Quantitative Reasoning, Complexity, and Additive Structures*

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Running Head: Complexity and Additive Structures

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures.
Educational Studies in Mathematics, *25*(3), 165–208.

Abstract

Six fifth-grade children participated in a four-day teaching experiment on complex additively-structured problems, which was followed by in-depth interviews of individual children. The teaching experiment was meant to investigate children's difficulties in holding in mind at once situations in which one or more items played multiple roles. Two important difficulties were identified: (1) distinguishing between "difference" as the result of subtracting and "difference" as the amount by which one quantity exceeded another; and (2) indirect evaluation of an additive comparison. Sources of these difficulties, along with pedagogical and curricular recommendations for addressing them, are discussed.

Quantitative reasoning is the analysis of a situation into a quantitative structure--a network of quantities and quantitative relationships (P. Thompson, 1989, in press). A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities. In that regard, quantitative reasoning bears a strong resemblance to the kind of reasoning customarily emphasized in algebra instruction.

A common application of algebra is to model complex situations. Many problems in algebra texts can be solved arithmetically, but algebraic techniques are taught as a method for handling complex relationships within a situation. Thus, one area needing research is children's emerging abilities to deal with complexity in situations and to deal with complexity in descriptions of situations.

There are two distinct sources of complexity. A situation can be complex because it requires a person to possess sophisticated conceptual structures in order to constitute it, such as the concept of rate (Thompson, in press; 1992). Also, a situation can be complex because it requires a person to keep multiple relationships in mind in order to constitute it. Of course, these two sources can be confounded—a situation can be complex because both of these requirements must be met in order for a person to constitute it.

Before proceeding I need to settle some matters of terminology. I give special, specific meanings to the words "quantity," "complexity," and "difference." A quantity is not the same as a number. A person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it (Thompson, 1989, in press). Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height, and the amount by which one of you is taller than the other without having to know the actual values. Quantities are more concrete than numbers.

By a "relationally complex situation" I mean one that is conceived as involving at least six quantities and three quantitative operations.[1] Also, I use "quantitative difference" to mean a quantity constituted by an additive comparison of two quantities. Comparing two quantities with the intent to find the excess of one against the other is a quantitative operation. The quantitative difference of two quantities, the amount by which one quantity exceeds the other, is the result of comparing them additively. I use "numerical difference" to mean the result of subtracting. Quantitative difference and numerical difference are not synonymous. A quantitative difference is not always evaluated by subtraction, and subtraction may be used to evaluate quantities other than a quantitative difference. The importance of distinguishing between quantitative and numerical difference will be evident later in this paper.

The site for this teaching experiment—additively structured situations—was chosen as a point of departure from other studies of addition and subtraction. Previous studies have focused on the development of numbers and numerical operations (Steffe, von Glasersfeld, Richards, & Cobb, 1984; Steffe & Cobb, 1988), children's addition and subtraction problem-solving strategies (Carpenter, 1985; Carpenter & Moser, 1982, 1983; Steffe, Thompson, & Richards, 1982), on children's conceptions of additive operations (Greeno, Riley, & Gelman, 1984; Janvier, 1983; Vergnaud, 1982), and generalizations of numerical operations to integers and integer operations (Dreyfus & Thompson, 1985; Thompson & Dreyfus, 1988; Vergnaud, 1982). While many of these studies acknowledged quantitative features of the settings in which children became engaged, the predominant focus was on the development of numerical operations with diminished attention having been given to children's constitution of situations in terms of quantities and relationships.

The primary purpose of this teaching experiment was twofold: (1) to investigate children's abilities to deal with relational complexity in situations and their descriptions, and (2) to investigate conceptions of difference as a quantitative structure. The two purposes are highly related, in that conceptions of difference cannot be investigated thoroughly in simple situations.

A secondary purpose of the teaching experiment was to familiarize children with the expectation that a primary instructional task was to discuss situations, understandings of situations, and methods that follow from understandings. Our previous experience with middle-school children, and recent research by Paul Cobb and his associates (Cobb, Yackel, & Wood, 1989) suggested that these children would be oriented to isolating and memorizing answer-getting procedures. I needed to change their expectations of what constituted legitimate instructional activities, and make clear our expectations of their conduct (listen to each other, discuss ideas, discuss methods, etc.) and our expectations of what they were to learn (concepts, relationships, methods, generalizations, etc.).

Children

Twenty-five children in an intact 5th-grade class were tested on complex additively-structured word problems. Items were scored according to the extent with which children dealt with the quantitative structure of the described situations: Appropriate Structure/Correct

Arithmetic, Appropriate Structure/Incorrect Arithmetic, Incorrect but relevant structure, and No structure. Four children were selected from this group: Two children with mid-range scores and two children with low scores. Two children (both mid-range on pre-test instrument) were included from a second fifth-grade class. I should be note that none of the tested children scored well on the pre-test.

The six children will be referred to in this report as Jill, Liz, Molly, Peter, Bob, and Don. They were three boys and three girls. Children were selected to provide a variety of levels in conceptual sophistication and school mathematics performance. Children's performance levels are given in Table 1.

School Mathematics Performance	Pretest Performance: *High*	Pretest Performance: *Medium*	Pretest Performance: *Low*
High		Jill Molly	
Medium		Peter Bob	Liz
Low			Don

Table 1. Children's performance levels.

The teaching experiment was held in a room on the campus of Illinois State University. Project staff collected the children immediately after school and brought them to the meeting room. Each session lasted 50 minutes, from 3:10 to 4:00 p.m. The sessions were held from October 16 through October 19, 1989.

Day 1

I explained to the children what the purpose of the project was. I stated that I wanted them to focus on good reasoning more than right answers, and I would remind them about this often. I also mentioned that they were to discuss each others' ideas, and that to do this well they were going to have to "practice their listening skills."

The session centered around of this problem and four variations of it: "Tom, Fred and Rhoda combined their apples for a fruit stand. Fred and Rhoda together had 97 more apples than Tom. Rhoda had 17 apples. Tom had 25 apples. How many apples did Fred have?" Each variation was discussed in terms of:

- Describing the situation (in words and by drawing diagrams that captured information and relationships);

- How a situation was related to a previously described situation (with the exception of the first);
- Appropriate operations (justify choice based on understandings of relationships in the situation and on what was known about the situation)

At first, children wanted only to state the arithmetic they would do (e.g., “I would add 17 and 97 and subtract 25”), regardless of the point of a discussion. Also, they were not disposed to discuss each others remarks. Rather, they wanted to say what they would do. The teacher (PT) had to instruct them to listen to each other and to base their remarks on what was said, not just on what they wanted to say.

Several children persistently conceived “Fred and Rhoda had 97 more apples … ” as if the statement were “Fred and Rhoda had 97 apples.” This confusion has deep conceptual roots, which will be discussed in this section’s summary.

Another activity had the children attempt to draw diagrams that reflected as much information as possible about the situation. The children diagrams are reproduced in Figure 1.

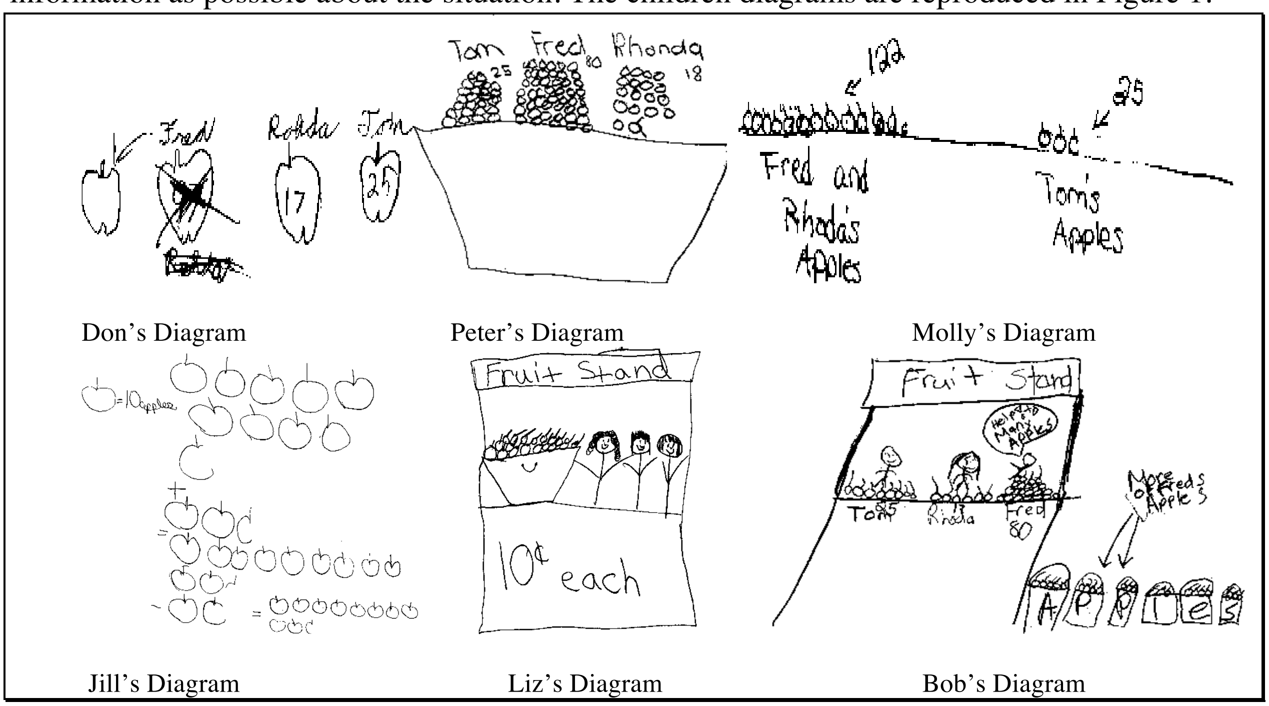


Figure 1. Children’s diagrams of Tom, Fred, and Rhoda setting.

Liz and Bob’s diagrams highlighted the social situation of apple-selling. Liz’s diagram reflects the barest of quantitative information—only that apples had been combined by the three children. Bob and Peter reflected their understanding that Fred and Rhoda together had 97 apples, and hence that Fred had 80 apples since Rhoda had 17. Jill’s diagram reflects the arithmetic she did to solve the problem. Jill's and Molly's diagrams are the only one that reflect any of the relational information presented in the text—that Fred and Rhoda together had 97 more apples than Tom—and even then they reflect their inference of how many apples Fred and Rhoda had together, instead of that Fred and Rhoda had 97 more than Tom.

PT: *Copies Bob's diagram.* How could we show this sentence … that Fred and Rhoda together had 97 more apples than Tom?

Liz: You could … have … Fred and Rhonda [sic] laughing at Tom.

Day 2

The second day of the teaching experiment was spent discussing the three problems listed below. The purpose of selecting these particular problems was to accentuate the notion of quantitative difference as a comparison of two quantities and the notion of numerical difference as being meaningful independently of knowing the values of the minuend and subtrahend.

Problem 1	Problem 2	Problem 3
Jim, Sue, and Tom played two games of marbles together. Sue won 6 marbles from Jim and 5 marbles from Tom. Jim won 3 marbles from Tom and 4 marbles from Sue. Tom won 12 marbles from Jim and 2 marbles from Sue.	Jim, Sue, and Tom played two games of marbles together. Sue won 6 marbles from Jim and 4 marbles from Tom. Jim won 5 marbles from Tom and 3 marbles from Sue. Tom won 2 marbles from Sue, and altogether he came out ahead 4 marbles.	Jim, Sue, and Tom played two games of marbles together. Sue won ___ marbles from Jim and ___ marbles from Tom. Jim won ___ marbles from Tom and ___ marbles from Sue. Tom won ___ marbles from Sue, and ___ marbles from Jim.
Compare the number of Tom's marbles before and after these two games.	How many marbles did Tom win from Jim?	Put numbers in the blanks so that, altogether, Sue came out *behind* by 3 marbles.

The problems were discussed individually. Problem 2 was presented after discussing problem 1; problem 3 was presented after discussing problem 2.

<u>Problem 1</u>

The format of Day 2 followed that of Day 1. First, we discussed the situation—how the game of marbles is played and what it means to win.

Children thought at first that they needed to know how much each person started with to make the comparison that was asked for. One child suggested that we could add up all the "wins" for each child and that would tell us how much they started with. In response, PT drew an analogy: "Suppose I play one game with Molly, and I win two marbles from her. Does that mean that I have two marbles?" Don responded "No, you have two more than you had." Jill said, "And Molly has two less than she had."

PT directed them to take the story one sentence at a time and to organize the information in a table. PT suggested summarizing the status of each player whenever possible. The table was constructed at the blackboard, and the construction was done through group discussion. The table they ended with is shown in Figure 2. The use of “minus” and “plus” in Figure 2 was done at the initiative of the children.

	Jim	Tom	Sue
	minus 6	minus 5	plus 11
	plus 7	minus 3	minus 4
so far →	plus 1	minus 8	plus 7
	minus 12	plus 14	minus 2
at end →	minus 11	plus 6	plus 5

Figure 2: Group’s table for Problem 1 (Day 2)

The children thought momentarily that when Jim lost 12 marbles (line 4 of Figure 2) the next summary should show that he had zero marbles, “because he had one and he lost 12, and you can’t take away 12 from 1.” Also, several children insisted that, at the end, we should say “Tom had six marbles.” Both positions show again these children’s common confusion of relative change and absolute amount.

Problem 2

Children were asked to compare Problem 2 with Problem 1 with the aim of describing how they were different and how they were the same. There was a quick consensus that the two situations were the same, that the question was different, and that the same method could be used for Problem 2 as was used for Problem 1. We filled out the table as far as we could, then figured how many Tom had to win from Jim for Tom to come out ahead by 4.

Problem 3

Problem 3 was presented with no teacher-led discussion. PT asked only, “What do you think about this one?” Children’s comments were: “We can put in anything we want!” “It’s a guess and check.” “Is there more than one way to do it?”

They were left to do it on their own. When finished, PT asked Peter to put his table on the board (Figure 3). PT asked, “Do you see anything that isn’t quite right?” Jill responded after awhile, “If Sue came out minus three, then someone had to win them from her, and nobody won.” A long discussion ensued, led by Jill, about the need for vertical *and* horizontal consistency, and what general principles apply to any line in the table.

Jim	Tom	Sue
+ 6	− 1	− 5
− 9	− 3	+ 3
− 0	+ 2	− 2
− 3	− 2	− 4

Figure 3: Peter’s table for Problem 3 (Day 2)

Day 3

The third day of the teaching experiment was spent discussing the three problems listed below. These problems were selected for the purpose of analyzing complex descriptions of situations, and highlighted the issue of relational structure of the situation. It turns out that any number can be “put in the blank” in Problem 2, and it turns out that no number can be “put in the blank” in Problem 3. These problems were given to children at the end of Day 2 so that they could think about them overnight.

Problem 1

Metcalf has two third grade rooms (C and D) and two fourth grade rooms (E and F). Together, rooms E and F have 46 children. Room C has 6 more children than room F. Room D has 2 fewer children than room E. Room F has 22 children. How many children are there altogether in rooms C and D?

Problem 2

Metcalf has two third grade rooms (C and D) and two fourth grade rooms (E and F). Together, rooms E and F have 46 children. Rooms C and D have 50 children together. Room C has 6 more children than room F. Room D has 2 fewer children than room E. There are __ children in Room E.

What number or numbers can go in the blank so that everything works out?

Problem 3

Metcalf has two third grade rooms (C and D) and two fourth grade rooms (E and F). Together, rooms E and F have 46 children. Rooms C and D have 48 children together. Room C has 6 more children than room F. Room D has 2 fewer children than room E. There are __ children in Room E.

What number or numbers can go in the blank so that everything works out?

Problem 1

The session began with Peter explaining his solution to Problem 1, saying why he decided to do whatever he did. Figure 4 shows the diagram Peter drew on the blackboard. His explanation was that since F had 22 and E and F together had to be 46, then E had to be 24. Since C had six more children than F, C had to be 28. Since D had two fewer than E, D had to be 22. So C and D together had 50 children.

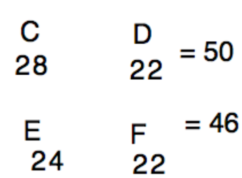


Figure 4. Peter’s diagram for Problem 1 (Day 3)

Afterward, Peter asked, “Is it right?” PT asked Peter, “Suppose everyone in the room were to say that you are wrong. Would you think you are wrong?” Peter answered, “No. But I still want to know if it’s right.”

Problem 2

Molly put a diagram on the blackboard to show her solution to Problem 2 (Figure 5). Peter asked, “What does ‘F+6=C’ mean?” Molly explained that it meant that however many people are in F, plus 6 more people is however many there are in room C.”[2]

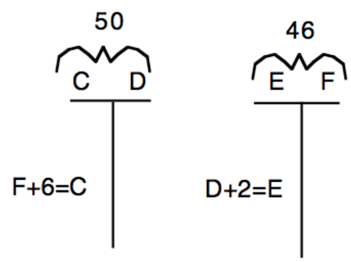


Figure 5. Molly’s diagram for Problem 2 (Day 3)

Molly’s answer to the question “What number or numbers can go in the blank so that everything works out?” was “I sort of guessed that there are 24 in E (she then concluded by way of her stated relationships that D must be 22, F must be 22, and C must be 28).” Why did Molly guess that E must be 24? “Because that’s what the first problem said.” Don disagreed in principle with Molly’s solution. The ensuing discussion is given below.

Don: She’s wrong. Well, she’s right but she’s wrong. Where it says ‘what number or num<u>bers</u>’ I figured that any number from 2 to 46 would work.”

Jill: I tried 19 and it didn’t work.

PT: Molly, try 19. Don says it should work and Jill says it didn’t work for her.

Confusion over who is to work at board. Molly turns chalk over to Don. He starts with E being 19, and everything works out.

Don: It works. Well, all of 2 through 46 work.

PT: Really? Did you try them all?

Don: Well, yeah, kind of, in a way. I tried 2 and 46 and I tried a bunch in the middle, and they all worked.

PT: Would zero work?

Don: No, because room D has 2 less, and you can’t have minus 2 people in a room.

Liz: Try 2.

Don: Is it all right to have 0 in a classroom?

PT: How many people would be in this room if everyone walked out?

Don: Okay. *Goes on to finish.* Two works.

Jill: I want to try one. *She tries 20. It works.*

Peter: You could put in any numbers as long as they add up to 46 and 50.

Jill: / Molly: The numbers also have to be 2 less and 6 more. *Ensuing discussion about how the numbers have to satisfy all the constraints in the situation.*

PT: Bob, you try one.

Bob: *Tries 4 for the number of children in room E. It works.*

PT: Would 47 work?

Group: No--because E and F have to have 46 together. They can't have 46 if E by itself has 47.

<u>Problem 3</u>

Only Don had done any work on Problem 3. The others were directed to listen to Don and ask questions any time they didn't understand his reasoning. It is worthwhile to recall that Don was the child who scored very low on the pretest and to note that he was one of the poorest performers in his class over the past several years.

Don: I figured something out.

PT: Don, you explain. The rest of you listen and ask questions whenever you don't understand.

Don: There are 48 children [in C and D] instead of 50 [as previously]. Everything else is the same. C has 6 more than room F and D has 2 fewer children than room E, so I figure a difference of 4 children between E and F and C and D. Rooms C and D have to have 4 more children than E and F.

PT: [To rest.] You follow that?

Peter: I think he has to work it out the way Molly did.

PT: [To Don.] Try putting it on the board so the rest can follow you.

Don: *Writes F=C+6* [sic]. Room C has 6 more than room F. D has 2 fewer children than room E, so there's a difference of 4 children between E and F and C and D. Rooms C and D together have 4 more children than E and F together.

Peter: / Liz: I don't get it.

PT: Let me try. Let's draw these as pictures, the way we were doing before [Figure 6]. Now combine C and D and combine F and E. Put the "D and E" figure on top of the "C and F" figure. When E settles down, how much is this (C and D) going to stick up? [Figure 7.]

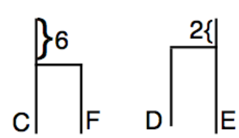


Figure 6. PT's diagram for Don's explanation of Problem 3 (Day 3)

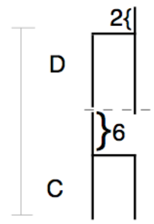


Figure 7. PT's transformation of Figure 6 in discussing the combination of C and D versus the combination of F and E.

Peter: Six.

PT: What will happen to E when D rests on top of C?

Several: E will slide down?

PT: Yes. E starts out sticking up 2 over D, and when it slides down to meet F, how will C and D compare against E and F?

Peter: }
Molly: } There will be 4 difference between C and D and E and F.

Molly: C and D will be up higher by 4.

PT: *Recaps process of combining C, D, E, and F.* Now let's look at the question. We just said that C and D together has to be 4 more than E and F together if we have this difference of 2 [between E and D] and this difference of 6 [between C and F].

Don: Can I show it up there [on overhead projector]? *Underlines 46 and 48.* There are 4 more in C and D than in E and F, but here there's a difference of 2 [between 46 and 48] so that means it would be impossible to do.

PT: *Recaps Don's argument.* Give Don a hand!

It was not clear if the other children followed Don's argument to the point of being able to reproduce it. Time was running out (parents were waiting to pick up their children). Before stopping, I hoped to get a sense of their feelings about the teaching experiments in relation to their school mathematics.

PT: Do you do these kinds of problems in math?

Group: No!!

Bob: Like four plus four.

PT: How are these different from the ones you do in math?

Student: They're story problems [in the teaching experiment].

Peter: Its the same, except these are hard story problems. If Don had 2 apples and Sam had 3 apples, how many apples would they have together?

PT: How would you describe the kinds of problems you do in math?

Several: Math problems are *borrrrring*.

PT: If things are harder, then usually they're not much fun.

Liz: They're funner!

Bob: They're more fun because it's more of a challenge.

Don: It takes more than 2 seconds.

PT: Do you like doing problems like these?

Group: Yeah!

Molly: They're a lot better than what we do in school.

Jill: In school we get 45 problems to do everyday.

Bob: They're so *easy.*

Day 4

The fourth day of the teaching experiment was intended to be spent discussing the three problems listed below. These problems were selected for the purpose of analyzing situations that involved a comparison of two comparisons. Problem 2 was meant to highlight quantitative difference as a structure; Problem 3 was meant to get at that notion even more pointedly.

Bob and Liz were absent from the Day 4 session.

Problem 1

Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B.

The two fellows argued about which one stood taller over his sister.

It turned out that Brother A won by 17 centimeters.

Brother A was 186 cm tall.

Sister A was 87 cm tall.

Brother B was 193 cm tall.

How tall was Sister B?

Problem 2

Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B.

The two fellows argued about which one stood taller over his sister.

It turned out that Brother A won by 17 centimeters.

Brother A was 186 cm tall.

Sister A was 87 cm tall.

Brother B was _____ cm tall.

Sister B was _____ cm tall.

Put numbers in the blanks so that everything works out.

Problem 3

Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B.

The two fellows argued about which one stood taller over his sister.

It turned out that Brother A won by 17 centimeters.

Brother A was _____ cm tall.

Sister A was _____ cm tall.

Brother B was _____ cm tall.

Sister B was _____ cm tall.

a) Put numbers in the blanks so that everything works out.

b) What has to be true about the numbers anyone puts in the blanks if everything is going to work out?

Problem 1

A review of the videotapes from Days 1 and 2 suggested that, with the exception of Don and possibly Jill, the notion of quantitative difference as a structure was quite fragile. As such, Problem 1 was not presented in its entirety. Instead, I displayed for discussion the first three paragraphs on an overhead projector. Peter read the statement aloud; PT said, "Okay, let's talk about what is going on here."

PT: What are they doing?

Several: Arguing.

PT: What are they arguing about?

Jill:
Peter: Who's taller.

Molly: Who's taller than Sister A and Sister B.

PT: Are they arguing about who's taller?

Several: No.

Don: Who was taller than their sisters.

PT: Who was taller than their sister?

Several: Yeah.

PT: Are they both taller than their sister?

Several: Yes.

Jill: Who was *more* taller.

PT: Who was more taller? What does that mean?

Peter: Like … here's Brother A and here's Brother B (borrowing pencils from others and holding them aside one another).

PT: How many pencils do you need?

Peter: Four. (Borrows one more pencil.) These two are brother and sister (holding up two pencils) and these two are brother and sister (holding up two more pencils in other hand).

PT: Show who's taller by sliding a pencil up.

Peter: (Does so.) … [inaudible] and he's taller than she is *pointing to a pair of pencils* and he's taller than she is *pointing to other pair* and they found out that Brother A won.

PT: Is that because Brother A was taller?

Peter: Yeah, Brother A was taller … no, Brother A was taller than his sister more than the other one was taller than his sister.

PT: Show a big winner.

Peter: *Adjusts pencils as shown in Figure 8; pencil bottoms are covered by Peter's hands.*

PT: Who can show me a winner where Brother A wins, but he is shorter than Brother B?

Peter: *Tries several configurations where Brother A is shorter than Brother B, but in each case Brother A does not win.*

Jill: I can do it. … Say it again?

PT: Make Brother A win, but make him so that he is shorter than Brother B.

Jill: (Many tries, but none is completed; ends with Figure 9; pencil bottoms are covered with paper at PT's suggestion.)

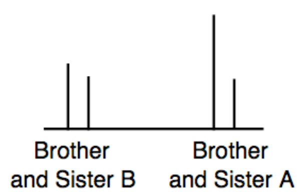


Figure 8. Peter's pencil representation of the two comparisons.

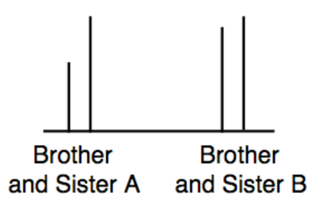


Figure 9. Jill's pencil representation.

PT: Is Brother A shorter than Brother B?

Jill: No. *Shortens Brother B and Sister B.*

Don: No -- now he's taller [Brother A is taller than Brother B.]

Jill: He's (Brother A) is supposed to be shorter?

Molly: Do it this way. (Slides Brother and Sister A down.] Now this one (Brother B) is taller than this one (Brother A), but this one (Brother A vs. Sister A) is more taller than this one (Brother B vs. Sister B).

Jill: Yeah.

PT: Okay. What are you showing us, Jill?

Jill: Even though Brother A is shorter than Brother B, he's still taller … than … ummm …

PT: What is it that we're comparing to see who wins?

Jill: How much … how … how taller each brother is than his sister.

PT: (To Don.) Is that good reasoning? Does it make sense?

Don: Yep.

The issue being forced in the above episode was that for a difference to be a quantity in and of itself, children must be able to conceive of it as being more than just the end result of comparing two quantities. They must be able to conceive of it both as the result of a comparison and as a quantity in relation to the quantities operated upon to get it. These children's difficulty in producing a depiction of Brother A as winner, but being shorter than Brother B, was that the two aspects of quantitative difference (result of a process and item in a structure) had to be coordinated simultaneously. This issue was revisited in Problem 2 (next section), and will be featured in the follow-up interviews and in the section discussion.

The fragility of these children's abilities to coordinate the two aspects of quantitative difference is shown in the subsequent discussions. The intent of this episode was to talk about how to read story problems so that you can make sense of them as you read, rather than trying to make sense of them after having read the story in its entirety.

As each sentence of Problem 1 was uncovered, the group discussed what they either knew or could figure out. Jill pointed out that the sentence, "Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B" did not really say which sister went with which brother. We discussed what assumptions we could make that were reasonable and that would let us proceed.

Upon uncovering the next sentence, "The two fellows argued about which one stood taller over his sister," we came upon another sign of how fragile the structural notion of quantitative difference was. It should be noted that the following excerpt occurred within three minutes of the pencil-depiction activity of the previous excerpt.

PT: What does this say? Don?

Don: They argued.

PT: What did they argue about? Who is going to win the World Series?

Don: No. They argued about which one was taller than his sister.

PT: Is that what they argued about? Which one was taller than his sister?

Peter: Which one stood over taller. Like, if they were the same size, or almost the same size, they could say, "Well, I'm taller than my sister." And the other could say, "Well, I'm taller than my sister … more than you.

PT: Okay. Compare the sentence there *#1: "The two fellows argued about which one stood taller over his sister"* with this one. *Writes #2: "They argued about which one stood taller than his sister."*

Jill: / Peter: / Molly: It's (*#2*) the same as the other [original] sentence.

PT: Is it the same?

Don: Same words.

PT: Oh, I meant this. *Changes #2 to "Which one was taller than his sister?"*

Don: Now they're the same.

PT: What kind of answer could you get from asking the question "Which one was taller than his sister?"

Peter: That one (revised #2) is better. That one you can more understand.

Jill: Wait … that one (#2) asks which one is taller. The other one (#1) asks which one is *more* taller.

Lengthy discussion of what it means for two differently-worded questions to be asking the same thing.

PT: *Uncovers the statement "It turned out that Brother A won by 17 centimeters."* What does this tell us?

Peter: That Brother A was 17 cm taller than his sister than Brother B was.

PT: Molly, how would you say that?

Molly: It answers the first question.

PT: Yes, it settles the argument. Can we say anything more?

Molly: It says that Brother A won by 17 cm?

PT: But does it tell you anything about them and their sisters?

Jill: Uh-huh (yes). It tells you that Brother A was 17 cm taller than his sister than Brother B was.

PT:\` *Hands out copies of full statement of Problem 1.* Let's do this. Draw a diagram that shows what's going on. Just put in essential details. You don't need to draw boys and girls. What is it that is *really* important? (*See Figure 10 for children's diagrams.*)

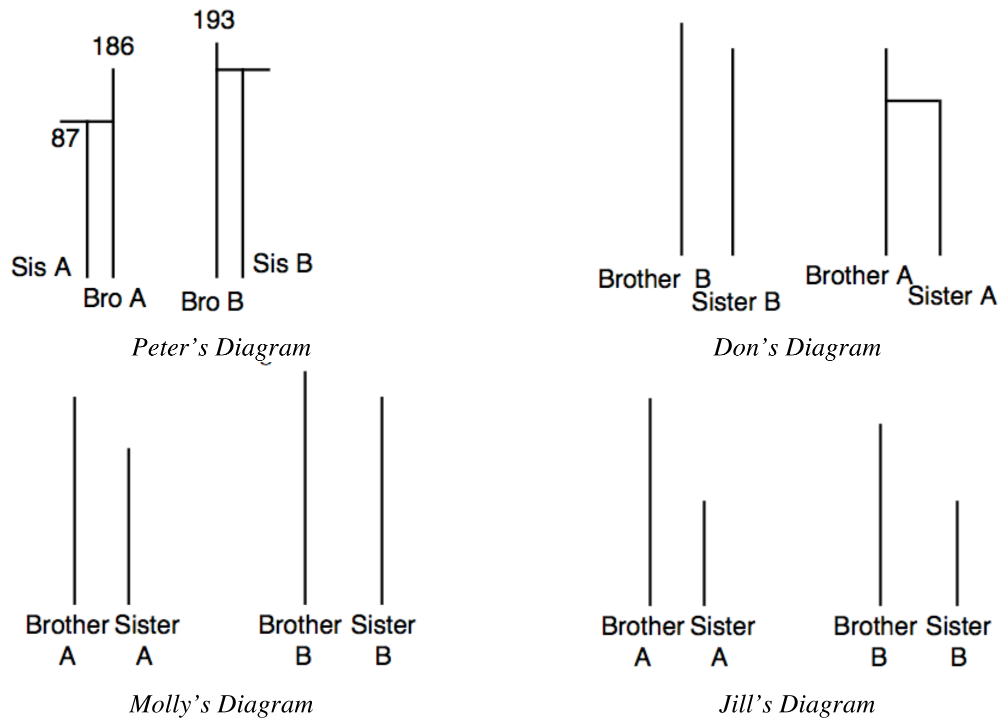


Figure 10. Children's initial diagrams for Brothers & Sisters problem

Peter: It says that Brother A won by 17 cm, so his sister is 17 cm shorter than he is.

Jill: No, Brother A won by 17 cm more than Brother B won.

Peter: Then he should be only 17 cm more than his sister.

Jill: See … Brother B is taller than his sister, too! And … say Brother B is 5 cm taller than Sister B, then Brother A is 21 cm taller than Sister A … I mean 22 cm taller. Because, he *won* by 17 cm.

Peter: I know he won by 17 cm. So he's 17 cm taller than Brother B.

Jill: He's taller than his *sister*.

Peter: Oh …

PT: I didn't hear how that was resolved. (To Peter) You were wondering if that means Brother A was 17 cm taller than his sister?

Peter: Yeah.

PT: Is that what it means?

Peter: No.

PT: What does it mean?

Peter: It means that he was 17 cm more than Brother B was than his sister.

PT: It's complicated to describe, isn't it? Do this. Try showing the 17 cm in your picture. (*See Figure 11 for children's modified diagrams.*)

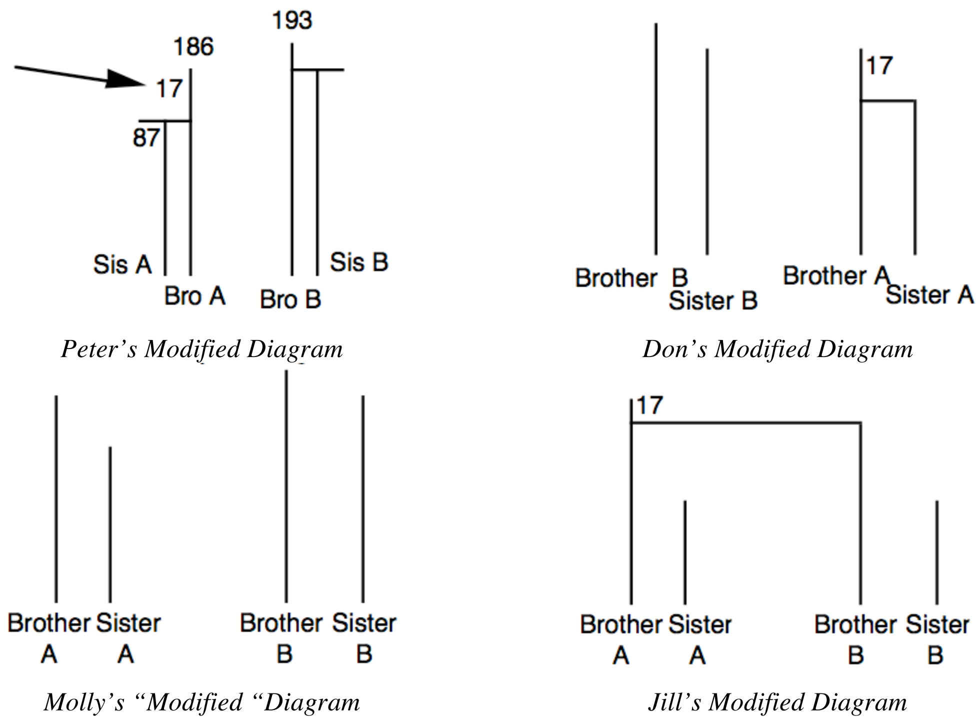


Figure 11. Children's placement of "17 cm" in their initial diagrams.

PT: Is this a tough one, Molly? Showing where the 17 is?
Don: I did it! [See Don's modified diagram.]
Jill: I don't think I have it.
PT: That's okay. Let me have your pictures. We'll do something else.

The fragility with which the children could identify 17 as resulting from a comparison of two differences is striking. One can see the "looseness" of the idea in the children's description of what 17 stood for: "Brother A was 17 cm taller than his sister than Brother B was." "Brother A was taller than his sister more than Brother B was taller than his sister." The fragility can also be seen in the difficulty children had in locating 17 in their diagrams—even after giving essentially correct descriptions of its meaning within this situation (e.g., Peter, Jill).

At one point, Jill actually instantiated the general comparison between the two differences ("If Brother B is 5 cm taller than his sister, then Brother A has to be 22 cm taller than his sister") yet she quickly lost this conception of the situation, and couldn't reflect that momentary understanding in her diagram. The discussion of Problem 2 will highlight a fundamental conceptual component to the notion of quantitative difference as a structure, a notion that was not well established in the children's thinking.

Problem 2

PT: Okay. Do this one as a team. Put numbers in the blanks so that everything works out.
Pause as children read the text.

Jill: … (inaudible) … Well, actually, Brother B has to be at least 17 cm shorter than Brother A.

Molly: But it says that Brother A won by 17 cm.

Jill: I know.

Molly: So that means …

Jill: So that means he (Brother B) has to be at least 17 cm shorter …

Molly: Yeah, at least.

Jill: No wait … no it doesn't, because … not it doesn't, because …

Molly: It says she could be shorter than him if the difference is small. *Pause.* But if Brother A won by 17 cm, that means that Sister B must be at least 17 cm taller than Sister A.

Jill: Uh-uh (no). If Brother B is shorter, then she can't be.
Pause.

Jill: Well we could put …

Molly: Like the last one we had.

Jill: Well, you could put that Brother B was 150 cm tall and Sister B was 149 cm tall.

Peter: How do you know that would work?

Molly: But Brother A still has to win by 17 cm.

Don: Oh, so Brother B … yeah would be 17 … oh no, he wouldn't … But he could be 17 cm shorter. He doesn't have to be, but he could be.
Pause.

Don: I don't think it matters what we put in there.

Jill: Ummm … Brother A had to win by 17 cm.

Don: (Quietly.) I don't think it matters.

Molly: }
Peter: } Yes it does. Brother A has to win by 17 cm.

Molly: As long as he wins by 17 cm, you could put anything in.

Peter: But what do you mean that he wins by 17 cm?

Molly: He has to be 17 cm taller than Sister A … than …

Jill: Brother B is than Sister B.

Peter: Oh, I know.
Pause.

PT: You have any numbers in there yet?

All: No.

PT: Good discussion! … Are you trying to figure out what the first number is?
Pause.

Jill: Wait. Lets find out the difference between Brother A and Sister A.

Don: It's easy.

Molly: It's 99.

Don: It's 98 … 99.

Molly: So, Sister B must be 17 cm … must be 99 minus 17.

Jill: She could be.

Molly: She could be, because he has to win by 17 cm.

Peter: 99 minus 17 … 83.

Don: It's 82.

Peter: She could be 82. But what would Brother B be?

Molly: So, she could be 82 cm tall. But she doesn't *have* to be 82 cm tall.

Jill: She could be 82 cm or lower.

Molly: He'd [she'd?] have to win by 17 cm …

Peter: Brother B?

Molly: Brother A.

Jill: But then Brother B can't be … we could make Brother B be 90 … we could make Brother B be the same thing.

Peter: You could make Brother B be 100.

Molly: No, you couldn't, because, then he'd be … I don't know.

PT: What does the 82 mean?

Jill: That's the difference between Sister A and Brother A.

PT: The 82 is?

Jill: } Don: } No, it isn't. 99 is.

Molly: But he had to win by 17 cm, so we took away 17 from 99 … So Brother B would have to be the same as Brother A. [To Jill] We could do it that way.

Jill: Uh-huh (yes) …

Peter: 186 …

Jill: Then Brother B would win.

Peter: No he wouldn't

Molly: Yes he would.

PT: Could I ask you to think about something … Try this. Think about what the 82 means.

Long pause.

Jill: *inaudible* … okay, so the difference between Brother A and Sister A and subtracted … uh … how much … he won … by (to Molly) …

Pause.

Molly: Huh?

Jill: … We took the difference between Brother A and Sister A. Does everybody understand that?

All: Yeah.

Jill: And then we took away how much Brother A won by.

All: Yeah.

Molly: Yes … yes … yes!

Peter: (Giggles at Molly.) So 186 …

PT: (To Jill) And you got 82, right? Now, what does the 82 stand for?

Pause.

Jill: Uhhhhh …

All: (Giggles.)

PT: Well, what does that 99 stand for?

Molly: How much the difference is … Oh, so the 82 could be the difference between Brother B and Sister B.

PT: Is that what it is?

Don: It could.

PT: Or is that how tall Sister B is?

Molly: It couldn't be how tall she is because then Brother B would win.

Peter: No it wouldn't. He could be 82 or more.

Jill: She has to be shorter than 90 cm … or taller …

Don: It doesn't matter.

Peter: This would work … 185 and 82.

PT: Would Brother A win by 17 cm, Peter?

Peter: … no.

Molly: He has to win by 17 cm, so you could have Sister B be 127 cm shorter than Sister A.

Peter: It could be 99. That would be 17 difference.

Pause.

PT: Let's think again. The 99 stands for what?

Jill: }
Don: } How much difference there is between Sister A and Brother A.

Molly: I know. I know. You can subtract 82 minus 186 and then you'd get 104. And 104 is 17 more than 87.

Pause.

Don: Huh?

PT: Explain it again, Molly. But instead of talking about the arithmetic, talk about what it is that you are thinking.

Molly: The first thing we did was the difference between Brother A and Sister A, and it was 99. And minus 17 is 82.

Jill: But what was the 82 for?

Molly: Huh?

Jill: What does the 82 stand for?

Molly: I don't know.

For ten more minutes the children occasionally flirted with the idea that 82 was the difference between Brother B and Sister B, but were never convinced of its necessity. They mostly worried about picking the "correct" height for Brother B, and several times returned to the notion that Brother A was 17 cm taller than Brother B, or that Sister B was 17 cm taller than Sister A—each time rejecting the idea because of the lack of any necessary height for the respective sibling.

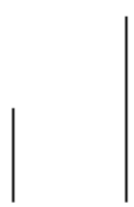
It appeared over and again that a major obstacle to the children's progress was the lack of coordination between two aspects of quantitative difference: Difference as the amount left over after a comparison (i.e., the result of a process) and quantitative difference as an item in a relational structure. Their conception of quantitative difference seemed predominantly to be "result of comparing." This seems to be a plausible explanation of why they had such difficulty identifying 82 (an *operand* of the comparison resulting in "won by 17 cm") as the measure of a quantity that itself resulted from a comparison.

Interviews

Each child was interviewed individually during the week following the teaching experiment. The interviews focused ultimately on children's ability to conceive of a quantitative difference independently of a process by which to evaluate it. As such, the interviews sought to take children as far as possible toward revealing such a conception.

The interview questions are given below (Table 2). Each question is discussed separately. Cross-sectional discussions of children's performance on each problem will highlight common features of their conceptions.

Interview Problem 1



Complete this picture to show a difference. Circle the part that is the difference.

Interview Problem 2

Jim had 34 more cents than Sally.
Sally had 23 more cents than Fred.
____ had 83 cents.
A
How many cents did ____ have?
B

Put a name in blank A and a name in blank B so that you end up adding to answer the question.

Interview Problem 3

Team 1 played a basketball game against Opponent 1.
Team 2 played a basketball game against Opponent 2.
The captains of Team 1 and Team 2 argued about which team won by more.
The captain of Team 2 won the argument by 8 points.
Team 1 scored 79 points.
Opponent 1 scored 48 points.
Team 2 scored 73 points.
How many points did Opponent 2 score?

Interview Problem 4

Team 1 played a basketball game against Opponent 1.
Team 2 played a basketball game against Opponent 2.
The captains of Team 1 and Team 2 argued about which team won by more.
The captain of Team 2 won the argument by 8 points.
Team 1 scored 79 points.
Opponent 1 scored 48 points.
Team 2 scored __ points.
Opponent 2 scored __ points.
Put numbers in the blanks so that everything works out.

Table 2. Problems used in post-teaching-experiment interviews

<u>Opening</u>

I opened the questioning by saying, "Last week I heard you use the word 'difference' a lot when we were working on problems. Sometimes it wasn't clear to me what you meant by that. Could you tell me what you mean by a difference?" Only one child (Jill) gave a description that was unambiguously quantitative and non-numerical. The rest made more or less direct reference to what you get from subtracting. Don attempted to use numbers as magnitudes, referring to their size, but spoke of the quantitative difference as "what's left over" after subtracting. The others shrugged their shoulders when asked if they could have a difference without numbers. This does not mean that they were incapable of non-numerical conceptions of a difference, it only means that it was not their predominant conception. Liz's answer was non-quantitative. Asked if she could have a difference without numbers, she said "Yeah. Like a black owl and a orange owl."

Interview Problem 1

The purpose of Problem 1 was to obtain information about their ability to identify a quantitative difference. None of the children had any difficulty identifying the excess of the longer line over the shorter as the quantitative difference between the two lines. This is not surprising, as this type of diagram was used many times during the teaching experiment.

Interview Problem 2

The intent of Problem 2 was to gain insight into how closely these children's tied the notion of quantitative difference to the result of a comparison as distinct from difference as the numerical result of subtraction.

I paid special attention to children's initial approach to the problem, and how they adjusted their effort after an inappropriate substitution. Each child, with the exception of Liz, initially substituted names in a way that resulted in subtraction being the operation used to answer the question. Liz's choice of names seemed to be entirely fortuitous. Her statements indicated that she was picking names randomly. Also, when asked why she added instead of subtracting to answer the question, Liz responded, "Because the question doesn't ask for a difference. It asks for how much **B** has." Only Jill adjusted to a structural analysis of the situation to find an appropriate substitution of names.

Jill: Jim had more than Sally, and Sally had more than Fred. So … well … um … [inaudible] more than Sally … Jim *… writes Jim in blank A …* and Sally. *Writes Sally in blank B.*

PT: Okay. Now, now pretend that you're just given this much with Jim and Sally's name in it. *Covers last sentence with hand.* Okay? Now answer the question. How many cents does Sally have? *Points to the question.*

Jill: Jim 83 cents, and he had 34 more than Sally. Then you take Jim, you take 83 minus 34. *Writes 83-34.*

PT: Okay. Yeah, you don't have to do the subtraction. But you would subtract. Right?

Jill: Yeah. *Nods "yes"*

PT: So do you end up adding to answer the question?

Jill: *Shakes head "no"* Well, you could put … *pause* … put Fred here, *pointing to blank A,* and Sally here, *pointing to blank B*, and, and Fred has less than Sally, so you'd take 83 plus 23, and you'd come up with, um, Sally.

Interview Problem 3

Problem 3 was structurally isomorphic to the Brothers A and B problem given during Day 4 of the teaching experiment, but with one change. Instead of subtracting to evaluate both subordinate differences, one must add to evaluate one of them. Also, in the Brothers A and B problem, distinguishing between the linguistic constructions "stood taller than his sister" and "stood taller over his sister" was at times subtle. Interview Problem 3 attempted to get at the same issue—a comparison of two differences—in a linguistically cleaner way. The interviews showed clearly that the difficulty was not primarily language. Rather, the primary source of difficulty was the fact that it was two differences being compared, and the evaluation of one of the differences (between Team 2's score and Opponent 2's score) must be done indirectly by combining two quantitative differences—the quantitative difference between Team 1's and Opponent 1's score with the super-ordinate quantitative difference.

The analysis of children's performance on Problem 3 is in two parts. First, I paid special attention to children understanding of what was being argued—of what the 8 stood for. If a child had difficulty conceiving the super-ordinate comparison, then I helped as indirectly as possible to sort out that issue so that I could attend to their understanding of what they found when they combined the quantitative difference between Team 1's and Opponent 1's scores with the super-ordinate quantitative difference.

In the interest of brevity, I have included an excerpt of only one child's interview in each part of the analysis. For the other children I have summarized their responses.

Problem 3 Analysis, Part 1: What was being argued?

Jill

Jill's initial understanding of the problem was that Team 2 beat Opponent 2 by 8 points [¶'s 1-4]. Her adjustment to the realization that they were arguing about *who won by more* was gradual [¶'s 4-8]. However, the notion that it was simply two amounts of points being compared was a hard one for her to discard.

1. PT: Could you read it for me? *Points to problem.*
2. Jill: Team 1 played a basketball game against Opponent 1. Team 2 played a basketball game against Opponent 2. The captains of Team 1 and Team 2 argued about which team won by more. The captain of Team 2 won the argument by 8 points. Team 1 scored 79 points. Opponent 1 scored 48 points. Team 2 scored 73 points. How many points did Opponent 2 score? *Pause.* Team 2 won by 8 and then Team 2 scored 73, so that meant, *rests elbow on*

table and hand to side of head, that you would have to take 73 minus 8 and you would get your answer.

3. PT: I see, now what about here *Underlines: Captain of Team 2 won the argument.* Did Team 2 win their game by 8. *Slides paper and pen to Jill.*
4. Jill: [Inaudible] Well, they won the argument, oh, won by more … because, he won the argument and they were arguing about which won by more.
5. PT: Does that change what you just said?
6. Jill: *Nods head "yes"* . Uh-huh. Let's see. You could have … you could have anything … you could have anything, behind 65? *Shrugs shoulders.*
7. PT: Anything behind 65? Could you explain that to me?
8. Jill: *Points to problem.* Well, um, if, okay, Team 2 won the argument by 8 points and then if, wait, um, [inaudible] because, I was thinking if Team 2 scored 73 points then you could take 8 away from that, and if you took away more, [inaudible]. *Pause.*
9. PT: Confused?
10. Jill: *Nods head "yes".*

Peter

Peter's conception was half-way to a comparison of quantitative differences. Initially, Peter thought that "8" stood for how many more points Team 2 scored than Team 1, but when the arithmetic didn't work out (79 is not 8 more than 73), he readjusted his conception. He recognized that the captains were arguing about quantitative differences, but when he came to evaluating them, he re-conceived the situation as Team 2 having 8 more points than the amount by which Team 1 beat Opponent 1.

Molly

Molly was the only child during Day 4 of the teaching experiment to develop the idea that the problem was about a comparison of quantitative differences, but at that time she was not confident of this conception and did not get beyond her question of "is the result a difference?" In the interviews, Molly conceived the comparison of quantitative differences straight-away, and moreover saw that to evaluate one of the compared quantitative differences (the amount by which Team 2 beat Opponent 2) she had to combine the other quantitative difference (the amount by which Team 1 beat Opponent 1) and the super-ordinate comparison.

Don

Don's initial conception of the quantitative situation was as if Team 1 played Team 2, and that "winning the argument by 8 points" meant that Team 1 played Team 2 and beat them by 8 points. It wasn't until his attention was brought back to the sentence describing the captains' argument that he noticed a similarity between this problem and the problem he'd seen during Day 4 of the teaching experiment .

Liz

The connection Liz made between the text and her unfolding conception of the situation was rife with misdirected references. She understood "captain of Team 2 won by 8" as meaning that Team 2 won by 8 points, and then combined that 8 points with Team 2's points. Eventually she saw that it was more appropriate to subtract than to add, but she still behaved as if Team 2 beat Team 1 by 8 points. When asked to explain how her answer made sense, she appeared to notice a conflict between the statement that Team 1 scored 79 points and her sense that she had just gotten Team 1's score (as she conceived it in relation to Team 2's score), it being 65 points. Her initial conception stayed with her, even though her accommodations to the text caused conflict with that conception.

Bob

Bob appeared to have several initial competing, incompatible references for the numbers in the text: the captains having points, the teams having points, the amount by which a team won a game, and the amount by which a captain won the argument. He began sorting out "won the argument by 8 points" from other references, but he did not conceive of that comparison being a super-ordinate relationship between the quantitative differences between pairs of teams. It became evident that the amounts by which Team 1 and Team 2 won their respective games were not related in Bob's conception of the situation. They both won by more.

Problem 3 Analysis, Part 2: What did you find by adding 31 and 8?

I attempted to ensure that each student came to a point where they realized that "31+8" was an appropriate calculation. When they had reached this point, I attempted to uncover their understandings of what quantity they had just evaluated—what they "had found" by adding 31 and 8.

As in part 1 of this analysis, I include only one major excerpt and summarize the rest. I also include diagrams drawn by the children as they explained their reasoning.

Jill

Jill developed a clear sense that the 8 points mentioned in the text referred to the quantitative difference between the two teams' quantitative differences. Her attempt to reflect that understanding in her diagram (Figure 12) was thwarted by her failure to separate (in her diagram) the quantitative differences between the teams' scores from the scores themselves [¶'s 1-10]. That is, this difficulty was more an artifact of her diagram than of her conception. Paragraphs 14 and 15 suggest that Jill had developed a clear set of relationships among the quantitative difference between Team 1's and Opponent 1's scores, the quantitative difference between Team 2's score and Opponent 2's score, and the super-ordinate quantitative difference. I should also point out that Jill's use of addition seemed not to reflect any thought that she was combining two quantities. As such, it would seem that she reasoned relationally—if a-b=8, then a=b+8.[3] In this case, though, she first had to determine a value for b.

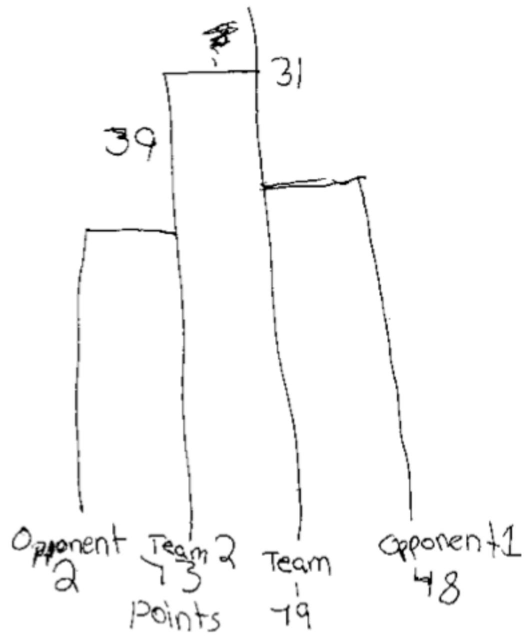


Figure 12. Jill's diagram for Interview Problem 3.

1. PT: Could you try to draw a picture?
2. Jill: Let's see. *Draws line, labels this Team 2. Draws line, labels this Team 1. Draws line, labels this Opponent 1. Speaking inaudibly throughout. Looks up.*
3. PT: Okay, where is Opponent 2 in all of this?
4. Jill: Well, Opponent 2 has to be under this. *Points to line Team 2.* Wait, [inaudible] 48, well, okay, Team 2 scored 79 … Team 2 scored 73, then 79. Wait. Then this would have to be less than, um, 60. *Shakes head.* Wait, 61, okay, because this, um, it is still confusing. *Looks up from paper.*
5. PT: Okay, what does the 8 points mean?
6. Jill: It means that um, Team 2, um, won by 8 points more than Team 1 did.
7. PT: All right, what did, could you show me the picture where, how much Team 1 won by.
8. Jill: *Draws horizontal line from top of line Team 2 to line Team 1.*
9. PT: Who did they play?
10. Jill: Oh, they played Opponent 1 [inaudible]. This … okay, this would have to be, this right here would be more than 8. *Points to the difference between lines Team 2 and Team 1.*
11. PT: More than 8?
12. Jill: Well, yes. Well, it is, uh, [inaudible] um. *Writes 79-48=31.* 31 points. 31. So you could take 8 away from that, [inaudible] because, this is how much Team 1 won by, *pointing to paper,*

and they, they argued and Team … *pause.* Opponent 2 would have to be 30, um, 20, um, 27 or less, 27 or less?

13. PT: 27 or less?
14. Jill: No, it would have to be … it would have to be 38 or more. No, it would have to be 39.
15. PT: 39. Now, what, what does the 39 go with?
16. Jill: Because, you take 31 plus the 8 … *pointing to line Team 1 and the text of the problem.*
17. PT: Yeah.
18. Jill: … and get 39.
19. PT: Yes, that's true. 39 what?
20. Jill: 39 points and this …
21. PT: *Interrupting.* Whose points are they?
22. Jill: They're, um, they're Team 2's.
23. PT: They're Team 2's points?
24. Jill: Uh-huh.
25. PT: Okay.
26. Jill: Because they're …
27. PT: That's how many points Team 2 scored?
28. Jill: No, it's how much more they scored than Opponent 2.

Peter

Peter ended the first episode thinking that Team 2's score was 8 more points than the quantitative difference between Team 1's score and Opponent 1's score, which resulted in his confusion over Team 2 having to have 73 points (as stated) and having to have 39 points (as he inferred). When asked to identify the referent of "39 points," Peter maintained that it really didn't stand for anything. Perhaps Peter meant that the 39 did not refer to anything directly stated. His later remarks indicated a difficulty in coordinating the comparison of quantitative differences, and also that his predominate meaning for "difference" was the numerical result of subtracting. Moreover, his remarks suggested that Peter could conceive of difference as a quantity only as the outcome of an additive comparison that he actually evaluated through subtraction. When Peter added 31 and 8 he did not find the quantitative difference between Team 2's score and Opponent 2's score.

1. PT: Okay, and then you added 8 to get 39. *Points to 31+8 .* Why did you add 8? You added 8 points to the number of points that Team 1 beat Opponent 1 by.
2. Peter: Um, because 31 equals, well, I did that because it had 8 more. Oh, I should, should I add 8? *Pause.* Well I added the 8 because this is 8 plus the how much they, um, they had, it would be 39. *Pointing to 31+8 =39.*
3. PT: Okay. Now what does the 39 stand for? *Pointing to 39.*
4. Peter: *Twisting pen cap.* How much … well, the 39 is … stands for the 31 plus the 8. *Points to 31+8. Smiles.*
5. PT: Okay, but does it stand for how many points somebody scored …
6. Peter: No.

7. PT: … or does it stand for how many points somebody beat somebody else by?
8. Peter: Um, I don't think so.

Following this episode, I asked Peter to construct a diagram (Figure 13). Peter constructed an incomplete diagram, and I then suggested that he fill in as much information as possible. Once Peter had the diagram, he identified 39 (*not* 31+8) as the difference between Team 2's and Opponent 2's scores but could not place "39" in it. His explanation for how "it makes sense" to label the pictured quantitative difference with "39" was that "this number [73] seems to be 39 more than this number [34]."

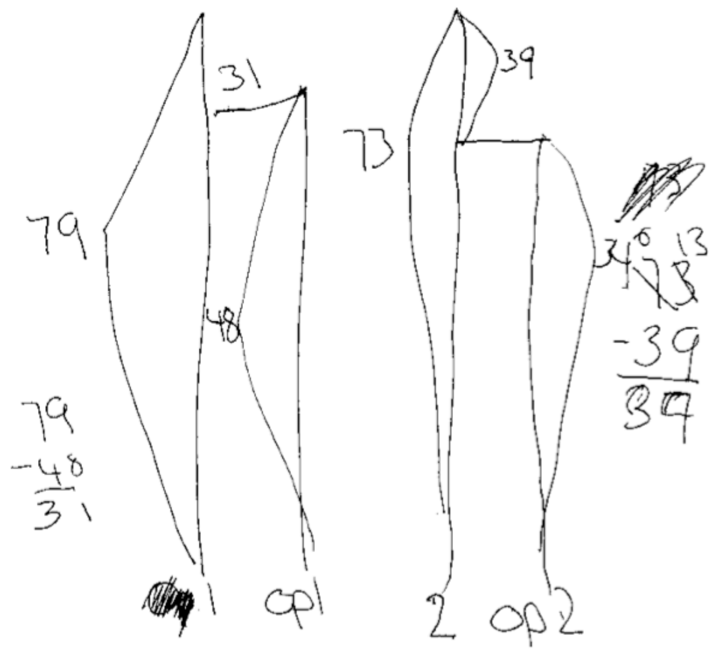


Figure 13. Peter's Diagram for Interview Problem 3.

Molly

Molly expressed her understanding of to what 31+8 referred while she explained what the argument itself was about. It seems that Molly was capable of conceiving of a quantitative difference independently of the process of evaluating it directly, and was capable of conceiving of it as being simultaneously an operand of one comparison and the result of another.

Don

Don ended the first episode recalling that he had seen a problem like this one while taking part in the teaching experiment. I was direct in asking Don to label the various parts of his diagram (Figure 14) and to consider the quantitative difference between Team 1's score and Opponent 1's score. After that, Don worked largely independently in identifying that 31+8 is the quantitative difference between Team 2's and Opponent 2's scores, and that Team 2's score minus that difference's value gives Opponent 2's score.

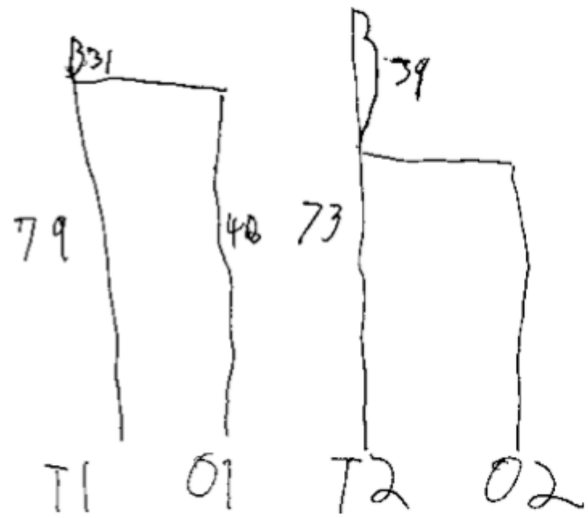


Figure 14. Don's diagram for Interview Problem 3

Liz

Liz did not go beyond her initial conception that the situation was about a single comparison. Also, Liz's diagram (Figure 15), depicting the situation, was consistent with her diagrams earlier in the interview and her diagrams during the teaching experiment: figurative details of the situation dominated, and quantitative relationships were absent.

Figure 15. Liz's Diagram for Interview Problem 3.

Bob

Bob continued to have trouble coordinating the two unevaluated differences between team's scores. When the interviewer provided the value of the quantitative difference between Team 1 and Opponent 1, which Bob evidently heard as "38" instead of "31," and which he also heard as the amount by which Team 2 beat Opponent 2, Bob saw immediately a way to use that information. But Bob's serendipitous listening circumvented his major difficulty—coordinating *two* quantitative differences. I had to be quite directive in focusing Bob's attention on the sentence "… argued about which team won by more" in relation to the sentence "The captain of Team 2 won the argument by 8 points," but even then Bob had a tendency to think that "… which team won by more" was the same as "… which team scored more". Instantiating one operand of the super-ordinate comparison with a value of 2 (between Team 1 and Opponent 1) at this point, and instantiating it later with a value of 31 points seemed to enable Bob to conceive of the quantitative difference of Team 2's score and Opponent 1's score as a comparison that yielded a value of 39. The most salient aspect of Bob's interview is that I had to intervene directly, guiding Bob's attention to the construction of the operands for the super-ordinate comparison. Once the last comparison (between Team 2 and Opponent 2) entered his conception of the situation, Bob knew what to do with it. But Bob did not once infer an operation that would provide an operand to the operation he was considering. That is, he never conceived of evaluating a substructure in order to obtain information needed for evaluating some other structure. Perhaps he failed to conceive of the situation as involving a structure of substructures.

Interview Problem 4

The purpose of Problem 4 was to highlight the structural relationship between Team 2's and Opponent 2's score, given that Team 1 scored 79 points, Opponent 1 scored 48 points, and the captain of Team 2 won his argument by 8 points. It was intended that children would come to the realization that Team 2 and Opponent 2 could have any scores as long as the Team 2 won and the numerical difference between their scores was 39 points. That is, it was intended that the children realize the structure of the comparison between the two differences.

Five children solved this problem. Liz was the only one who did not solve it. Three used diagrams (Jill, Peter, and Don). The two who did not use diagrams (Molly and Bob) explicitly connected the necessity of captain 2 winning the argument by 8 points with the necessity that Team 2 beat Opponent 2 by 39 points. Don and Bob initially attempted a "guess and check" strategy, and then readjusted to a more general, structured analysis when they saw that they had violated the constraint that captain 2 win the argument by 8 points.

Jill

Jill's solution was structured and straightforward. She focused almost exclusively on the quantitative differences between pairs of scores (Figure 16) and on the constraint that Team 2 had to beat Opponent 2 by 8 points more than the amount by which Team 1 beat Opponent 1.

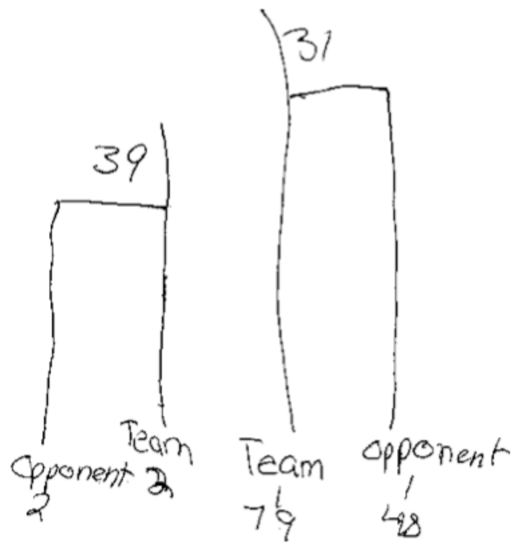


Figure 16. Jill's diagram for Interview Problem 4

1. PT: Okay, that's good. Now one last question. Remember these kinds of problems where it says put numbers in, so that everything works out.
2. Jill: Uh-huh.
3. PT: We did one last week , right?
4. Jill: *Nods head "yes".*
5. PT: So, this is the same problem that you just did except, now, instead of telling you how much Team 2 scored, it just says that they're blank. And they're going to ask you what kind of numbers can you put in there, so that everything works out.
6. Jill: Okay. *Pause.* Team 1 scored 79, so Team … okay, the difference of these, 31 and [inaudible] … draw a picture again. *Draws two vertical lines, labels them Team 1 and Opponent 1.* Team 1 … Opponent 1. And this has to be, let's see, this is 79, and this is 48, and this has to be 31. *Labels difference between the two lines as 31.* Okay, so … Team 2, [inaudible] … Team 2 scored … Let's see. *Draws two lines, labels them Team 2 and Opponent 2. Indicates the difference between the two lines.* So, this … 39. *Writes 39 by the difference between Team 2 and Opponent 2.* And it could, it could be almost anything as long as the difference was 39.
7. PT: Very good. Thank you very much. Now, how was it you went from this [problem text] to that picture again.
8. Jill: You look up here, *pointing at problem ,* and find these [scores for Team 1 and Opponent 1] and then you find the difference, *pointing to difference in lines Team 1 and Opponent 1.*

9. PT: All right.
10. Jill: And then, um, Team 2 has to win 8, 8 more, by 8 more than, than Team 1 did so …
11. PT: All right. And you don't know what these numbers are so you say, just as long as they have a difference 39. *Jill nods "yes".* Can I ask you something back there [refers to Problem 3]? *Brings out Jill's diagram for Problem 3.* Do you think it confused you when you tried tying Team 2 together with Team 1 with that difference of 8? *Jill nods "yes". Points to diagram on previous problem. Pause.* See down here you didn't tie them together did you. *Points to diagram for current problem.*
12. Jill: *Shakes head "no".*
13. PT: Team 2 and Team 1. You didn't try tying them together with a difference, but up here you tied them together with a difference of 8.
14. Jill: *Nods "yes".*
15. PT: What, how do you suppose that led you astray?
16. Jill: Well, I thought that, um, that Team 2 won over Team 1 by 8 points, but they didn't play each other.

Peter

Peter was aware of the constraint that Team 2 had to win by 8 more points than did Team 1 [¶ 6], but this awareness did not translate directly into a relationship between Team 2's score and Opponent 2's score. Peter initially attempted a "guess and check" strategy, choosing 80 and 40 to put in the blanks. However, as soon as he realized that he had violated a constraint [¶'s 14-16], Peter readjusted his strategy so that it accommodated to the constraint that the result of subtracting had to be 39 [¶'s 16-22].

1. PT: Now, I'm going to give you just one more. We're not going to go all the way through it, but I just want you to think about it. *Pause.* Okay? *Peter begins to read the problem quietly to himself.* Why don't you read it out loud.
2. Peter: Team 1 played basketball game against Opponent 1. And Team 2 played a basketball game against Opponent 2. The captains of Team 1 and Team 2 argued about which team won by more. The captain of Team 2 won the argument by 8 points. Team 1 scored 79 points. Opponent 1 scored 48 points. Team 2 scored zero …
3. PT: Blank points.
4. Peter: … scored blank points. Opponent 2 scored blank points. Put the numbers in the blanks so everything works, so everything works out.
5. PT: *Sneezes.* Pardon me. What do you have to, can you tell me something about the numbers that you need to put in there. All right, you know two of them right?
6. Peter: You need to put them in so that these, so that those two difference, well, you need to have those two so that they, so that Team 2, Team 2 wins by 8 points more than he wins by that, *pointing to parts of the problem as he explains.*
7. PT: Oh. Okay. Can you put in more than one number?
8. Peter: Um …
9. PT: Well, I'm sorry that's a vague question. Go ahead and just, go ahead and put in … try to put in some number.
10. Peter: Any number?
11. PT: If you like.

12. Peter: Okay. Um.
13. PT: Put in one that makes it really simple.
14. Peter: Um-hum. *Pause. Peter writes 80 in the first blank and 40 in the second blank.*
15. PT: 80 and 40?
16. Peter: That'd be, that'd bring the … look at those two. Team 2 would have to win by 8 more points. So 79 minus 48. *Writes 79-48 =31.* [Inaudible] … minus 79 … [inaudible] … and 31. So that has 31 … The difference is 31 so this would have to be 39.
17. PT: Okay. And 80-40 is 39?
18. Peter: What? 80-40 is not 39 so …
19. PT: Oh. Suppose that I were to say, "Okay, well..."
20. Peter: *Interrupting* [inaudible] … I think … what is it?
21. PT: Oh, all right.
22. Peter: Is it, um … so you have to have the number and a number. The number and a number minus and then you have to get 39.
23. PT: Okay.
24. Peter: Uh-huh.
25. PT: Well, that's all right. I know that, I know that you know how to do it. Okay. So now it's just a matter of picking numbers.

Molly

Molly evidently reasoned according to the quantitative structure of the problem, without the aid of a diagram, and concluded that "you could have it mostly any way … just as long as Team 2 wins by 39 points."

1. PT: *Presents Problem 4.* Now this one's just a little bit different. It's, uh, let me, uh, save some time, and I'll just explain to you that this is just like it was before. All right? Now with the basketball games and the captains of the teams arguing, and the captain of Team 2 won the argument by 8 points, and Team 1 still scored 79, and Opponent 1 still scored 48, but now, these are both blank. *Points to problem as he explains it.* Okay?
2. Molly: *Nods "yes".* Um-hum.
3. PT: And it says "Put numbers in the blanks so that everything works out." *Points to problem as he speaks. Molly nods "yes"..* Okay? And could you kind of speak out loud while you're thinking?
4. Molly: Okay. Um, well, these two are the same. *Pointing to problem as she explains*. Team 1 scored 79 points, and Opponent 1 scored 48 points. That would mean that there was still a 31 point difference between this, and that would mean that there would, um, Team 2 would have to win the argument by 8 points and 31 plus 8 is 39 so they would have to win by 39 points. So … *writes 98-39 =59* … you could have it 98 and 59. *Pointing to 98, then 59.*
5. PT: 98 and 59?
6. Molly: You could have it mostly any way.
7. PT: Mostly any way. *Molly nods "yes".* Just as long as …
8. Molly: Just as long as, um, Team 2 wins, um, by 39 points.

Don

Don immediately assumed a "guess and check" strategy [¶'s 4-6]. Evidently, his "guess" was influenced by his awareness that there was a numerical difference of 8 somewhere in the situation. Don drew a diagram to check himself, and in constructing the diagram he foresaw that 48 for Team 2's score and 40 for Opponent 2's score would not work [¶'s 10-12]. In explaining why he foresaw that 48 and 40 would not work, the relationship "8 more" appeared to become prominent once again [¶ 22], but Don avoided difficulty by referring to his work on Problem 3 [¶'s 22-26]. Once he had re-established the relationship of "differed by 39" between Team 2's and Opponent 2's scores [¶'s 26-34], Don saw that any pair of numbers would work as long as they differed by 39 [¶'s 42-44].

1. PT: And this one won't take as long. *Presents Problem 4.* Okay, now this is the same thing like you just worked on …
2. Don: Okay.
3. PT: Except both Team 2 and Opponent 2 have blanks. *Pointing to problem as he explains.* And the idea is tell me about the numbers you can put in those blanks so everything works out.
4. Don: Well, okay. [Inaudible] Well, first you, you could just try some numbers.
5. PT: Okay.
6. Don: And see if it would work out … *Pauses to watch PT as PT leaves room to investigate loud noises coming from adjacent room …* 48 … 40. *Writes 48 in first blank and 40 in second blank. PT returns.* Okay.
7. PT: You're trying 48 and 40?
8. Don: Yep. So let's see. Team 2, they were arguing about … . *Begins to draw.*
9. PT: Here, I'll hold it for you. [Referring to holding the paper in place]
10. Don: *Draws line and labels it TEAM 1* … Team 1 … 79 points. *Writes 79 by line TEAM 1.* Let me see, Opponent 1 … *Draws line and labels it O1* … They had 48 points … *Writes 48 by line O1.*
11. PT: Could you speak just a little louder, Don?
12. Don: Oh, okay. Um, okay. Opponent 1 has 48 points. Team 2 had 48 points … Oh. I can see that it won't work out already.
13. PT: You can see already that it's not going to work out?
14. Don: Yeah. Well, I'll just try it just to make sure.
15. PT: All right.
16. Don: Team 2 and Opponent 2 … *Draws two lines. Labels them TEAM 2 and O2* … 40 and 48 … *Writes 40 by line O2 and 48 by line TEAM 2.* Okay, so … this … I don't think this will work out. (*See Figure 17*).

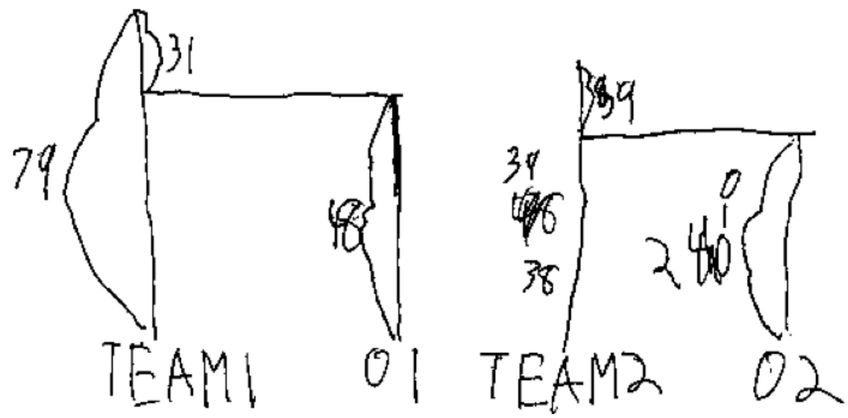


Figure 17. Don's Diagram for Interview Problem 4

17. PT: Okay.

18. Don: *Draws a perpendicular lines from TEAM 2 to O2 indicating difference.* Huh-uh. *Shakes head "no".*

19. PT: Why isn't it going to work out?

20. Don: Well, because 79 minus 48 would be … *Begins to write 79-48. Pause.*

21. PT: I think you already did that in the last problem. It was 31.

22. Don: Okay. Then that would be 31. That would make that … This difference is 31. *Writes 31 by the difference between TEAM 1 and O1.* Wait a minute. [Inaudible] … You get 31 and, … *writes "8" aside difference between Team 2 and Opponent 2* … oh, boy … Did I do that in the last problem? [inaudible]

23. PT: So this difference is 31, *pointing to the difference between TEAM 1 and O1.*

24. Don: Yeah.

25. PT: In the last problem? *Retrieves Don's diagram for Problem 3,* you said this difference was 39, *pointing to the difference between TEAM 2 and O2 in Don's Problem 3 diagram.*

26. Don: Hmm. Okay. *PT removes Problem 3.* That's the one you were pointing to *looking at Problem 3 … pause … Writes "8" then "39" aside difference between Team 1 line and Opponent 1 line.* Hmm. Let's see … Would that work? … I think any, any number would work.

27. PT: How about 1 and 2?

28. Don: One and two? Okay. *Writes 1 by line TEAM 2 and 2 by line O2* … Oh, well … no, that wouldn't work because there's a difference of 39 here, *pointing to the difference between TEAM 2 and O2.* There's only one …

29. PT: Oh.

30. Don: Here it says there's only a difference of one. *Pointing to TEAM 2 and O2 respectively.*

31. PT: Okay.

32. Don: So that wouldn't work.

33. PT: So that wouldn't work … So what has to be true about those two numbers?

34. Don: Oh, so there has to be at least, I think the lowest number that would work would be … 39 and 0. *Writes 39 by Team 2 and 0 by O2.*

35. PT: All right. So 39 and 0 …

36. Don: Yeah.

37. PT: Any others?

38. Don: Well, I think just about any, well, we could also try 38 and 1, and you know.
39. PT: 39 and 0 and 38 and 1?
40. Don: 38 and … *Writes 38 by Team 2 and 1 by O2.* 38 and 1. Oh, no … no, that wouldn't work … oh, oh, 4, 40, and 1.
41. PT: Oh, okay. So 40 and 1.
42. Don: 40 and 1 and 42, 4, 41 and 2, 43 and 4 …
43. PT: Okay. Good. All right. Great.
44. Don: You just keep on going.

Liz

Liz began much differently than in the previous segment, wherein she conceived of only one quantitative difference. In analyzing Problem 4, Liz isolated most of the key elements [¶'s 2-5], and seemed to have them in the proper relationship. But she refused the correctness of her (*correct*) solution when she noticed that Opponent 2's score was not 8 more than the Team 1/Opponent 1 numerical difference [¶'s 5-7]. It is notable that Liz had essentially solved Problem 3 as a special case of Problem 4, yet had made few of the numerical identifications. For some reason, Liz thought that she should subtract a number from 73 that would give a number that was 8 more than 31 [¶'s 8-17], and that *73-39* had no significance in the problem [¶'s 18-27].

1. Liz: *Writes her name on the paper.* Okay, um, Team 1 played a basketball game against Opponent 1. Team 2 played a basketball game against Opponent 2. The captains of Team 1 and Team 2 argued about which team, which team won… by more. The captain of Team 2 who won the argument by 8 points. Team 1 scored 79 points. Opponent 1 scored 48 points. Team 2 scored blank points. Opponent 2 scored blank points. Put numbers in the blanks so that everything works out… *Rests chin on hand, on the desk…* Hmm… um… *pause.*
2. PT: What are you thinking of, Liz?
3. Liz: I'm not sure. What if we try the same number as last time, 73, in. *Writes 73 in first blank.*
4. PT: Okay.
5. Liz: And then subtract,73 subtract some number, and we can write it in [inaudible]… Um, more than, has to be 8 points more than this number subtract. So… 79 subtract, *Writes down 79-48.* It is 31, just like I guessed. Okay, let's see, it has to be 39, so… 73 subtract 39 equals… *Writes it down as she says it.* Umm… Oh, well… okay… 34… That doesn't make it by 8 points. So… about 43.
6. PT: You said what doesn't make what by 8 points?
7. Liz: This, *points to 34*, doesn't meet this, *points to 31,* by 8 points.
8. PT: All right. Well, where did you get that 39?
9. Liz: Um, I guessed it from here, *pointing to 31.*
10. PT: Why 39 and not 36 or something like that?
11. Liz: Because, uh, 39 is 8 more than this, *pointing to 31*, but it doesn't… This, *pointing to 34,* has to be 8 more than this, *pointing to 31.*
12. PT: 39 is 8 more than 31, isn't it?

13. Liz: Yes, but…
14. PT: Oh, you're saying that, that when you subtract 39, *pointing to 73-39,* you should get a number that is 8 more than that one, *pointing to 31*?
15. Liz: Yes.
16. PT: How come?
17. Liz: Because that's, because Team 2 won by 8 points.
18. PT: What does that 34 mean? *Points to 34.*
19. Liz: Um, nothing really.
20. PT: Nothing really?
21. Liz: It's not the right answer.
22. PT: What does the 73 mean? *Points to 73.*
23. Liz: How many points Team 2 scored. *Points to reference in problem.*
24. PT: And if you subtract 39, then what do you get? *Pointing to 73-39=34.*
25. Liz: 34.
26. PT: Yes, I know you get 34, but you get a number that… Whose points are those? *Liz checks her watch.*
27. Liz: Um, those are no one's points. *Taps pencil on paper.*
28. PT: No one's points? Who, what , what points are these, 39? *Points to 39.*
29. Liz: Um, it could be how many Opponent 2 scored.
30. PT: That, well if that's how many Opponent 2… Oh, I see. So, that's how many Opponent 2 scored, and then…
31. Liz: Well, it's not how many Opponent 2 scored because this number, *pointing to 34,* has to be 8 more than this number, *pointing to 31*, and its' 3 more than, than this number.
32. PT: I see. Okay.
33. Liz: So, let me try it this way. *Tries 73-43.* Oh, no, that don't work. It's getting lower… Oh… *Tries 73-27* … 46. Let me see. No… About… Oh… *Tries 73-30…Taps pencil on table.* Another one. It's closer, but it still doesn't work.

Bob

Bob's solution of Problem 3 seemed to leave him with a structured conception of the situation, with the exception that Bob forgot that the Team 2/Opponent 2 numerical difference needed to be 8 more than the Team 1/Opponent 1 numerical difference.

1. PT: Now, one last quick one. Okay. *Writes Bob's name on paper and gives him Problem 4.* This one's just like the previous one. Same information, *pointing at problem.* The two teams, played opponents. The captains argued. Captain of Team 2 won the argument by 8 points.
2. Bob: Okay. Yeah.
3. PT: Team 1…
4. Bob: Since I got that last one, I think I'll get this one.
5. PT: All right. Now it says put numbers in these blanks for Team 2's scores and Opponent 2's scores so that everything works out. *Points to last sentence of problem.*
6. Bob: Okay. [Inaudible] Okay. *Writes 79-48=31.* Okay. Team 1, 79 minus…31, 9. So Team 2 won by 39 points, and so it means this got 30…wait no, yeah, 30, 48 minus, um, 8, yeah. *Writes 48-8=40.* [CR (the camera operator) brings a Kleenex for Bob] Oh, thanks.
7. PT: Thank you, CR.

8. Bob: 40 points. Um, what did I just do? Okay, oh, okay…7…this is 31, *pointing to the difference 79-48,* so it'd be…they won by 38 points and…It's about the same thing. If I could just, is it the same thing from last time, I could just write in. The last time I…

9. PT: Well, yeah. You could just write the same numbers in, uh, but it would be, you know, if you could tell me something about the numbers that you have to…

10. Bob: Yeah. Okay. Team 2, I wish I went to this game. Um, Team 2 scored the blank number of points. Team 2… Opponent 2 scored 40 points. I say 40. *Writes 40 in the second blank.* So if they won by 31, plus 31 means they have 71. *Adds 40+31.* Means this team got 71. *Writes 71 in the first blank.*

11. PT: Okay. Now did you mean to write 31? *Pointing to 40+31=71.*

12. Bob: Uh. Yeah. wait.

13. PT: *Pointing to 79-48=31.* That's how much Team 1 won by.

14. Bob: Yeah. And no. Now I see what I did. I need to put 39. *Changes 31 to 39.* No. Yeah. 39. So that would be 79. *Changes 71 to 79 in his work and in the blank.*

15. PT: Okay. Now could you tell me real quickly could you tell me two other numbers you could put in there. *Points to blanks.*

16. Bob: 78 and 39.

17. PT: All right. Two others.

18. Bob: Okay. 80 and 41. 81 and 42.

19. PT: Just, so you put lots of different numbers in there just as long as what?

20. Bob: Yeah, as long as there's that 39 space gap.

Discussion

The teaching experiment was intended to explore children's abilities to conceive of complex relationships in settings and in textual descriptions of settings, and to explore their conceptions of difference as a quantitative structure. As a result of the teaching experiment and interviews another issue became important: the roles of language and notation.[4] We are also able to draw tentative conclusions about pedagogy and curriculum that pertain to issues of relational complexity and about quantitative structure.

Complexity

The primary difficulty had by children of this study in dealing with relationally complex settings appears to have been the need for conceiving of an operand of a quantitative operation as simultaneously being an operand for a super-ordinate quantitative operation and the result of a different quantitative operation. The type of problem that forced this issue was one having the calculational structure shown in Figure 18. It needs to be made clear, however, that calculation was not the source of difficulty. Rather, it was conceiving of comparisons independently of the customary calculations done to evaluate them. For example, in the discussions of Brothers A and B, it was evident that children found it to be very difficult to conceive of a quantitative difference between two persons' heights as being found by subtracting the measures of two other quantities.

Similarly, it was not straightforward for children to conceive of a winning margin in a basketball game as being found by adding two numbers of points. Many times children would perform an appropriate calculation, but would be unable to proceed—because they could not place the result of the calculation in the situation about which they were reasoning. Typically, the result that they could not place in the situation was an indirect evaluation of (what was to us) the result of a quantitative operation. To these children, this double-identification was often not possible because, to them at the moment they were reasoning, a quantity could not be the result of a quantitative operation independently of actually performing a customary calculation, with the values of the quantitative operands, to evaluate it.

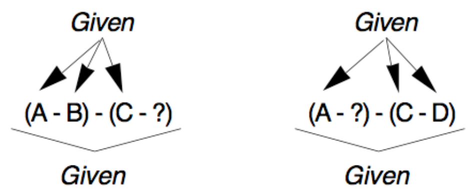


Figure 18. Calculational Structure of Study's Complex Difference Problems

My claim about the significance of being able to conceive of a quantitative structure independently of customary calculations done to evaluate it would be stronger had I asked children to solve, and were they able to solve, a related problem: Find the super-ordinate comparison given all the primary information. For example, the question for the "Brothers A and B" problem would have read something like: *By how much more did Brother A stand taller than his sister than Brother B stand taller than his sister?* The question for the Teams 1 and 2 problem of the interviews would have been something like: *By how much more did Team 1 win than did Team 2?* In both cases, the calculations done would have been customary for the quantitative operations involved in children's conceptions of the situation, and children would not have had to conceive of the quantitative operations independently of their customary evaluations. Had children found these problems easier, I would be able to claim that the primary source of difficulty in the investigated problems, beyond conceiving of a quantitative difference of quantitative differences, is evidently the necessity of conceiving a quantitative difference independently of any particular calculation employed to evaluate it. I do note, however, that children *did* have difficulty saying what the sum of (*C-D*) and (*A-?*)-(*C-D*) stood for (namely, (*A-?*)), even after they had identified what (*C-D*) and (*A-?*)-(*C-D*) stood for.[5]

We should note that these children showed an increasing level of competence in dealing with complex situations over the period of the teaching experiment and subsequent interviews. This suggests that lack of occasion to reason about complex relationships was one factor in why they met the cognitive obstacles they did.

Language and Notation

English does not provide linguistic devices for referring to the product of a process in relation to the process. This is reflected in the complex sentence constructions needed to discuss the Brothers A and B situations and the Teams 1 and 2 situations. It could be argued that limitations in language were the primary source of children's difficulties. I disagree with this position. I would argue instead that we do not have such linguistic devices because it is hard to conceive the situations that would give rise to them in the first place. Of course, if I were to become aware of a vernacular in which such devices occur commonly I would change my position.

It might also be suggested, as one reviewer did, that it would be productive to introduce notational devices that support dual interpretations, such as A-B versus (A-B) to distinguish between the process of calculating a difference and the result of subtracting. I see two problems with this. First, the predominant problem had by students was not in conceptualizing the dual nature of calculational representations. Rather, their difficulties stemmed from conceptualizing the dual roles played by quantities that calculations were meant to evaluate. To focus on calculational matters would distract children from addressing the sources of their difficulties. Second, children's school experience already orients them to reference-less operations on numbers. I suspect that an even greater emphasis on calculational subtleties would produce even greater distractions regarding children's conceptualizions of relationally-rich situations. Instead, my recommendation is that we give greater attention to orienting children toward the building of quantitative operations which will allow them to construct something in their reasoning they might wish to represent. One avenue to that end is simply to have them talk about situations with the intent of building and expressing coherent images—including their constituent relationships and inferences we can make about them. Calculational subtleties, addressed later, could then be kept in proper perspective and still be made reasonable topics of discussion.

On the other hand, it may be possible to devise computer programs that will allow children to put iconic representations of quantities into multiple roles, a possibility not allowed by the children's pencil-and-paper diagrams. I cannot, at this moment, imagine what such a program might look like, but the potential benefits are such that it appears a productive area of research.

Concept of Quantitative Difference

These children were able to reason in terms of differences as quantitative operations—additive comparisons between two quantities—but it appeared that they did not distinguish between the *quantitative* operation of comparing two quantities additively and the *arithmetical* operation of subtraction. In fact, it was common for them to confound the two (although for some this was true more in the early part of the teaching experiment than in later parts and in the interviews). I suspect that the confounding of quantitative and arithmetical operations is symptomatic of all quantitative operations—ratio, additive and multiplicative combinations, and multiplicative compositions—not just of difference. Also, I suspect that confounding the two aspects of difference is more an artifact of children's schooling than of anything intrinsic to quantitative reasoning.

Two aspects of the concept of difference were problematic for the children of this study. One was difference as an invariant *numerical* relationship. The difficulty occurred primarily in the variations discussed in Day 2 of the teaching experiment (two games of marbles). Children felt that they needed to know initial numbers of marbles before they could speak of wins and losses. This aspect of difference is a cognitive root of the concept of integer—an equivalence class of number pairs, each pair in the class having the same numerical separation. The second aspect of the concept of difference that was problematic, and the one having direct bearing on quantitative reasoning, is the notion of difference as an additive comparison of quantities. I saw many instances where children calculated a quantitative difference—the amount by which one quantity exceeded or fell short of another—and yet could not identify it as such. This points to the necessity of being able to conceive of a quantitative difference independently of a particular arithmetical calculation employed to evaluate it.

Pedagogy

Four pedagogical issues emerged during the teaching experiment and during interviews: the need to hold instructional conversations, the need to keep explicit attention on numbers as values of quantities ("This is a number of what?"), the need to keep the task in mind ("What are you trying to find?"), and the need to identify the quantity that has been evaluated by an arithmetic calculation ("What did this calculation give you?").

Instructional Conversations

If children are to develop the complex reasoning patterns that constitute quantitative reasoning, teachers need to engage children in *instructional conversations* on a regular basis. This means that teachers must expect children to take significant responsibility for explaining their reasoning, raising questions about tasks, and making decisions about appropriate assumptions, decisions, and conventions. At the outset of the teaching experiment, children explained themselves only by way of saying what arithmetic they had done or would do. It appeared that they relied almost completely on intuition, and only the arithmetic that they would do was in their awareness. When this occurred, children had no way to judge the appropriateness of their implicitly-made decisions—their understandings of situations were implicit, and hence were not objects of reflection. The technique of holding instructional conversations about "what is going on" tended to orient children away from rash judgments about arithmetic. Children got better at holding these conversations even within the brief time of the teaching experiment.

This is a Number of What?

It often happened during both the teaching experiment and the interviews that children would be at a loss as to what to do, and the teacher or interviewer would ask what various numbers stood for. In constructing explanations of what each number stood for, it was common for children to become aware of possibilities that had eluded them before. Evidently, these children were not in the habit of thinking in terms of measured quantities in their initial analyses of problem texts, and that the practice of having them identify what various numbers "stand for" oriented them to a more quantitative analysis of the situation being described in text. Were this practice a standard characteristic of instruction, and were children to understand that they are expected to analyze text in the same way that the teacher exemplifies, then the likelihood that children interiorize this practice will be increased. If children in fact interiorize the practice of reasoning in terms of quantities and values, then it sets the stage for them to reason about quantities in the absence of knowing their values.

What Are You Trying to Find?

This question, "What are you trying to find?", is typically asked at the outset of problem analysis and refers to "the answer." I have a different intention—that this question be asked *every time an operation is considered*. Children in this teaching experiment often went directly from an initial understanding of a situation and problem goal to doing arithmetic, without considering what quantity they were evaluating by their choice of arithmetic operation. It seems self-evident that such practice is debilitating with respect to developing quantitative reasoning. Teachers must have patience in laying the foundation for long-term development of quantitative reasoning. Having children engage in the cycle of analysis-implementation-reflection may initially appear to be time consuming, but the long-term benefit will be more than worth the initial investment.

What Did This Calculation Give You?

The question “What did this calculation give you?”, or put another way, “What did you find by doing this calculation?” might seem to be redundant with the question “What are you trying to find?” discussed in the previous paragraph. However, it was not uncommon for children in the teaching experiment and interviews to have an idea of what they were trying to find by a particular calculation, yet not be able to say what they found after calculating.

The paradox of children knowing what they want to find, yet not knowing what they have found, seems to be related to an observation made in the interview analyses—that children need to be able to conceive of a quantity as an item in multiple relational structures. For example, in Day 4 of the teaching experiment, the children had the idea that they should subtract 17 (the amount by which Brother A won his argument with the Brother B) from 99 (the amount by which Brother A was taller than Sister A). They knew that this gave them the number that Brother B lost with in his argument. However, they did not identify the numerical difference 99-17 as an evaluation of the quantitative difference between Brother B’s height and Sister B’s height. That is, children knew what they were trying to find when they calculated 99-17, yet they did not know what they found in relation to the rest of the problem. Evidently, their experience with subtraction was that one subtracts the operands of a difference to evaluate the difference. In this case, they needed to understand that they were subtracting *relationally* to evaluate a subordinate difference (i.e., if c=a-b, then b=a-c). The children experienced dissonance when they subtracted two values and tried to relate the numerical result of subtracting to another quantitative difference. The numerical difference they obtained was derived (relationally) from two quantities that were not part of the comparison which made the quantitative difference to which they were trying to ascribe their result of subtracting.

Curriculum

I have found three implications for the design of curriculum in regard to complexity and additive structures.[6] These have to do with the need to have children deal with complex situations on a regular basis, principles for problem construction and presentation, and the need to highlight quantitative relationships as a predominant *problématique* (Balacheff, 1990).

Complex Situations

Typical "complex" problems included in U.S. 5th-grade texts are at most two-step problems that are intended to have children do two calculations to get a final answer. Also, the "answer" is typically the result of calculating, it is not the answer to a question. But complexity of situations is determined more by the number of relationships that must be dealt with, not by the number of calculations that need to be done. Adding up the amounts of money had by 10 children requires 9 additions, but the *situation* is not complex. A slight variation of this example increases its complexity significantly: "Three children, Jim, Bob, and Siree compared their savings accounts. Jim had $15 more than Bob and Siree had three times as much as Jim. Siree had how much more money than Bob?"

Another aspect of complexity has to do with the relationship between quantitative operations and numerical operations. Quantitative operations have associated *canonical* numerical operations when the values of the operand quantities are known (e.g., subtract to evaluate a difference; divide to evaluate a ratio). However, it is often the case that we do not evaluate a quantity with its canonical operation. For example, in Interview Problem 3 (Teams 1 and 2), one *added* to evaluate the quantitative difference between Team 2's score and Opponent 2's score. Thus, the situation was "complexified" for children by their need to distinguish between the quantity being evaluated (a difference) and the arithmetic being done to evaluate it (addition). Thus, an important aspect of problem complexity is the extent to which one has to distinguish between quantities as structures to be evaluated, "typical" meanings associated with numerical operations, and the canonical numerical operations associated with specific quantitative relationships. To distinguish among these aspects of complexity is a non-trivial achievement, and to make these distinctions children must experience the obstacles caused by not having made them. If children do not experience the need to make these distinctions in arithmetic, they will have little hope of making them when they are absolutely necessary for success—in algebra.

Problem Variations

One type of problem that appeared to orient children in productive directions was a "problem" that asked no question, but instead was presented with the intent that children discuss "what is going on here." That is, the *problématique* (Balacheff, 1990) was one of interpretation and understanding, not of calculation or answer-giving. In some cases problem texts that contained questions were used in the same format; the question was ignored. Ignoring the question in a problem's textual description is common in algebra. Successful algebra problem-solvers routinely ignore the question and concentrate on writing expressions that model the situation. Problems-without-questions in the arithmetic curriculum would seem to support children's development of inclinations to analyze situations independently of the "question" being asked, and would seem to support children's development of the mental operations necessary for such analyses in algebra. Of course, the situations being presented for analysis must be chosen judiciously; they must provide occasions for children to "bump into" obstacles which lead them to productive reflection.

The format used in the teaching experiment and interviews wherein a problem was made more general by relaxing restrictions on numerical information seemed to work well in regard to getting children to think more generally about reasons for doing the arithmetic that was done. It also supported conversations about what relationships constrained our choice of numbers, which will be discussed at greater length in the next section.

Highlight Quantitative Relationships

There is a variation principle I did not use in this teaching experiment, but which I think will be productive in highlighting quantitative relationship independently of arithmetic. The task would be to have children "rewrite" the problem so that the situation stays the same, but different information is given about quantities in the situation and a different question is asked about it. More generally, children would be given the task of explaining under what conditions questions about how much each team scored are answerable and under which conditions they are not. For example, the "Teams 1 and 2" interview situation is "solvable" under these (among other) conditions: (1) we are told which captain won the argument and by how much, and we are told any three of the four team's scores; (2) we are told a difference between one team/opponent pair and one of those scores, the winning argument-amount, and one score from the other pair; and so on. Such analyses require children to constitute a situation in terms of quantities and relationships independently of numerical information about them.

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Footnotes

[*] Research reported in this paper was supported by National Science Foundation Grants No. MDR 89-50311 and 90-96275, and by a grant of equipment from Apple Computer, Inc., Office of External Research. Any conclusions or recommendations stated here are those of the author and do not necessarily reflect official positions of NSF or Apple Computer.

[1] Each result of a quantitative operation is also a quantity. So, the simplest additively-structured "complex" situation would be a situation involving three quantities and three differences or combinations (for a total of six quantities), such as a situation modeled by the expression [(A compared-with B) compared with (B compared with C)]. The quantities here are A, B, C, (A compared-with B), (B compared-with C), and [(A compared-with B) compared with (B compared-with C)].

[2] I do not know the source of Molly's formulas. She understood what they were saying, and they expressed what she was trying to say, but she reported that she had never used formulas in this way before.

[3] Had I not been intent on pursuing the matter of what quantity Jill understood "31+8" to evaluate I would have given her more opportunity to explicate her reasoning for adding 31 and 8.

[4] I am indebted to two anonymous reviewers for raising this issue.

[5] My use of expressions to identify quantities is somewhat misleading. I use them here referentially, not calculationally. I am not saying that the students had actually identified a super-ordinate difference by evaluating an expression. Rather, I am saying only that the quantity customarily evaluated with $(A\text{-}?)\text{-}(C\text{-}D)$ was the one they identified. See Thompson (1989, 1990) for alternative methods for representing quantities and quantitative relationships that do not rely on the use of expressions.

[6] Actually, these implications are not particular to additive structures. They also apply to extensive multiplicative structures as well as ratios and rates (Thompson, 1990, 1992, in press).