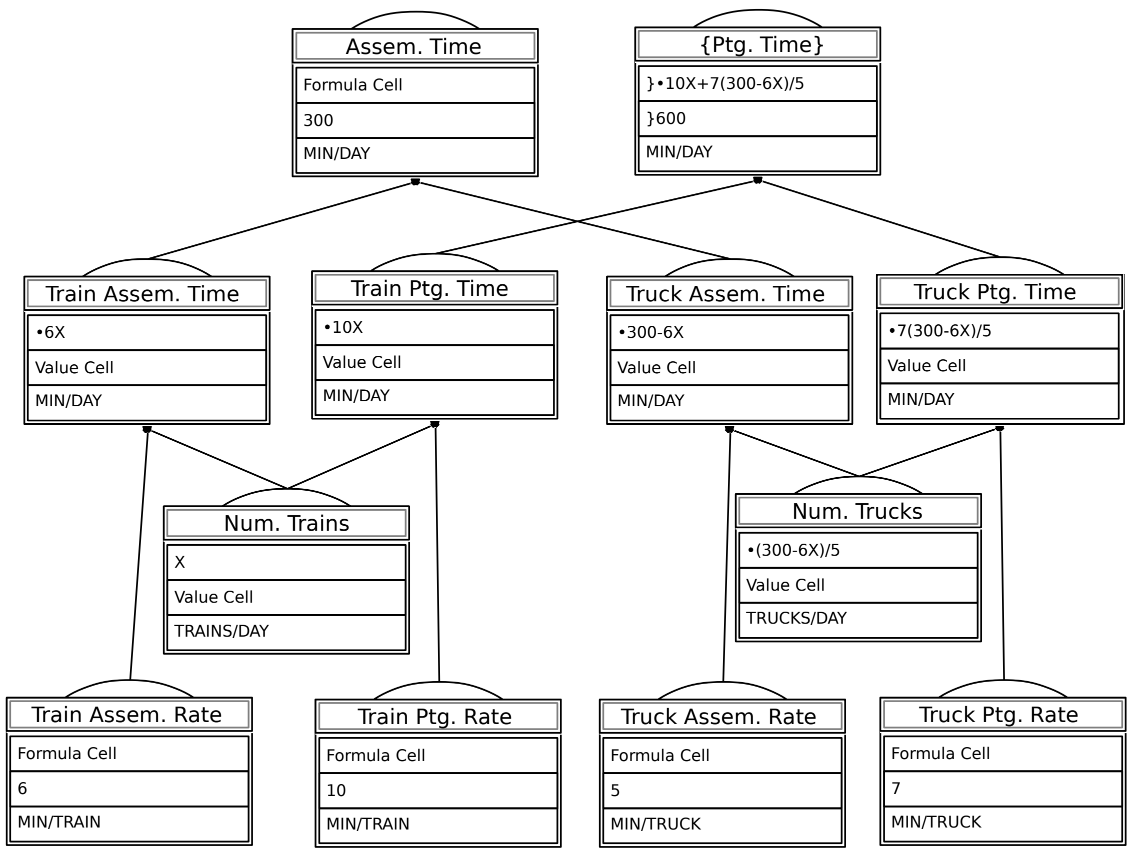


Figure 31

Subtleties of Propagation

Different effects of when we say, "They are equal"?

(Me and Brother discussion)

When and how to stop propagating

Discussion

From Quantity to Algebra

Missing: A Theory of Quantification

Adequacy of the Model's Notation

Formal Algebra and Quantitative Algebra

Other Research

Greeno, Shalin

The model of quantitative reasoning in arithmetic and algebra expressed in this paper was inspired by the work of Greeno (1987) and Shalin (1988). The notational system displayed by the computer implementation of the model is an extension of one developed by Shalin (Shalin & Bee, 1985; Shalin, 1988). I am indebted to them; the creation of powerful notation is a non-trivial achievement.

Shalin's intent in developing her notational scheme was to isolate and investigate the influence of problem structure on students' abilities to solve two-step arithmetic word problems (Shalin, 1988). She found a discernible influence. Latency response data suggested that persons parsed descriptions of situations in ways consistent with predictions based on her analyses of problems' quantitative structures. The specific constructions her subjects made, however, were not analyzed.

Shalin's research did not have an underlying theory of quantity, except for the arithmetic of quantity developed by Schwartz. The word problems students solved involved only rates or discrete, extensive quantities. Multiplicatively-structured quantities were not considered in Shalin's investigation.

There are three significant difference between Shalin's model and the model presented here. The first is that the model presented here includes a theory of propagation. The theory of propagation is the foundation of the model's hypotheses about students' transitions from quantity-based arithmetic to quantity-based algebra. Second, Shalin did not make a distinction between quantitative operations and arithmetic operations, which resulted in the confounding of type of quantity with arithmetic operation (e.g., describing a quantity is a difference simply because, in a particular situation, subtraction is used to calculate its value—see Greeno, 1987, p. 77). Third, Shalin's model did not have an underlying theory of quantity, except for the arithmetic of quantity developed by Schwartz. An arithmetic of quantity, however, is not a theory of quantitative operations. It is a codification of expressions of quantitative operations, whereby units are treated as if they are numbers or algebraic variables. Arithmetics of quantity are symptomatic of quantitative reasoning, but they are not descriptions of it.

Hall

Situated cognition (Brown, Lave)