Images of Rate[†]

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Running Head: Images of Rate

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In previous papers (Thompson, 1990; Thompson, in press-a; Thompson, in press-b) we have presented details of a theory that addresses the reasoning that might underlie a person's competent, flexible use of visible algebra. We have called this theory a theory of quantitative reasoning. We should note something about the word quantity. In mathematics, it is common for "quantity" to be used loosely, almost synonymously with "number." We use the word quantity here more in the physical sciences tradition, as something to be measured. Quantitative reasoning, then, is reasoning about things and their attributes. This is not a precise representation of the theory, but it is sufficient for now.

In this paper we will focus on the concepts of ratio and rate, approaching them from the point of view that they are constituted by mental operations grounded in images of multiplicative comparisons and dynamic change. We will then explore the idea that there is a developmental, reflexively constitutive relationship between students' images of (what we see as) rate-like phenomena and their mental operations entailed in proportional reasoning. We will also explore potential relationships between students' understandings of rate and their understandings of function and change in calculus.

Ratio and Rate

That we have two terms "ratio" and "rate" would suggest that we have two ideas different enough to warrant different names. Yet, there is not a conventional distinction between the two, and there is widespread confusion about such distinctions. Confusion is not limited to school classrooms. It can be found even in the research literature. Ohlsson (1988) wrote the following as an analysis of ratio:

> … If I get 8 miles per gallon out of my car during the first leg of a journey but for some reason get only 4 miles per gallon during the second leg, then the correct description of my car's performance over the entire trip is <u>not</u> (4+8)=12 miles per gallon. Similarly, if one classroom has a ratio of 2 girls per 3 boys and another classroom has 4 girls per 3 boys, the combined class does not have 2/3 + 4/3 = 6/3, or 6 girls per 3 boys....The correct analysis of the examples is, I believe, the following. The fuel consumption during the first leg of the journey in the first example was 8 miles per gallon. We represent this with the vector (8,1). The fuel consumption during the second leg of the journey was (4,1). Adding these vectors … gives (12,2), which is equivalent to (i.e., has the same slope as) the vector (6,1). Vector theory predicts that the fuel consumption

> during the entire trip was 6 miles per gallon, which is correct. (Ohlsson, 1988, p. 81; emphasis in original).

Claims of correctness notwithstanding, Ohlsson's analysis of his car's mileage is correct only if he traveled ($8x$) miles in the first leg of his journey and ($4x$) miles in the second leg. Otherwise, it is incorrect. He evidently took a rate of consumption (8 miles per gallon) for actual consumption (went 8 miles and used 1 gallon). Without knowing the relative number of miles travelled or relative number of gallons used in each leg of his journey, we can say nothing about his car's rate of fuel consumption.[1]

Distinctions between ratio and rate have traditionally been treated as unproblematic, although the definitions of each have varied among researchers. Perhaps the lack of conventional distinction between ratio and rate is the reason that the two terms are used often without definition. Lesh, Post, and Behr noted that "… there is disagreement about the essential characteristics that distinguish, for example rates from ratios … In fact, it is common to find a given author changing terminology form one publication to another" (1988, p. 108). The most frequent distinctions between ratio and rate appearing in the literature are:

1) A ratio is a comparison between quantities of like nature (e.g., pounds vs. pounds) and a rate is a comparison of quantities of unlike nature (e.g., distance vs. time (Vergnaud, 1983; Vergnaud, in press)).
2) A ratio is a numerical expression of how much there is of one quantity in relation to another quantity; a rate is a ratio between a quantity and a period of time (Ohlsson, 1988).
3) A ratio is a binary relation that involves ordered pairs of quantities. A rate is an intensive quantity—a relationship between one quantity and one unit of another quantity (Kaput, Luke, Poholsky, & Sayer, 1986; Lesh, Post, & Behr, 1988; Schwartz, 1988).

While there is an evident lack of consensus about distinctions between ratio and rate, each of these distinctions seems to have at least some validity. My explanation for the lack of consensus is that these distinctions have been based largely upon situations per se instead of

[1] Suppose, for instance, that the first leg of the journey was 99.99 miles, and the second leg of the journey was 0.01 miles. In this instance it is evident that the car's rate of fuel consumption is very close to 8 mi/gal.

being based on the mental operations by which people constitute situations. When we shift our focus to the operations by which people constitute "rate" and "ratio" situations, it becomes clear that situations are neither one nor the other. Instead, how one might classify a situation depends upon the operations by which one comprehends it. Rates and ratios are in the mind of a person conceiving a situation. They are not in a situation independently of someone thinking of it.

In (Thompson, 1989; Thompson, 1990) we illustrate how "objective" situations can be conceived in fundamentally different ways depending on the quantitative operations available to and used by the person conceiving them. When we take the perspective that ratios and rates are the products of mental operations, classification schemes for separating situations into "rate" and "ratio" categories are no longer of great importance. What is important is that we characterize those mental operations in ways that are helpful for shaping instructional actions and curricular materials so that students have ample occasions to build them.

Images and Mental Operations

We have found ourselves paying greater attention to imagery in people's reasoning. We wish to make clear that by "image" we mean much more than a mental picture. Rather, we have in mind a mental image as being constituted by experiential fragments from kinesthesis, proprioception, smell, touch, taste, vision, or hearing. It seems essential also to include the possibility that images can entail fragments of past affective experiences, such as fearing, enjoying, or puzzling, and fragments of past cognitive experiences, such as judging, deciding, inferring, or imagining.[2] We admit that this meaning for the word image is too broad, but that is where our thinking is now, and it has afforded us the ability to hear much more in children's expressions of their reasoning than we used to. Nevertheless, this formulation does suggest that a person's actual images can be drawn from many sources, and hence individual's actual images are going to be highly idiosyncratic.

The roots of this overly-broad characterization of image go back to Piaget's ideas of praxis (goal-directed action), operation, and scheme. we discuss this connection more fully in other

[2] We are aware of the recursiveness in this formulation.

papers (Thompson, 1985b; Thompson, 1991b; Thompson, in press-a). For this paper we will focus on Piaget's idea of an image and its relationship to mental operations.

Piaget distinguished among three general types of images. The distinctions he drew were based on how dependent upon the image were the actions of reasoning associated with it. The earliest images formed by children are an "internalized act of imitation … the motor response required to bring action to bear on an object … a schema of action" (Piaget, 1967 p. 294). By this we take Piaget to have meant images associated with the creation of objects, whereby we internalize objects by acting upon them. We internalize them by internalizing our actions. His characterization was originally formulated to account for object permanence, but it also seems especially pertinent to the creation of mathematical objects.

A later kind of image people come to create is one having to do with primitive forms of thought experiments. "In place of merely representing the object itself, independently of its transformations, this image expresses a phase or an outcome of the action performed on the object. … [but] the image cannot keep pace with the actions because, unlike operations, such actions are not coordinated one with the other" (Piaget, 1967 p. 295). It is advantageous to interpret Piaget's description broadly. If by actions we include ascription of meaning or significance, then we can speak of images as contributing to the building of understanding and comprehension, and we can speak of understandings-in-the-making as contributing to ever more stable images.

A third kind of image people come to form is one that supports thought experiments, and supports reasoning by way of quantitative relationships. An image conjured at a moment is shaped by the mental operations one performs, and operations applied within the image are tested for consistency with the scheme of which the operation is part. At the same time that the image is shaped by the operations, the operations are constrained by the image, for the image contains vestiges of having operated, and hence results of operating must be consistent with the transformations of the image if one is to avoid becoming confused.[3]

> [This is an image] that is dynamic and mobile in character … entirely concerned with the transformations of the object. … [The image] is no longer a necessary aid to thought, for the actions which it represents are henceforth independent of their physical realization and consist only of transformations grouped in free, transitive and reversible combination … In short, the image is now no more than a symbol of an operation, an imitative symbol like its precursors, but one which is constantly outpaced by the dynamics of the transformations. Its sole function is now to express certain momentary states occurring in the course of such transformations by way of references or symbolic allusions." (Piaget, 1967, p. 296).

Kieren and Pirie (Kieren & Pirie, 1990; Kieren & Pirie, 1991; Pirie & Kieren, 1991) draw heavily on the notion of images in their analyses of the structure of mathematical reasoning and mathematical learning. Their theory focuses on the fractal nature of mathematical experience from the large perspective of mathematical knowing. We have taken a complementary approach by trying to attend to characteristics of specific images and how they provide a ground for application of mental operations involved in quantitative reasoning.

In the remainder of this paper we will attempt to make concrete the benefits of attending to the images students might be working with as they grapple with reasoning about situations through the operations of ratio and rate. The definitions of ratio and rate that we will work from are, in fact, highly imagistic.[4]

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[3] The Latin root of "confused" is *confundere*, to mix together. Thus, one way to think of being in a state of confusion is that we create inconsistent images while operating.

[4] This creates a considerable problem. To convey images through text, or even through figures, is highly problematic, for aside from the lowest level of imagery the dynamic aspect of an image cannot be adequately captured in language. Nevertheless, I will give it my best attempt.

Ratio and Rate

A ratio is the result of comparing two quantities multiplicatively. If the quantities being compared are thought to be the same attribute (e.g., length), then this is like addressing the question "how many times as big is this than that." If the quantities are thought to be different attributes, then it is their segmentations that are being compared.

This definition is in accord with most given in the literature. One slight difference is that it does not specify how the result of a multiplicative comparison is denoted or expressed. For example, a collection of 3 objects can be compared multiplicatively against a collection of 2 objects in either of two ways: a comparison of the two collections per se, or a comparison of one as measured by the other (Figure 1). The first comparison is of the two collections as wholes. The second comparison is of one quantity measured in units of the other. Both are expressions of a multiplicative comparison of the two quantities. The second is propitious for concepts of fraction, and may be more sophisticated than the first. We should note something about the second comparison: Even though the result is expressed in the same way as what is often called a "unit rate," the comparison described is a multiplicative comparison between two specific, non-varying quantities, and hence is a ratio comparison.

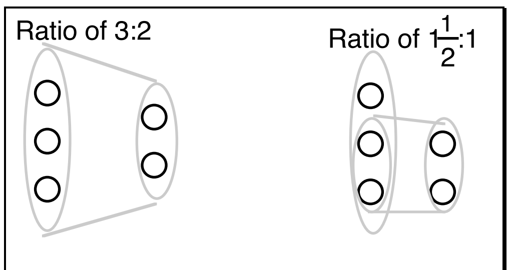

Figure 1

When we focus on the mental operation of multiplicative comparison, evidently it makes no difference if the quantities are of the same dimension or not. What matters is that the two quantities are being compared multiplicatively. If the quantities being compared are measured in the same unit, then the comparison happens to be a direct comparison of qualities. If the quantities are measured in different units, then it is segmentations (measures) of their qualities that are being compared. In either case, the salient mental operation is multiplicative comparison of two specific quantities, and the result of the comparison is a ratio.[5]

A rate is a reflectively abstracted constant ratio. This is in the same sense that an integer is a reflectively abstracted constant numerical difference (Thompson, 1985a; Thompson & Dreyfus, 1988). A specific numerical difference, as a mental structure, involves a minuend, a subtrahend, and the result of subtracting. An integer, as a reflectively abstracted numerical difference, symbolizes that structure as a whole, but gives prominence to the constancy of the result—leaving minuend and subtrahend variable under the constraint that they differ by a given amount.[6] Similarly, a specific ratio in relation to the quantities compared to make it is a mental structure. A rate, as a reflectively abstracted constant ratio, symbolizes that structure as a whole, but gives prominence to the constancy of the result of the multiplicative comparison.[7]

Between ratio and rate are a succession of images and operations. We have identified, in principle, four levels in the development of children's ratio/rate schemes. The first level, *ratio*, is characterized by children's comparison of two taken-as-unchanging quantities according to the criterion "as many times as". The second level, *internalized ratio*, is characterized by children's construction of co-varying accumulations of quantities, where the accrual of the quantities occurs additively, but there is no conceptual relationship between ratio of accumulated quantities at iteration x and the ratio of accumulated quantities at iteration $x+1$. The third level, *interiorized ratio*, is characterized by children's construction of co-varying amounts of quantities, where the amounts vary additively but with the anticipation that the ratio of the accumulations does not change. The fourth level, *rate*, is characterized by children's conception of constant ratio variation as being a single quantity–the multiplicative variation of a pair of quantities as a

[5] Values of compared quantities need not be known in order to *conceive* them as being compared multiplicatively. The process by which one produces a numerical value for the ratio is the quantification of the ratio.

[6] An idea that needs examination is that understandings of integers is one source of understanding rate, in that "constant difference" as a result of an additive comparison entails the idea that the two compared quantities can vary in value, but only by the same amounts (i.e., at identical rates). Put another way, "y=x+2 means that *y* is always 2 greater than *x*, because however much *x* changes *y* changes by the same amount."

[7] This is one way to explain why a rate has the feel of being a single quantity.

measure of a single attribute. A *rate* is a reflectively-abstracted conception of constant ratio variation. Another viewpoint on this development is how children think of the unit of accrual, how they think of total accumulation, and on how they think of accruals in relation to total accumulation. The matter of accrual and its relation to accumulations will be taken up in the next section.

Images of Speed

We will discuss speed as a canonical instance of rate, not because it is canonical, but we suspect that children develop *some* canonical image of rate. It also happens that we have more data on images of speed than on instances of other rates.

In four teaching experiments with individual children—one fifth-grader (Thompson, in press-a) and four sixth-graders (Thompson & Thompson, 1992; Thompson, 1991a)—and one whole-class teaching experiment with 32 seventh-graders we have seen a consistent pattern in children's development of sophisticated concepts of speed.

Image 1: Speed is a distance and time is a ratio

Children begin thinking of speed as a distance—how far one goes (in one time unit).[8] The unit of time is implicit in the stated speed-distance. This conception is often revealed in the way they express a speed in context—"He is going 30 feet," omitting the time unit (e.g., per second)—or expressing an elaborated unit, "He goes 45 miles in an hour." Children thinking of speed in this manner answer questions about how much time it will take to go some distance at a given speed by imagining that they are measuring the total distance to be traveled in units of the given speed-distance, each iteration of a speed-distance implying a time-unit. Some complexities arise as to how to interpret non-integral measures, but they are the same difficulties in principle as how to interpret the part of a ruler that sticks out past the end of an object whose length one is measuring.

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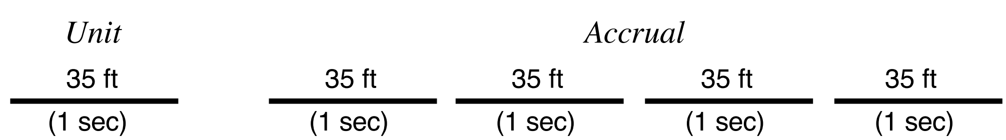
[8] One child, when asked if we could measure the speed of a car in miles per century, said no. "Either you would die or the car would rust away before a century had gone by." When asked to think about it he did not see anything to think about.

Thinking of speed as a distance keeps students from seeing a multiplicative relationship between speed, time, and distance. When asked at what speed someone would have to run to go 100 feet in 6 seconds they often resort to a guess-and-check strategy (Thompson, in press-a). Their conception of the task is tantamount to choosing a ruler with the appropriate length so that when they measure 100 feet with it they end up putting the ruler down six times. This image of speed is completely additive and does not involve time as an explicit quantity. In fact, an image of time can only be constructed by iterating speed-distances, and constructed time ends up being a ratio for those children who can hold total distance and speed-distance in multiplicative comparison.

Figure 2 is our attempt to capture children’s images of speed at this level. The heavy line segment is mean to suggest that distance is prominent in the child’s awareness. Putting the time-unit in parentheses is mean to suggest that time is implicit in the speed-distance. It is how much time it takes to go that distance. Image 1 of speed is a pre-ratio image.

Figure 2. Unit of time is implicit in speed-distance. Image of accrual is that of putting out speed-distances.



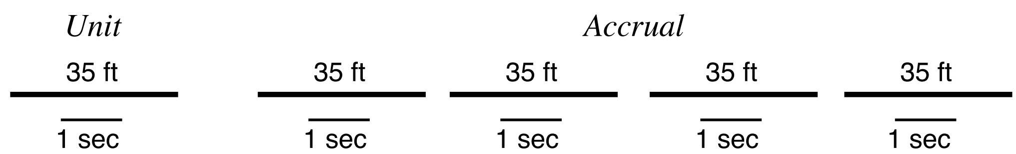
Speed as a distance . Student is thinking only of distance. Time is implicit—how long it takes you to go that far.

Image of accrued distance. Distance accrued discretely and additively (in jumps or part of a jump). Time is implicit in number of jumps.

Image 2: Speed-distances accrue; time follows

Through the repetitive experience of determining the amount of time it takes to travel some distance at some speed, students begin to differentiate speed so as to produce accumulated distance and accumulated time as two separate quantities. They come to anticipate that traveling a number of speed-distances produces an identical number of time-units. However, distance is still predominant in their awareness. That is, thinking of a number of time-units does not automatically evoke an image of an identical number of speed-distances. We have depicted this image in Figure 3. Image 2 is the earliest case that can support a ratio-image of speed, but the student must first reconceive a completed trip in its totality. While imagining the trip happening, the student is not comparing the accumulations of distance and time. Also, there are no signs of proportionality in this image, except perhaps within a unit—some fraction of a speed-distance implies traveling for the same fraction of a time-unit. This early sign of proportional reasoning (e.g., half of a speed-distance takes half of a time unit) could be based on Image 1 as well, but it would be more of a nominal association than a sign of proportion.

Figure 3. Unit of time is explicit, but still dependent upon first imagining having gone a speed-distance. Time accrues explicitly, but accrues as a result of putting out speed-distances.



Speed as a distance moved in one time-unit. Time is explicit in student's awareness, but distance is prominent. You move that distance; time happens because of moving

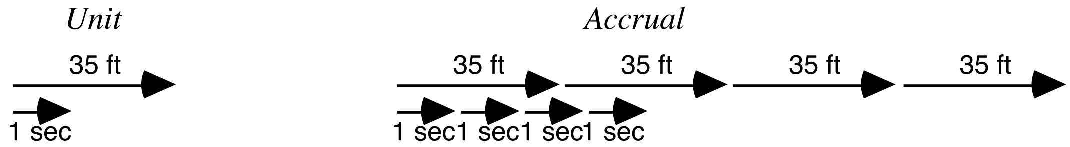
Image of accrued distance. Time is explicit in student's awareness. Distance accrued discretely and additively (in jumps or part of a jump), and time accrued discretely (each completed speed-distance made one additional time-unit). Total accumulation is not explicit in student's awareness.

Image 3: Distance and time accrue simultaneously

Image 3, the image that distance and time accrue continuously and simultaneosly as an object moves, is the first occasion wherein students reason about speed with the seeds of multiplicatively. The multiplicativity in their reasoning is not numerical. Rather, it is logical—distance and duration co-occur (Piaget, 1970; Piaget & Inhelder, 1969). Numerically, distance and time still accrue additively, as in previous images. Students operating within this image can find occassions to stand back and view completed distance and completed duration in multiplicative comparison, but the total accumulations are not part of their image. Only accrual is part of their image of motion. We have attempted to capture these aspects of simultaneous accrual in Figure 4.

Note that Image 3 provides the first occasion where students can view speed as involving a constant ratio. The constancy is in the simultaneous accrual of the same amount of distance and same amount of time in each imagined iteration of the distance-time accrual. However, this image does not support an awareness that total distance and total time stand constantly in the same multiplicative comparison as do the distance-speed and time-unit.

Figure 4. Distance and time accrue simultaneously and continuously, but still additively. Each new time-unit, or part thereof, corresponds to one speed-distance, or proportinal part thereof. Similarly, each new speed-distance, or part thereof, corresponds to one new time-unit, or proportional part thereof.



Distance and time accrue simultaneously. Moving for one time-unit implies moving one speed-distance; moving through one speed-distance implies moving for one time-unit. Also, having moved one speed-distance implies having moved for one time unit; having moved for one time unit implies having moved one speed-distance.

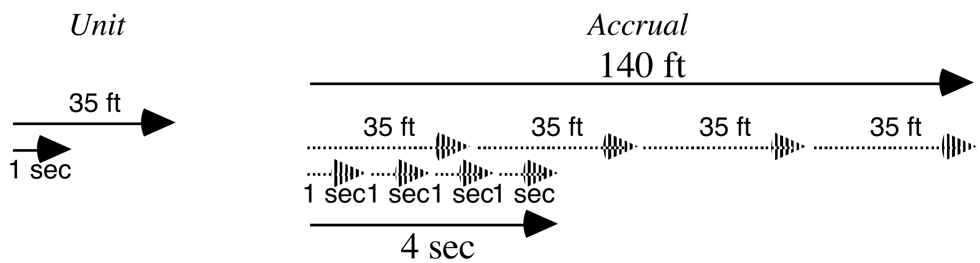
Image of accrued distance. Distance and time accrued simultaneously and additively. Moving for some amount of time automatically implies having moved some distance. Moving over some distance automatically implies having moved for some amount of time. Total accumulation is not explicit in student's awareness.

It is possible for students having Image 3 to solve problems about at what speed one must travel to go 100 feet in 6 seconds, but the reasoning will be elaborate. First, they must imagine that the 100 feet was made in six jumps, thus reconstituting 100 feet as being cut up into six parts. Then they might call upon a previous image (from schooling) that "cut up means divide." This reasoning was suggested to us by a student who drew such a picture and, after having done the numerical division on a calculator, reported, "You have to go 16.666667 feet." Note that she reported her answer as a number of feet.

Image 4: Speed as a rate

The first image of speed that supports the kinds of reasoning that we (as mathematics educators) see as being involved in a mature understanding of speed is an image that always entails two levels of simultaneity: the simultaneity of accruals and the simultaneity of total accumulations during accruals. We have tried to capture this image in Figure 5. The image of total accumulations of both distance and time growing simultaneosly with accruals of each is similar to the notion of progressive integration in children's counting (Steffe, von Glasersfeld, Richards, & Cobb, 1983), but with the exception that the integration is continuous and it simultaneously involves two quantities.

Figure 5. Speed as a rate. Distance and time accrue simultaneously and continuously, and accruals of quantities stand in the same proportional relationship with their respective total accumulations. This image supports proportional correspondence—$\frac{a}{b}$ ths of one accumulation corresponds to $\frac{a}{b}$ ths of the other accumulation.



Distance and time accrue simultaneously and in proportional correspondence. Speed-distance is made while moving for 1 time-unit. Moving for one time-unit implies moving one speed-distance. Similarly, having moved one speed distance implies having moved for one time-unit, and vice versa.

Distance and time accrued simultaneously, continuously, and in direct proportional correspondence. Each speed-distance is a fractional part of the total accrued distance. Each time-unit is a fractional part of the total accrued time. Fractional parts of completed time and completed distance are in direct proportional correspondence.

Here are some indications that a student is reasoning with a scheme of operations based on an image like Image 4:

a How fast do you need to go in order to go 100 feet in 6 seconds? *Let's see … one second is one-sixth of the time, so he will go one-sixth of the distance in one second.*

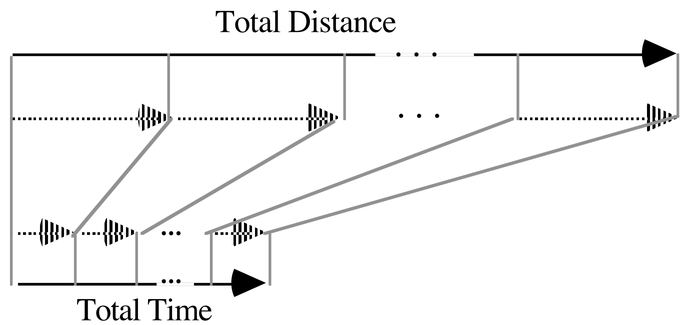
b How fast do you need to go in order to go 100 feet in 6.27 seconds? *You are breaking the total time into 6.27 pieces, so you are breaking 100 feet into 6.27 pieces. Each piece of distance will be 100÷6.27 feet long, so you will go that many feet each second.*

c It is 110 miles from Jim's office in San Diego to his company's office in Los Angeles. Jim traveled at 22 miles per hour from San Diego to Los Angeles. How fast must he return so that his average speed for the round trip is 45 miles per hour?" *Well, he needs to go 220 miles altogether, and if he is going to average 45 miles per hour he needs to take some number of hours altogether (220/45.0=4.9 hours). His total time on the round trip will be his time going up together with his time coming back. But he took 5 hours just to go from San Diego to Los Angeles, so he can't do it.*

A more refined image of speed as a rate involves the general, numerically nonspecific relationships among accruals and total accumulations. We try to capture this image in Figure 6.

Figure 6 is probably a more appropriate depiction of the kind of image upon which the above examples might be based.

Figure 6. A more general depiction of Image 4. The image implies that each accrual of distance and time extends their respective total accumulations and at the same time imposes a segmentation of them.



Students who reason with the aid of something like Image 4 have many avenues by which to approach problematic situations involving speed. They may reason about rates of accrual, they may reason about comparisons between total accumulations, or they may use one to represent the other, all with the operational security that everything is proportional so it doesn't really matter how they think about it. For instance, in example (c) given on the previous page, a student might reason that Jim cannot average 45 miles per hour because at a speed of $\frac{45}{2}$ miles per hour Jim will use up all of his time going one way. Traveling at a speed that is half of 45 miles per hour takes twice as much time as traveling at 45 miles per hour, and traveling both ways at 45 miles per hour uses up all of the time.

We hasten to add that students are not always so kind as to adhere coherently to one of these images of speed. Other factors often intervene, one of them being beliefs developed through past experiences of schooling. For instance, on one occasion we asked students to time one of us (PT) while walking at a steady pace for 20 feet (it took 7 seconds), and to decide at what speed PT walked. One student insisted that PT sped up and slowed down, and maintained his insistence even after watching a repeat performance. His reasoning was that PT must have sped up and slowed down because 7 does not go into 20 evenly. This student might have reasoned from the basis of an image of speed like Image 1. Having come to know this student well, it is also quite possible that he accommodated a more advanced image to his strong belief that the numerical operation of division must always produce whole numbers.

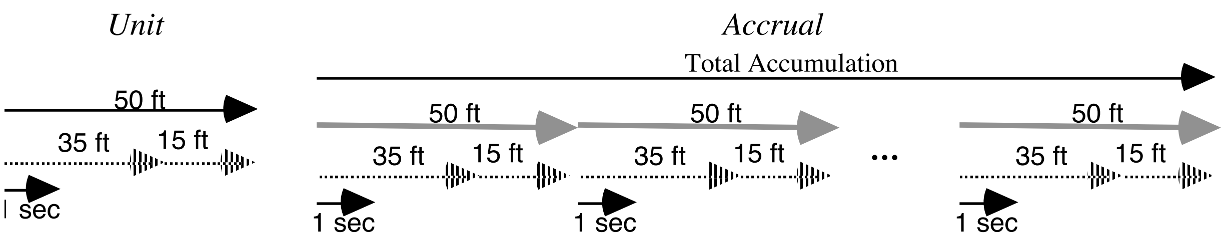
Extensions and Generalizations

We will extend our depiction of images of speed to images of combined speeds and to images of acceleration. This extension is speculative[9] and we do not have data on it. We will also generalize our discussion of speed to discussions of other rate-like phenomena, with particular attention being given to the area of ratio-as-measure.

Combined rates

The notion of combined rates (e.g., the stereotypical “upstream/downstream” problems and “work” problems in algebra) appears to be problematic for a fairly simple reason: the accrual needs to be reconceptualized as a combination of accruals, and the accumulation of the combined accruals needs to be seen in relation to total accumulation (Figure 7). We have omitted some detail in Figure 7. With all pertinent details it would look like Figure 6 except with the added layer of the combined quantities.

Figure 7. Combined rates—Combined speed-distances accrues with each time unit.



With work problems (e.g., “Jim can mow a lawn in 3 hours. Tom can mow the same lawn in 2 hours. How long will it take them to mow this lawn if they start at different ends and mow at the same time?”) there is the added complexity that they must draw an inference that mowing can be measured in hours per lawn or lawns per hour. The connection between these measures is the quantitative operation that we have called “inverting a rate” (Thompson, 1990). Other formal processes could also come into play during a student’s reasoning (e.g. distributivity, substitution

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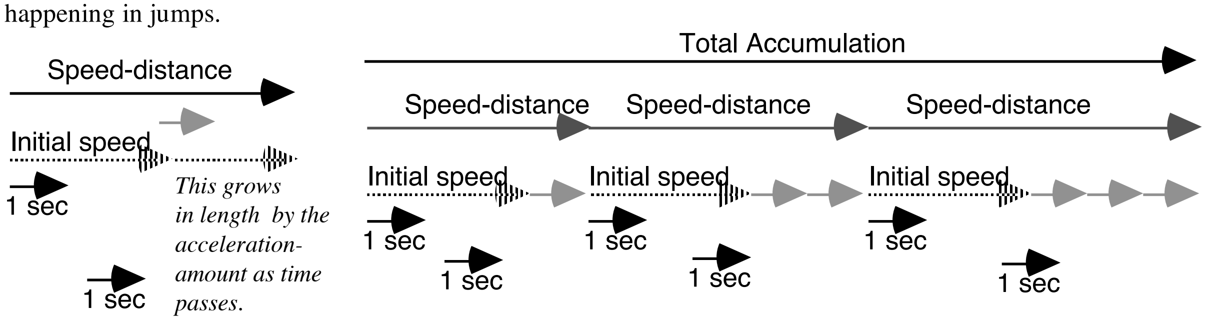
[9] As if the preceding is not speculative.

of variable) which would affect the actual chain of reasoning he or she produces, but we feel such reasoning will be guided by an image like the one we have depicted.

Acceleration

An image of acceleration is that "speed grows with time." We have depicted this image in Figure 8. The quantification of acceleration is the determination of by how much the speed-distance grows with each passing unit of time. The complication that acceleration introduces in students' comprehension of situations is not so much in the accrual as in imagining the accumulation.

Figure 8. Acceleration—the rate at which the speed-distance per time-unit grows. Image is of acceleration happening in jumps.



We depicted the accumulation in Figure 8 as happening only in whole-increments of time. This depiction seems justified to us not as an accurate portrayal of the most sophisticated understanding of acceleration, but as an intermediate image that becomes refined through formal processes typically developed in studying calculus. Below we relate one seventh-grade student's work on a problem having to do with acceleration that is consistent with the image depicted in Figure 8. This student could reason about speed in a way that suggested to us that she possessed a scheme of operations entailed in Image 4. She had completed a unit that focused on developing an understanding of speed consistent with our depiction of Image 4 (Thompson, in press-a).

Excerpt 1.

Pat: Imagine this. I'm driving my car at 50 mi/hr. I speed up smoothly to 60 mi/hr, and it takes me one hour to do it. About how far did I go in that hour?

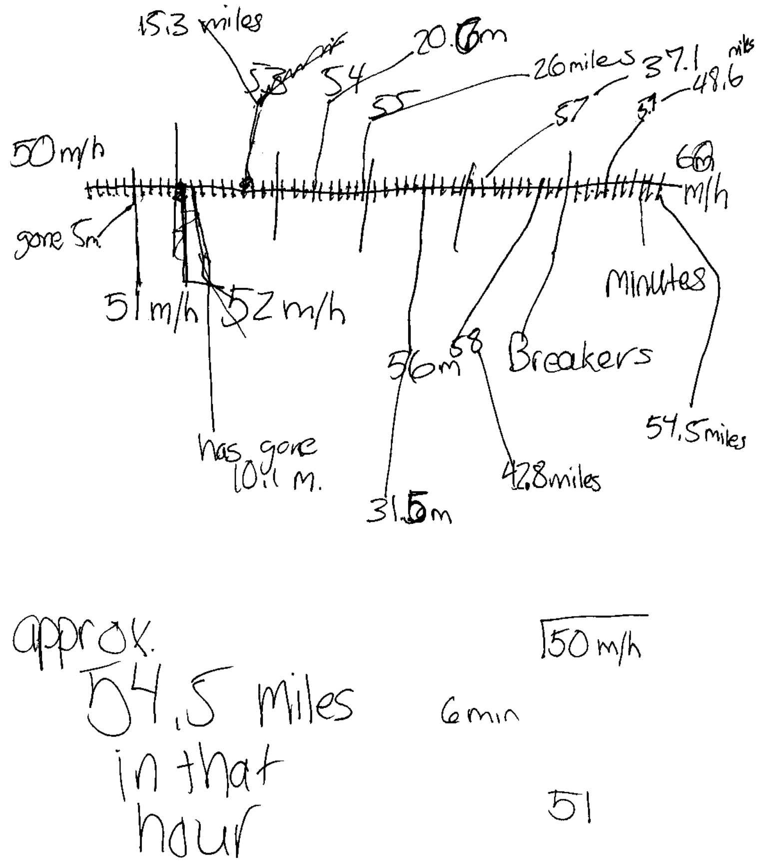
S: (*Long pause. Begins drawing a number line.*)

Pat: What are you doing?

S: I figure that if you speed up 10 miles per hour in one hour, that you speeded up 1 mile per hour every 6 minutes. So I'll figure how far you went in each of those six minutes and then add them up. (*See Figure 9.*)

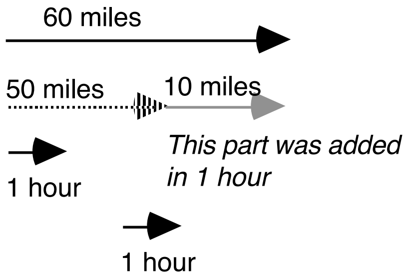
Figure 9 shows this student's work. She assumed Pat went 50 mi/hr for 6 minutes (one-tenth of an hour), then 51 mi/hr for 6 minutes, etc., and determined how far Pat would go in each one-tenth hour at the assumed speed for that time interval.

Figure 9. Student's scratch work for "How far did I go when I took one hour to speed up from 50 mi/hr to 60 mi/hr?"



This student's solution to estimating the distance traveled while accelerating has the structure of a Riemann sum. What we wish to focus upon, however, is her initial inference that Pat's speed increased by one mile per hour every one-tenth of an hour. This seems to us to be the crucial inference that got her going, and this inference seems to be based on an image of total acceleration like that shown in Figure 10.

Figure 10. A student's image of increasing a car's speed from 50 miles per hour to 60 miles per hour as being the result of increasing the speed-distance by 10 miles at a uniform rate of 1 mile every one-tenth of an hour.



Her inference was that since 10 mi/hr was added to Pat's speed in one hour, this was the same as adding 1 mile per hour to Pat's speed every one-tenth of an hour.[10] We see this as suggesting the intermediate image depicted in Figures 8 and 10 as falling between a concept of speed and the concept of continuously accelerated speed.

Ratio as measure

A number of studies have examined contexts that involve ratios as measures of attributes (see Tourniaire and Pulos (1985) for a review) but two recent studies in particular have focused on this idea explicitly. One study is by Harel, reported in this session. The other is a study by Simon and Blume (1992). Each raises interesting questions about ideas of ratio and its relationship to rate. We believe that the framework outlined here gives one entry into understanding the phenomena they have reported.

Harel asked a group of middle-school students whether a large amount of orange juice would taste the same as a small amount of orange juice from the same container. We were all surprised to hear that many students thought that they would not taste the same. Harel relates their explanations, which even when they involve ideas of ratio do not involve the idea of constancy of ratio across amounts of mixture.

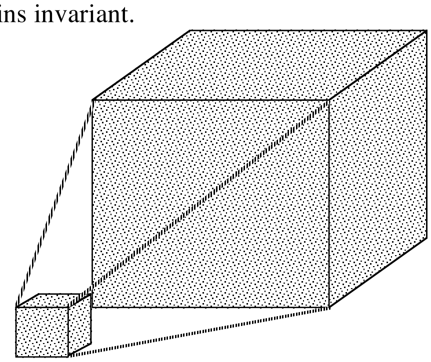
What images might support the obvious (to us) answer that, if shaken well and sampled at the same time, size of container should not make a difference in taste of mixture? Rather than

[10] In a later problem, this student, in the context of answering the question "about how far does a rock fall on the moon in its fourth second of falling if on the moon falling things speed up at the rate of 6 ft/sec every second" concluded that at the beginning of the fourth second the rock would be falling 18 ft/sec, and that each one-tenth of a second thereafter the rock would speed up by 0.6 ft/sec.

answer that question directly, we will first consider an analogy between mixture, concentration, and density.

The idea of mixture, or emulsion, seems to involve the idea of particles floating in a fluid where the particles are distributed uniformly throughout the fluid (i.e., spaced evenly). Even though we might know that the pulp is floating around, we might imagine it in a stationary state. A measure of "oranginess," or taste, or concentration, would be the ratio of mass of pulp to mass of fluid in that amount of the mixture that goes into our mouth. Figure 11 depicts an image that might lead to the conclusion that the amount we put into our mouth should not matter (excepting for orders of magnitude differences, like buckets versus individual molecules) in regard to the mixture's taste. The notion we are trying to capture is that this is a rate-like transformation, wherein all accruals of amount of pulp and volume remain in constant ratio throughout the transformation, and all total accumulations of amount of pulp and volume remain in constant ratio. When viewed this way, the similarities between images of speed and images of concentration both have definite rate-like characteristics. The difference between the two is that the quantities that accrue, and the notions of accumulation, are more complex in the case of concentration than in the case of accumulation of distance and time. Volume itself must be well established as a quantity, which is a nontrivial accomplishment for students. Also, in the case of concentration, time does not play an explicit role, although the metaphor of motion that is present in imagining an enlargement does give the transformation a temporal characteristic.

Figure 11. An image of a smooth enlargement (or shrinking) of a cube so that the concentration of orange pulp remains invariant.



Simon and Blume (1992) asked prospective elementary teachers how they might measure which of two fields viewed from an airplane might be “more square” and which of two ski slopes might be “more steep.” Many of the proposed measures were additively-based (“find the difference in their side lengths” for squareness; “find the difference in how much it drops and how far it goes forward” for a ski slope). However, even those students who thought that a ratio of two component quantities might be an appropriate measure of squareness or steepness questioned whether any number you computed “really meant anything.”

The idea of measuring squareness or steepness clearly involves the geometric notion of similarity. We suspect that images that support similarity (e.g., dilation) are constituted by the same sorts of operations that constitute a concept of rate. When we imagine an object changing size but retaining the same shape, how do we imagine the object adjusting incrementally so that we have the confidence that shape is retained? It seems that the constraints by which we adjust our images are those that constrain a rate-like understanding of motion: accruals in the same proportion among linear dimensions, which entails the result that accumulated accruals remain in constant proportion. That is, internal relationships among linear dimensions change in a rate-like relation to one another.

At a recent meeting of science educators we heard the lament that once students understand ratio they can understand science. Now, to us it became evident that what they meant was what we call ratio-as-measure (e.g., concentration, density, etc.), and we now feel strongly that ratio-as-measure is the idea of rate in disguise. We end with a call for others to examine these and other areas of mathematics and science concepts for possible linkages to fundamental images of rate.

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