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Yes, Virginia, Some Children Do Grow Up To Be Mathematicians

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If in the distant future an archaeologist were to try to build an image of mathematics education based on artifacts from the educational research community, she might conclude that, as late as 1990, mathematics had not progressed past proportional reasoning.

All joking aside, it is fair to say that the preponderance of mathematics education research has been on elementary mathematics, and this research is often carried out with such a tight focus that it is hard to imagine by what mechanisms one level of concept development can possibly be transformed into more sophisticated levels. There might be several reasons for this, but I suspect a primary one is that much of mathematics education research has not been conducted within a larger perspective of mathematical thinking—a perspective that keeps firmly in mind that children, over time and when taught appropriately, often do learn sophisticated and advanced mathematics. For example, models of children's competent additive reasoning will tend to be quite different when the researcher focuses only on their ability to solve simple addition and subtraction problems than when she keeps clearly in mind that the knowledge structures presently being imputed to children must provide a foundation for their conceptualization of integers (Steffe, Cobb, & von Glasersfeld, 1988; Thompson, in press; Thompson & Dreyfus, 1988; Vergnaud, 1982). Similarly, models of children's multiplicative reasoning will tend to be quite different when the researcher focuses only on their ability to solve simple multiplication and division problems than when she keeps clearly in mind that the knowledge structures presently being imputed to children might provide a foundation for children's conceptualization of multiplicative variation—viz., direct and inverse variation, linear function, exponential function, and concomitant rates of change (Harel & Confrey, in press).

Until recently, mathematics education researchers had to rely mainly on their own mathematical experience and introspection to situate their research on elementary mathematical concepts within a larger curricular and conceptual perspective. The bits of research on advanced mathematical reasoning and understanding were scattered and not widely accessible. On the other hand, while mathematics education researchers have focused largely on the development of elementary concepts, university mathematicians have seen little relevance of this research to their practice or to their teaching. With the publication of <u>Advanced Mathematical Thinking</u> this situation is changed considerably. This book can help mathematics educators place their research within a broader perspective of the development of mathematical reasoning, and it can help mathematicians see the deep connections between mathematics education research and their own teaching.

<u>Advanced Mathematical Thinking</u> is the product of five years of collaboration among the authors, under the auspices of the Advanced Mathematical Thinking Working Group of the International Group for the Psychology of Mathematics (PME). The authors met each year as part of the PME's annual meetings and communicated among themselves between meetings.

David Tall introduces the book with an overview and closes the book with reflections on its contents. Tall's introduction, besides giving an overview, sets the tone for the book with a masterful weaving of current cognitive, pedagogical, and curricular issues with a historical account of tensions within the community between mathematical practice and teaching. I especially enjoyed his quotes from famous mathematicians, such as Poincaré, which served as a reminder that thoughtful reflection on learning and teaching are not unique to mathematics education. In his introduction, Tall summarizes the contents of book quite nicely.

> In the remainder of the first part of this book we consider the cognitive <u>processes</u> involved in advanced mathematical thinking and the two complementary attributes of the discipline: <u>creativity</u> in generating new ideas and the mathematician's notion of <u>proof</u> in convincing his peers of the truth of his assertions.
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> In the second part of the book we turn to cognitive theories that are proving of value in analysing the difficulties that students face and providing insights into the learning process that can be used in designing new ways of helping students construct mathematical ideas for themselves. First the differences between <u>concept definitions</u> and students' <u>concept images</u> are considered, then the

> nature of the mental objects which mathematicians construct: the conceptual entities that are the essence of advanced mathematics. This leads to the theory of reflective abstraction in which processes are encapsulated as mental objects which prove to be easier to manipulate at higher levels of abstraction.
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> In the third part of the text we review various advanced mathematical concepts from a cognitive viewpoint, showing the cognitive obstacles that can occur during their development from cognitive viewpoints. These involve the central ideas of function, limit, more advanced concepts of analysis, infinity, and proof. We then move on to look at the new paradigm: the use of the computer and its cognitive effects in advanced mathematical thinking. (p. 21)

The book's structure is nice. The first part portrays that which needs explication—advanced mathematical reasoning—setting it apart from elementary mathematics while at the same time making it evident that it can be a natural outgrowth of issues entailed in elementary mathematics. The second part provides a more technical focus on the elements of advanced mathematical reasoning, making specific what was portrayed more generally in the first part. The third part focuses on the learning and teaching of important conceptual domains.

The chapters on the nature of advanced mathematical thinking are by Tommy Dreyfus, Gontran Ervynck, and Gila Hanna. Dreyfus' ostensible task is to explicate the dialectic between representing and translating as advanced mathematical processes. But he does much more. He also illustrates that this dialectic is present when students reason competently in elementary mathematics, and argues convincingly that teachers must cultivate this dialectic early on and consistently for students to progress to advanced levels. Dreyfus also does an excellent job of making it evident that many university students often "succeed" in advanced courses without these processes because of an instructor's emphasis on correct performance instead of deep understanding. Dreyfus' observations are completely consistent with an emerging research trend that shows correct performance cannot be taken as an indicator of understanding (Seldon, Mason, & Seldon, 1989). While Dreyfus focuses on reasoning processes, Ervynck attempts to elucidate the leaps of insight that we swear happen but have little control over—either personally or pedagogically. Although Hadamard's (1954) account of invention gives a better testament to creativity than does Ervynck's, Ervynck assumes the enormous task of exploring how we might shape instruction to engender students' creativity, a task that was completely aside Hadamard's objective. The section closes with Hanna's brief discussion of the necessity to consider social dimensions of mathematical proof when attempting to understand proof either cognitively or historically.

The chapters on cognitive theory are by Shlomo Vinner, Guershon Harel and Jim Kaput, and Ed Dubinsky. Vinner expands his and David Tall's distinction between concept definitions and concept images (Tall & Vinner, 1981; Vinner & Dreyfus, 1989) to provide insight into college students' understandings of important ideas like function and derivative. It is important to note that the distinction between concept definition and concept image does not explicate competent understanding of any specific concept. Rather its importance is that it enables us to make constructive sense of students' frequent inability to reason coherently from the basis of a technically-defined vocabulary and gives hints as to pedagogical and curricular directions that might support students' development of these abilities. Harel and Kaput's chapter, on conceptual entities and symbols, continues an emerging line of research on students' constructions of mathematical objects (Dubinsky & Lewin, 1986; Greeno, 1983; Harel, 1989; Sfard, 1991; Sfard & Linchevski, in press; Thompson, 1985). Their chapter is a welcome synthesis of this line of theory development, presented in the format of case studies of specific concepts. It also extends prior work on mathematical objects by explicating the important role of symbolization in the entification process. While Cajori's (1929) analysis of mathematical notations is crucial to understanding the development of mathematics historically, Harel's and Kaput's chapter goes in a complementary direction. They highlight the importance of a dialectic among students' creating mathematical objects, their use of notation to express their reasoning, and notational characteristics that can facilitate or obstruct students' achievements in both regards. For example, the derivative of a function f might be expressed as $\frac{\mathrm{d}f(x)}{\mathrm{d}x}$ or $\mathrm{D}_x(\mathrm{f})$, but the two often do not coincide in terms of what mathematicians have in mind when using them. Attending to students' internalization of notations as a vehicle for expression is quite different from the community's adoption of customary and conventional notational systems (Thompson, 1992). Ed Dubinsky closes this section with an excellent discussion of Piaget's notion of reflective abstraction. His chapter, together with von Glasersfeld (1991), should constitute basic reading for anyone wishing to gain an understanding of this difficult construct.

The last section, on teaching and learning advanced mathematical thinking, is where many of the ideas brought out in the first two sections get applied to the undergraduate curriculum. Aline Robert and Rolph Schwarzenberger open the section with a general overview of research issues in undergraduate mathematics education. Ted Eisenberg discusses research on functions, Bernard Cornu discusses research on concepts of limit, Michèle Artigue discusses research on students' learning of functional analysis (including differential equations), Dina Tirosh discusses students' concepts of infinity and transfinite cardinal numbers, and Daniel Alibert and Michael Thomas discuss research on proof. Eisenberg, Cornu, and Artigue each draw heavily on the Tall/Vinner/Dreyfus notion of concept image to gain insight into students' difficulties within their various areas of focus. Alibert and Thomas continue Hana's earlier discussion of social dimensions of proof, but also address cognitive obstacles to students' understanding of proof by relating Uri Leron's very interesting approach to the structuring of proof presentation (Leron, 1983; Leron, 1985). They close with a discussion of Grenoble's creative experiment in teaching mathematics by structuring courses so that students engage in "scientific debate." By "scientific debate" they mean that students generate mathematical propositions and then debate their validity; acceptance of validity is determined by vote—where the vote is about whether someone's demonstration is sufficiently convincing. It is interesting that false statements, along with their counterexamples, and validated statements are treated with equal importance.

The final chapter (aside from Tall's reflections), by Ed Dubinsky and David Tall, is about advanced mathematical thinking and computers. On the one hand, it is largely about Tall's <u>Graphic Calculus</u> and Dubinsky's <u>ISETL</u> programming language. On the other hand, it is about a fundamental re-thinking of the calculus (including differential equations) and students' engagement with the concept of function. I suspect that anyone whose interest is piqued by this chapter will need to go to original sources before they can appreciate the power of these uses of computers.

To return to my introductory comments, I consider it imperative that the mathematics education community regain the sense that mathematics is a deep and abstract intellectual achievement. Lest I be misunderstood, I should also say that I do not support the view that school mathematics should focus on preparing students for college. But at the same time, we must acknowledge that school mathematics must provide an adequate foundation for advancement, and without a vision for what they might grow into, it is highly unlikely that students' intellectual preparation will be appropriate.

The importance of Advanced Mathematical Thinking for school mathematics education is more than its revelatory value. The book's authors are all mathematicians, but at the same time most have conducted research in school mathematics education. This shows itself in the book's continuing emphasis that advanced mathematical thinking does not begin after high school. As far as professional mathematics education can influence it, it must begin in first grade.

My final comment is that, if the book has a shortcoming, it is that it is the first of its kind and has little to build on. Tall remarked that the preponderance of cited literature is published within the previous ten years. As such, it is understandable that the content has more to do with the authors' emerging reconceptions of mathematics and less to do with actual thinking. David Tall acknowledged as much in his reflections. Put in perspective, however, this is hardly a shortcoming, for research on thinking is highly influenced by the kind of thinking researchers seek, and this book has the potential of changing our image of what to seek.

Finally, I give two quotations from David Tall's reflections that he offered to university mathematicians. They are appropriate to mathematics teachers at all levels.

> It is no longer viable, if indeed it ever was, to lay the burden of failure of our students on their supposed stupidity, when now the reasons behind their difficulties may be seen to be in part to be due to the epistemological nature of mathematics and in part to misconceptions by mathematicians of how students learn. We often teach certain skills because we know that these will bring visible, albeit limited, success, but we now know, somewhat furtively, that the acquiring of those skills may develop concept imagery that contains the seeds of future conflict. (p. 251 ff.)
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> We cheated our students because we did not tell the truth about the way mathematics works, possibly because we sought the Holy

> Grail of mathematical precision, possibly because we rarely reflected on, and therefore never realized, the true ways in which mathematicians operate. (p. 255)

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