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Students, Functions, and the Undergraduate Curriculum†

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Someone, I cannot remember who, paraphrased Winston Churchill by saying that mathematics and mathematics education are two disciplines separated by a common subject. The mathematician is primarily concerned with doing mathematics at a high level of abstraction. The mathematics educator is primarily concerned with what it is that one does when doing mathematics and what kinds of experiences are propitious for a person's later successes. Until recently mathematics education research has focused predominantly on the learning and teaching of early mathematics in the school curriculum, so it is natural that practicing mathematicians have found it difficult to relate to mathematics education research. I suspect that the current interest in calculus reform [**21, 63**] and the broader rethinking of the undergraduate curriculum, together with the advent of the AMS/MAA Joint Committee on Research in Undergraduate Mathematics Education, will lead to a wider recognition that mathematics and mathematics education are fundamentally dependent upon one another.

My purpose in writing this paper is to discuss research on students' understanding of functions and its importance for the undergraduate curriculum. Much has been written recently about concepts of function that goes into far greater detail than I will (see [**33, 46, 57, 64**] for extensive reviews). I will impose a somewhat idiosyncratic structure upon this literature to present an overview of research on concepts of function and to highlight issues I believe need greater consideration than they have so far received.

As a matter of background, I should say a few words about the perspective I bring to this task. We cannot speak strictly about the development of a single concept, such as function. If we have learned anything in mathematics education research it is that a person's thinking does not respect topical boundaries. When analyzing students' concepts of function, we need to keep in mind that the imagery and understandings evoked in students by our probing is going to be textured by their pre-understandings of such things as expressions, variables, arithmetic operations, and quantity. We also need to keep in mind that their mathematical learning has, for the most part, happened in schools, which means that our interpretations of students' performance must be conditioned by our knowledge that they are taught by teachers with their own images of what constitutes mathematics, and that both the learning and teaching of mathematics are conditioned by the cultures (school, ethnic, and national) in which they occur [**5, 12, 56, 77**]. I have tried

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to capture these background relationships in Figure 1. While ignoring this issue might simplify matters enormously for us as teachers and mathematics education researchers, we do so at the peril of losing generalizability and validity of our interpretations and conclusions.

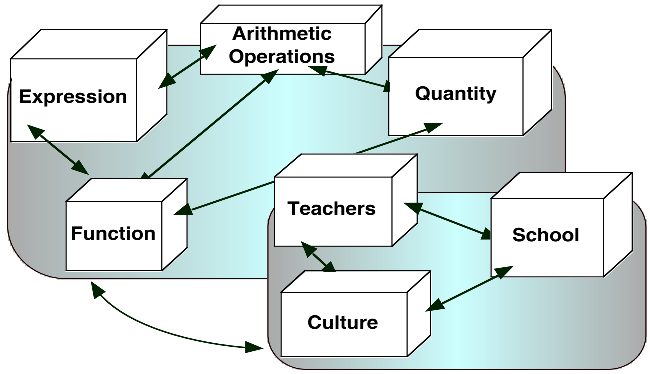

Figure 1. Students' concepts always emerge in relation to other concepts they hold, and in relation to teachers' orientations and in relation to cultural values expressed by teachers, peers, and families.

Another perspective I bring is that concepts emerge over time, much as a dynamical system. The actual form a concept takes (or that it fails to take) in a student's reasoning can be tremendously influenced by seemingly trivial deviations from valid understandings of mathematics they learned much earlier. For example, it is common that elementary school students have impoverished understandings of whole-number numeration. This in itself makes it difficult for them to develop what is often called "number sense" [**32**], and also makes it virtually impossible for them to make sense of standard arithmetic algorithms. This contributes substantially to their developing an orientation toward memorizing meaningless symbol manipulation. They develop this orientation as a mechanism for coping with an otherwise intolerable situation—not having a clue as to what the teacher is talking about, but nevertheless being expected to perform.

School students' common orientation toward "remembering what to do with marks on paper" eventually shows up in our college classrooms, perhaps showing itself only vestigially as ungrounded formal reasoning. This is what Sfard [**59**] refers to as disconnected reification—students turning what are offered, by us, as representations into the actual objects of their reasoning. I mention Sfard's notion of disconnected reification for a purpose. Tinkering with instruction or curriculum to emphasize functions will be insufficient if we fail to address students' common orientation to ungrounded symbol manipulation. I will return to this point later.

I will shape my discussion of research on function concepts around six themes. These are:

- Concept image and concept definition
- Function as action, as process, and as object
- Function as covariation of quantities and function as correspondence
- Understanding phenomena and representing phenomena
- Operations on numbers and operations on functions
- Emergent issues

One theme I will not discuss, except to explain why not, is the literature on multiple representations. In this regard I will express my opinion as to why this line of research needs to be rethought.

I selected these six themes because of their emergence in the literature as constructs around which a stable consensus seems to have developed regarding their importance for students' understandings of function. When examined closely, these themes are highly related, but they nevertheless seem to provide a useful organization for entry into the issues of learning and teaching the concept of function and using the concept of function as an organizing construct in the curriculum.

The distinction between students' concept images and the notion of concept definition arose as a way to understand how students expressed reasoning that was inconsistent with taught definitions of function, limit, derivative, etc. The distinction between function as process and function as object emerges from a variety of traditions, both philosophical and psychological. One way to think of this distinction is to reflect on the formulation $\int_a^x f(t)dt$ in the First Fundamental Theorem of Calculus. We must think of integration as the culmination of a limiting process, but at the same time consider that process, applied over an interval of variable length, as producing a correspondence. The third theme, regarding covariation and correspondence, highlights a tension, both in students' learning and among researchers, regarding what a function is. I'll attempt to make it evident that this tension can be both natural and productive. The fourth theme, regarding phenomena23 and representation, is one over which I will linger. It has to do with students' conceptions of the stuff that mathematics is often about, and I don't just mean physics or chemistry or some domain of application. The fifth theme, having to do with operations on numbers and operations on functions, reflects my feeling that a distinction needs to become part of the fabric of instruction in secondary and early undergraduate mathematics—the distinction between seeing arithmetic operations as operations on numbers and seeing those same operations as operations on functions. The last theme points to issues that emerge from research on function concepts that have yet to be investigated directly.

Concept Image and Concept Definition

The distinction between concept image and concept definition arose originally in the work of Vinner, Tall, and Dreyfus [**66, 83, 85**]. In their usage, a concept definition is a customary or conventional linguistic formulation that demarcates the boundaries of a word's or phrase's application. On the other hand, a concept image comprises the visual representations, mental pictures, experiences and impressions evoked by the concept name.

In lay situations, people understand words through the imagery evoked when they hear them. They operate from the basis of imagery, not from the basis of conventional constraints adopted by a community. People understand a word technically through the logical relationships evoked by the word. They operate from the basis of conventional and formal constraints entailed within their understanding of the system within which the technical term occurs. Vinner, Tall, and Dreyfus arrived at the distinction between concept image and concept definition after puzzling over students' misuse and misapplication of mathematical terms like function, limit, tangent, and derivative. For example, if in a student's mathematical experience the word "tangent" has been used only to describe a tangent to a circle, then it is quite reasonable for him to incorporate into his image of tangents the characteristic that the entire line lies to one side or the other of the curve, and that it intersects the curve only once [**83**]. Notice that this image of tangent—uniquely touching at one point—has nothing to do with the notion of a limit of secants. It is natural that a student who maintains this image of tangent is perplexed when trying to imagine a tangent to the graph of $f(x)=x^3$ at (0,0), or a tangent to the graph of $g(x)=x$ at any point on its graph.

A predominant image evoked in students by the word "function" is of two written expressions separated by an equal sign (Figure 2). We might think that only neophytes hold this image of function. I suspect it is far more prevalent than we acknowledge. An example will illustrate my suspicion and at the same time illustrate how Tall, Vinner, and Dreyfus envision the influence of concept images over concept definitions.

Figure 2. A concept image of "function." Something written on the left is "equal to" something written on the right.

My wife, Alba Thompson, teaches a course designed to be a transition from lower- to upper-division undergraduate mathematics. It focuses on problem solving and proof. Students are supposed to take it after calculus and linear algebra, but a fair portion of the class typically have taken at least one term of advanced calculus or modern algebra. In the context of studying mathematical induction she asked one student to put his work on the board in regard to deriving and proving a formula for the sum $S_n = 1^2+2^2+\ldots+n^2$. The student wrote $f(x)=\dfrac{n(n+1)(2n+1)}{6}$. Not a single student thought there was anything wrong with this formulation. It turned out, after prolonged probing by Alba, that this formulation was acceptable because it fit within students' concept image of function, which I've presented in Figure 3.

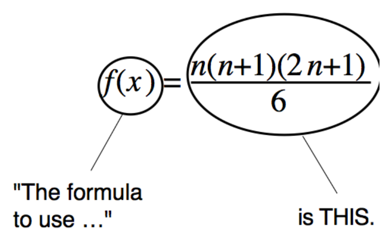

Figure 3. Students' acceptance of an ill-formed function representation because it fit their concept image of function.

An important point to draw from Figure 3 is that mal-formed concept images are insidious. They keep showing up in the strangest places. On the other hand, we could not function intellectually without having concept images. The key point is that mathematical "experts" come to use concept images and concept definitions dialectically [**83, 84**]. Over time, their images become tuned so that they are consonant with a conventionally accepted concept definition, which in turn allows intuition to guide and support reason. Not every student of mathematics attains equilibrium between definitions and images, however. We can increase their chances of success by giving explicit attention to imagery as an important aspect of pedagogy and curriculum. In the next sections I discuss important aspects of well-formed concept images of function.

Function as Action, Process, and Object

It is well known that elementary school students have difficulty conceiving of arithmetical expressions as anything beyond a command to calculate [**13, 19, 35**]. They typically do not think of, say, 4(12-(4+5)) as representing a number. Similarly, algebra students often think of, say, $x(12-(x+5))$ as a representing a command to calculate. When they come to think of an expression as producing a result of calculating, they have what several researchers have called an *action* conception of function [**6, 24**]. This conception is as a recipe to apply to numbers. Students holding an action concept of function imagine that the recipe remains the same across numbers, but that they must actually apply it to some number before the recipe will produce anything. They do not necessarily view the recipe as representing a result of its application.

When students build an image of "self-evaluating" expressions they have what is called a *process* conception of function. They do not feel compelled to imagine actually evaluating an expression in order to think of the result of its evaluation. From the perspective of students with a process conception of function, an expression stands for what you would get by evaluating it.

It is surprising that achieving a process conception of function is a non-trivial achievement for students, and that for many students it is not achieved without receiving instruction that focuses explicitly on its development [**24, 30**]. Dubinsky and his colleagues [**6, 25**] have developed an instructional approach using ISETL, a set-theoretic programming language, that shows promise as an instructional environment for students' development of a process conception of function. They use ISETL to write named processes, which then serve as function definitions. Students can then direct that a function be applied to individual numbers or to numbers in a pre-specified set, in both cases by using the name of the function in place of its defining process.

A process conception of function opens the door to a wealth of imagery. Students can begin to imagine "running through" a continuum of numbers, letting an expression evaluate itself (very rapidly!) at each number.[1] I should note that to become skilled at conjuring such an image students must practice conjuring it [**18**]. Goldenberg and Lewis [**29, 30**] have developed visual supports for students to envision functions as processes applied over a continuum.

Once students are adept at imagining expressions being evaluated continually as they "run rapidly" over a continuum, the groundwork has been laid for them to reflect on a *set* of possible inputs in relation to the *set* of corresponding outputs. I will say more about this idea in the section *Covariation and Correspondence*.

At the point where students have solidified a process conception of function so that a representation of the process is sufficient to support their reasoning about it, they can begin (I emphasize *begin*) to reason formally about functions—they can reason about functions as if they were objects. To reason formally about functions seems to entail a scheme of conceptual operations which grow from a great deal of reflection on functional processes. Primary among these is an image of functional process as *defining* a correspondence between two sets: a set of possible inputs to the process and a set of possible outputs from the process.

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[1] I am not speaking literally. Rather, I am speaking from the point of view of the student.

The many paths by which students achieve an object conception of function are long and complex[**2**], and explanations of it draw on a long tradition in philosophy and epistemology regarding the notion of reflective abstraction [**4, 23, 53-55, 69, 86**]. One hallmark of a student's object conception of functions is her ability to reason about operations on sets of functions. I should quickly point out that it is easy to be fooled—to think that students are reasoning about functions as objects when it is actually the function's literal representation (i.e., marks on paper) that is the object of their reasoning [**58, 59, 61**]. I suspect that the kinds of intellectual operations that go into operating meaningfully on functions have a considerable overlap with the kinds of operations that enable students to reason about such things as operations on cosets in quotient groups—behind their visible operations is a tacit image of completed element-by-element operations.[2]

A question raised by several reviewers, and also raised by Carolyn Kieran [**43, p. 232**], is whether students must first develop process conceptions of function before developing object conceptions of function. This question was raised in the context of discussions of computer environments that ostensibly allow students to interact directly with function graphs, or to manipulate situations and see real-time changes in associated graphs. My remarks in the section *Multiple Representations* are pertinent to the matter of multi-representational computer environments. For now I will say that only that, as a matter of word usage, I would prefer not to talk about students interacting with functions as objects until I am assured that the *students* have conceived the objects they interact with as functions or representations of functions. As Jim Kaput said, "What is being represented, for a knowledgeable third party observer, is NOT what is being represented for the person living in the representational process" [personal communication]. I think we easily confuse perspectives when we say that students interact with functions as objects. A more veridical description might be that students interact with automatically generated "things" (e.g., wavy lines on a computer screen) that they come to make sense of in relation to the situations the things are tied to and in relation to their progressive internalization of the conventions by which the things behave. So, to answer the question of process/object precedence, I see every reason to believe that in an individual student's construction of function, process conceptions of function will precede object conceptions of function. What has changed because of technological advances are the kinds of experiences we can engender in the hope that students eventually create functions as objects.

Function as Covaration and Function as Correspondence

One way to think of the evolution of today's many ways to think of functions is as the current state of a long battle to conceptualize our world quantitatively. Clagett [**9**] relates Oresme's attempts to capture the variational nature of a quality's "intensity" (e.g., temperature) over position and time. Kaput [**41**] extends Clagett's analysis to trace the

[2] Intuitionists might complain that such an image is impossible over infinite domains [**7, 36, 80**]. But I do not mean that people imagine actually completing all possible element-by-element operations. Rather, they just imagine that it is done—in the same way that they might imagine passing over all possible points on a path between their easy chair and their television set.

evolution of today's ideas of variable and variability in the calculus, concluding that today's static picture of function hides many of the intellectual achievements that gave rise to our current conceptions.

The current standard definition of function highlights correspondence over variation—elements in one set correspond to elements in another so that each element in the first corresponds to exactly one element in the second. Since the 1930's this ordered-pair notion of function has been taken as the "official" definition of function, largely because it solved many problems introduced historically by people like Fourier who wished to define functions by a limit process [**31, 44**]. The ordered-pair definition has received strong criticism on pedagogical grounds [**8, 28, 45, 90**]—that it can be meaningful only to people who recognize the problems it solves, but not to a student who is new to the idea of function. On the other hand, we can point to many natural occurrences of correspondences that cannot be expressed analytically or imagined as the product of covariation but which we still would like to call functions (e.g., person's name to person's social security number in a relational database), so a non-correspondence understanding of function is too restrictive in regard to relationships we would like say are functional relationships.

The tension between thinking of function as covariation and of function as correspondence is natural. They are both part of our intellectual heritage, so they show up in our collective thinking. Poincaré put the matter nicely when he said:

> Perhaps you think I use too many comparisons; yet pardon still another. You have doubtless seen those delicate assemblages of siliceous needles which form the skeleton of certain sponges. When the organic matter has disappeared, there remains only a frail and elegant skeleton of certain sponges. True, nothing is there except silica, but what is interesting is the form this silica has taken, and we could not understand it if we did not know the living sponge which has given it precisely this form. Thus it is that the old intuitive notions of our fathers, even when we have abandoned them, still imprint their form upon the logical constructions we have put in their place. (Poincaré, 1913, p. 219 , quoted in [**65 p. 16**]).

Function as covariation is one of those "old intuitive notions of our fathers" of which Poincaré spoke. It is natural that vestiges of it show up in our mathematical culture. But we still face the question of how to reflect our heritage within a curriculum in a way that is coherent in regard to a conceptual development of the subject and at the same time respects current mathematical conventions. One way is to reflect the historical development within the curriculum—emphasize function as covariation in K-14, and then introduce function as correspondence as the need arises (e.g., differential equations; pointwise and uniform convergence of function sequences). This would also respect current thinking about the development of function conceptions through the levels of action, process, and object.

I wish to mention quickly that in today's K-14 mathematics curriculum there is <u>no</u> emphasis on function as covariation. In fact, there is no emphasis on variation. I examined the most recent editions of two popular K-9 text series and found that the closest they come to examining variation is to have students construct tables of data, and even then there is a profound confusion between the ideas of random variable and variable magnitude.[3] This is in stark contrast to the Japanese elementary curriculum [**37**] which repeatedly provokes students to conceptualize literal notations as representing a continuum of states in dynamic situations.

Finally, I am surprised that so little has been investigated in regard to students' concepts of variable magnitude—the focus instead being on variable as literal representation of number [**1, 79, 87**]. It seems, to me anyway, that a progressively more abstract notion of covariation rests upon a progressively more abstract image of variable magnitude.

Understanding Phenomena and Representing Phenomena

Zorn, in a report of a NSF-sponsored conference on the state of calculus reform, said that common complaints were represented by one physicist's remark that, whether having had calculus or not, "his students were 'as innocent as newborn babes' about acceleration and velocity" [**91 p. 1**]. Many mathematicians respond to this complaint by saying that we teach mathematics, not physics. The larger issue, I believe, is to what extent we should expect our students to understand the "stuff" that mathematics is about in its applications.

A debate about whether we should teach physics or chemistry in mathematics class will not be productive. A more productive debate will center around the extent to which we orient students toward conceptualizing the situations our mathematics is about at the moments we use it, and, to relate this debate to functions, what role conceptions of function might play in supporting or inhibiting students' conceptualizations of situations.

Introductory calculus is a natural site to begin discussions of situational conceptions in relation to curriculum, pedagogy, and student learning. For most students, it is the first time they meet functions as models of quantitative situations.[4] The research by Monk [**50, 51**] suggests that students' difficulties with applications run much deeper than their difficulties with the visible mathematics taught in class.

Monk investigated students' conceptualizations of two classical situations having to do with related rates: (1) A person is walking toward a street lamp; students are asked to relate changes in the length of the person's shadow to changes in his distance from the lamp [**51**]. (2) A ladder is sliding down a wall; students are asked to relate changes in one end's height above the floor to changes in the other end's distance from the wall [**50**]. Monk provided physical models for students' experimentation and asked questions about the situations that encouraged students to reason with the physical devices. We should take special note of one aspect of students' reasoning in Monk's reports: Their difficulty in developing a coherent conceptualization of a physical model as a system of

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[3] The confusion is between, for example, my height as it varies over a sample of 1000 moments in time and the heights of a sample of 1000 people. The 1000 measurements of my height are values of a variable magnitude that can be thought to covary with time; the heights of 1000 people at one moment in time can be thought of as values of a random variable or one value of a vector-valued function, but thinking of them either way would not justify speaking of "height as a function of age" as these texts do.

[4] This is not to say that subject matter in school mathematics could not be cast as involving functions as models. Rather, the instruction received by most students rarely emphasizes even thinking about situations, let alone of functions as models of situations.

dependencies among quantities whose values vary—even while holding the devices in their hands and playing with them. For example, in [**51**] one student, who was one of the more adept at symbolic mathematics, was also quite certain that (in the street lamp situation) the tip of a person's shadow moves at an increasing rate while the shadow's length changes at a constant rate. This suggests to me that whatever students have in mind as they employ symbolic mathematics, it often is not the situation their professors intend to capture with *their* symbolic mathematics. It also suggests to me that we need to pay much closer curricular and pedagogical attention to students' pre-symbolic actions, such as imagining dynamic situations so that their images adhere consistently to systems of constraints. In [**73**] I propose that imagining situations as being functionally constituted is also part of seeing generality in geometric diagrams, and that we can actively promote this ability in schoolchildren with carefully crafted curriculum and instruction. Perhaps the same types of activities would be productive for college students.

The importance of attending to students' conceptualizations of situations applies to more than physical phenomena and physical quantities. It applies whenever we use mathematical notation referentially. In informal investigations of senior mathematics majors' and secondary mathematics teachers' understandings of the problem shown in Figure 4, I have found, in principle, three categories of conceptualizations. The first is that they imagine moving one corner of the rectangle and all other parts adjust automatically to conform to the problem's constraints (Figure 5a). These students tend to express functional descriptions of area in relation to lengths of Side1 and Side2, and to express the lengths of Side1 and Side2 in relation to the length of some segment. Another category is reflected in Figure 5b. It is that the initial rectangle is made into other rectangles by moving each corner independently of the others. These students don't even reach a point where they consider what quantity area might be a function of—if they attempt to think of area as a function of anything then they are obstructed by the fact that, in their conception, it is a function of four variables. The third category (Figure 5c) is that they focus their attention on one side, imagining they are moving the rectangle much as do students in the first category, but they do not attend to anything but the side as a whole. They were obstructed from conceiving area as a function of a something by the fact that, within their current conception of the situation, they felt there was nothing they could quantify as a measure of the aspect they imagined themselves manipulating.[5]

[5] I included in the first category those students who thought of moving a side while thinking of measuring the distance between a vertex on the triangle and a vertex on the rectangle.

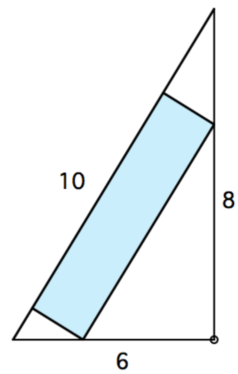

Figure 4. Diagram to accompany the problem, “A rectangle is situated within a triangle as shown above. Find dimensions of the rectangle that maximize its area.”

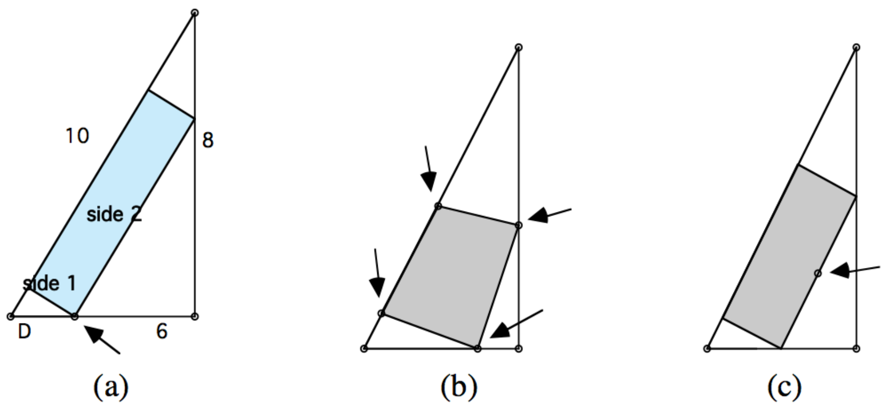

Figure 5. Three categories of conceptualizing the underlying situation: (a) Move one vertex, everything moves accordingly, lengths of sides, and hence area, are a function of the distance from vertex D to corner of rectangle. (b) Move each of the rectangle’s corners to get another rectangle. (c) Move a side of the rectangle; everything else moves accordingly. Area is somehow a function of “where the side is.”

A pedagogical implication of these examples is that, for students whose conceptualizations fall into the second two categories illustrated in Figure 5, an instructor who fails to understand how they are thinking about the situation will probably speak past their difficulties. Any symbolic talk that assumes students have an image like that in the first category will not communicate. These students need a different kind of remediation, a remediation that orients them to construct the situation as one of constrained variation. Only then will they be in a position to understand the task as originally intended, to represent analytically a covariation of magnitudes.

The examples given by Monk, as described previously, and the example I gave above should not be dismissed as somehow pathological. A growing body of evidence suggests that this kind of miscommunication—instructors erroneously assuming students have a principled understanding of an underlying situation—is far from uncommon. Alba Thompson and I have found this to be the case in the teaching of rate in middle school [**67, 68**] and in the teaching of calculus [**75**]. Research by White and by Ueno [**78, 88, 89**] suggests quite strongly that this type of miscommunication also occurs frequently in the teaching of physics.

Operations on Numbers and Operations on Functions

The process conception of function described by Dubinsky and his colleagues [**2, 6, 24**] emphasizes arithmetic operations as operations on numbers, so that, for example $f(x) = x^2 + 3x$ is the function determined by evaluating the sum of a number squared and three times the number. This seems non problematic. We can even consider the family of functions $f_a(x) = x^2 + ax$. A common exercise in secondary school algebra is to show that the graphs of any function in this family is a translation of the graph of another (Figure 6).

An article by Dugdale [**27**] inspired me to ask students in one course for senior mathematics majors to examine the influence that changes in the linear coefficient has on the behavior of functions in the family $g_a(x) = x^2 + ax,\ \ a,x\ \ \varepsilon\ \ \Re,\ \ n\ \ \varepsilon\ \ \mathrm{N}$. They quickly discovered that thinking of one graph as being the translation of another gave little insight into the general effect that changing the value of *a* has for *n*>2 (Figure 7).

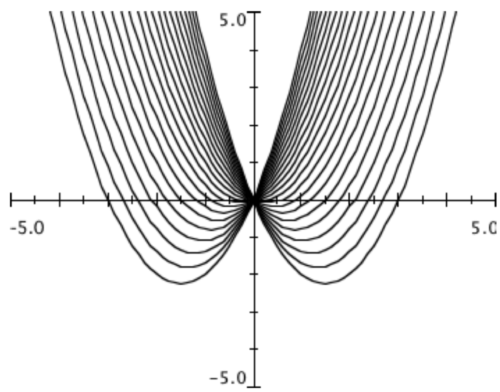

Figure 6. The family of functions $f_a(x)=x^2+ax$ as a ranges from -3 to 3 in increments of 0.3. Each graph in the family is a translation of any other graph in the family.

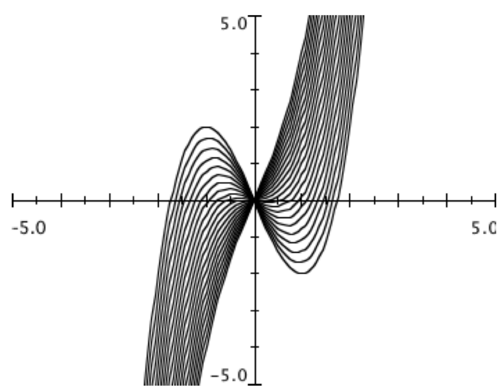

Figure 7. The family of functions $f_a(x)=x^3+ax$ as a ranges from -3 to 3 in increments of 0.3. The graphs are *not* translations of each other.

A little reflection makes their difficulty clear. Thinking of the graph of one quadratic as being a translation of another draws only on pointwise correspondence of points in the Cartesian plane. It is somewhat coincidental that, in the case of two quadratics *f* and *g*, there exist numbers *a* and *b* that will relate two subsets of the plane defined by $\{(x,y)|y=f(x)\}$ and $\{(u,v)|v=g(u)\}$ so that $f(x)+b = g(x+c)$ for some real numbers *b* and *c*. This relationship does not generalize to polynomials of degree greater than 2. It was only after these students came to think of the expression x^n+ax as a sum of functions instead of as a sum of numbers that they gained insight into the effect of varying the linear coefficient in a way that generalized across n. Figure 8 shows their generalized image of the effect of varying a in the case of $g_2(x)$. By varying the linear coefficient we change the slope of the line upon which segments of length x^2 are placed, but we continue to add the same values of x^2. Seeing the effect this way makes it clear why the family $g_3(x)$ appears as it does in Figure 7—the values of x^3 remain unchanged, but are being "attached" to a line of varying slope.

I suspect that an orientation toward viewing arithmetic operations as operations on numbers supports students' natural inclination to view graphs as pictorial objects *sans* points. I often hear even mathematics majors speak of a graph as "stretching" or "getting skinnier" or "being squished" without any thought being given to an underlying dynamics of functional relationship. Goldenberg [**30**] notes that a similar orientation toward casting change of scale and change of axes as operations on functions tends to direct students away from thinking of functional relationship and toward thinking of graphs as objects in and of themselves.[6]

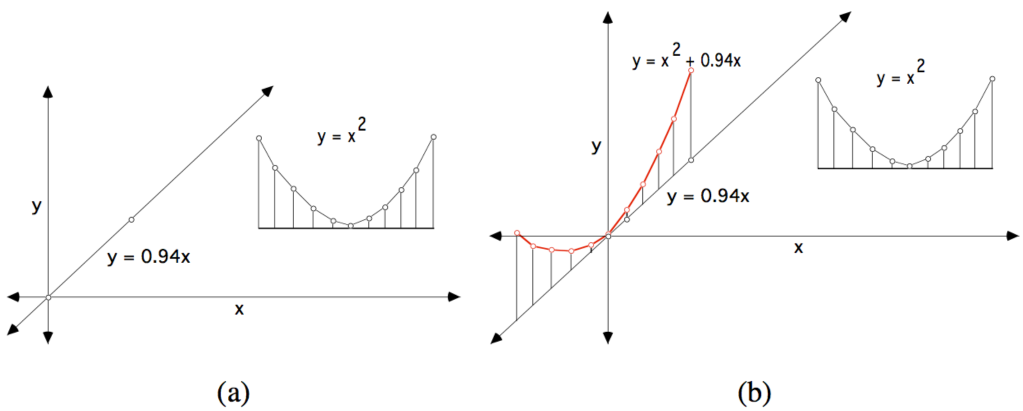

Figure 8. (a) The functions $f(x)=0.94x$ and $g(x)=x^2$. (b) The function $h(x)=x^2+0.94x$ as the sum of $f(x)$ and $g(x)$. Varying the linear coefficient varies the slope of the graph of f, but the increments due to g remain constant—they just get moved up or down as the linear coefficient varies.

The notion of derivative and integral as operators (e.g., on products, sums, quotients, compositions of functions) is based on seeing expressions as being comprised of operations on functions. A curricular and instructional emphasis in algebra and precalculus on having students develop images of arithmetic operations in analytically-defined functions as operations on functions would seem to prepare them for a deeper understanding of this aspect of the calculus. At the same time, a conception of operations in expressions as operating on numbers and not on functions would seem to be an obstacle to understanding the derivative and integral as linear operators. These are empirically testable hypotheses; I would welcome research on them.

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[6] One reviewer interpreted this comment as being denigrative of graphs as mathematical objects. This is not the case. Rather, I am speaking about students thinking of graphs as *non*mathematical objects—as if they were a piece of string or a rubber band. I see no benefit in students holding such conceptions of graphs.

Another investigation suggested by this line of argument would be to assess the conceptual requirements for understanding expressions as being comprised of operations on functions. I suspect they would entail, at least, an image of function as process completed over a domain. But, again, this is an empirically testable hypothesis—one which would be ideally suited for a research/development project, since to investigate this question one would need to develop curriculum and pedagogy aimed at having students learn to think of functions as something to be operated on.

Emergent Issues

A number of issues emerge from the literature on function concepts that have not been directly researched, but nevertheless seem important. These are:

- Are specific kinds of intellectual operations required to conceptualize different kinds of functions?
- To what extent are students' difficulties a product of instructional obstacles?
- To what extent are students constrained by our misunderstanding their practical realities?

Development of specific functions

Recent research in students' understanding of multiplicative structures [**16, 17, 74, 81, 82**] suggests that students who develop strong concepts of function begin doing so by building images of quantitative covariation. But at the same time, it is becoming evident that "quantitative covariation" is not a unitary construct. My own research [**74, 76**] suggests that concepts of linear function emerge from deep understandings of rate. Earlier in this paper I argued that understanding polynomial functions entails special conceptualizations. Dreyfus' research [**22**] highlights the need for students to conceptualize periodic phenomena if they are to develop more than a superficial understanding of trigonometric functions. The work by Confrey's research group [**17, 62**] suggests that students must first comprehend recursive processes in order to conceptualize exponential functions.

The case of exponential functions is especially instructive. I have shared the graph shown in Figure 9 with a number of secondary mathematics teachers and university professors, asking "during what period of time was inflation the greatest?" Responses, vary, but they almost uniformly include the period from 1978 to 1980. When asked to pick the period with the next highest inflation rate, responses vary considerably more—but they have yet to include the period from 1945 to 1948, the period which includes the *actual* highest inflation rate.

After having been told that 1945-1948 contains the actual highest inflation rate, many people say something like, "Oh, of course—you have to compare one year's price as a percent of the previous year's price." But even more (so far, none have been university professors) could not reconcile my comment with their reading of the graph. It seems that they were looking at inflation as increase in price per year (i.e., rate of increase with respect to time), which translates into slope, instead of as percentage change, which is a recurrence relationship. Confrey and her group argue that the notion of recurrence is common to conceptualizations of situations that entail exponential growth. They also note that this style of thinking is absent in the school mathematics curriculum. I suspect that many calculus instructors routinely assume that it is non-problematic for students to envision exponential growth when it seems few are inclined to do so without a great deal of orientation on the instructor's part.

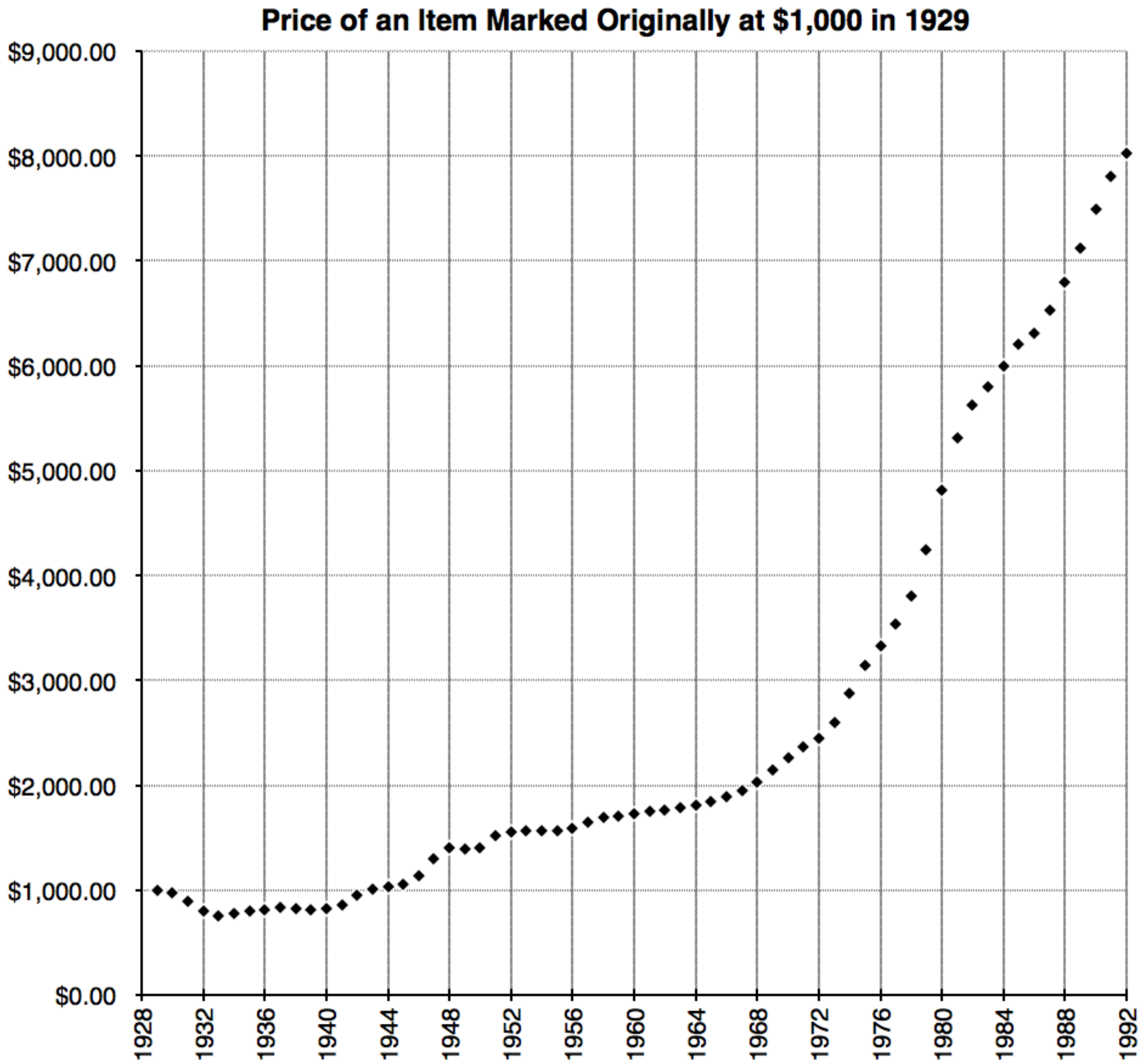

Figure 9. The price of an item marked in 1929 at $1,000 during the years 1930-1992, adjusted for inflation.

Our tacit expectations: Cognitive and instructional obstacles

Herscovics [**34**] explicated the notion of cognitive obstacle as it relates to learning mathematics. A cognitive obstacle is a way of knowing something that gets in the way of understanding something else. For example, thinking of a graphics construction as producing only what is immediately envisioned can be an obstacle in regard to conceptualizing the construction of a fractal [**70**]. An instructional obstacle is instruction that promotes new cognitive obstacles or supports or is neutral in regard to students' existing cognitive obstacles. Sierpinska [**61**] discusses a number of instructional obstacles to the understanding of functions. I will attempt to illustrate how we often contribute unthinkingly to our students' difficulties.

Many calculus texts begin their section on implicit differentiation with a discussion of how an equation in two variables somehow hides a function—one variable being a function of the other. *Behind this approach lurk three conceptions of function, and students are not alerted that there is something subtle going on*. The first conception of function is of a multi-variable equation—$f(x,y)=c$. That is, we are looking at the pre-image of a level curve. The second conception is that of function as rule. It is suggested that from the equation we can, or at least would like to, rewrite the equation to get a rule for obtaining values of one variable from values of the other. The third conception is that of function as correspondence between sets. Any subset of the pre-image that determines a univocal mapping from one variable to the other is a function.

My point in relating this common opening to implicit differentiation is twofold: First, it is often evident that what the author really has in mind is getting to the rule for implicit differentiation with no real expectation that students understand the notion of implicit function. Second, if the intent is for students to understand the setting under which implicit differentiation happens, then they should be alerted to how complex the setting is. "Warning, Go slow! We're going to look at a pretty sophisticated set of ideas here."

I entitled this section "Our tacit expectations." Do we expect students to really understand what we teach? If no, then we need to say so. If yes, then we need to expect understanding, communicate that expectation, and provide curricular and pedagogical support for students to meet our expectations.

Students' practical realities

We must keep in mind that our college students have spent 12 years in school learning that mathematics is a ritualistic behavior, and that often their expectation of us is to "show them how to do it." Their mathematical experiences did not include learning how to use notation thoughtfully and reflectively; notation is something to be seen, not to be interpreted. But, to change students' orientation requires a "renegotiation of the didactic contract" [**3, 49**]. Students need to know that we know where they are coming from, and that ritualistic performance is not satisfactory. At the same time, we must assume the responsibility for shaping our instruction so that it *can* be understood conceptually, and to do that we must attend constantly to matters of imagery and understanding.

Multiple Representations

While the importance of students' understanding expressions, tables, and graphs has been common knowledge for at least a century, it is only since the early 1980's that they have been seen cognitively and pedagogically as alternative windows on a central idea [**38, 39, 42**]. Even though a semblance of multiple representations can be seen in Diene's original idea of multiple embodiments of mathematical concepts [**20, 47**], the notion of multiple representations has today become a powerful motor of curricular research and development largely because of access to increasingly powerful computers and graphing calculators. I refer you to [**57**] and [**46**] for state-of-the-art reviews of research on graphs, tables, and expressions as they relate to the matter of multiple representations of function. I will instead give a cautionary note regarding an important missing element in this line of research and development.

I believe that the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of representation.[7] Tables, graphs, and expressions might be multiple representations of

[7] This is entirely parallel to the situation in information processing psychology—no one

functions to us, but I have seen no evidence that they are multiple representations of *anything* to students. In fact, I am now unconvinced that they are multiple representations even to us, but instead may be areas of representational activity among which, as Moschkovich, Schoenfeld, and Arcavi [**52**] have said, we have built rich and varied connections. It could well be a fiction that there is any interior to our network of connections, that our sense of "common referent" among tables, expressions, and graphs is just an expression of our sense, developed over many experiences, that we can move from one type of representational activity to another, keeping the current situation somehow intact. Put another way, the core concept of "function" is not *represented* by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance.

I do not make these statements idly, as I was one to jump on the multiple-representations bandwagon early on [**71, 72**], and I am now saying that I was mistaken. I agree with Kaput [**40**] that it may be wrongheaded to focus on graphs, expressions, or tables as representations of function. We should instead focus on them as representations of something that, from the students' perspective, is *representable*, such as aspects of a specific situation. The key issue then becomes twofold: (1) To find situations that are sufficiently propitious for engendering multitudes of representational activity and (2) To orient students toward drawing connections among their representational activities in regard to the situation that engendered them. The situation being represented must be paramount in students' awareness, for if they do not see something remaining the same as they move among tables, graphs, and expressions, then it increases the probability that they will see each as a "topic" to be learned in isolation of the others. Dugdale [**26**] provides an excellent example of a productive and powerful coordination of situation and representation.

Reflections

Much of the literature on students' concepts of function highlights what they do not know about functions and why that might be the case. One lesson we can learn is that students can use abstract function concepts only if they build connected abstractions from "functional" reasoning—reasoning about constrained covariation, reasoning about representations of quantitative and numerical relationships, and reasoning about properties of relationships. Curricula that are constructed logically, but which do not attend to transitional conceptualizations, can put students at risk of having to cope with demands of performance by turning our representations into their objects of learning [**48, 59, 60**]

Another lesson to draw from the foregoing is that our success in vitalizing the undergraduate curriculum is highly dependent upon a great deal more conceptual development happening in schools. If students come to us with impoverished concepts of function, then there is not a lot we can do except accommodate to the constraints we find.[8] We must support, collectively and individually, the efforts of NCTM [**14, 15**] to reform school mathematics. We cannot succeed unless they succeed.

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has bothered to question what is meant by "information" [**10, 11**].

[8] This is not to say that we cannot do anything. It means that whatever we try must take

Finally, we need to broaden our notion of appropriate curriculum. Logical developments of content look especially good to people who already know it, but they are "logical" precisely because they express the logic we have constructed in our understandings of the subject. I propose that we orient ourselves toward developing *conceptual* curricula—curricula that are mathematically sound, but nevertheless are constructed from the start with an eye to building students' understandings, and are constructed to assess skill as an expression of understanding. I take this as our major challenge over the coming decade.

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into account the low level of experience in grounded, functional reasoning which our students will commonly bring to the classroom.

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