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Problems of Reification: Representations and Mathematical Objects

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Had Bishop Berkeley, as many fine minds before and after him, not criticized the ill-defined concept of infinitesimal, mathematical analysis — one of the most elegant theories in mathematics — could have not been born. On the other hand, had Berkeley launched his attack through Internet, the whole foundational effort might have taken a few decades rather than one and one-half centuries. This is what we were reminded of when starting our discussion. Like Berkeley, we were dealing with a theory that works but is still in a need of better foundations. Unlike Berkeley and those after him, we had only a few months to finish, and we had e-mail at our disposal.

Needless to say, the theory we were concerned with, called *reification*, was nothing as grandiose and central as mathematical analysis. It was merely one of several recently-constructed frameworks for investigating mathematical learning and problem solving. The example of Bishop Berkeley taught us there is nothing more fruitful than a good disagreement. Thus, we decided to play roles, namely to *agree to disagree*. Since we are, in fact, quite close to each other in our thinking, we sometimes had to polarize our positions for the sake of a better argument.

The subject proved richer and more intricate than we could dream. Inevitably, our discussion led us to places we did not plan to visit. When scrutinizing the theoretical constructs, we often felt forced to go meta-theoretical and tackle such basic quandaries as what counts as acceptable theory — or why we need theory at all.

Above all, we enjoyed ourselves. We also believe it was more than fun, and we hope we made some progress. Whether we did, and whether our fun may be shared with others, is for you to judge.

Pat:

The following excerpt appears in *Research in Collegiate Mathematics Education*. In it I speak about the "fiction" of multiple-representations of function—I do not speak about reification as such. However, I think what I said about functions is applicable in the more general case of reification, too: That we experience the subjective sense of "mathematical object" because we build abstractions of representational activity in specific contexts and form connections among those activities by way of a sort of "semantic identity." We represent to ourselves aspects of (what we take to be) the same situation in multiple ways, and we come to attribute logical identity to our representations because we feel they somehow represented the "same thing."

A number of fuzzies are entailed in what I said above, such as matters of scheme and matters of abstraction. I'm sure these will come out as we go along.

> I believe that the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of representation.[1] Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us, but instead may be areas of representational activity among which, as Moschkovich, Schoenfeld, and Arcavi (1993) have said, we have built rich and varied connections. It could well be a fiction that there is any interior to our network of connections, that our sense of "common referent" among tables, expressions, and graphs is just an expression of our sense, developed over many experiences, that we can move from one type of representational activity to another, keeping a current situation somehow intact. Put another way, the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance.
>
> I do not make these statements idly, as I was one to jump on the multiple-representations bandwagon early on (Thompson, 1987; 1989), and I am now saying that I was mistaken. I agree with Kaput (1993) that it may be wrongheaded to focus on graphs, expressions, or tables as representations of function, but instead focus on them as representations of something that, from the students' perspective, is *representable*, such as some aspect

[1] This is entirely parallel to the situation in information processing psychology—no one has bothered to question what is meant by "information" {Cobb, 1987 #429; Cobb, 1990 #38} .

> of a specific situation. The key issue then becomes twofold: (1) To find situations that are sufficiently propitious for engendering multitudes of representational activity and (2) Orient students to draw connections among their representational activities in regard to the situation that engendered them.
>
> (Thompson, 1994b, pp. 39-40)

Anna:

How daring, Pat! After all, the idea you seem to be questioning is quite pivotal to the research in math education right now. A comparable move for a physicist would be to say that he or she doubts the soundness of the concept of force or energy. Indeed, what can be more central to our current educational project than the notion of representation? What could be more fundamental to our thinking about the nature of mathematical learning than the idea of designating mathematical entities in multiple ways? Your skepticism does not sound politically correct, I'm afraid. But, I'm glad you said this. In fact, I have had my doubts about the "careless" way people use the notion of representation for quite a long time now. Obviously, when one says that this and that are representations, one implies that there exists a certain mind-independent *something* that is being represented. Not many people, however, seem to have given a serious thought to the question what this something is and where it is to be found.

Some methodological clarifications could be in point before we go any further. I remember the discussion that developed in August 1993, when Jim Kaput decided to forward your blasphemous statement to his Algebra Working Group.[2] Many people responded then to the challenge, but my impression was that each one of them attacked a different issue, and everybody was looking at the problem from a different perspective. Somehow, the disputants seemed to be talking past each other rather than disagreeing. For example, David Kirshner interpreted your statement as a rejection of introspection. He said:

> Amen to Pat!
>
> I am deeply supportive of perspectives that challenge the presumed connection between our introspections about our knowledge and the actual underlying representations. Understanding consciousness as a mechanism

[2] The Algebra Working Group is an affiliation of mathematics educators who communicate regularly via Internet on matters pertaining to learning and teaching algebraic reasoning at all grade levels. The AWG is managed by Dr. James Kaput, JKAPUT@UMASSD.EDU, under the auspices of the University of Wisconsin's National Center for Research in Mathematical Sciences Education.

> that allows us to maintain a coherent picture of ourselves for the purposes of interacting within a social milieu, shows introspection as an extremely unreliable guide to our actual psychology …

Thus, for David your message was mainly methodological: it dealt with internal rather than external representations and with the problem of how to investigate these representations rather than with the question of the existence and the nature of their referents. Ed Dubinsky, on the other hand, understood the issue as mainly epistemological. While taking the use of the term "representation" for granted, Ed translated your dilemma into the question how we come to know and how we construct our knowledge:

> I think that Pat raised … the epistemological question of existence and representation. No one seems to have trouble with various forms of representational activities, but if one speaks of representation as a verb, then its transitivity forces one to ask the question what is being represented. Actually, Pat is asking the deeper question, is anything being represented?

Although these two interpretations are miles apart, they seem to share a tacit ontological assumption. This assumption was also quite clear in the language you used yourself. As I already noted, the very term "representation" implies that it makes sense to talk about an independent *existence* of certain entities which are being represented. The expression "multiple representation" remains meaningless unless we believe that there is a certain *thing* that may be described and expressed in many different ways. I am concerned about the fact that the discussion whether this implication should be accepted or rejected took off before the disputants explained what kind of "existence" each one of them had in mind. One may agree or disagree with the claim about the existence of mathematical objects, but if the meaning of the world "existence" in this particular context is not made explicit, our discourse will never rise above the level of a mere word game.

Let me present you with two options (by no means exclusive). First, it seems to me that the default interpretation of the whole issue would be as follows: we should view our problem philosophically rather than psychologically, and the Objectivist outlook should be taken as a point of departure. Let me explain.

Objectivism was defined by the American philosopher Putnam as a view grounded in two assumptions:

> 1. there is a clear distinction to be drawn between the properties things have “in themselves” and the properties which are “projected by us;”
>
> 2. the fundamental science … tells us what properties things have “in themselves.” (Putnam, 1987, p. 13)

In this description, Putnam refers to science rather than to mathematics. In the case of mathematics, the problem is somewhat more difficult, as the distinction between the knowledge and the object of this knowledge is much less clear than in the case of physics or biology. Even so, the very fact that representation is a central motif of our discourse shows that we do view the mathematical realm as independent from the way we think or talk about it.

An alternative position would be psychological, and not philosophical. We could concentrate on what people have in mind and disregard the problem of the “objective” existence of mind-independent abstract entities. But then, of course, the question must be answered what we have in mind when we talk about things “that people have in mind.” Did you notice the circularity in this last sentence? It seems we cannot escape it, just as we cannot escape talking about mind. A truly sticky issue, isn’t it?

Pat:

A sticky issue indeed!

Let me, for the moment, side-step the philosophical matters you raised and speak about my motive for saying what I did. My motive was pedagogical. The multiple-representations movement often translated into a particular kind of instruction or a particular kind of curriculum: Show students several representations and tell them what they mean—or worse, have them “discover” what they mean. To the person doing the showing, the representations always represented something—a function, a structure, a concept, etc. That is, the person doing the showing has an idea in mind, and presents to students something that (to the person doing the showing) has that idea as its meaning. This creates an impenetrable loop—impenetrable by students, that is. So, the background motivation for my opening statement was largely pedagogical, and its thrust was psychological. I was calling for taking students’ reasoning and imagery as preferred starting points for discussions of curriculum and pedagogy instead of taking (fictitiously) unitary constructs, like function (or division, fraction, rational number, etc.), as preferred starting points.

I hope you don't interpret these remarks as saying that we must abandon adult mathematics and be satisfied with whatever mathematics children create. Rather, I was saying that we must be more clever. Rather than teaching the mathematics we know, we should understand students' construction of concepts *from their assimilations and accommodations over long periods of time*, and to be open to the realization that what students end up knowing never will be a direct reflection of what we teach. Neither David nor Ed, in the AWG excerpts you presented earlier, picked up on the last paragraph in my opening statement — that we cease our fixation with representations of (our) big ideas and instead focus on having students use signs and symbols only when they (students) have something to say through them (symbols).

I propose that we force ourselves to speak in the active voice—that when we speak of a representation, we always speak of to whom it is a representation and what we imagine it represents for them. When we speak of, say, "tables as representations of functions," we say for whom we imagine this to be true, what we imagine it represents for them (the idea they are expressing in a table when producing it, or the meaning they are reading from the table if it is presented to them), and something about the context in which this is all happening.

What, you ask, does this have to do with reification? It is this: Whenever I observe people doing mathematics and constrain myself to speak in the active voice, and constrain myself to be precise in my use of "representation," I don't see objects in people's thinking. Instead, I see schemes of operations and webs of meaning. Sometimes these schemes are ill-formed or in the process of formation. Sometimes they are well-formed and highly integrated. In the latter case, the people possessing these schemes maintain that they are thinking of "mathematical objects."

Anna, your turn!

Anna:

Easier said than done, Pat.

When you insist that we should "cease our fixation with (our) big ideas and instead focus on having students use signs and symbols only when they (students) have something to say through them (symbols)," you seem to have a more or less clear image of what you want to say and where it all is supposed to lead us to. But it is not that obvious to me. I still feel that this conversation will not proceed if we don't make it clear what this discussion is about.

You already seem to have given your response. If I understood you well, you are questioning the concept of representation (which, in the present context, is meant to refer mainly to an external representation, right?), and you do it on the grounds of a claim that in the eyes of many students, nothing is being represented by a graph or a formula; the abstract objects that would unify the clusters of symbols supposed to refer to the same thing are absent from student's mind only too often. I agree whole-heartedly that we should reconsider the concept of representation. This will force us to focus on the notion of mathematical object, and try to examine its possible meaning and uses. I will stress right away that, for me, the notion of "mathematical object" can only function as a theoretical construct, and it should only be used as such if a good theory may be built around it.

Someone may ask whether anything as elusive as the idea of mathematical object has a chance to turn into a scientific concept at all. The notion of science, however, has greatly evolved in the last few decades, and it became a lot more flexible than it was when cognitive approaches made their first steps toward general acceptance. Many factors brought about this change. One of them was the growing dissatisfaction with the information-processing account of the functioning of human mind. Another was the evolution of philosophers' vision of human knowledge. The notions about what is scientific and what cannot be regarded as such underwent dramatic transformations. In fact, the demarcation line between science and non-science, once so clear to everybody, was irreversibly blurred and in certain domains became almost impossible to draw. Today, the majority of those who view themselves as scientists are prepared to deal with concepts that would once be discarded by them without hesitation. Varela et al. (1993) admit that still

> most people would hold as a fundamental truth the scientific account of matter/space as collection of atomic particles, while treating what is given

> in their immediate experience, with all its richness, as less profound and true (pp. 12-13).

Cognitive science, however, cannot discard all the elements of human immediate experience any longer, and thus it is

> Janus-faced, for it looks at both roads at once: One of its faces is turned toward nature and sees cognitive processes as behavior. The other is turned toward the human world (or what phenomenologists call the "life-world") and sees cognition as experience (Varela et al., 1993, p.13).

Today, nobody is really afraid anymore of talking about such immeasurable entities like concept images and abstract objects — the entities that can only be seen with our minds eyes. To great extent, it is the growing abandonment of the Objectivist epistemology that made us more daring than ever in our theorizing about the human mind and about its functioning. Indeed, we came a long way since the times when "mind" itself sounded somewhat dirty. Nowadays, people are no longer concerned with the objectivity of knowledge — with the question how well a given scientific theory reflects the "real" state of affairs. There is no belief anymore in the "God's eye view" of reality. The concern about the truthfulness of our representation of the pre-given world has been replaced with pragmatic questions of usefulness (Lyotard, 1992) and of "intersubject agreement" (Rorty, 1991). One of the central criteria for evaluation of scientific theories is the question whether they are likely to generate many interesting ideas:

> … the justification of scientific work is not to produce an adequate model or replication of some outside reality, but rather simply to produce *more* work, to generate new and fresh scientific … statements, to make you have "new ideas" … (Jameson, 1993, p. ix)

Thus, if I somewhat disagree with your critique of the notion of (multiple) representation, it is not because I wish to keep this notion with its traditional meaning intact. On the contrary, I think that the use of the word in the context of cognitive science is somehow misleading. But for me, it is misleading not so much because of the fact that the referent of the symbol may be absent from the student's mind, but because when construed in the traditional way, it seems to reinforce an Objectivist approach. It is misleading because it implies an existence of an objectively given state of affairs even within the human mind itself (like in the case when we say, for example, that such abstract concept as function, represented by a graph and a formula, is inaccessible to a student).

I am not sure whether your protests against the traditional approach to the issue of representations stemmed from the disillusionment with the Objectivist epistemology, but my doubts about this notion *are* the result of such disillusionment. This does not mean that I will not talk about "abstract objects hiding behind symbols." I will. But when I ask whether an abstract object exists or not, it will not be a question about any *real* existence, which can be proved or disproved in a rigorous way. The only criterion I will use will be that of theoretical effectiveness: I shall make claims about existence or non-existence of abstract objects in the learner's mind only if it helps me in making sense of observable behaviors. Indeed, you yourself, in your last statement, gave me a perfect example of a situation in which *I* would say that I can see objects in peoples' thinking — just when you say the opposite. Let me remind your own words:

> … I don't see objects in people's thinking. Instead, I see schemes of operations and webs of meaning. Sometimes these schemes are ill-formed or in the process of formation. Sometimes they are well-formed and highly integrated. In the latter case, the people possessing these schemes maintain that they are thinking of "mathematical objects."

What else do you need to at least try using the notion of an "abstract object" as a potentially fruitful theoretical construct? Your own description makes it clear that this idea could help us in pinpointing the difference between different mathematical behaviors in a concise and productive way. You seem to me still quite afraid of being accused of making ontological statements (about some kind of *real* existence of the abstract objects). Free yourself from these fears — go theoretical and be brave! After all, theory is the way we speak, not an attempt to say that our abstract constructs mirror reality.

Pat:

I must chuckle. This is the first time I am chided for appearing to *fear* being theoretical. I am often accused of being *too* theoretical. I've even called for members of PME to take theory more seriously (Thompson, 1991a).

Perhaps it will help to make clear on what we agree before going further. I agree with you that modern notions of science no longer are concerned with whether theories are *true*, only with whether they are *coherent* and *useful*. This point is well-articulated in the writings of Kuhn (1962; 1970a), Popper (1972), and Feyerabend (1988) in the philosophy of science and in the writings of, among many others, Brouwer (1949; 1952), Lakatos (1976; 1978), and Wilder (1968; 1981) in the philosophy of mathematics. In earlier publications I, too, have said that what matters most is that we develop useful ways of thinking about aspects of teaching, learning, and experiencing mathematics (Thompson, 1979; 1982; 1991b) — useful in the sense that greater insight into problems leads to more informed and efficacious action. One of my favorite sayings is Dewey's: *There is nothing more practical than a good theory* (Dewey, 1929). We have no quarrel on this matter.

Finally, you object to my criticism not because you disagree with it (I know you don't), but because you see my criticism as being misleading. You said,

> But for me, it is misleading not so much because of the fact that the referent of the symbol may be absent from the student's mind, but because when construed in the traditional way, it seems to reinforce an Objectivist approach. It is misleading because it implies an existence of an objectively given state of affairs even within the human mind itself.

I agree completely that the notion of "representation" as implying a symbol-referent relationship is highly problematic. In fact, following von Glasersfeld's (1991) and Cobb, Yackel and Wood's (1992) examples, I tried being quite careful to make my usage of "representation" reflect the context of someone attempting to convey or impute *meaning*. My criticism is of people using "representation" too loosely, without mentioning a person to whom some sign, symbol, or expression has some meaning. I think you put it quite nicely in another publication:

> While Objectivism views understanding as somehow secondary to, and dependent on, predetermined meanings, non-Objectivism implies that it is our understanding which fills signs and [notations] with their particular meaning. While Objectivists regard meaning as a matter of a relationship between symbols and a real world and thus as quite independent of the human mind, the non-Objectivist approach suggests that there is no meaning beyond that particular sense which is conferred on the symbols through our understanding. (Sfard, 1994, p. 45)

Part of our miscommunication is due, I suspect, to the various stances we take naturally when speaking about mental processes and to the various perspectives we take, again naturally, when speaking theoretically. In regard to the first, Donald MacKay (1969) makes a useful distinction between “actor language” and “observer language.” We speak in actor language when speaking for ourselves or in an attempt to speak for another. We speak in observer language when speaking as an observer of another or others. It is very difficult to remain within one or the other. In the previous sentence I adopted neither stance — and illustrated my point about the dangers of writing in the passive voice. Did I mean that *I* find it difficult to remain within one or the other, or did I mean that *anyone* will find it difficult to remain within one or the other? You cannot tell.

In regard to perspectives we take when speaking theoretically, I find Alan Newell’s (1973) discussion of “grains of analysis” quite useful. In one example he compares different analyses of teeth. On one level, teeth are quite structure-like. They are stable biological structures which we use to gnash and to chew. When examined on another level, teeth are constantly changing shape, eroding, and regenerating. We could say they are the same “things,” only viewed with different grains of analysis.

It seems that when you quoted my passage beginning “I don’t see objects in people’s thinking …” and said that, indeed, you could see objects, we used different grains of analysis. What were the objects you saw me speaking of? Schemes? At the grain of analysis I had in mind, I would say those were *my* constructions (actor language regarding me, observer language regarding the people I observed). At a more distant grain of analysis I could say, yes, those schemes were objects in their thinking. But to whom are they objects? They are objects to me. There may be something in their thinking that are objects to them, but I would not automatically attribute “objectness” to the schemes I identified (speaking in actor language regarding the people I observe). To hypothesize what they took as objects at the time of my observing them would require a different analysis.

Anna:

Shall I let you have the last word on the first question? Oh well, I will. After all, you managed to show that there is more agreement between us than argument. So let me start pondering our second issue.

After all the explanations regarding the non-Objectivist vision of knowledge and of scientific theories, I feel it is my duty now to show that *abstract object* is a useful theoretical construct. For the sake of enlightening the discussion, I invite you to try to make my life difficult also on this point. To put things straight, however, let me precede the defense of the theory of reification with a more thorough clarification of what "abstract object" means to me.

First, may I remind you that my use of the term is quite different from that of a Platonist. For me, abstract object is nothing 'real', nothing that would exist even if we did not talk about it. As Putnam (1981) put it,

> "Objects" do not exist independently of conceptual schemes. We cut up the world into objects when we introduce one or another scheme of description. (p. 52)

In other words, objects — of any kind whatsoever — are, in a sense, figments of our mind. They help us put structure and order into our experience. My approach is no different from Putnam's: for me mathematical objects are theoretical constructs expected to help in making sense of things we see when observing people engaged in mathematical activity. What counts as a good theoretical construct? Something that makes it possible for you to have more insights and generate more knowledge out of a fewer basic principles. Something that helps you to build an effective theoretical model.

I see such notions as *abstract object* in psychology of mathematics — and, for that matter, as *energy* in physics or as *mind* in cognitive science — as a kind of link (a glue, if you wish) we add to the observables in order to make the latter hung together as a coherent structure. These additional "somethings," being our own inventions, cannot be directly observed, and cannot be identified with any specific discernible entities. Their "presence" can only manifest itself in certain well-defined clusters of phenomena — phenomena which, in fact, wouldn't appear as in clusters and wouldn't make much sense if it wasn't for these special "somethings" that we invented.

As it often happens, the nature and function of the special element unifying many different situations may best be scrutinized in pathological cases: in situations in which it is missing. Indeed, Pat, I agree with what you said in your opening piece: for many people, certain "representations" may be empty symbols that do not represent anything. But while saying this, you only made a stronger case for the notion of abstract object! It was thanks to the notion of mathematical object that in my studies on the notion of function and on algebraic thinking I was able to see many kinds of student's faulty behavior as different symptoms of basically the same malady: student's inability to think in structural terms. A failure to solve an inequality, an unsuccessful attempt to answer a question about a domain of a function, a faulty formulation of an inductive assumption on the equality of two sequences of numbers, a confusion about the relation between an algebraic formula and a graph — all these diverse problems combined into one when I managed to see them as resulting from learner's "blindness" to the abstract objects called functions.

Needless to say (but I'll say it anyway, just to be sure that you don't accuse me of overlooking this important aspect), theoretical notions are not stand-alone constructs. One can only justify their use if they become a part of a theory — theory which neatly organizes the known facts into a coherent structure and, maybe even more important, has a power of generating new insights and turning into clearly visible things that would otherwise escape researcher's attention. As Sherlock Holmes once nicely put it, without the special alertness which can only come from a good theory, "you see but you do not observe."

Well, abstract objects did become a part of a theory some time ago. Many mathematics education researchers worked in parallel on theoretical frameworks which took the concept of abstract object seriously. The list is quite impressive: it includes both Piaget and Vygotsky (whose theories may be viewed as incompatible in some respects, but who nevertheless seem to be in agreement on the points which match our present interest), Dubinsky (1991), Harel and Kaput (1991), Gray and Tall (1994), Douady (1985) … and the list is still quite long. Of course, both participants of this dialogue are among the most devoted members of this school — or, at least, one of them was and still is, and the other one was known to be before this dialogue began; see e.g. Thompson (1985), Sfard (1991; 1992); Sfard and Linchevski, (1994). I will later give an outline of my favorite variant of such theory — the breed which we call here a *theory of reification*.

Right now, however, I can feel it's time to cut this flow of theorizing and meta-theorizing with some enlightening illustrations which would take us back to the mathematical thinking itself. I'm going to present you with two short examples which hopefully will make it clear how the notion of abstract object works as an explanatory device. First, I will present you with a situation which, when seen through the lens of the theory of reification, displays the presence of an abstract object in the learner's thinking. It will then be contrasted with a case in which abstract objects are conspicuous in their absence.

My first example comes from a study recently completed by Carolyn Kieran and myself in Montreal. In our experiment, 12 year old kids made their first steps in algebra. Our approach was functional and the learning was massively supported with computer graphics. I'm far from saying that in the study everything went according to our expectations and that our special approach brought a solution to all the problems the teachers always grappled with. But some nice things did happen. In the final interview, a boy named George was asked to solve the equation $7x+4 = 5x+8$. The children did not learn an algebraic method of solving equations, but they did learn to see linear functions through formulae such as $7x+4$ or $5x+8$. Here is our exchange with George:

G: Well, you could see, it would be like, … Start at 4 and 8, this one would go up 7, hold on, 8 and 7, hold on … no, 4 and 7; 4 and 7 is 11 …. they will be equal at 2 or 3 or something like that.

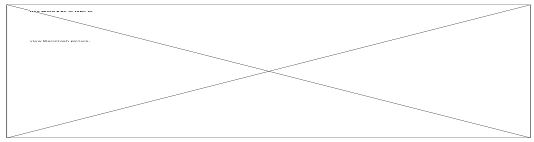
I: How are you getting that 2 or 3?

G: I am just graphing in my head.

For me, it is clear that George was able to see more than the symbols — more than the formulae and the graphs. He was able to imagine abstract objects called functions. Why abstract objects and not just graphs? Because "graphing in one's mind" is one thing, and being able to make smooth transitions between different representations (I hope, Pat, that you agree with my use of the term representation in this case) means that there is something that unifies these representations. What I call "linear function $7x+4$" is such a unifying entity (it is neither the formula, nor the graph — it's an abstract being).

Here comes my second example. A 16 year old girl — let us call her Ella — was asked to solve a standard quadratic inequality: $x^2 + x + 1 < 0$

At this stage, Ella could solve any linear inequality and was quite familiar with quadratic functions and their graphs. The girl approached the problem eagerly and in a few minutes produced the following written account of her efforts:



Was the written solution the only source of teacher's insight into Ella's thinking, he would certainly reward her with a high score. As it happened, however, he talked and listened to Ella when she was working on the problem, and the things he heard prevented him from praising her. Let us have a look at a fragment of this dialogue.

E: [After she wrote line (1) above] There will be no solution for *x*, because here [points to the number under the radical sign] I've got a negative number.

T: O.k., so what about the inequality?

E: So the inequality isn't true. It just cannot be …

T: Do you know how to draw the parabola..?

E: The parabola of this [expression]? But there is no *y* here … how can one draw parabola when there is no *y*?

T: Do you know the relationship between a parabola and the solutions of such an inequality as this?

E: Of an inequality? No. Only of an equation. But maybe it is the same.. Let's suppose that this is equal zero [points to the inequality symbol and makes a movement as if she was writing "=" instead of "<"]. But how can there be a parabola if there is no result here [points to the expression she wrote in (1)], no solution?

T: So what is your final answer ? What is the solution of the inequality?

E: There is no solution.

Do I have to add anything to convince you that we are dealing here with the case of a girl who cannot see *through* symbols and can only see the symbols themselves?

Pat:

I fail to see how my argument provides a case for "abstract objects." It seems you are saying if something is meaningless for students, it is meaningless because they do not possess the abstract objects which would give it meaning. This doesn't follow. Chinese characters are meaningless to me. Does that imply that my possession of certain "abstract objects" will render them meaningful? No. My inability to read Chinese characters means only that I do not possess the many grammatical, rhetorical, and perceptual schemes which I need to read Chinese. On a similar note, I recently picked up a physics text which uses notational conventions unfamiliar to me, and on top of that it employed poor rhetorical style. I felt like I was reading Chinese, but I certainly understood the physics about which the text's author spoke. To say that I am blind to certain "abstract objects" is a poor explanation of my inability to understand either Chinese characters or a particular physics text.

Our agreement on the importance of theoretical constructs should be clear, so I'll say nothing further on that. However, not just any theoretical construct is a good one. I find the construct of an "abstract object" problematic in two ways: its internal coherence and your use of it as an explanatory device.

You haven't said what you mean by an *abstract* object. I think I understand what you mean by an object, or at least my understanding is not incompatible with what you've said. It seems that by object you mean what Piaget had in mind when he spoke of children's construction of object permanence (Piaget, 1950; 1976; 1985) — people construct objects by building and coordinating schemes of action or thought to form a locally-closed, self-regulating system which they can re-present to themselves in the absence of the network being activated *in toto* (von Glasersfeld, 1991). As a matter of methodology, to characterize someone's construction of a *particular* object (especially, an object to which we might assign a name like *function*), I would think it necessary to say something about that person's schemes of action or thought which we presume constitute it.

As for "abstract" objects, if a person has constructed an object, then it would seem this object, to that person, will be concrete. I don't know what you mean by an *abstract* object. I know what Steffe et al. mean by an *abstract unit* (Steffe, Thompson, & Richards, 1982; Steffe, Cobb, & von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983), but they use this as a technical term to denote something that a child has constructed through reflective abstraction — an *abstracted* unit, so to speak. They do not use "abstract" as a tack-on adjective, as if there are objects and then there are abstract objects. If my characterization of "object" is satisfactory to you, it would help me considerably were you to explain how the adjective *abstract* adds anything to its explanatory power.

If in an explanation of some student's behavior you say "she has constructed function as an object," I would still have to ask what schemes comprise this object for that student, for objectness comes from her possessing coordinated schemes — but not necessarily the schemes you wanted her to construct. Lee and Wheeler (1989) found a large number of students for whom expressions, proofs, and rules were "objects," but they were objects to these students in the same way that this sentence might be an object to my 8 year-old daughter. She knows about sentences — that they are to be read, interpreted, that they have a beginning and an end, they generally communicate a single thought, and so on. But that sentence is not the same object to her as it would be to a linguist who takes it apart according to systems of grammar or pragmatics. Why? Because the schemes which constitute sentence-objects for my daughter are very different from the schemes which constitute sentence-objects for the linguist. You cannot say that a sentence is an object for one but not for the other. Rather, they are different objects to two different people.

Another example that "objectness" cannot be taken at face value is Kuhn's account of a debate between a chemist and a physicist (Kuhn, 1970b, p. 50). The chemist maintained that a helium atom is a molecule, because it behaves as a molecule should according to the kinetic theory of gases. The physicist maintained that a helium atom was not a molecule, because it displays no molecular spectrum. Looked at one way, they were arguing about what label to apply to some object. Looked at another way, their argument reflected that the term "molecule" *pointed to different (i.e., non-identical) objects for these two people*. The mention of "molecule" activated different schemes of operations in them. Is one person correct? No. In fact, the question can be misleading.

Your examples illustrate the difficulty I have with the way you use "abstract object" as an explanatory device.

[*Solve* $7x + 4 = 5x + 8$]

G: Well, you could see, it would be like, … Start at 4 and 8, this one would go up 7, hold on, 8 and 7, hold on … no, 4 and 7; 4 and 7 is 11 …. they will be equal at 2 or 3 or something like that.

I: How are you getting that 2 or 3?

G: I am just graphing in my head.

You said,

> For me, it is clear that George was able to see more than the symbols — more than the formulae and the graphs. He was able to imagine abstract objects called functions. Why abstract objects and not just graphs? Because "graphing in one's mind" is one thing, and being able to make smooth transitions between different representations … means that there is something that unifies these representations.

I agree with you that George "saw more than symbols." But it is a leap I cannot make to say he was able to "imagine abstract objects called functions." What does that mean? Did he imagine a domain? A range? A correspondence? Did he imagine two variables covarying continuously? Did he understand that going 3/11 of the way between x_0 and x_0+1 would correspond to an increase of 3/11 of 7? What is a variable to him? What kinds of operations can George perform on these objects called functions? Can he compare them? Combine them? Compose them? I suspect he can do none of these. If I am correct, then I have a difficult time understanding what these objects called "functions" are to him. I do not mean he is not thinking of objects which he calls "functions." Rather, I mean we do not know what comprises those objects.

It seems a more "explanatory" explanation would be: When he said "Start at 4 and 8, this one would go up 7 … " he was thinking something like, "I need to find a value for *x* so that the two expressions have the same value. As I start at 0 and go over 1 in the left-hand expression (as in moving on a horizontal axis) I go up 7 (as in moving on a vertical axis) and as I start at 0 and go over 1 in the right-hand expression, I go up 5. So going over 1 in the left-hand-side is 4+7 …" [then, to himself, "going over 1 in the right-hand-side is 5+8"]. I have too little information to guess at his reasoning in regard to his saying, "They will be equal at 2 or 3," but what I've postulated certainly fits the information you presented. What constructs would I use to enrich my explanation? Constructs like imagery, scheme, etc. How would I explain the connections he seemed to make? I would appeal to constructs like assimilation and generalizing assimilation (Thompson, 1994a). I see no need to appeal to such a vague notion as his imagining "abstract objects called functions" or to posit that, because he made some connections, that "there is something that unifies these representations." When we appeal too quickly to grand ideas, we lose sight of the richness and intricacy of students' reasoning.

Actually, I would have tried not to be in the position of so boldly guessing George's reasoning. Had I conducted the interview I would have looked to get different information than what you presented. The question asked of George, "How are you getting that 2 or 3?" moved the discussion in the direction of explaining answer-getting actions instead of discussions of what he had in mind when thinking about the task (Thompson, Philipp, Thompson, & Boyd, 1994). I suspect the conversation would have produced more useful information had the immediately succeeding question been something like, "You said: Start at 4 and 8, this one would go up 7. What did you mean by that?" with subsequent questions sustaining that emphasis.

Your example involving Ella is even more problematic. At this point I must be brief, so I'll just say that I do not understand how it furthers our understanding of students' mathematics to explain their reasoning in terms of the *absence* of various abstract objects in their reasoning. I can understand attempts to compare where students are with where we would like them to be, but to explain where they are by saying they are not where we want them is a non-explanation. I think a richer explanation of Ella's behavior might be found by speaking about her assimilation of certain figural forms to an action-schema which has "replace '<' with '=' and solve" as its first part. That is, explanations of students' behavior which try to capture students' experience and which posit what students *do* understand add more to our understanding than do explanations which explain their behavior by stating what they do not know.

You end Ella's example by stating "… we are dealing here with the case of a girl who cannot see *through* symbols and can only see the symbols themselves." To a great extent I agree with your statement. I do not agree that Ella's example buttresses your case that "abstract objects" is a useful theoretical construct.

Anna, my complaints might seem methodological, but they are methodological at the level of *research programme* (Lakatos, 1978), for they address basic orientations we bring to our work of theorizing and they raise the question of the kinds of theories we value most.

Anna:

Wow, Pat! You do seem to have taken the invitation to make my life difficult seriously! You might even have overdone it a little bit. But it's good. A fight will force us to sharpen our theoretical weapons and to elicit points inadvertently glossed over.

Your reaction to what I said sounds convincing: the often observed "meaninglessness" (I dislike this term) of mathematics is not, *per se*, a proof for the usefulness of the notion of mathematical object. I agree, and the fact is I never made this illogical claim. The only aim of the episodes and phenomena I brought earlier was to exemplify situations in which a person *who looks through the lens of theory of reification* would spot either the presence or the absence of abstract objects. The examples were not, and could not be, meant to show the objective necessity of the notion of abstract objects as means of explaining the phenomena.

Since we seem to agree that there is more to understanding mathematics than knowing the rules of symbol manipulations, the question arises what is this additional something. This may be translated in the question what we mean by "meaning." No, don't expect me to explore the morass of this time-honored philosophical puzzle. Let me tell you one thing, though. You say "Chinese characters are meaningless to me." And you ask, "Does that imply that my possession of certain 'abstract objects' will render them meaningful"? Of course it doesn't; but although "meaninglessness" certainly does not *imply* the necessity of abstract objects, having abstract objects is one way of explaining how people make certain expressions meaningful. If the sentences you are dealing with happen to be built around a noun, such as, say, a chair, a gremlin or a function, then having the ability to think about the objects hiding behind the words is what we call "grasping the meaning." Sometimes, like in the case of a chair, the referent of a noun is a tangible material object. Sometimes, like with gremlins, the existence of such object could, theoretically, be ascertained with our eyes, provided it really existed. Sometimes, like in the case of function, the nature and the existence could not, even in theory, be explored with our senses. In this last case I say that the objects we are talking about are *abstract*. I hope this answers your question what I mean by the adjective "abstract" in this context.

Not quite yet? You might be right. Well, I have more to say about that. You claim for example, and rightly so, that "If a person has constructed an object … [then] to that person it will be concrete." I can even help you with this. Some mathematicians I have recently talked to used expressions like "concrete," "tangible," and "real" when referring to the things they were manipulating in their minds when thinking mathematically. Here, one starts to wonder what the word "concrete" means. Like in the case of the notion of 'meaning', it is much too loaded a problem to be dealt with in this short exchange. But let me refer you to an insightful essay by Wilensky (1991) in which, in one voice with Turkle and Papert (1991), he suggests a "revaluation of the concrete." The need for the revaluation arises at the crossroads of two current trends: constructivism and emergent-AI. Wilensky analyses the "standard view" of the concrete through new glasses and arrives at the conclusion that "concreteness is not a property of an object but rather *property of a person's relationship with the object*" (Wilensky, 1991, p. 198). If so, it may be helpful here to follow your suggestion and distinguish between two perspectives: that of an actor and that of an observer. The mathematical object may be concrete to the former, and at the same time abstract to the latter.

I once tried to capture the difference between these two perspectives in the metaphor of mathematics as a virtual reality game. Have you ever seen a person wearing a computerized helmet and a glove, engaged in a virtual reality game? Wasn't it quite amusing? Could you make anything of this person's strange movements? Probably not, but I bet that it never occurred to you that the funny fellow might be out of his mind. And if in addition you had been told that, for instance, he is trying to transverse a heavily furnished, messy room, then, quite likely, you instinctively tried to imagine the kind of objects he could be moving around. Like this virtual reality game player, a person engaged in a mathematical activity seems to be dealing with objects nobody else can see. You and I, as observers, do not have a *direct* access to what the actor thinks he is playing with. But assuming that he does see some objects helps us in being tolerant toward his strange movements and makes us believe that the funny behavior has an inner logic. Trying to figure out what the player sees is the most natural way to make sense of what he is doing. Thus, we recognize the existence of the objects the player is dealing with, but while for the latter they are quite concrete, for us they are abstract. Since in this conversation I am speaking mainly from observer's perspective, I refer to the mathematical objects as abstract.

As a side-effect, this parable brings home to us that the notion of mathematical object is a metaphor that shapes the abstract world in the image of tangible reality. Hopefully, it is also more clear now how important a role a perceptually-based metaphor plays both in our mathematical thinking and in our thinking on mathematical thinking. As we both agreed, the actors themselves, when looking at what they are doing, would usually admit that some objects are present in their thought. At least the best, the expert actors would say so. You mentioned it yourself in the beginning of this dialogue, remember? "The people possessing [highly integrated schemes of operations] maintain that they are thinking of 'mathematical objects'." My work with mathematicians brought lots of further evidence that, indeed, the inner world of a mathematizing person may look very much like a material world, populated with objects which wait to be combined together, decomposed, moved and tossed around (Sfard, 1994; Thurstone, 1994).

The fact that according to the actors themselves the metaphor of object is ubiquitous in mathematical thinking is hardly surprising. What can be better known to us than our perceptual experience, than the physical world that surrounds us? The mathematical objects we can see with our mind's eyes are metaphors that *constitute* the mathematical universe in the first place, and then make it possible for us to move around it in ways similar to those in which we move in the physical world. The embodied schemes generated by our physical experience are deeply engraved in our minds and this is thanks to them that we often find our way in this world intuitively, without reflection (Johnson, 1987, p. 102). By using such schemes to help ourselves move in the virtual reality of mathematics we inject mathematical thinking with the meaningfulness of our physical experience.

The metaphoric use of "object" is by no means restricted to mathematics. Here is an account of a physicist:

> [W]hen analyzing physical phenomena, people like to put into play "objects." Beside real objects, they ascribe a realistic character to physical concepts or models. They build their reasoning on these "objects" as if they were material. (Viennot, in press)

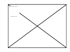
Finally, the picture will not change in a substantial way if we climb to the meta-level — the level of thinking about mathematical thinking, the level of an observer. The metaphor of mathematics as dealing with as-if-material objects has a special appeal for the researcher. Speaking about mathematics in terms of abstract objects and processes on this objects makes mathematics in the image of the world we know best: the material world. Whatever we know about the former — and we know an awful lot about it — has a potential of bringing insights about the nature and function of the latter. Those concerned with the methodology and psychology of scientific innovation have agreed a long time ago that scientist is “an analogical reasoner” (Knorr, 1980) — that resorting to our knowledge of things with which we are familiar and which are somehow similar to those we find in the new domain may be for the scientist the most powerful, albeit “unofficial,” way to get moving in untrodden territories.

At a certain point you say, Pat, that “‘Objectness’ cannot be taken at its face value.” I couldn’t agree more. Objects have many faces and our knowledge of them can never be “full.” What your daughter knows of “sentence-objects” seems to be partial to rather than different from what the linguist knows. Your second example is even more enlightening: you say that for the chemist and the physicist “the term ‘molecule’ *pointed to different … objects*.” In mathematics, things like that are happening all the time. For example, through one algebraic formula, say $3x$+b, one may see quite a number of different mathematical objects: a number, albeit unknown, a linear function, a family of linear functions. The expression may, of course, be also taken at its face value and treated as nothing more than a string of symbols. You don’t have to deal with a number of different people in order to have all this interpretations; on the contrary, it should be teacher’s goal to help her students construct a scheme which will include all the possibilities. Such scheme is necessary for the *flexibility* of thinking which was called “the hallmark of [mathematical] competence” (Moschkovich et al., 1993) and which can be described as the ability to match an interpretation to the context in which the formula is used. You also remark that “objectness comes from possessing coordinated schemes — but not necessarily the schemes you want [the student] to construct.” I agree again. More often than not, the scheme built by the student is only partial. Worse than that, it often includes only one object: the “opaque” formula, a formula which is taken as an object in its own right and through which no other object can be seen. My colleagues and I once called this kind of conception (scheme) *pseudo-structural* (Linchevski & Sfard, 1990; Sfard & Linchevski, 1994). In our studies, we had a chance to observe numerous phenomena which can be interpreted as adverse effects of the lack of flexibility inevitably accompanying such impoverished conception.

Now to the most important part of your critique: the questions about my interpretation of George's and Ella's behavior. What happened here is a result of talking "from the middle," of bringing just one piece of a greater whole and hoping that this one element will speak for all the rest. Obviously, things do not work in such simple way. It is only when the stories of George and Ella are viewed within the context of a theory that they may become truly meaningful. Talking about the presence or absence of abstract objects without tying the notion to a theoretical framework is like using the term "energy" in physics while talking to a person knows nothing about mechanics. It would be quite futile if one said to this person that one of two stones has much kinetic energy and the other has none. Such labeling is only useful if, by connecting a given situation to a theory, it immediately increases the amount of information — if, for example, it conveys the message that one stone is in motion while the other rests. I'll try now to make up for my mistake by showing how my explanations draw on the theory. Incidentally, I wouldn't like to sound presumptuous when I call this particular framework "a theory." I am using this word only to be brief.

Let me begin. You just said that "to explain where [students] are by saying they are not where we want them to be is a non-explanation." It is certainly not the case when you are within a theory which provides you with information about the possible alternatives to the student's being where we want her to be, about the consequences of being in this other place, about the possible reasons of the situation and about the means that can be taken to change it. I tried to explain George's success by conjecturing that he did construct a certain object, and I ascribed Ella's failure to the absence of this object. This kind of explanation is common in science. When two kids come in contact with a case of chicken pox but only one of them gets infected, the doctor is likely to say that the first one had antibodies whereas the other did not. This statement has an explanatory power, since it ties both cases to the same underlying mechanism; it has a predictive power, because it gives a basis for expecting what will happen to the children if they come in contact with the chicken-pox again; finally, it may serve as a basis for some medical decisions. In a similar way, my interpretation of the two episodes, if supported by a theory, may have an explanatory, predictive, and prescriptive power.

Let me elaborate on the explanatory aspect. Both children were presented with situations with which they were not well acquainted. The parable of the messy room highlights the importance of "seeing" object for a person who is supposed to move in an unfamiliar setting. One salient feature of objects — whether material or abstract — is the fact that they tend to preserve their identity and are easily recognizable in different contexts. You are right in claiming that we cannot say much about the kind of conception George has developed and our information is too scarce to know whether he had the multifarious structural (object-oriented) understanding of the formula I was talking about earlier. But we have reasons to conjecture that he was able to deal with the non-standard situation because he had a good sense of the particular object he was dealing with (a liner function) and thanks to that he could adjust his actions to the new needs. Ella obviously could not see the objects with which she was supposed to deal and, as a result, the only thing she could do was to repeat the standard movements she once learned by watching and mimicking people engaged in the game (e.g. the teacher). To use a description by Dörfler (in press): When the "adequate image schemata have not developed to supply meaning through metaphor [of object], the discourse will instead be used in a parrot like and rote manner and will not be flexible or extensible." Ella's problem was that the standard behaviors she learned (you are right: most probably, it was solving quadratic equations) were inappropriate in the new situation — but, not having access to the "virtual reality" of functions, she could not see the change.

The idea of reification may give us an even deeper insight into Ella's plight: it can help in figuring out the reasons of her inability to think in terms of abstract objects. More generally, the theory provides its own explanation for the fact that the kind of deficient conception displayed by Ella is evidently very common. The first thing that must be explained is how mathematical objects come into being. According to the theory, *these objects are reified mathematical processes*. To understand this statement, one has first to notice an inherent process-object duality of mathematical concepts: such notions as -3,  or function 3*x*-2, although clearly referring to *objects*, may also be viewed as pointing to certain mathematical *processes*: subtracting 5 from 2, extracting the square root from -1, a certain computational procedure, respectively. Historical and psychological analyses of concept formation led to the conclusion that operational (process-oriented) conceptions usually precede the structural. Let me build the rest of the outline around the testimony of a mathematician:

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurstone, 1990, p. 847)

If the “compression” is construed as an act of reification — as a transition from operational (process-oriented) to structural vision of a concept (it doesn’t *have* to be construed in this way, but such interpretation is consonant with what was said before about structural conceptions), this short passage brings in a full relief the most important aspects of such transition. First, it confirms the developmental precedence of the operational conception over structural: we get acquainted with the mathematical process first, and we arrive at a structural conception only later. Second, it shows how much good reification does to your understanding of concepts and to your ability to deal with them; or, to put it differently, it shows the sudden insight which comes with “putting the helmet and glove on” — with the ability to see objects that are manipulated in addition to the movement that are performed. Third, it shows that reification often arrives only after a long struggle. And struggle it is! Numerous studies suggest that whether we are talking about functions, numbers, linear spaces or sets, reification is difficult to attain (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Sfard, 1992; Sfard & Linchevski, 1994). The main source of this inherent difficulty is what I once called *the (vicious) circle of reification* — an apparent discrepancy between two conditions which seem necessary for a new mathematical object to be born. On one hand, reification should precede any mention of higher-level manipulations on the concept in question. Indeed, as long as a lower-level object (e.g. a function) is not available, the higher-level process (e.g. combining functions) cannot be performed for the lack of an input. On the other hand, before a real need arises for regarding the lower-level process (here: the computational procedure underlying the function) as legitimate objects, the student may lack the motivation for constructing the new intangible "thing." Thus, higher-level processes are a precondition for a lower-level reification — and vice versa! It is definitely not easy to get out of this tangle.

It seems that Ella's predicament was like that of many other students who fall victim to the inherent difficulty of reification. The explanation provided by the theory presented Ella's story as a special case of a general phenomenon. Understanding the underlying reason of Ella's poor performance on the given task makes it now possible to predict what kind of situations will be most problematic for the girl in the future. Indeed, she may be expected to fail time and again when confronted with tasks that require having a function as an object. Thus, while she may be quite skillful in solving all kinds of equations, she will probably be helpless if asked to deal with, say, a singular equation or a parametric equation (Sfard & Linchevski, 1994).

Finally, the theory has a prescriptive power and it does provide its answers to the question how we should teach in order to cope with the difficulty of reification, in order to prevent the situations like the one Ella got into. Since I already talked longer than I should, I will confine myself to one more remark (see the literature quoted above for more on the didactic implications of the theory). You opened this debate with the statement that we should "cease our fixation with representations of (our) big ideas and instead focus on having students use signs and symbols only when they (students) have something to say through them." I suppose that mathematical objects, such as function, are the "big ideas" you were talking about. In our dialogue we tried to clarify this term, so when we now approach pedagogical questions we hopefully know a little better what these "big ideas" are all about. I agree that it is highly unlikely that the student will construct mathematical objects right away. The theory I just presented points to the great difficulty of the undertaking and explains why reification does take time. I also agree that "finding situations that are sufficiently propitious for engendering multitudes of representational activity" may be very helpful indeed, and once again, the theory supports this view. But if the upshot of what you said is that we should give up striving for fully-fledged structural conceptions in our students, I hope that what I said will make you soften this position. The ideas I just presented support the view that structural conceptions — the ability to "see" abstract objects — are difficult to attain, but having them is most essential to our mathematical activity at all ages and at every level.

After all I said here you may be surprised that I have no wish to argue with your alternative interpretations of the two episodes. I won't do it because I don't think there is a real discrepancy between us. You just chose to look at things from a different vantage point, and I do see the merits of this other approach. After all, accepting my point of view does not necessarily imply rejecting yours. I hope you agree that the same phenomena may admit different interpretations when scrutinized with different theoretical tools, and that such different interpretations should often be regarded as complementary rather than mutually exclusive. I hope you agree that the theory filled the notion of abstract object with meaning just like geometrical axioms fill the primary geometrical concepts (point, line) with meaning.

Let me finish with a few words on the place of theories in our project as researchers. With all my preference for theorizing in terms of abstract objects, nothing could be farther from my mind than claiming an exclusivity — than saying that the resulting theory is an ultimate answer to all the questions about mathematical thinking people have ever asked. Two theories are sometimes better than one, and three are better than two. To quote Freudenthal (1978, p. 78), "Education is a vast field and even that part which displays a scientific attitude is too vast to be watched with one pair of [theoretical] eyes." Like in physics, where a number of ostensibly contradictory theories exist and flourish side by side, so in mathematics education there is certainly a room for several research perspectives. Theory of reification, like any other model, brings in full relief certain aspects of the explored territory while ignoring many others. The only thing I wanted to convince you about is that what it does show is important enough to make this particular theoretical glasses worth wearing, at least from time to time. We have agreed that good theory is a theory which may become a basis of a rich research program. Does the framework built around the concept of reification stand up to the standards? I believe it does. In fact, I *know* it does. It already proved itself in the past, when it spawned numerous studies, as well as many useful pedagogical ideas, and enabled a synthesis of much of the existing research on the development of central mathematical concepts. One day, it will probably exhaust its power to generate new research, like any theory. But not quite yet. At the moment, when the world of cognitive science is more and more fascinated by new theories of perceptual-metaphorical sources of all human thinking, the notion of mathematical object may have more appeal than ever.

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