(1991). In L. P. Steffe (Ed.), *Epistemological Foundations of Mathematical Experience* (pp. 260-281). New York: Springer-Verlag

TO EXPERIENCE IS TO CONCEPTUALIZE: A DISCUSSION OF EPISTEMOLOGY AND MATHEMATICAL EXPERIENCE[1]

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When I accepted the invitation to comment on the papers in this volume, I had little idea of their diversity. Yet, that very same diversity, while being initially overwhelming, turns out to be a considerable strength of the collection. It is remarkable that these papers, written within diverse traditions and disciplines, reflect a coherent theme: that experience and conceptualization are inseparable.

In assimilating sense data or accommodating to it, we cannot experience “the world” without already “knowing” something about it. This is not to say that what one knows is correct, true, or even viable. Rather, it says only that we must already know something with which sensation or conception resonates.

In the same vein, we cannot experience the world mathematically without using mental operations we would call mathematical. Let me anticipate two interpretations of this statement. The first is that the mathematics anyone comes to know is innate, ready to emerge over time, awaiting appropriate environmental “triggers.” The second is that no one may know or come to know mathematically, which is evidently absurd. Neither interpretation is consistent with a constructivist epistemology, yet the statement that led naturally to them *is* a hallmark of constructivism.

Thus, there are two principal challenges implied by a constructivist epistemology of mathematics. The first is to provide compelling arguments that it is *possible* for adult mathematics to emerge as the product of life-long constructions, where those arguments are painstakingly, evidently, non-Chomskian. The second is to hypothesize *constructive mechanisms* by which specific knowledge might be made, and to give detailed accounts of those constructions. The papers by Bickhard, Cooper, Steffe, and von Glasersfeld address these challenges with remarkable clarity.

A third challenge to a constructivist epistemology of mathematics is more practical, and at the same time it is the more important challenge. This is the challenge of framing curriculum and pedagogy within a constructivist tradition. It sounds quite non-constructivist to say that, as mathematics educators, what we try to do is shape students’ mathematical experiences. Yet, that is what mathematics educators working within a constructivist framework try to do. We attempt to provide occasions where students' experiences will be propitious for expanding and generalizing their mathematical knowledge. Not just any experience is satisfactory.

Five papers take up the second challenge. Steffe and Dubinsky address models of students' knowledge in specific mathematical domains. Hatfield addresses pedagogy. Confrey and Cooper address both. Each paper informs our attempts to characterize curriculum and pedagogy within a constructivist tradition.

The paper by Kieren and Pirie can be profitably viewed as belonging to a separate, important category--methodology. I do not mean methodology in the limited sense normally used in experimental psychological research. Rather, I mean it in the sense of what is needed to give sensible accounts of observations. Kieren and Pirie are concerned with methods of explanation, and at the same time they use their language of explanation (recursion) to describe desirable experiences had and to be had by students.

Dubinsky raises issues quite relevant to this notion of methodology, but his paper is not an analysis of method. Instead, it is a dialectic. He analyzes the processes of delimiting a class of phenomena that need explanation while constructing a framework for describing them.

Lewin's paper was the most difficult for me. It provides a clear demonstration of methodology in that he uses his theoretical foundations to explain the sense-making activities of students' readings of literary text. However, it is more profitably viewed as a challenge to mathematics educators to cast mathematics education as *paideia*, as being fundamentally concerned with the formation of character. In later remarks I will suggest that this challenge is entirely consistent with mathematics educators' concern with the provision of occasions for students to have rich, meaningful mathematical experiences.

Construction of Mathematical Thought

Constructivism is commonly thought of as an epistemology--a theory of knowledge. Constructivism has another face--it is a theory of the genesis of knowledge. It is emerging as a theory of learning. It makes specific the claim that anyone's knowledge is the life-long product of constructions.

As a learning theory, however, constructivism is in its infancy. It is seen by many as being more useful as an orienting framework than as an explanatory framework when investigating questions of learning. To say only that we are constructivists because we believe that knowledge is constructed and not received is less than compelling, and it is clearly not useful. It is also insufficient to argue that a psychological theory is invalid if it presupposes direct access to "reality." This argument has been around since the Skeptics. What we need is a *technical* constructivism. We need a technical constructivism that allows its proponents to form precise, testable hypotheses and that allows its opponents the opportunity to refute them, and to refute them on the basis of the adequacy and viability of the *system* of explanations constituting constructivism.

Taken as a collection, the five papers by Bickhard, Steffe, von Glasersfeld and Cooper constitute a primer in constructivist learning theory. Bickhard lays a theoretical foundation for a constructivist learning theory, and Steffe proposes mechanisms for the construction of arithmetical knowledge. Reflective abstraction, which is central to both papers, is clearly explicated by von Glasersfeld, as are several other explanatory constructs. Cooper investigates the roles of repetition and practice as constructive mechanisms in learning. These papers are fundamental reading for any student of mathematical learning and cognition.

Foundations of a Constructivist Learning Theory

Bickhard and Steffe let us glimpse a technical constructivism. Their papers attack the problem of the *possibility* of learning in general and of learning mathematics in particular. That the possibility of learning is problematic can be appreciated when considering how an individual might construct knowledge that is not made by associations of existing concepts. Both Bickhard and Steffe respond to Fodor's anti-constructivist argument: If learning must involve the construction of new representations, then learning cannot happen; that "some basic set of representations, combinatorically adequate to all possible human cognitions, must be innately present" (Bickhard, this volume, p. _).

Bickhard claims that Fodor's argument is valid only if one accepts encodingism--the idea that humans somehow have mental symbol systems that are isomorphic to features of reality. He argues that encodingism is, in fact, an incoherent position, and that in this regard Fodor's argument for innatism is fallacious.[2]

Steffe takes a different approach to refuting Fodor's innatist conclusion. He attempts to offer a counter-example to Fodor's argument. It is debatable whether Steffe's example is a counter-example to Fodor's argument or is instead a demonstration of an alternative framework having its roots in constructivism. But this is a minor point. What is clear is that Steffe offers an interpretation of one student's learning that is not subject to Fodor's argument.

Both approaches succeed by denying naive realism--the idea that somehow we are imprinted with knowledge of the real world--at the foundation of their theories of representation. Instead of characterizing representations as representing something about the real world, Bickhard and Steffe characterize them as "mental stand-ins," *made by the individual doing the representing,* for interactions with a world of objects or ideas. That is, we can represent by re-presenting. These ideas are new and old. They are new because of context and specificity. They are old in the sense that Piaget (1968) anticipated the need to address issues of representation to make specific his claim that language is just one expression of "the general semiotic function" (Piaget, 1950).

Piaget delineated three forms of representation: indices, signs, and symbols. An *index* is a re-presentation of an experience by recreating parts of it in the absence of the actual experience (e.g., imagining an exchange of hands in a "promenade round" to re-present a square dance, or rhythmically nodding one's head to re-present the experience of counting).

A *sign* is a figural substitute--something that captures an essential aspect of a class of experiences, but which is only analogous to them in its similarity. The perception of a wavy line on a yellow board alongside a highway signifies to many people that a bend in the road lies ahead. The wavy line in Figure 1 has nothing to do with one's experience with roads as such, yet it suggests a feature of one's experience of driving on winding roads. Similarly, the underline character in "2 + _ = 7" is not usually part of one's experience in carrying out arithmetical operations, yet it suggests that something is missing.

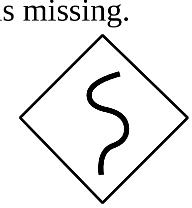

Figure 1: A road sign.

Signs are inferentially linked with their referents, but the inference is much less direct than is the case with indices. I suspect that, were we to look closely, we would find that even the most sophisticated knowers of mathematics make abundant use of signs in organizing their mathematical knowledge.

A *symbol* represents something only by way of association. Symbols have the qualities of arbitrariness[3] and, in the case of symbols which serve a communicatory function, conventionality (Hockett, 1960; von Glasersfeld, 1977, this volume). It would be presumptuous of me to try to improve upon von Glasersfeld's (this volume) discussion of symbols.

While Bickhard focuses primarily upon issues of representation, Steffe focuses primarily upon issues of learning, in particular on accommodations that can account for learning. Before judging the success of Steffe's attempt, we should remind ourselves of his principal goal--to establish that learning is *at least* inductive inference, and to give an existence proof that it can be more. To judge it as successful, we need to answer two questions affirmatively: Do we accept that Steffe's ideas of engendering and metamorphic accommodation as viable, explanatory constructs? Does metamorphic accommodation account for a change in Tyrone's behavior that inductive inference cannot? These are non-trivial questions. When answering them for yourself you will come face-to-face with the core of Steffe's theory of units and operations.

One thing that seemed missing from Steffe's analysis was specificity. This might sound like an odd comment, especially given the extremely small segments of behavior analyzed in great detail by him. But the kind of specificity I have in mind is different from what Steffe gives us. I would like to have an image of Tyrone's *knowledge*. It is evident that Steffe has such an image, but it is not well communicated by natural language. I believe we can take advantage of decades of research and methodology in artificial intelligence and information processing theory. Models expressed in natural language are notoriously poor at facilitating precise thought and communication. Also, they are extremely cumbersome when trying to capture the dynamics of functioning systems. I am reminded of Cobb's (1987) well-known remark that "it would be a tragedy if all serious students of cognition felt compelled to express their creativity solely within the confines of particular formalisms such as computer languages." It would be just as tragic if all serious students of cognition eschewed formalisms such as computer languages.

Reflection and Repetitive Experience

Reflective abstraction is an idea that is central to constructivism. Bickhard's and Steffe's arguments would have gotten nowhere without appealing to reflective abstraction. Without something like it, constructivist theories of learning are dead in the water. In constructivism, reflective abstraction is the motor of accommodation, and hence of learning.

To say that reflective abstraction is central to constructivism is one thing; to say what it *is* is quite another. At times, discussions of reflective abstraction take on the character of describing the homonculus--the little man in the mind that does all the nasty work not accounted for by a cognitive theory. We have been much more successful in describing mechanistic models of the products of reflection than we have in describing how people reflect.

All this notwithstanding, we need to have a clear idea of how any model of reflective abstraction needs to behave, and we need to have a clear idea of the phenomena we wish to ascribe to the operations of reflective abstraction. Von Glasersfeld gives us a portrait of reflective abstraction in these regards: its necessity in constructivism, its character, its history as developed in Piaget's genetic epistemology, and the similarities between reflective abstraction in Piaget's theory and in Locke's empiricism.

What von Glasersfeld makes clear is that reflective abstraction, re-presentation, and representation are inseparable aspects of cognitive functioning. If I can add to von Glasersfeld's contribution, it is this: Piaget gave considerable prominence in his earlier work to the development of intuition, and I believe it was for a good reason. By focusing on intuition, we gain additional clarification of the ideas of reflective abstraction and at the same time push the homonculus farther into the background.

As is frustratingly common with so many terms appearing in Piaget's writings, he failed to give a clear definition of what he meant by "intuition." The clearest statement I have found is in (Piaget, 1950).

> We see a gradual co-ordination of representative relations and thus a growing conceptualization, which leads the child from the [signific] or pre-conceptual phase to the beginnings of the operation. But the remarkable thing is that this intelligence, whose progress may be observed and is often rapid, still remains pre-logical even when it attains its maximum degree of adaptation; up to the time when this series of successive equilibrations culminates in the "grouping," it continues to supplement incomplete operations with a semi-symbolic form of thought, i.e. intuitive reasoning; and it controls judgments solely by means of intuitive "regulations," which are analogous on a representative level to perceptual adjustments on the sensori-motor plane. (p. 129)

To draw out the significance of intuition, I need to digress briefly. One modern interpretation of constructivism is in terms of autopoeitec systems (Maturana & Verela, 1980; Maturana, 1978) and as cybernetic systems (MacKay, 1969; Powers, 1973, 1978; von Glasersfeld, 1976, 1978). Within these perspectives, cognition is viewed as the product of a nervous system's attempts to control and regulate its functioning. Of course, "its functioning" is not something that happens with no exogenous intrusions.

The primary aspect of autopoeitec or cybernetic systems is the fundamental, overriding principle of control: the elimination of perturbations within the system, the resolution of unmet or unattainable goals. Intelligence progresses through the development of Cooperative systems, or schemes, for eliminating classes of perturbations ("classes" from the cognizing organism's perspective). These schemes--systems for controlling cognition--while emerging, fit roughly with Piaget's description of intuitive thought.

Intuitive thought, then, is the formation of *un*-controlled schemes which themselves function to control aspects of cognitive functioning. But these un-controlled schemes are themselves part of the organism's cognitive functioning, and hence are something to be controlled. They become regulated as *their* controlling schemes reach the level of intuition, whence the schemes controlled by them become equilibrated. That is, intuitive thought is actually the fodder of operative thought. Figure 2 illustrates this discussion: the emergence of intuitive thought, and then intuitive control of intuitive thought--operative thought.

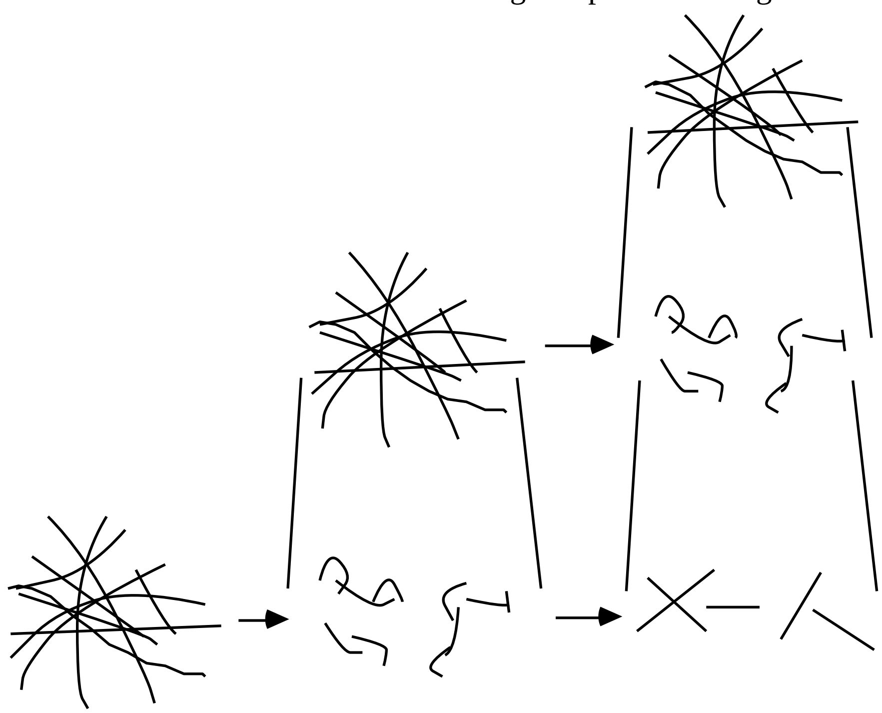

Figure 2. Intuitive thought.

Attention to intuition has two benefits. First, the homonculus is now extrinsic to our picture of the emergence of knowledge. The homonculus is in the principles of cybernetics. The question now is one of architecture--how are biological systems organized that they might or might not behave like this? Second, understanding intuition as unregulated schemes provides a connection between, on the one hand, von Glasersfeld's characterizations of reflective abstraction and reflected thought and Steffe's characterizations of metamorphic accommodation, and, on the other hand, Cooper's descriptions of the dramatic influence of repetitive experience. Intuitive thought develops through recurring experience by way of functional accommodations; it is transformed into operative thought through reflective abstraction and metamorphic accommodation.

Curriculum and Pedagogy

"Constructivist curriculum" and "constructivist pedagogy" sound like oxymorons. It seems paradoxical that, on one hand, we maintain that we are, of necessity, in the dark about how and what people think--that people will make of their social and physical environs what they will, while on the other hand we plan what students are to learn and attempt to design "effective" instruction.

The paradox is in appearance only. We put on the hat of constructivism *so that* we have more coherent visions of what *might* be happening when students evidently learn and understand mathematics and think mathematically. It is from a basis of coherent visions that we are positioned to have greater confidence in our plans.

Cooper, Hatfield, and Confrey each inform our attempt to enrich our understanding of what it means to learn, understand, and teach mathematics. Cooper sets out to convince us that appropriately-conceived repetitive experience can provide a strong foundation for reflective abstraction, and hence that it is a crucial element of students' mathematical learning.

Hatfield reminds us that experience is a private affair, and that there are many affective components to mathematical experience. Nevertheless, it seems that a common thread to his arguments is that "good" experiences lead students to feeling in control of situations, or at least lead them to feel confident that they can come to be in control. His discussions of student programming and simulations explicate the powerful idea that one avenue toward building control over ideas is to operationalize them.

Confrey makes concrete the adage that we cannot understand students' behavior without understanding at least one student. She attempted to understand the thinking of one student, named Dan, in the relatively complex domain of exponential functions. Her analyses of this one student's difficulties gives insight into the machinations of "correct" knowledge of exponentiation. To paraphrase Hersh (1986), it is not until we see students who provide counter-examples to our implicit assumptions about the constitution of specific concepts that we recognize them and make them explicit .

Practice Space and Mathematical Knowledge

Cooper declines to use the word practice because "the implication of the term practice is that what is being practiced is what is being learned" (Cooper, this volume). This is a wonderful distinction. "What is being practiced" is normally in the mind of an adult observer. We *intend* something to be learned, and we have students "practice" it. What they actually learn can be quite a different matter.

We can still use the notion of practice in our theoretical and pedagogical analyses, however. Instead of beginning with the statement "Students practiced *X*," we should begin with the observation "We *intended* that student practice *X*," and then continue by asking the question, "What did they *actually* practice?" What they actually practiced is probably what they actually learned.

The examples given by Cooper are compelling. They also help to clarify an important relationship between knowledge and reflected knowledge. The more tightly woven intuitive knowledge is, the richer is knowledge constructed as a reflection of it. The "map"--the set of inter-relationships among situations to which Cooper refers--is what is reflected. The denser the populated areas, the more relationships in the map. The fewer the populated areas, the sparser the map.

This raises an issue. If repetitive experience generates "richly interconnected spaces," providing a foundation for reflection and reflected knowledge, then as mathematics educators we have a responsibility to describe "spaces" we hope get constructed. This is a curricular issue, and one not settled by Cooper's paper. Cooper makes evident the need for rich and varied repetitive experience in students' schooling. However, to settle on the interconnections we wish children to generate through repetitive experience, we must clarify what we hope they achieve. We need to describe *cognitive objectives* of instruction, and we need heuristic guidelines for organizing instruction and curriculum so as to have some confidence that the objectives can be achieved. How shall we vary situations to map a space? How shall we decide whether two variations are within the same space?

Cooper's analyses prompted me to recall comparisons of American, Japanese, Taiwanese, and Soviet mathematics textbooks (Fuson, Stigler, & Bartsch, 1988; Stigler, Fuson, Ham & Kim, 1986). The gist of these comparisons was this: American textbooks rarely have word problems of any complexity, the problems are commonly of the lowest order of conceptual difficulty, and problems within a set hardly vary in their solution procedure (Porter, 1989). What do students practice when they “work” these problems? At best, they practice “getting answers.” At worst, they practice ignoring such things as context, structure, and situation. In any case, students do not have occasions to generate the “richly interconnected spaces” that Cooper has identified as being crucial for constructing mathematical knowledge. They end up with islands of superficial knowledge without a canoe to get from one to another.

Algorithmics and Mathematical Knowledge

A common view of “skill” in mathematics is to know a large number of procedures for solving a similarly large number of problems. The word “algorithm,” interpreted from this viewpoint, justifiably strikes fear among teachers and students. Here is a brief list of “school-math” algorithms:

- whole-number addition
- whole-number subtraction
- whole-number multiplication
- whole-number long division
- whole-number short division;
- all of the preceding with fractions instead of whole numbers;
- all of the preceding with decimals instead of whole numbers.

Add to this list all the variations that school texts commonly promote: addition with and without trading, subtracting “across zero,” division by a single-digit number, division by a two-digit number, and so on. We soon see a combinatorial explosion in the number of “algorithms” students meet. To ask anyone to learn such a large number of isolated, ostensibly unrelated procedures is inhumane.

We are urged to view algorithms from a different viewpoint, to consider that significant mathematical learning takes place when students’ *create* algorithms and when they investigate them systematically. The task Hatfield has taken on is difficult. He must communicate the richness contained in his idea of algorithmics when many in his audience have not experienced this approach.

Hatfield’s call for the inclusion of algorithmics--the creation and study of algorithms--in school mathematics might sound reminiscent of discovery learning (Bruner, 1963; Hendrix, 1961), but it is actually quite different. Discovery learning emphasized the “uncovering” of concepts and principles, as if they were to be found by turning over a rock. Algorithmics emphasizes the routinization of problem solving by the creation of schemes. At first glance, to routinize problem solving sounds self-contradictory, but it is a standard activity when *doing* mathematics. Schemes contain concepts and principles. To create schemes, one must also create component concepts and principles, but that in itself is insufficient. One must also create operations and relationships between and among them.

Hatfield also concentrates on student programming as a vehicle for bringing algorithmics into school mathematics. His efforts are in the tradition the Computer Assisted Mathematics Project (Johnson, Hatfield, Le Blanc, & Kieren, 1968). As recently as five years ago I was a fervent supporter of having students study mathematics via computer programming. While I still have a great deal of empathy and respect for this approach, I now have reservations about programming as a vehicle for learning mathematics. The reservations come from the practicalities of having to teach a programming language and to teach programming constructs in order to teach mathematical concepts. If it is done well, it can work. However, I have strong suspicions that the requirements for making it happen are beyond the capabilities of the vast majority of teachers. To use programming as a vehicle for algorithmics requires more than knowledge of mathematics and knowledge of programming. It requires that that they be reflections of one another, and this is an intellectual achievement of the highest order.

I remain open to being shown how wrong I am. In fact, I would like to be wrong. I agree with Hatfield on the potential benefits of student programming. However, there are many open questions as to what might constitute a proper balance among the components of content, curriculum, and programming, and there are many open questions about the long-term curricular and cognitive implications of student programming as a vehicle for learning mathematics.

There is another approach to introducing algorithmics into mathematical instruction. It also uses computers in non-traditional ways, but students do not write computer programs. Instead, they use programs that have been designed with the aim of making explicit the mathematical constraints of situations while imposing as few methodological constraints as possible. Students are asked to solve problems, posed independently of the program, but solve them within the constraints imposed *by* the program. A student’s task is threefold: interiorize the mathematical constraints, develop methods for solving classes of problems, and construct a *recording scheme* that, at each step of his or her solution, reflects the state of the situation and a history of the steps taken so far. A student’s recording scheme, viewed as an object in itself, is his or her personally-constructed algorithm for solving problems of a particular class.

However algorithmics is injected into mathematical instruction, one benefit seems clear. It is that students have occasions to experience the mutually-defining relationships between problems and methods. A slight modification in a “standard” problem may cause one to re-conceptualize one’s methods for solving problems of that kind. Likewise, an analysis of a method may cause one to re-conceptualize one’s understanding of the “kind” of problems for which the method was originally constructed. That is to say, a “class” of problems is determined by a method, but the boundaries of that class may not be as evident as originally thought. Likewise, problems give rise to methods, but methods may be more or less general than originally thought. Both realizations are propaedeutic for the construction of substantive mathematical knowledge.

Conceptions, Misconceptions, Teaching, and Learning

When reading Confrey’s paper I was reminded of a recent event. Though my three-year-old daughter, Nicole, knows how to swim, when enrolled for summer swimming lessons she could not get past “Station 1.” Station 1 is where the component skills of swimming were taught. The reason she could not pass was this: Instruction was focused so closely on component skills that Nicole never recognized them as part of her experience of swimming. Moreover, since instruction was focused so narrowly, the instructors never had an opportunity to see Nicole attempt component skills in the context of trying to swim. The instructors were teaching something of which Nicole was already capable, but Nicole never realized that she was already capable of what was being taught. As a result, instructors never had occasions to recognize areas where Nicole actually needed improvement. Gagné’s reductionism has found a home in swimming instruction.

Confrey gives us a holistic view of one students’ attempts to make sense of the world of multiplicative structures, in general, and of exponential functions, in particular. She could have taken a reductionist, Gagné-like approach and interviewed Dan on “component” tasks, where her task analyses determined the component concepts. Had Confrey done this, the result could easily have been analogous to Nicole’s swimming lessons. We would have seen what Dan could do under the artificial constraints of an imposed segmentation of the concepts--a segmentation emanating from an analysis of instructional learning objectives. But we would not have known Dan’s capabilities, nor would we have seen the simultaneous richness and poverty of Dan’s cognitions in regard to multiplicative structures.

Confrey encouraged Dan to extend himself, to grope, to contradict himself, to make sense of nonsense and nonsense of sense. The picture we get of Dan’s knowledge is not clean; it is quite messy. But in *our* making sense of Dan's struggles and inconsistencies, we learn more about the concept of exponential function. We see the tension between Dan's inclination to see the world additively (e.g., focusing on successive differences instead of on successive ratios) and the tension this causes when dealing with problems that, to him, were ostensibly multiplicative.

One task given to Dan ("Draw a picture of 5^3"), and Confrey's subsequent analysis, raises an important question. To what extent is it profitable to portray exponentiation as repeated multiplication? If multiplication is portrayed as repeated addition, and if exponentiation is portrayed as repeated multiplication, then exponentiation reduces to repeated addition. If we are speaking about *calculating* some power of a whole number, the calculation does reduce to repeated addition. But calculation is not the issue. *Conception* is the issue. If one is attempting to conceive of an attribute of an object that we would normally call multiplicatively-structured, and if that conception is fundamentally additively-structured, then one cannot conceive the attribute. We see this continually--children treating area as what we would normally call perimeter, or treating volume as what we would normally call surface area. We saw this in Dan's initial attempt to draw a picture of 2×3×4. We also saw his progressive "dimensionalization" of multiplication as he formed a product, and then created copies of that product. His pictures were still two-dimensional, but his comments indicated that he began imposing a structure on them that made his *conception* more like a Cartesian product.

Confrey's analysis prompted me also to think about how $5^{\sqrt{3}}$ would fit with the portrayal of exponential functions as dimension-building objects. I do not know how people make sense of $5^{\sqrt{3}}$, if they make any sense of it at all.[4] Confrey has opened an intriguing and important domain of inquiry. I suspect it will be a long and fruitful research program.

Methodology

Kieren and Pirie have taken the considerable task of developing a framework for viewing the evolution of a person's mathematical knowledge. As they say, "What is needed is an insightful way of viewing the whole of a person's growing mathematical knowledge and understanding built through this knowledge." They propose the use of recursive description as one way to capture the self-referentiality of knowing while remaining within the constraints of their view that humans are autopoietic systems.

Their contribution is evidently ground-breaking, and as such requires careful reflection and analysis to realize its importance. It is a major contribution to the endeavor of providing useful and powerful visualizations of what we mean by "knowing mathematically." As with any work that breaks new ground, there are aspects to it that are problematic. This remark does not diminish the importance of Kieren and Pirie's contribution. Rather, it frames my task of discussing their current work.

Explanations vs. Descriptions

There is a subtle, yet significant difference between an explanation of an observation and a description of an observation. A description tells what happened; an explanation tells why what happened happened (and implicitly, why other things *didn't* happen). That is to say, an explanation must be framed by a theoretical context and oriented toward a class of possibilities, whereas a description is framed by an observation. Explanations and descriptions are evidently related, as descriptions are made out of the same theoretical stuff as explanations; but the intent of describing is different from the intent of explaining.

It is crucial to keep in mind the distinction between a description and an explanation as we struggle to understand students' mathematical thinking, for it is all too easy to unduly impute some of our constructs to our students. I am reminded of the recent literature on children's arithmetic, where early-on the counting strategy of "start with the larger" was described as children's growing awareness of the commutativity of addition (Ginsburg, Baroody, & Waxman, 1983). Lately this same strategy is being described in terms having nothing to do with the mathematical principle of commutativity (Baroody & Ginsburg, 1986). Did children begin to know less while behaving the same? No. Researchers began to do a better job of separating their mathematical knowledge from descriptions of children's mathematical knowledge.

The reason for dwelling on differences between descriptions and explanations is this: If our language of description is too powerful, or is used indiscriminately, our explanations may, in the end, be descriptions of our mathematics instead of children's mathematics (Steffe, 1988).

Recursion vs. Repetition

Recursion and repetition are the primary constructs upon which procedural descriptions are founded. Dijkstra (1976) has given the classic distinction between repetition and recursion:

> The semantics of a repetitive construct can be defined in terms of a recurrence relation between *predicates*, whereas the semantic definition of general recursion requires a recurrence relation between *predicate transformers*. This shows quite clearly why I regard general recursion as an order of magnitude more complicated than just repetition. [Italics in original]
>
> Dijkstra, 1976, p. xvii

Dijkstra used "predicate" to mean a proposition, or a state of a computing mechanism and he used "predicate transformer" to mean a rule by which to derive one class of predicates from another. We can think of a predicate as a conception of a situation and a predicate transformer as a mental operation constituting a general relationship between classes of situations. In these terms, the distinction between repetition and recursion is between repeatedly applying an action to situations within a class (where attention is focused on the operation's "inputs") and the application of a mental operation to transform a class of situations (where attention is focused on the general relationship between the class of inputs and the class of outputs).

Unadvised use of recursive descriptions can result in more being attributed to students than we might wish. It is not uncommon to see recursion used where repetition would be more appropriate. I agree with Dijkstra when he says,

> Although correct [to define repetition in terms of recursion], it hurts me, for I don't like to crack an egg with a sledgehammer, no matter how effective the sledgehammer is for doing so.
>
> Dijkstra, 1976, p. xvii

The instance of Kieren and Pirie's interpretation of Simon's and Alison's remarks in solving the handshake problem illustrates how the language of recursion can overpower observation. My understanding of *transcendence* is that it involves conceiving of a process as if it were completed. However, by this criterion, neither Simon nor Alison transcended Joanne's original idea of having each of 35 people count handshakes and report their individual totals. Evidently, Simon constructed a solution method: compute the sum 34 + 33 + 32 + … + 1. Alison evidently generalized Simon's method to *n* people, the solution being $n + (n\text{-}1) + \ldots + 1$. But neither appeared to transcend Joanne's original conception of each person walking down the line of people counting handshakes as they went. Appropriate attribution of transcendent recursion to a student's conception of the handshake problem would require evidence of thinking something like this:

> Each person in line leaves to drink a Pepsi, saying to the person on his or her left, "You and all the people on your left do all your handshakes. When you're done, tell me how many people were in your group and how many handshakes your group made. I'll get my answer by adding the number of people in your group to the number of handshakes."

It is important to realize that, were this method actually carried out, there would not be any actual handshakes--even though each person gives an explicit directive to do handshakes and count their number.

The difference between iterative and recursive conceptions of the handshake problem resides in how a solver thinks of a partial tally. In Simon's description, the person closest to the door computes the running tally, and only when the last person walks out is the tally identifiable with the total number of handshakes among some group of people. In the Pepsi-drinkers' method, each person in line computes a partial tally, and *any* partial tally is the total number of handshakes among some group of people. The distinction is significant.

Conventions for Attributions of Meaning

It is not clear what types of behavior can be taken as indicative of transcendent recursion. Watkins & Brazier (1985) reported that two students wrote procedures to produce derivatives of functions. Their procedures seemed to be highly recursive, yet it turned out that neither student had even the most rudimentary comprehension of recursion (Brazier, 1985).

How do we explain this apparent paradox? By appealing to Kieren's and Pirie's conditions for transcendence. The students (using Logo) programmed a formula, $[f(g(x))]' = f'(g(x))\cdot g'(x)$. The need for these students to transcend their initial conception of a derivative (as the output of the Logo function they were then defining) was short-circuited by their "knowledge" that the derivative already existed (as a *formula*) independently of the function they were defining. In short, they were not constructing a recursive function to make derivatives. They were translating a textbook formula into Logo.[5]

The larger issue is this: By what convention can we minimize the probability that we use too-powerful constructs to explain students' behaviors or to describe students' understandings? Steffe's group faced this problem in the course of developing what ultimately became known as counting types (Steffe, Thompson, & Richards, 1982; Steffe, von Glasersfeld, Richards, & Cobb, 1983). The methodological convention adopted was to attribute no more to a child's understandings than what is minimally necessary to account for their behavior on the task being performed, taking into consideration interpretations made of his or her performance on related tasks. Of course, to make this approach work, one must be conservative. To be conservative means to assume, at first, that a student does not fully understand the dimensions of the task. As an interpreter of behavior, one must always build a case for interpretations made.[6]

Cognitive Modeling

Steffe and Dubinsky have developed theoretical frameworks for modeling students' knowledge and to describe its construction. However, the goals of their frameworks are quite different. The constructs in Steffe's framework are used to explain children's construction of mathematical concepts within a specific domain at a high level of conceptual detail. In this volume, he used the notion of a unitizing operation to describe one student's construction of a number sequence. Dubinski's framework, on the other hand, appears to have a different intent. It has the flavor of a "universal" framework for modeling the construction of many types of mathematical knowledge, from universal and existential quantification in the predicate calculus to mathematical induction.

Dubinski's approach is to apply his framework to his personal understandings of a concept. By doing this he obtains a "genetic decomposition" of the concept, and takes that as his initial model for investigating students' concepts in the same domains. As Dubinski himself states, a genetic decomposition is only a starting point, to be discarded as one learns more about what students understand and how they came to these understandings. Even with this caveat in mind, however, I found myself asking for more than this framework provided.

The framework, at least in the description given in Dubinski's paper, is too general to support precise interpretations of students' performance on tasks. I frequently found myself providing alternative interpretations to those given by Dubinski, and the alternatives had more to do with the nature of students' understandings of specific concepts than with the broad operations of encapsulation, reversal, etc. Perhaps my understanding of the framework was not sufficiently precise to constrain my interpretations to those offered by Dubinski. It might have helped were the central constructs of the framework somehow operationalized independently of their use in interpreting students' behavior. For example, what does it mean to "reverse a process." Is it like running a motion picture backward? Is it like saying the alphabet backward? Or is it the performance of an operation that has the effect of undoing the effect of another? These are all very different from one another.

Mathematics Education as *Paideia*

The literary education portrayed by Lewin aims to develop students' character by having them appropriate the literary artifacts of others. Through the dialectic of appropriation--viz., interiorization and internalization--students transform literary text into personally meaningful correlates of their own life experiences. In the process of appropriation, Lewin says, students' personal schemas are likewise transformed.

> We have access not to [others] in the moments in which they create, but to their artifice, not to a life lived in its ongoing fullness, but to a stable presence, the artifact, generated out of that ongoingness. We ask students to engage these artifacts, enriched, educated in the literal sense of finding themselves drawn out" (Lewin, this volume, p. _).

How different is this from practices in mathematics education? On one hand it is not different at all. Students are bombarded with artifacts in mathematics: definitions, theorems, proofs, and algorithms to name a few. On another hand, it is very different. The "objects" with which students are bombarded are not *presented* as artifacts. They are presented too often as "the truth," as being engraved in stone, having just arrived from the Mount.

What kind of mathematics pedagogy would parallel the pedagogy described by Lewin? In teaching a definition we would first need to make sure of its author.[7] Discussions would include why the author worded his or her definition this way and not another, why the boundaries were placed as they were.[8] Discussions of theorems would include why the conditions of the theorem are as given and not more, less, or differently restrictive. Discussions of algorithms would include the possible dependence of the algorithm's validity on the notational scheme in which it is expressed. In short, for mathematical pedagogy to parallel that described by Lewin, teachers would have to address the history of mathematics and the culture of the mathematical community explicitly. These issues traditionally are not an integral part of mathematics instruction or mathematics teacher preparation programs.

Lewin gives us an extended example, wherein he describes in concrete terms the process of appropriation as a dialectic of internalization and interiorization, and the subsequent transformations of self in relation to an objective Romanticism. As a reminder, here is the task he set for his students: (1) Write a passage explaining what it would mean for someone or something to be called "romantic." (2) Read "Mock on, Mock on, …". (3) Try to apply your description of "romantic" to see whether or not it fits the poem.

Lewin mentioned an essential ingredient of this task only in passing. *Students were told that Blake's poem is generally considered a good example of Romantic poetry.* Without their acceptance that Blake's poem "really is" romantic, students would only have had an occasion to classify it by their already-held criteria. They would not have found an occasion to reflect on what they understood by "romantic" in relation to meanings negotiated among the group of scholars who use the term precisely. That is, unless students accept characteristics of an artifact as being *objective*, they have little cause to constrain or reshape their understanding of it.

Students easily take notations as being objective (after all, they *see* them), but rarely do they consider that notations can have objective correlates.[9] That students eventually develop objective correlates of notation and notational transformations is a major aim of mathematics instruction. However, we are at a disadvantage. Students must relate experientially to anything before they can begin to objectify it, to apprehend it as immediately given. There are few mathematical artifacts that can be taken by students as objective independently of the notations in which they are expressed. One way to remedy this situation is to construct computer microworlds where things are made to happen by way of a notational scheme, but whose behaviors are constrained according to the mathematical systems we wish students to construct. To interiorize the microworld is to interiorize the mathematical system by which the microworld's behavior is constrained (Thompson, 1985, 1987).

The major significance of Lewin's paper is in its implications for the preparation of mathematics teachers. Practitioners of mathematics primarily need to do mathematics. They do not need to relate to mathematical knowledge in the same way that a mathematics teacher needs to relate to it. Teachers must be able to express deeply principled knowledge "softly," so that its expression does not overpower a learner and yet is sensitive simultaneously to the learner's current constructions and to long-term curricular objectives. That is, a teacher of mathematics needs to be capable of imagining a construction of concepts that he or she already possesses and can be accessed only in their current moment.

It is in this regard that I see the greatest relevance of Lewin's notion of *paideia*, the formation of character.

How might we affect future teachers' reconstruction of the mathematics curriculum from a sequence of topics to a fabric of constructions? A slight paraphrase of one passage in Lewin's paper addresses this question with remarkable acuity.

> [We] ask students to juxtapose their pre-existing ideas about [the curriculum] with their growing classroom familiarity of it in order to facilitate the examination, and perhaps revision, of their pre-suppositions. Unless a student can find the relevance of [the artifact], can construct its meaning for her or his own life [as a teacher], there is no possibility for *paideia*, for an engagement that enriches the self. (p. _)

The key to affecting future teachers' reconstruction of the mathematics curriculum is in the creation of appropriate *artifacts* that somehow reflect a coherent vision of it as already recast into a fabric of constructions. A text (e.g., a book, an article, etc) that elucidates an end-product of such a re-creation would be one possibility. A text, however, would suffer the same disadvantages as when expressing the end-product of a mathematical construction in a notational system. Many students will have difficulty objectifying the ideas presented in the text; they will have difficulty constructing meaning for his or her own life as a teacher.

Another possibility is to extend the approach taken with computer microworlds. This is one that I am currently researching (Thompson, 1989a, 1989b). The general idea is to reflect the development of a conceptual field within a computer program, where "development" is reflected in transformations of constraints on the program's behavior. The program is designed so that it can be used to solve problems normally studied within the conceptual field, but the constraints on its behavior forces students to view the problems through the eyes of a student whose thinking is likewise constrained. The task ("Solve these problems") and the program are immediately taken by future teachers as objective. In the dialectic of internalizing and interiorizing the program, brought about through their use of it in solving problems, these future teachers have an occasion to reconstruct the conceptual field being studied.

Postscript

The papers in this volume clearly are concerned with mathematical experience. In many cases they describe "good" experiences to be had. However, we have not been informed about what *is* mathematical experience. As commentator, I could have taken "mathematical experience" as a theme to be drawn out from each paper. I chose not to try. The word "experience" was used to mean anything from a vague awareness to an exhilarating insight. I am not accusing the authors of using vague language. The problem is with common usage, which has strong realist overtones. "Experience" is sometimes used as if referring to an event (as separate from an experiencer), and at other times as if referring to the act of living through an event. Clearly, the latter was of concern to authors in this volume. To muddy the waters more, "experience" can be used as if referring to a totality of events lived through by someone, as in "Experience tells Paul that this will not work," and it can be used as if referring to the cumulative effect of living through those events, as in "Paul's experience is that this will not work." Perhaps "experience" is not a very useful word when trying to understand mathematical thinking through constructivism. Its non-technical usage overpowers any attempt at precision.

Notes

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[2] Johnson-Laird (1988, pp 134-137) notes, as does Bickhard, that one sign that Fodor's argument is wrong is it proves that logic and concepts cannot even be innate, because they cannot have evolved. However, Johnson-Laird takes a different approach than Bickhard to refuting Fodor's argument generally, and does so without explicitly denying encodingism. Instead, he draws a distinction between an increase in a cognizing organizism's computational power (as modeled by a Turing machine) and an increase in its conceptual power. He argues that while the former cannot happen, the latter can.

[3] The word "arbitrary" sometimes is given the same sense as "random." Clearly, this is not what is meant here. Instead, it means that there is nothing intrinsic to a symbol that would cause a loss of meaning by systematically substituting another one for it. We might find some symbols more practical or more convenient than others, or easier to relate to other notations, but this is beside the point. For example, in about 20 minutes most adults can recast their number-name and numeration schemes using the notational and verbal scheme "a, b, c, d, e, f, g, h, i, ton, ton-a, ton-b, ton-c, ton-d, ton-e, ton-f, ton-g, ton-h, ton-i, b-ty, b-ty a, b-ty b, …" That their original naming schemes were symbolic is suggested by the fact that they can perform their addition and subtraction algorithms using the new naming scheme. For children, number-names may

or may not be symbols. They could be only indeces of counting experiences, or signs of having counted.

4 By "make sense" I do not mean to make formal sense, e.g. $5^{\sqrt{3}}$ as an equivalence class of Cauchy sequences. Rather, I wonder how they envision it as being the same sort of object as 5^3 .

5 It is curious how different notations make it more or less difficult to translate the chain rule into Logo. Operator notation, i.e. $D_x (f \circ g) = [D_x(f) \circ g] D_x(g)$, seems easiest to translate, but it is the most difficult for students to understand. Leibniz' notation, i.e. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, seems easiest for students to understand but it is the hardest to translate into Logo. The prime notation used in this narrative seems somewhere in the middle on both counts.

6 On the other hand, interpreting will always be an art, and there are no guarantees against attributing too much or too little knowledge to another person's understanding.

7 Definitions do have authors, but their names are normally not mentioned in matheamtics texts. The tradition in mathematics is that an author's name is attached only to important theorems or to important algorithms. Perhaps this is because of working mathematicians' implicit Platonism, that these things are "there," so only the hardest to find are named after the explorer finding them.

8 For example, why, in any definition of prime number, is 1 excluded from being prime? Because if it were included, the Fundamental Theorem of Arithmetic, which says that factorizations of integers are essentially unique, would have to be reworded in order to remain true, and the rewording would be very clumsy.

9 I deliberately use "notation" instead of "symbol." A notation is a mark of some kind; a symbol is something that has a referent. Students can see notations. If a student gives a notation meaning, for that student it is a symbol.

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