Bridges between Mathematics and Science Education[†]

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Running Head: Bridges between Mathematics and Science Education

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I wish to thank the organizers of this conference for inviting me to participate in *Project 2061*'s exciting, and daunting, effort to transform science education as it exists in the United States. I am not a science educator, and I realize that I run the risk of sounding naïve regarding issues of "scientific literacy." Nevertheless, I am excited about the prospects of school science and mathematics coming to complement each other at levels deeper than each being a context or tool for the other.

I will briefly develop three themes, and later show that the first two are aspects of the third. The first theme is that mathematics and science education share a significant common problem: the improvement of elementary and secondary teacher education programs, with the former being the more urgently needed. The second theme is that imagery built up by students through their experiences in elementary school has a far deeper influence on their ability to construct mathematical and scientific concepts than we had thought. The third is that school science and mathematics instruction and curricula, including those envisioned in the *Benchmarks*, need to pay greater attention to students' conceptualization of physical quantities and quantitative reasoning than is presently the case.

Problems of Teacher Education

Reform in science education must face a question also faced by math education: What do we take as evidence of meaningful reform? Shall we be satisfied if teachers adopt new methods of teaching or new methods of assessment? Shall we be satisfied if teachers exhibit a newfound excitement for mathematics or science? These are often taken as indicators of reform, but upon closer examination it can easily turn out that what a teacher teaches remains unchanged when implementing "reformed" curricula and methods of instruction (Boyd, 1992; Cohen, 1990). Moreover, it is easy for us to think that activity at the national level parallels activity at the local level. The *Benchmarks* opens with this introduction:

> *Benchmarks for Science Literacy* does this not by offering a standard curriculum to be adopted locally but by providing educators in every state and school district with a powerful tool to use in fashioning their own curricula. We believe that they will respond with zest and imagination to the opportunity to bring their own

> insights and experience to bear on the task of moving towards the grade-level standards for science literacy presented in this report from the American Association for the Advancement of Science.
>
> This has been truly a grass-roots effort. As the Participants list shows, an unprecedented number of elementary-, middle-, and high-school teachers, school administrators, scientists, mathematicians, engineers, historians, and learning specialists participated in the development of *Benchmarks* and in its nationwide critique. (American Association for the Advancement of Science, 1993, p. vii)

While the production of the *Benchmarks* was indeed a landmark event involving large numbers of people, a little proportional reasoning and back-of-the-envelope calculating is sobering. Suppose each person involved in developing the *Benchmarks* represents 500 active educators. This would mean approximately 20,000 people are knowledgeable of *Science for All Americans* and the *Benchmarks for Science Literacy*[1]. According to NEA statistics, there are approximately 3.5 million teachers, 1.75 million of whom teach elementary school and 200,000 specialize in teaching science at the junior- and senior-high levels. By this reckoning, you are influencing at most 1.1% of the nation's elementary teachers and 10% of the nation's science teachers. My figures are rough, but I suspect they are useful. I'm sure AAAS and Project 2061 would find it instructive were they to gain accurate estimates of influence and to study how *Science for All Americans* and *Benchmarks* are being assimilated by those of the nation's teachers who learn of them.

We have all heard the adage, "Teachers teach the way they were taught," which people often take to imply that we will move toward solving our educational problems if we can get teachers to teach mathematics and science differently. It is instructive, however, to remind ourselves of what "adage" means: "A saying that sets forth a general truth *and that has gained credit through long use*" (Wordstar International, 1993). The statement that teachers teach the way they were taught seems so credible, on the surface, that it generally goes unquestioned. Many mathematics and science courses for prospective elementary and secondary teachers have been designed around this idea. Unfortunately, there is growing evidence that few elementary or secondary teachers teach

[1] This is not the same as saying that 20,000 people have a copy of either *SAA* or the *Benchmarks*.

"the way" they were taught in their specially-designed education or content courses (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Brown & Borko, 1992; Brown, 1993; Harel & Behr, 1995; Olson, 1994), and when practicing teachers do change their curriculum or teaching methods, it seems the essential character of *what* they teach often remains largely unchanged (Boyd, 1992; Cohen, 1990).

I believe we can now make a more useful statement: "Teachers teach *what they know*," using "know" broadly to include images of mathematical activity and beliefs about the enterprise of learning and teaching mathematics (Thompson, 1984; 1991; 1992a). If the mathematics a teacher knows is ritualized performance, then regardless of what methods he uses, his image of what constitutes successful learning will be of ritualized performance, and his classroom actions will be textured around that image (Boyd, 1992). If the mathematics a teacher knows is bound up with rich imagery derived from reflection, analysis, and expressions of systems of meaning, then his image of successful learning is likely to entail the expectation that students speak sensibly and coherently about their reasoning, their judgments, and the context giving rise to them.

A major problem faced by mathematics education, and I presume the same problem exists in science education, is that many, perhaps most, school mathematics teachers do not know their subjects in a way that allows them to speak coherently about their reasoning, their judgments, and the context giving rise to them. That is, they are unable to hold grounded conversations about mathematical ideas and mathematical thinking. There are relatively few studies examining teachers' and prospective teachers' capabilities to construct coherent explanations of their mathematical activities, but those that do (Ball, 1990; Post, Harel, Behr, & Lesh, 1991; Simon, 1993; Thompson, 1994c) suggest we face a problem of immense proportions. For example, Post et al. (1991) gave several versions of a ratio and rate test designed to reflect concepts covered in the intermediate (4-6) mathematics curriculum to 218 intermediate mathematics teachers in Illinois and Minnesota. Teachers scored, on average, between the mean 13-year-old NAEP average and the mean 17-year-old average on items drawn from the 1979 NAEP; overall performance varied widely across test versions, but average performance scores ranged from 60% to 69% across test versions, and more than 20% of the teaches scored less than 50%.

Most troubling about the Post et al. results is teachers' performance on problems like this: "Melissa bought 0.46 of a pound of wheat flour for which she paid $0.83. How many pounds of flour could she buy for one dollar?" (Post et al., 1991, p. 193). Only 45% of the teachers answered this question correctly and 28% left the page blank or said "I don't know." Teachers were also asked to explain their solutions as if to a student in their class. In the case of the "Melissa" problem, *only 10% of those who answered correctly could give a sensible explanation of their solution*. The authors concluded:

> Our results indicate that a multilevel problem exists. The first and primary one is that fact that many teachers simply do not know enough mathematics. The second is that only a minority of those teachers who are able to solve these problems correctly were able to explain their solutions in a pedagogically acceptable manner. (Post et al., 1991, p. 195)

They went on to draw implications for mathematics teacher education.

> We fail to understand how teachers without a relatively firm foundation could possibly be in a position to present and explain properly, to ask the right question at the right time, and to recognize and encourage high levels of student mathematical thinking when it occurs. Mathematics courses in teacher education programs, especially preservice mathematics courses, to date, have generally been concerned with rather superficial treatment of lofty content domains rather than a relatively deep treatment of elementary topical areas. (Post et al., 1991, p. 197)

The Post et al study is often interpreted as saying that elementary mathematics teachers need to take more mathematics courses. I interpret their study, and their conclusions, as calling for a radically *different* kind of mathematical preparation than the standard "mathematics for elementary teachers" courses that pervade elementary teacher education programs—a new kind of preparation which focuses on future teachers developing the *conceptual* structures we wish they had developed and which we hope they will foster in their students. A fundamentally important research question in mathematics education, which few people are examining, is how to educate young adults in ways of thinking they should have developed as elementary and secondary students, and at the same time have students elevate those ways of thinking to a level of reflective awareness so that, as teachers', their images of what-is-to-be-learned provide principled guidance for their instructional actions.[2]

The problems I've mentioned regarding mathematics teachers may be even more acute in regard to teachers of science at the elementary and middle grades. My newly-coined adage, "Teachers teach what they know," may be more evident in science teaching than in mathematics teaching, if only because elementary- and middle-school teachers' encounters with science commonly are few and are just as disconnected from their lived-in lives as is school mathematics.

The Impact of Elementary Experiences on Learning Secondary and College Mathematics and Science

We have ample evidence that the mathematical knowledge children construct in elementary school has a dramatic, debilitating effect on their capabilities to make sense of mathematical ideas encountered in high school and college. In a review of research on the early learning of algebra, Kieran (1989) concluded that students' difficulties with high school algebra are attributable largely to cognitive obstacles created by the way they learned arithmetic. Students learn that expressions are meant to be calculated, that "=" means "And the answer is," that word problems are occasions for them to guess what to do with numbers embedded in text, and that expressions have no structure (i.e., you read them from left to right and collect operations as you read).

A number of researchers have documented students' widespread belief that letters (as in "*x*") are objects to move around without representational significance (Herscovics, 1989; Sfard & Linchevski, 1992; Sfard & Linchevski, 1994; Wagner, 1981), and that letters in expressions are tantamount to labels (as in "$6S=P$ means there are 6 students for every professor" (Clement, 1982; Clement, Lochhead, & Monk, 1981; Rosnick & Clement, 1980)).

Carpenter and Moser (1983), as a by-product of their longitudinal study of children's learning of addition and subtraction, found that they were better problem solvers at the beginning of first grade than they were at the end of third grade. At the beginning of first grade, children's

[2] In (Thompson, 1994c) I found evidence that the problems discussed by Post et al., Ball, and Simon with elementary teachers is, in principle, just as acute with prospective secondary mathematics teachers.

tendencies were to solve problems by understanding them and acting appropriately to their understandings. At the end of third grade, children searched for a rule to fit a stereotype and were stymied if their memory failed them. Somewhere along the line they had learned that they were not supposed to solve problems by first understanding them. I suspect they were taught this view conscientiously, but not consciously.

The reasons for Carpenter and Moser's findings are clear. Porter (1989) reported

> In the two studies of Michigan teachers [48 teachers in grades 4 and 5], 70% to 75% of mathematics instruction was spent teaching skills—essentially how to add, subtract, multiply, and divide—and occasionally how to read a graph. .... Of the time not spent on skill development, teachers in both studies spent about half of it developing conceptual understanding and the other half teaching problem-solving skills, primarily story problems. One caveat is necessary here: In these analyses, applications consisted of work in which the operations necessary to solve a problem were not explicitly stated but were implied in the presentation of the problem. Nevertheless, all work on story problems was counted as applications, even if students worked on a page in which every story problem involved the addition of single-digit numbers. ... Two hundred sixty of the 288 possible topics in the content catalog were taught at least a little by at least one teacher, yet not one teacher spent one minute teaching applications involving percents. A full two-thirds of the fourth-grade teachers spent less than one hour across the full school year on story problems involving multiplication of whole numbers, yet multiplication skill development typifies the fourth-grade curriculum. (Porter, 1989, pp. 10-11)

In this same regard, Fuson, Stigler, and Bartsch (1989) found that elementary mathematics curricula in Japan, China, and the Soviet Union place far greater emphasis on drawing mathematical concepts *from* substantive, non-routine applications—and on developing skill through dealing efficiently with applications—than is the case in the United States. As I will argue in the next section, an image built from drawing mathematics *out of* situations, instead of applying mathematics *to* situations, is propitious for conceptual development of both mathematical and scientific reasoning, whereas the reverse too often is debilitating for both.

In short, as a result of their experiences in elementary school mathematics, many of our students build an image of mathematical activity as being: senseless, about nothing of substance, about performing rituals when commanded to do so, and about coping with demands for performance when they are clueless regarding what they are being asked. This is not a sound basis for science education reform, and it will impinge mightily on efforts to make science education more substantive.

A final note: As I watched my own children's introduction to science, it seems that science is taught as a language art.[3] Science lessons in elementary school often entail having children read some part of the text and then answer questions based upon what they have read.[4] The questions are often *designed* so that they can be answered by searching the text. The *Benchmarks* will be drastically transformed when implemented in settings where students and teachers work from an image of science as an extension of the reading period.

A Need for Greater Emphasis on Quantity and Quantitative Reasoning

Mathematics is presented in the *Benchmarks* as dealing only with numbers and shapes, with an emphasis on logical reasoning. Abstraction, as a mathematical activity, is presented as the process of weeding out details and using letters to represent what is left. This is, to put it mildly, a less than powerful image of school mathematics.

In my estimation, the *Benchmarks* presents a view of mathematics that is devoid of conceptual imagery. This view is expressed repeatedly, as in:

> Mathematics is the study of any patterns or relationships, whereas natural science is concerned only with those patterns that are relevant to the observable world. Although mathematics began long ago in practical problems, it soon focused on abstractions from the material world, and then on even more abstract relationships among those abstractions. (American Association for the Advancement of Science, 1993, p. 29)

While the above quote may be an accurate characterization of practicing mathematicians' mathematical activity it is highly problematic as a characterization of a subject which students can come to know. It suggests that even though mathematics originated in practical matters a long time ago, today it deals primarily with matters removed from our everyday world—abstract matters like number, shape, and relationship. This view is problematic because it suggests that numbers, shapes, and relationships are *given*, that they are the primary starting point for mathematical

[3] I owe this observation to my wife, Alba Thompson.

[4] I was dismayed to see how this pattern is internalized by teachers. I attended a workshop for teachers on "Developmental Learning in Mathematics and Science." The workshop leader organized us in discussion groups, asking each group to read short excerpts from articles and answer discussion questions presented after the excerpts. The first question was, "What are the major problems young children experience in learning mathematics?" As I spoke in response to the question, I noticed my group's blank stares. Eventually, one teacher said, "I think the answer they want is right here, on page so and so."

inquiry. Everything I know about curriculum and learning says that mathematical instruction which begins from such a starting point creates major obstacles for students. It puts students in the position of studying "objects" which, to them, do not exist.

The *Benchmarks* presents a highly symbol-oriented view of mathematical activity. While symbolic representation and formal (symbolic) reasoning are essential aspects of mathematical activity, this activity must be carefully cast. If symbols are too soon the objects of students' attention, then they become opaque and no longer serve a representative role—they become, from the students' perspective, what their activity is about. When that happens, what we see as symbols are, from the students' perspectives, merely letters with which they must "do things" (Sfard & Linchevski, 1992; Sfard & Linchevski, 1994). As Lynn Steen said, the ontological relationship between mathematical reasoning and symbolic activity should be like that between music and its expression in a score: first comes the music, then comes the score (Mathematical Sciences Education Board, 1989, p. ??).

The *Benchmarks* also contains misguided advice regarding symbolic activity. Specifically:

> Students should begin to assign letters as temporary names of objects—mathematical or not—for purposes of discussing these objects when no other name is known. Gradually the notion of a symbol standing in for a *particular* unknown can be extended to its standing for any of a *collection* of possible unknowns. (American Association for the Advancement of Science, 1993, p. 37)

The treatment of letters-as-names leads directly to what is called "fruit salad" mathematics (Philipp, 1992), as in "$3x+5x=8x$ in the same way that 3 apples plus 5 apples is 8 apples." A direct consequence of fruit-salad mathematics is the widespread occurrence of students writing $6S=P$ in response to "There are six students for every professor at this university. Write an equation that expresses this relationship using S for the number of students and P for the number of professors" (Clement, 1982; Clement et al., 1981; Lochhead, 1980; Lochhead & Mestre, 1988; Rosnick & Clement, 1980; Wollman, 1983).

My complaints revolve around this theme: the *Benchmarks* present mathematics and, to a lesser extent science, as being *non-quantitative*. I do not mean that neither is presented as dealing with numbers. Rather, numbers are not tied to anything, and scientific ideas are not quantified. I cannot go into great detail here about notions of quantity and quantitative reasoning. I have developed these ideas elsewhere in areas of additive reasoning (Thompson, 1993), multiplicative reasoning, ratio and rate (Thompson, 1994b; Thompson & Thompson, 1992; Thompson & Thompson, 1994), calculus (Thompson, 1994c), and the relationship between quantitative reasoning and symbolic activity (Thompson, 1992b; 1994a; 1995). In all this work the notion of quantity is, put simply, anything that can be measured, and students abstract (i.e., "pull") mathematics from their activity of reasoning about quantities and relationships among quantities, and from their activity of representing their reasoning. Another theme throughout this work is that mathematical reasoning is grounded in imagery (Thompson, 1996), where an essential role of imagery is that it supports students' constructions of the quantities themselves—the things about which students initially reason (and which they carry along with their reasoning) and from which they develop their mathematics.

Rather than trying to relate the entirety of this research program, I will give several examples: Examples of quantitative reasoning in mathematics and examples of how the same approach might be applied in science.

Quantitative Reasoning in Mathematics

The problem below is typical of present day high school algebra.[5]

> I walk from home to school in 30 minutes, and my brother takes 40 minutes. My brother left 6 minutes before I did. In how many minutes will I overtake him? (Krutetskii, 1976, p. 160)

An algebraic approach would be to set up an equation, such as $(6+t)\frac{d}{40} = t\frac{d}{30}$, and then solve for t. This is a customary approach taught in first-year algebra, and U.S. students would not see a problem like this until then.

On the other hand, this problem can be approached in a much more "situation-sensitive" manner. Here is an example of one such chain of reasoning:

[5] At the time of Krutetskii's publication, this problem was in the Soviet sixth-grade mathematics curriculum.

- Imagine myself and brother walking: What matters is the distance between us and how long it takes for that distance to become zero.
- The distance between us shrinks at a rate that is the difference of our walking speeds.
- I take $\frac{3}{4}$ as long as brother to walk the same distance, so I walk $\frac{4}{3}$ as fast as brother.
- Since I walk $\frac{4}{3}$ as fast as brother, the difference of our speeds is $\frac{1}{3}$ of brother's speed.
- The distance between us shrinks at $\frac{1}{3}$ of brother's speed, so the time required for it to become zero is 3 times what brother took to walk it.

  I will overtake brother in 18 minutes.

The reasoning behind this stream-of-consciousness example[6] is based on a sophisticated understanding of speed as a rate of change of distance with respect to time. It also reflects an orientation to reason from a sophisticated image of two people walking and of quantitative relationships involving relative motion. It is the *image* of the situation (distance between them growing and shrinking), which itself entails quantities and quantitative relationships (distance to school, my walking time, brother's walking time, my walking speed, brothers walking speed, my speed in relation to brother's, inverse relation between speed and time given a constant distance, etc.), which supports the chain of reasoning actually expressed.

In (Thompson, 1994b) I relate a teaching experiment in which a fifth-grade child constructed a concept of constant rate of change by first constructing a deep conceptualization of speed as a rate. Students' concepts of speed as a rate entail a scheme of mental operations which supports imagery of two growing amounts in relation to one another, and each in relation to the amounts by which they grow. By building this scheme in relation to speed, this child was able to comprehend complex situations involving price and flow as rates of change.

Following that teaching experiment I worked with two sixth-grade children, repeating the same development, and extended it to address the idea of acceleration. The following is their introduction to acceleration:

[6] Contrast the reasoning in this stream-of-consciousness example and the example in the *Benchmarks*, page 218.

Pat drops a paper clip several times while standing on a chair and holding the clip against the ceiling (8 ft above the floor). Kids take turns timing the paper clip's fall with a stop watch.

Pat: About how long does it take to drop?

Kids: Around .76 seconds.

Pat: Does the paper clip fall at the same speed all the time?

Kids: No. It speeds up.

Pat: What does that mean?

Kids: It goes faster each second.

Pat: But it doesn't even fall for one second. How can it go faster each second?

Kids: (Thinking about this for a long time.) It goes faster each tenth of a second.

Pat: Just each tenth of a second?

Kids: Well, no. You could do it for each hundredth of a second if you wanted.

Pat: Okay … The ceiling is 8 ft above the ground. What was the paper clip's average speed?

Kids: [Thinking.] (8 ÷ .76) ft/sec. (Calculator: 10.53 ft/sec).

Pat: What does this mean? How can it go an average of 10.53 ft/sec if it only goes 8 feet?

Kids: (Pause.) It doesn't need to go 10 feet. It means if it went for one second at this speed it would go 10.53 ft.

Pat: If we dropped it from 10.53 feet, would the paper clip take one second to hit the floor?

Kids: (They're not sure about this. It appears they want to say that it would take one second if it fell at 10.53 ft/sec, but they suspect that it wouldn't have an average speed of 10.53 ft/sec if it were dropped from 10.53 ft.)

Pat: Here's some graph paper. Now, you say that the paper clip speeds up each tenth of a second, and that from 8 feet it will take .76 seconds to fall. This is what I want you to do: Sketch a graph of the paper clip's distance above the ground at each tenth of a second while it is falling. Make sure that your graph has the paper clip starting 8 feet above the ground and has it hitting the ground in .76 seconds.

Kids: (They construct a graph which resembles the descending half of an inverted parabola; intersects axes at (0,8) and (.76,0)

Notice three things in this example: The word "acceleration" never appeared, the discussion emphasized paradoxes regarding average speed and non-constant speed, and the question I asked at the end involved nothing more than sketching a graph. Yet, while constructing their graph of the clip's distance from the floor versus the amount of time it had fallen, these children determined that "to go faster each tenth of a second" meant that in each tenth of a second the clip had to fall farther than in the previous tenth of a second, and at the same time it had to intersect the x-axis at (.76,0). That is, they negotiated between constraints, several times modifying their graph until the constraints were satisfied. In the process, they made explicit the notion of acceleration as a rate of change of a rate of change.

I also asked this question:

Pat: Imagine this. I'm driving my car at 50 mi/hr. I speed up smoothly to 60 mi/hr, and it takes me one hour to do it. About how far did I go in that hour?

Sue: (*Long pause. Begins drawing a number line.*)

Pat: What are you doing?

Sue: I figure that if you speed up 10 miles per hour in one hour, that you speeded up 1 mile per hour every 6 minutes. So I'll figure how far you went in each of those six minutes and then add them up. (*See Figure 1.*)

Pat: (*After Sue is finished*.) Is this the exact distance I traveled?

Sue: No … you actually traveled a little farther.

Pat: How could you get a more accurate estimate?

Sue: (*Pause.*) I could see how far you went every time you sped up a half mile per hour.

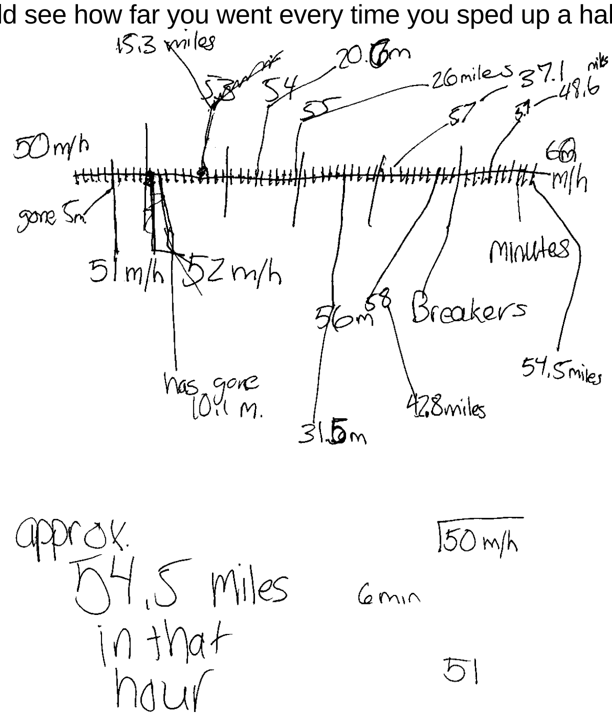


Figure 1. Sue's Diagram

Figure 1 shows Sue's work. She assumed that I accelerated at the rate of $10\frac{mi/hr}{hr}$, which would be equivalent to $1\frac{mi/hr}{{}^{1}\!/_{10}\,hr}$. She then assumed I drove for one-tenth of an hour (6 minutes) at 50 mi/hr, then one-tenth hour at 51 mi/hr, and so on. She then determined how far I would go in each of these one-tenth hour periods.

Sue's solution to estimating the distance I traveled while accelerating has the structure of a Riemann sum. It would be expressed formally as

$\Delta V = final\ speed - initial\ speed$ (a number of miles per hour)
$\Delta T = final\ time - initial\ time$ (a number of hours)
$\Delta t = \frac{\Delta T}{\Delta V}$ (a number of hours)[7]
$\Delta v = 1$ mi/hr
$n = \Delta V/\Delta v$
$d = \sum_{i=0}^{n-1}(initial\ \ speed + i\Delta v)\Delta t$ (a number of miles),

My point in these examples is to illustrate how a conceptual development of mathematical ideas, beginning with notions of quantity and quantitative reasoning, can pay off over time in advancing students' mathematical reasoning. I chose my examples also to illustrate the merging of traditionally scientific concepts (speed, acceleration, rate of change) with a mathematical orientation. I also suggest that you contrast this orientation toward speed and rate of change with the *Benchmark's* discussions of motion.

Deep treatments of motion and rate of change also are propitious for students' development of variable magnitude and functional relationship (Confrey, 1994; Confrey & Smith, 1994; Confrey & Smith, 1995; Kaput, 1992; 1994).

In the next section I outline other possibilities, not as an exhaustive list but as indicators of a point of view that seems promising for research on curriculum/learning/teaching in mathematics and science education.

[7] It is important to note that, formally, the unit of $\Delta T/\Delta V$ should be *hr/(mi/hr)*, but Sue evidently reasoned that $(1/\Delta V)$*ths* of the total change in velocity should correspond to $(1/\Delta V)$*ths* of the time in which the change in velocity occured. Therefore each increment of the total time would be $(\Delta T/\Delta V)$*ths* of one hour.

Quantitative Reasoning in Science

Density and Specific Gravity

Figure 2 comes from a 7th-grade science assignment. It was at the end of a unit on density and specific gravity. It occurred to me that this unit provided an excellent occasion to merge the scientific concept of density with matters of ratio, rate, and percent. But for that to happen the unit needed to be recast so that it attended to matters of quantification: What qualities of what kinds of objects might people have been concerned with so that *density*, as an attribute of something, was important? How did the *idea* of density provide a solution to their practical concerns? We all know the story of Archimedes in his bathtub, but density and specific gravity did not become "mainstream ideas" after his discovery. According to Klein (1974), measurements of density and specific gravity became matters of practical concern because of tax assessors' need to measure the alcohol content of beers and spirits.

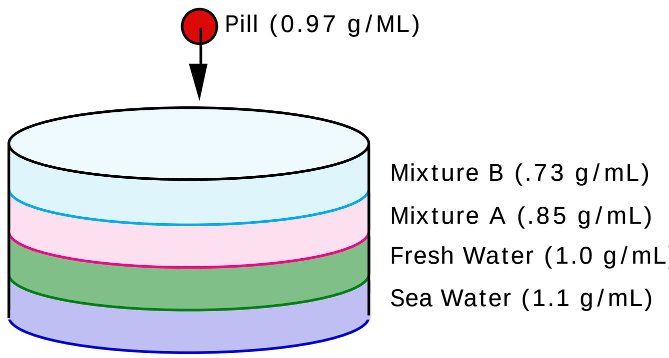
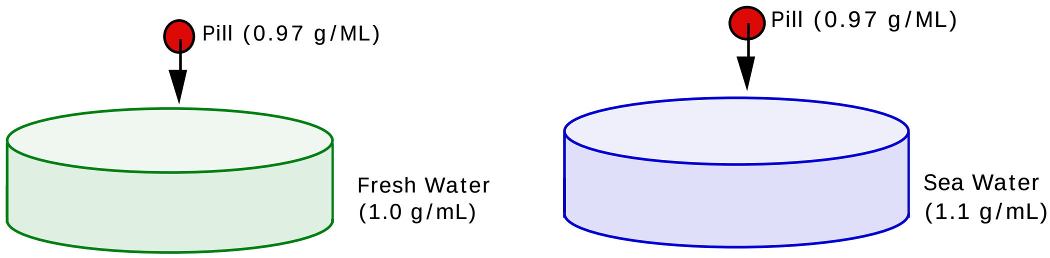


Figure 2. Where will the ball float?

> The annals of densitometry are still instructive, not because they present a pattern of how metrological progress should be made, but rather because they are so intertwined with familiar human habits, vices and misfortunes. The habits and the vices are apparent enough, when we note that a very large part of work on densitometry has been in the category of alcoholometry, the measurement of the relative alcohol content of spirituous beverages by means of their densities. The unfortunate aspect arises from the fact that such measurements were demanded, not for the health or well-being of the imbibers, but for the benefit of the excisemen who were about to tax those spirits. (Klein, 1974, p. ??)

For students also to make connections among density, specific gravity, ratio and rate, they need *problematiques* (Balacheff, 1990)—occasions which provide a setting where those connections prove useful. The original setting (Figure 2) and question demand only a "float/sink" decision. They do not demand a quantification of anything or connections among quantifications. Here is one set of questions which I think might be more useful in that regard.



• The pill has a volume of 1 ml. What fraction of the pill's volume will be *out* of water when it floats in water?

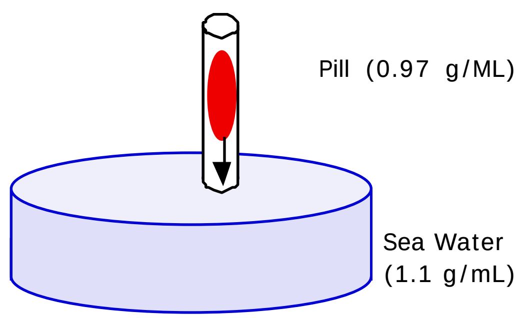
• Suppose the pill has density 0.97 g/ml and has a volume of 3.7 ml. What fraction of the pill's volume will be out of water when it is put into water?

• Suppose the pill has density 0.97 g/ml and has a volume of 8.92 ml. What fraction of the pill's volume will be out of water when it is put into water? Does the pill's actual volume matter in regard to what fraction of its volume will be out of water? Why?

• Suppose the pill has density 0.97 g/ml and has a volume of 2.4 ml. What fraction of the pill's volume will be out of water when it is put into in sea water?

• Suppose the pill has density x g/ml and has a volume of y ml. What fraction of the pill's volume will be out of water when it is put into a fluid having density z g/ml?[8]

• A 2 in × 3 in × 7 in block of wood is floating upright in sea water (2 inch × 3 inch side on top). It sticks up 0.75 inches above the water line. What is the block's specific gravity?



Also, qualitative aspects of "floating" are important to address explicitly from a quantitative point of view: Suppose the pill is oblong instead of spherical, its density is 0.97 g/ml, its volume is 8.2 ml, and it is forced to stay upright in sea water by being held within a plastic tube. Will this make the pill's "out of water" volume be different than when the pill is allowed to float freely? Why?

[8] The pill might not float. Your "rule" must take this into account.

Relative Motion/Relative Time

Here is a situation we have used to investigate ideas of rate of change and frames of reference:

> A “pitching machine” is a machine that shoots baseballs to simulate a baseball pitcher. Patrick Henry High School’s pitching machine shoots baseballs at a speed of 90 ft/sec, and it is set to shoot balls every 2 seconds.
>
> One day the machine was loaded with “gravity resistant” baseballs—they don’t fall to the ground, they just go in a straight line at 90 ft/sec—firing one every 2 seconds. The machine shot gravity resistant balls at José, the center fielder, while he ran straight at the machine at a speed of 10 ft/sec. He caught each ball fired at him, and then released it as he continued running.
>
> José’s coach thought that the time intervals between catches should get smaller as José got closer to the machine, since successive balls had smaller distances to travel. José’s father thought that José should catch one ball every 2 seconds, since that is how rapidly the machine fired them.
>
> What is your opinion? Explain.

Pressure

I don’t trust tire pressure-gauges. I have 5 and, on the same tire, 3 give a reading of about 26 lb/in^2 and the other two give a reading of about 31 lb/in^2. My owner’s manual says the tire pressure should be 29 lb/in^2 in each tire. It also says that the car weighs 2950 lb with a full tank of gas. I have inflated the tires so that, by one gauge, they all have the same pressure. How can I get a trustworthy measurement of my tires’ air pressure without using a tire gauge?

Force

I urge science educators to see Hans Freudenthal’s (1993) posthumously-published article on didactic phenomenology and the teaching of mechanics. It is entirely consistent with notions of Piaget that mental operations emerge by reflecting on internalized actions, and with Johnson’s (1987) claim that images of bodily action are, developmentally, at the basis of meaning—even formal meaning. Freudenthal’s *didactic phenomenology*, put simply, emphasizes the importance of helping students develop ways of thinking about ideas that are propitious for conceptual development and formalization. For example, the *Benchmarks* says

> Like inertia, the action-equals-reaction principle is counterintuitive. To say that a book presses down on the table is sensible enough, but then to say that the table pushes back up with exactly the same force (which disappears the instant you pick up the book) seems false on the face of it. (p. 87)

Freudenthal might have suggested having students "be the table"—lay on their backs with hands held above them parallel to the floor—and have someone place a book on their hands. The discussion would center around when the "table" did and did not exert an upward force, and how much upward force the "table" exerted when the book was placed on it and when the book was removed.

Didactic phenomenology as a pedagogical philosophy (Freudenthal, 1983) has shown considerable promise as implemented within mathematics curricula in The Netherlands. The approach is being piloted in Milwaukee schools under the direction of Tom Romberg. It would an interesting research program which investigates curricular possibilities and students' learning in settings that implement didactic phenomenology in science education.

Gravity

What would be the gravitational force exerted by one body in the absence of any other bodies? ZERO!

I am continually amazed by school science texts which speak about the gravitational force exerted by the earth on some object without speaking about the gravitational force exerted by that object on the earth. Here is a recap of a conversation I had with my daughter when she was in the seventh grade and trying to reconcile Newton's law of action-reaction with her common-sense idea that a steel ball is "too small" to pull the earth.

Her: How can this little ball pull the earth? You mean the earth moves too when I drop it?

Me: Well, suppose you had two equal masses, A and B, at a distance. How far would each move in one second?

Her: The same distance.

Me: Suppose that B has 1/10 the mass of A. How far would A and B move in one second?

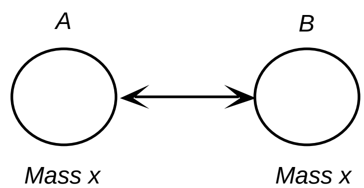
Her: A would move 1/10 as far as B.

Me: Suppose B has 1 billionth the mass of A. How far would A and B move in one second?

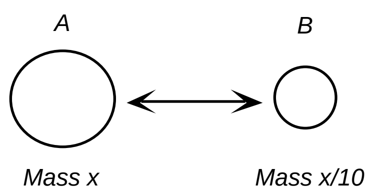
Her: A would move one billionth as far in one second.

Me: The earth is how many times more massive than this ball?

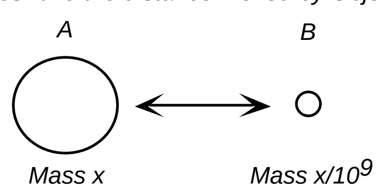
Her: About a zillionth. … So, the earth would move about a zillionth of a foot if the ball moved one foot!



Object A moves the same distance as Object B in 1 second.



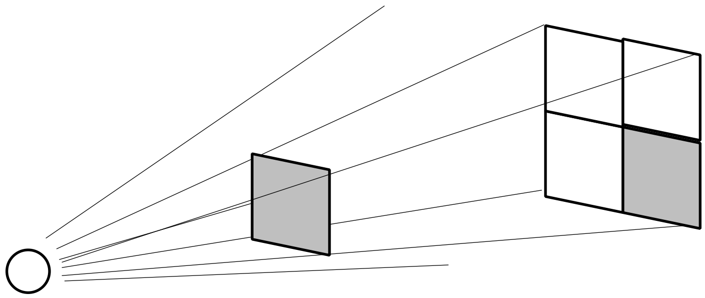
Object A moves 1/10 the distance moved by Object B in 1 second.



Object A moves $1/10^9$ the distance moved by Object B in 1 second.

Even though the reasoning is technically inaccurate (I should have asked her to think about accelerations rather than distances), the point is still that by basing the development on an invariant feature of the system (mutual attraction), it became sensible to her that, indeed, a tiny steel ball pulls on the earth with the same force that the earth pulls on it. As an aside, the reasoning called for by this development is precisely that required to conceptualize balance as an equilibrium of torques. You have two quantities in multiplicative relation that constitute an equilibrated system.

In the same vein, the notion that the gravitational attraction exerted by two bodies on each other is inversely proportional to the square of the distance between them might be made concrete in the same way that intensity of light can be made concrete. Of course, one would need to be careful that students don't understand this as saying gravity is due to "gravity rays".



Double the distance between a light source and an object. The same amount of light now covers 4 times the area, so the same object recieves 1/4 the amount of light as before.

In mathematics, we could diverge from talking about light intensity or gravity to talk about scaling factors, similarity, and dimension in any projective system.

Summary

I developed three themes, and attempted to show that the first two are aspects of the third. The first theme was that mathematics and science education share a significant common problem: the improvement of elementary and secondary teacher education programs, with the former being the more urgently needed. The second theme was that imagery built up by students through their experiences in elementary school has a far deeper influence on their ability to construct mathematical and scientific concepts than we have thought. The third is that school science and mathematics instruction and curricula, including those envisioned in the *Benchmarks*, need to pay greater attention to students' conceptualization of physical quantities and quantitative reasoning than is presently the case.

The examples in the previous section point out that if, in science education, greater attention were paid to physical quantities as measurable attributes of objects, there would be many occasions to pull mathematics out of science and to put mathematics into science. I suspect students would end up with deeper understanding of both scientific and mathematical concepts.

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