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Technology and Algebra Curriculum Reform: Current Issues, Potential Directions, and Research Questions

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The infusion of computers and calculators into mathematics instruction raises new possibilities for teaching and understanding algebra, as well as rethinking the nature of algebra itself. In this paper we characterize algebra as a way of reasoning involving three broad, interrelated themes: (a) variables and functional relationships, (b) generalization and modes of representation, and (c) mathematical investigation and argument. We examine curricular goals and consider the contributions of technology to reconceptualizing algebra in terms of these three themes. We suggest directions for future development of tools and implementation models, possibilities for a research agenda, and implications related to assessment and evaluation.

INTRODUCTION

In considering the roles of technology in algebra curriculum reform, two immediate questions arise. First, what are we including as "technology," and second, what do we mean by "algebra?" Of the technological innovations that have entered the algebra classroom in recent years, the calculator, the computer, and the graphing calculator (pocket computer) stand out as tools that have potential for dramatic impact on the teaching of algebra. It is primarily these three tools and their uses that shaped the discussions on which this document is based. Of course it is possible that other technologies, such as interactive video, will prove to have substantial impact in the future, but such possibilities are not addressed in this document.

Although the choice of what technology to address may be somewhat arbitrary and based on personal experience and interest, the second question, "What do we mean by algebra?" has no simple answer. In fact, the growing use of computers in mathematics may, in itself, be causing this question to take on new meaning. However, we have found reasonable consensus around a succinct set of themes that we can use to characterize the scope and spirit of what we intend to include as "algebra." We suggest that algebra be considered as a way of reasoning involving:

- *Variables and functional relationships* (including various representations of functional relationships—equations, tables, graphs). These ideas need not be confined to high school algebra courses. Arithmetic can (and probably should) include experience with variables and functional relationships.

- *Generalization and modes of representation.* This includes development and use of formulas, translation of ideas to and from various representations, and manipulation of those representations. Traditionally, the focus has been on algebraic-symbolic representation. Present applications of technology have enlarged the traditional focus to include a new emphasis on graphical representations and possibly others as well.
- *Mathematical investigation and argument.* In addition to formal algebraic proof, this includes mathematical reasoning in the sense of noting patterns, making conjectures, testing and justifying conjectures, and refining reasoning as necessary. Technology has broadened the scope of mathematical investigation and argument by facilitating the manipulation of representations (notably graphical representations) and the correlation of various representations.

The habit of mathematical reasoning should be fundamental throughout mathematics education. Spending the first 7 or 8 years of mathematics on practice of rote procedures makes it difficult to shift gears and expect mathematical reasoning to blossom with high school algebra. By that time, students already think they know what "math" is, and how it is approached. These assumptions and misconceptions can be difficult to unlearn or replace (McCaslin & Good, 1992; McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers, & Cooney, 1987; Porter, 1989; Resnick & Omanson, 1987; Thompson, 1992).

We have deliberately chosen each of the three broad themes above to imply no limitation to the use of (or predominance of) a particular mode of representation. Each theme relates to various modes of representation. For example, one dramatic effect of technology on the algebra curriculum has been to facilitate the manipulation of graphs, raising the possibility of graphical representations taking a more equal footing with the more traditional algebraic-symbolic representations. In fact, for solving equations, inequalities, and so forth, graphical techniques can be considerably more powerful than algebraic-symbolic techniques.

Graphical solution methods can be both more direct and more informative about a problem situation, and they can make interesting problems accessible earlier in the curriculum. For example, compare a traditional calculus solution and a graphical solution for the following problem (as discussed by Wagner, 1989):

Kathy is in a horse race. From start S to finish F is 20 miles diagonally across a rectangular field 12 miles wide. (See Figure 1. If point P is directly opposite S, then $PS = 12$). She knows that she can travel 6 mph through the field, but she can travel 10 mph on the harder ground on the opposite side of the field. Find the point X between P and F that would give Kathy the fastest time for the race. What would be her best time?

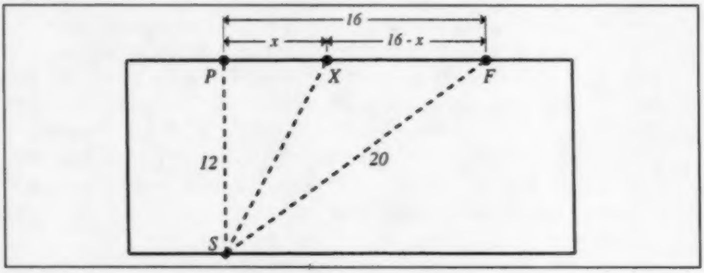

Figure 1. Diagram for Kathy's horserace

Either solution method (symbol manipulation or graphical) requires development of a function to express the time needed for the race. Using the Pythagorean Theorem, the distance PF is found to be 16 miles. Letting the distance $PX = x$, then $XF = 16 - x$, and the total time t for the race is the sum of the time across the field (segment SX) at 6 mph and the time on harder ground (segment XF) at 10 mph. Hence, the function:

$$t = f(x) = \frac{\sqrt{12^2 + x^2}}{6} + \frac{16 - x}{10}$$

From here, the calculus solution requires taking the first derivative to determine the relative maximum or minimum. Taking the second derivative can then confirm whether it is a maximum or a minimum. The result is a value for x that minimizes the function. This value is then substituted into the function to calculate the minimum value for the function. Solution: Kathy's best time would be 3.2 hours, when point X is 9 miles from point P.

Alternatively, the function can be graphed, as shown in Figure 2. Because the distance PF is 16 miles and a negative distance is not meaningful for the problem, x must be between 0 and 16 miles. For $0 \leq x \leq 16$, the function values appear to stay between 3 and 4 hours. Zooming in to examine the graph within these bounds, as shown in Figure 3, it becomes apparent that the minimum time is about 3.2 hours and that this time occurs when point X is between 8 and 10 miles from point P.

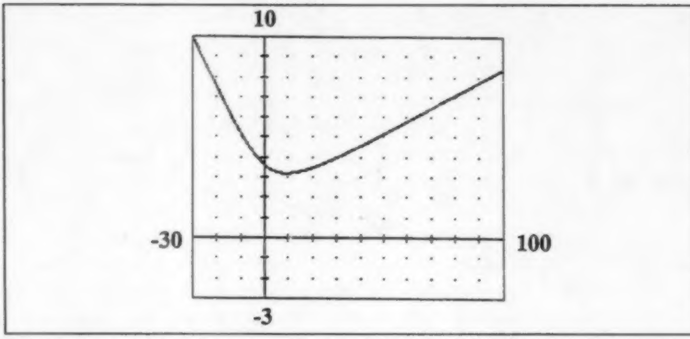

Figure 2. A graph of the function, $t = f(x) = \text{sqr}(12^2 + x^2)/6 + (16 - x)/10$, from $x = -30$ to $x = 100$

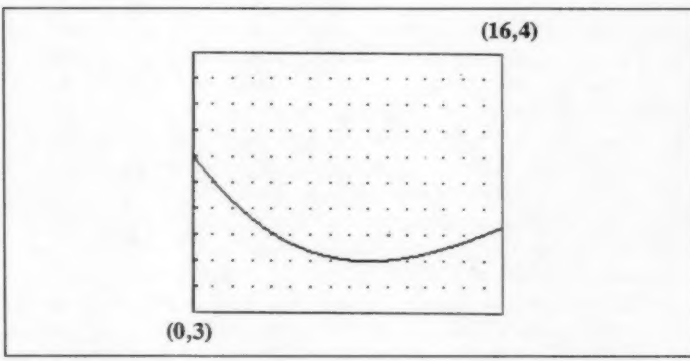

Figure 3. Zooming in to examine the graph for $0 \leq x \leq 16$ and $3 \leq t \leq 4$, the problem situation can be analyzed.

Zooming in further can determine the minimum more precisely, and the natural question is, "How close to the exact minimum do we have to get?" To answer this question, it is necessary to relate the graph to the problem situation. Because the graph models the *entire* problem situation (as opposed to producing a single answer), the graph is useful as an indication of how important it is for Kathy to make a perfect angle at just the right point X. If the function were very steep, a small change in x would make a big difference in t, and precision would be important. But in this case, for $8 \leq x \leq 10$, t appears unchanged on a scale where one screen pixel represents less that 30 seconds on the vertical axis. (With ten grid intervals shown per hour and about 16 pixels evident between grid dots, each pixel represents about 23 seconds.)

In considering whether a difference of less than 30 seconds is important, it is natural to examine the relationship of the mathematical model to the real situation. For example, Kathy's horse could not maintain a constant rate over the distance (time is required to speed up), the horse would not make an instantaneous change of speed at point X, and the horse could not make the perfect angle SXF. Hence, in the real situation, it is more realistic to head for some point X between 8 and 10 miles from point P, than to expect to travel the route indicated by the calculus solution.

A computer graphing utility makes this typical calculus problem accessible to any algebra student familiar with the Pythagorean Theorem and the formula $d = rt$. A graph can reveal insights into the problem situation that are not readily revealed by symbol manipulation alone. We believe that encouraging students to make connections among the problem situation, a graphical representation, and an algebraic-symbolic representation can facilitate a broader perspective on the problem and provide a base of experience for later treatment of similar problems in calculus.

Although historically there has been a strong focus on symbol manipulation in the algebra curriculum, the advent of computers and calculators in the classroom facilitates a new approach—one where the focus is on reasoning with a variety of representations and understanding the relationships among those representations. The characterization of algebra outlined at the beginning of this paper is intended to reflect the technologically-facilitated trend away from a strong focus on symbol manipulation toward an emphasis on reasoning with a variety of representations and the relationships among those representations. Although the need for algebraic symbol *manipulation* is considerably reduced by technology, algebraic-symbolic *representation* is still very important. For example, in the problem discussed above, expressing the situation as an algebraic function was a useful step toward producing a graphical representation.

If we accept the notion that the algebra curriculum can be improved, and significantly so if we use computers and calculators appropriately, we still need to work toward some consensus on the curricular goals we ought to achieve. For example, a new fourth-year high school course might be created that addresses the prerequisites of "the new" calculus courses that will result from the current calculus reform movement. Three widely agreed on features of the "new" calculus are: (a) decreased emphasis on the by-hand manipulative aspects of calculus that computer algebra systems can do much better, so that more time can be devoted to concept development; (b) involvement of students in the development of the mathematics in an exploratory, problem-solving instruction model; and (c) increased attention to communicating, including writing about mathematics and establishing connections within and outside of mathematics.

A fourth year course and a revised three-year curriculum should be consistent with the desired instructional model of the calculus reform movement. Possible goals are discussed below, structured around the themes used above to characterize the meaning of "algebra."

CURRICULAR GOALS AND CURRICULUM REFORM

The focus of the current algebra curriculum in many schools is, at best, largely symbol manipulation and, at worst, a total neglect of any real algebraic understanding. The current curriculum helps students develop algebraic symbol manipulation skills—for example, solving equations and simplifying expressions. Any new curriculum should help students acquire deeper algebraic understanding—for example, recognizing properties of related functions that are invariant, understanding how operations on equations affect their related graphical representations, and simply understanding "function" as a mapping. The challenge is to balance these different goals and decide what the curriculum should emphasize and in what order.

Not only can computers and calculators assist students in learning traditional algebraic concepts, they can provide support for learning concepts previously unaddressed in the algebra curriculum. For example, computer graphing software enables students to explore the effects of scale changes on the appearance of a function's graph (and therefore on its apparent properties). Other new curricular opportunities made possible by technology include the study of recursively defined functions, the study of function families, and explorations involving the graphical manipulation and construction of functions. (See, e.g., Confrey, 1990; Cornell & Siegfried, 1991; Dugdale, Wagner, & Kibbey, 1992; Harvey, Schwartz, & Yerushalmy, 1989; Maxim & Verhey, 1991.)

In addition to introducing new curricular opportunities, technology has facilitated a departure from the traditional algebraic skills practice in favor of a more active student role in applying algebraic ideas, planning strategies, and reasoning with and about mathematics. Used well, computers have excelled at helping students approach mathematics as an exploratory and participatory activity. One important contribution of computers has been to provide environments in which it is easy to learn from mistakes—analyze errors, try again, and experiment within the constraints of a mathematically-accurate environment. Computer problem-solving activities are also useful in helping students understand the limitations of machine precision. For example, how accurate can solutions obtained by computers be? Understanding new, complex material comes from experience: fiddling with it, testing what works and what doesn't, and applying it to something.

Unfortunately, much attention to date appears to have been directed toward using computers to enhance or support the traditional algebra curriculum. More attention should be paid to how the curriculum should and could change, and how computers and calculators could facilitate that change. Less emphasis should be placed on learning rule systems, or the algebraic equivalent of "arithmetic drill," while more attention could be directed toward helping students develop mathematical reasoning skills and helping students build mathematical models.

Variables and Functional Relationships

The development of a sense of number which lends itself to expression as a variable should start in the primary grades. Tables can be used to teach process, build the concept of variable, provide numerical examples of function, and develop deeper understanding about variable. For example, suppose that a store is having a 12% off sale. The table in Figure 4 can be constructed.

Original Price	Discount	Sale Price
10	.12(10) = 1.2	10 -.12(10) = 8.80
20	.12(20) = 2.4	20 -.12(20) = 17.60
30	.12(30) = 3.6	30 -.12(30) = 26.40
P	.12*P*	*P* -.12*P*

Figure 4. A table showing *Discount* and *Sale Price* as functions of *Original Price*

For a child, building this table is simply a matter of arithmetic. Extending a table to include a variable shows how variables allow for generalizing arithmetic. In particular, it goes beyond the understanding that a letter stands for a single number to be determined. The current curriculum gives students plenty of practice with the use of a letter as a placeholder for a single number to be determined. Little or no attention is given to the use of a letter to stand for a set of numbers as illustrated by the table above. This deficiency in the curriculum may cause students later to find the concept of function difficult to understand.

An immediate curricular goal should be to establish function as the central theme of algebra. Functions can be represented numerically (tables), geometrically (graphs), and analytically (symbolic expressions). In the example above, students create two different numerical functions of *Original Price*: *Discount* and *Sale Price*. Analysis of this process allows them to generalize to a symbolic representation of each of these functions.

The numerical entries in the first two columns of the table can be used to draw a graph that shows how *Discount* depends on *Original Price*. Figure 5 displays the three points determined by the numerical entries of the first two columns, as well as a graph that reveals the behavior of the function, $y = .12P$, for $P \geq 0$. The graph provides a geometric representation of the *Discount* function. Students should now be able to experiment with this problem situation. For example, one might want to change the discount rate and see how this affects the related quantities. Or one might even want to conjecture about how the graph would change when this particular parameter is changed. These experiments can be performed quite easily by using software that links the graphical, symbolic, and tabular representations of function and also allows one to make changes in any one of these representations and show the effects on the others.

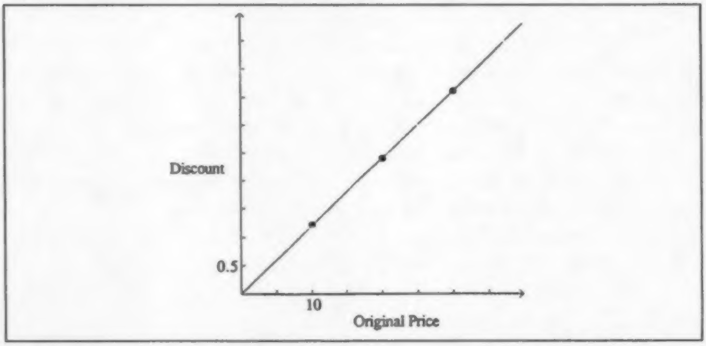

Figure 5. A graph of the *Discount* as a function of the *Original Price*

A technologically enhanced curriculum provides an environment in which students can acquire deeper understandings about functions. Classes of functions (e.g., quadratics, or more generally, polynomials) can be investigated more easily and quickly with the aid of computers. Properties of function families can be explored and conjectures posed. (See, e.g., Dugdale, Wagner, & Kibbey, 1992.)

Although most of the functions considered in the algebra curriculum are those that correspond readily to equations, the conceptualization of *function* in a more global sense is also important. Familiar physical phenomena that are easily modeled by graphs are often not as readily modeled by equations. Some efforts of the past decade have focused on the potential for computer-based experiences to develop a qualitative understanding of functional relationships (Dugdale, 1993; Phillips, 1986). For example, using Microcomputer-Based Labs (Mokros & Tinker, 1987) or other "probeware," a probe attached to a computer can be placed in a pot of water, and the temperature of the water can be graphed as students adjust the heating and cooling of the pot. Experience with qualitative interpretation of graphs that do not readily correspond to equations is important for:

- establishing a more global sense of functional relationships encountered in everyday life.
- confronting the limitations of algebraic-symbolic representations in describing real situations.
- approaching the idea of and need for *approximating* real situations with algebraic equation models.

Generalization and Modes of Representation

The 12%-off sale example given above illustrates the use of a variable to generalize the relationship between the *Original Price* and two associated entities: the *Discount* and the *Sale Price*. Further, it relates the problem situation to three modes of representation: tabular, symbolic, and graphical. Generalizing functional relationships, translating ideas to and from various representations, and exploring the connections among those representations are powerful tools for mathematical modeling.

Important topics in algebra can be introduced as models of interesting and realistic problems. Consider the following example:

Squares of side length x are cut from each corner of a 30" by 45" rectangular piece of cardboard, and the cardboard is folded to form a box with no top. The volume of the resulting box is given by the function:

$$V(x) = x(30 - 2x)(45 - 2x).$$

Questions that can be posed about the resulting box give rise to many different models. For example, if we want the box to have volume 2000 in^3, we need to solve the equation $V(x) = 2000$. This is not easily done by algebraic symbol manipulation, but it is easily solved numerically with a calculator, graphically with a function plotter, or by using an equation solver.

Figure 6 gives a complete graph of the function $y = V(x)$. (A *complete* graph is one that shows all important behavior.) In the process of obtaining a graphical solution to the above problem, students must decide which points of the graph of $y = V(x)$ make sense in the problem. They do this by exploring the connections among the symbolic representation $V(x) = 2000$, the geometric representation given the graphs of $y = V(x)$ and $y = 2000$, and the problem situation (Demana & Waits, 1990). The values of x that make sense in the problem situation are $0 < x < 15$. Thus, Figure 7 gives a graph of the problem situation in the restricted domain.

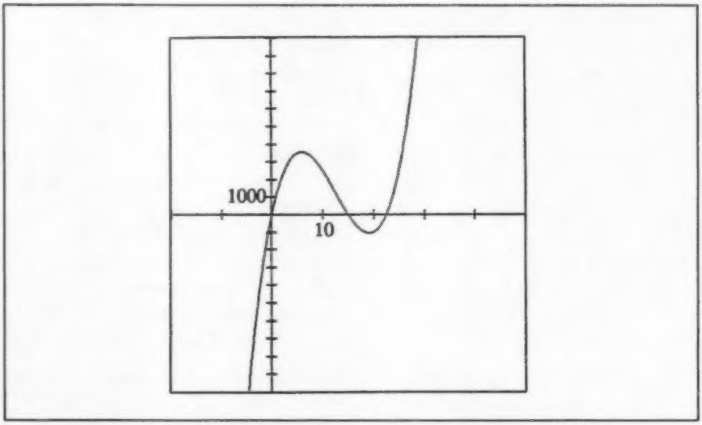

Figure 6. A complete graph of the function $V(x) = x(30 - 2x)(45 - 2x)$

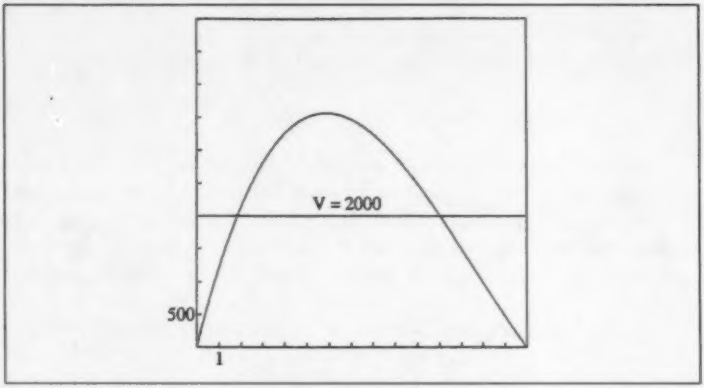

Figure 7. A complete graph of the problem situation

In a computer environment, problem exploration can be further encouraged by raising such questions as, "How does the solution, or even the *number* of solutions, to this problem change as the volume of the box (initially specified as 2000 in^3) is changed?" Continuing with this exploration, it is natural to ask what value of x produces the box of maximum volume—a problem usually reserved for calculus. Studying this problem in precalculus foreshadows the later study of this problem in calculus. In this example, technology eliminates traditional constraints of the algebra curriculum by removing limitations on the kinds of problems that can be posed.

Further, technology facilitates attention to some important, but heretofore unaddressed, issues in representing functions. For example, the ready availability of re-scaling when graphing functions can help bring to students' attention new and important understandings about the nature of graphical representations. Consider, for instance, the following case from Goldenberg and Kliman (1991). When students were asked to describe "the graph of a quadratic function," they all said, without hesitation, that it would be a parabola. They were then asked to describe the graph of a *specific* quadratic function: $f(x) = x^2 + 7x + 1$. Although different students approached this in different ways, they still all produced parabolas. They were then shown the graph in Figure 8 and told that this was a graph of the function $f(x) = x^2 + 7x + 1$. When asked to explain this, many students—good students—found themselves startled, confused, and generally unable to reconcile this image with what they *knew* was the graph of a quadratic function. This confusion existed even for students experienced with the use of computer graphing tools.

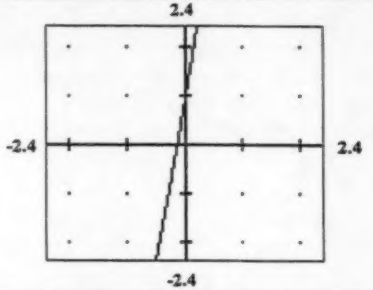

Figure 8. The function $f(x) = x^2 + 7x + 1$, graphed with both axes scaled from about -2.4 to 2.4

Mathematical Investigation and Argument

The exploration questions raised in the box volume problem above illustrate some fertile opportunities for mathematical investigation in a computer environment. In addition to formal proof, mathematical reasoning includes noting patterns, making conjectures, testing and verifying conjectures, and refining reasoning as necessary.

Technology has broadened the scope of mathematical investigation and argument by facilitating the manipulation of representations and the correlation of various representations. Students can note patterns in tables and make conjectures based on their observations. They are also able to view many graphs quickly with the aid of technology and make conjectures based on those observations. Sometimes they are able to verify the conjectures by algebraic symbol manipulation.

For example, computer exploration can help convince students that the graph of $f(x) = (x^3 - 10x^2 + x + 50)/(x - 2)$ in Figure 9 is *complete* (i.e., that the graph shows all important behavior). One component of this understanding is the end behavior of f for x large in absolute value. Using zoom-out to obtain the graph in Figure 10 (Waits & Demana, 1989), students can conjecture that $y = x^2$ is an end behavior model of f. (The function g is said to be an *end behavior model* of f, if f/g approaches 1 as the absolute value of x approaches infinity.) Division of polynomials can be used to establish that $f(x) = x^2 - 8x - 15 + 20/(x - 2)$ and see that $g(x) = x^2 - 8x - 15$ is the quadratic asymptote of f (i.e., that f approaches g arbitrarily closely as the absolute value of x approaches infinity). Overlaying the graphs of f and g in Figure 11 gives geometric understanding about the end behavior asymptote of f. Carefully choosing the degrees of the numerators and denominators of rational functions allows students to conjecture the behavior for large x, then long division of polynomials can be used to verify and understand the geometric observations (Demana & Waits, 1990).

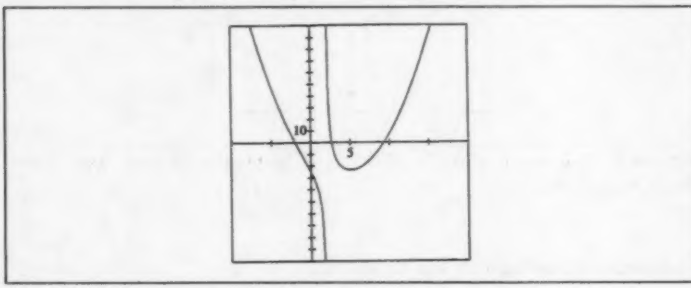

Figure 9. A complete graph of $f(x) = (x^3 - 10x^2 + x + 50)/(x - 2)$

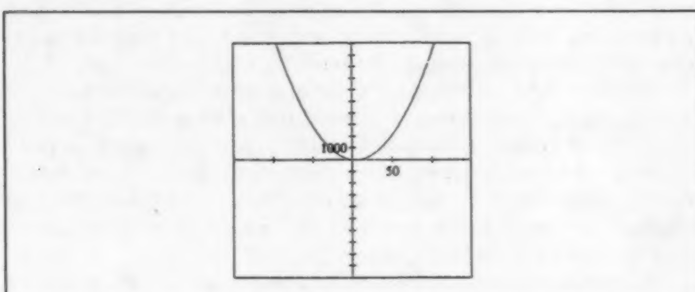

Figure 10. Geometric evidence that $y = x^2$ is an end behavior model of f

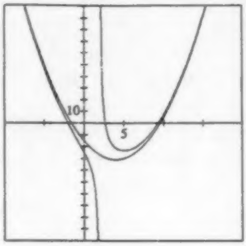

Figure 11. Geometric evidence that $g(x) = x^2 - 8x - 15$ is the quadratic asymptote of f

Further, the computational, visual, and interactive nature of computer environments enables the creation of new ways of observing and investigating mathematical behavior. Entirely new understandings of function may be possible through experimentation with "controlled dynamic phenomena."

For example, current work at Education Development Center (Goldenberg, Lewis, & O'Keefe, 1992) is concerned with the development of students' understanding of function as a mapping, and in particular, understanding of mappings from R^1 to R^1. Most graphing software accepts an algebraic-symbolic notation (e.g., $f(x) = x + 2$), and graphs the specified function in the real plane—that is, in R^2. However, the dynamic-interactive properties of computer interfaces enable *other* graphical representations that offer different views of functional relationships. Instead of using only the traditional perpendicular system of axes, the functional mapping $f(x) = x + 2$, from R^1 to R^1, can be represented on two *parallel* (or even coincident) number lines similar to those shown in Figure 12. With a single value of x plotted on one number line, and its image, $f(x)$, plotted on the other number line, a student can use a mouse to move the indicated x-value on its axis, causing the image, $f(x)$, to move simultaneously on its parallel axis according to the functional relationship specified.

This "kinesthetic" approach to investigating functions brings new intuitions and understandings to the fore. The simple function $f(x) = x + 2$ can be seen more obviously as an "adding of 2" to the value of x when investigated on the parallel axes than when graphed in the usual R^2 plane. Even more intriguing is the behavior of fixed points and relative maxima and minima. Also, various calculus concepts, such as limits and rate of change, become more directly observable. In fact, the behavior of functions with asymptotes, for example $f(x) = 1/(x - 1)$, can bring forward dramatically new intuitions about infinity. As a student moves the plotted x-value back and forth across the asymptote boundary, its image, $f(x)$, shoots off the screen toward positive infinity and back onto the screen from negative infinity instantly. One gets the distinct impression that positive and negative infinity are "connected back there somewhere!"

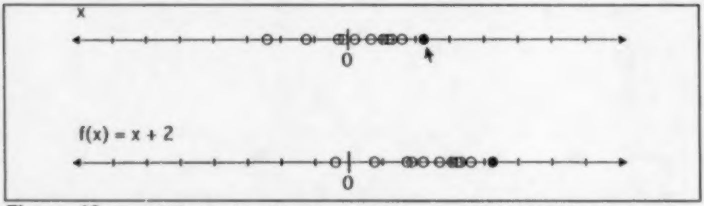

Figure 12. Parallel number lines represent x and $f(x)$. Clicking to mark a point on the x number line also marks its image on the $f(x)$ line.

Introducing technological innovations into even the most traditional curriculum can have a dramatic effect on what is taught, what is learned, and the very fabric of classroom discussions. The following example uses an equation-solving program, *Eureka* (Borland, 1988), to solve optimization problems typical of those found in calculus texts, such as Anton (1988).

A rectangle is to be inscribed in a right triangle having sides of length 6 in, 8 in, and 10 in. Find the dimensions of the rectangle with greatest area assuming the rectangle is positioned as in Figure 13.

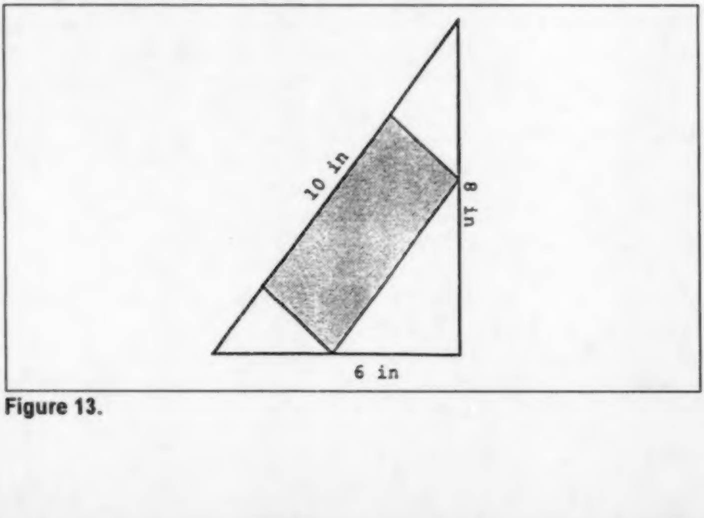

Figure 13.

In *Eureka,* as in most equation-solving programs, one enters lists of equations and a directive which commands the program to perform some tasks relative to that list. For the problem given above, the following can be entered into *Eureka*:

```
Hypotenuse = 10 ; '=' denotes assignment
Side1 = 6
Side2 = 8
; Let x represent distance of rectangle's "Side1-vertex"
; from intersection of Hypotenuse and Side1
area(x) := width(x) * height(x)              ; ':=' denotes function
width(x) := Hypotenuse * (Side1-x) / Side1        ; similar triangles
height(x) := Side2 / Hypotenuse * x   ; similar triangles
A = area(x)        ; store area for any value of x
$max(A)   ; find value of x that maximizes area
W = width(x)    ; store rectangle's width at maximum area
H = height(x)    ; store rectangle's height at maximum area
```

These equations reflect a conception of the triangle as embedded within a Cartesian coordinate system (Figure 14), where the rectangle's height and width are expressed as functions of the distance between the origin and the rectangle's lower vertex. As x varies, the height and width of the rectangle vary. The directive "$max(A)" tells *Eureka* to find values of variables that maximize the variable called A.[1]

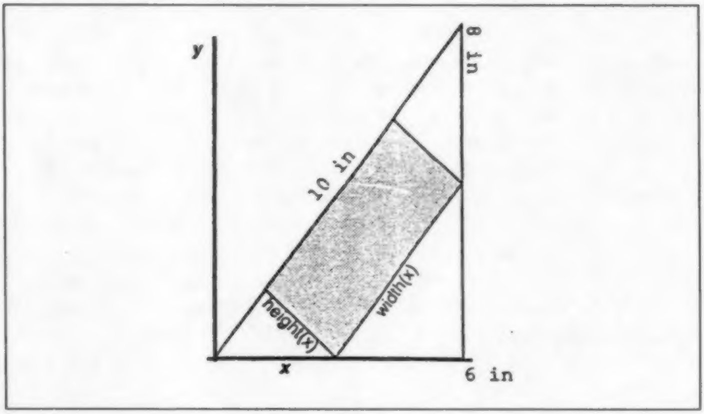

Figure 14.

Upon selecting "Solve," *Eureka* displays this information:

Solution

Variables		*Values*
Hypotenuse	=	10.000000
Side1	=	6.0000000
Side2	=	8.0000000
x	=	3.0000000
A	=	12.000000
H	=	2.4000000
W	=	5.0000000

Confidence level = 98.8% All constraints satisfied.

We can quibble about the "relevance" of this problem, and about whether more realistic problems might be dealt with more appropriately. But there is a more fundamental issue raised by this example. In creating a model of a situation, a student must think of the situation in terms of functional relationships between quantities whose values vary (e.g., area of rectangle as a function of a vertex' changing location on a coordinate axis) and develop the model so that it reflects those functional variations. An orientation toward developing models of functional relationships and variation is a drastic change from current algebra curricula, yet employing an equation solver puts such modeling within the reach of secondary school algebra students.

Imagine that students in an Algebra II course were to have a steady diet of problems that involved the use of algebra as a language for constructing mathematical models. Two questions arise: (a) What would we do with these students in their first year of university-level mathematics? and (b) What would we need to do in 9th and 10th grades to prepare them for Algebra II? We do not have answers to these questions. We do believe that this scenario deserves serious study, and that the implications will extend into elementary and middle-school curriculum. One implication of which we are sure is that mathematics curricula at all levels will need to change from "backward-looking" to "forward-looking" (McKnight et al., 1987)—that instead of continually honing skills supposedly taught at lower grades, these curricula will need to focus on developing attitudinal and conceptual foundations for more advanced work.

FUTURE NEEDS

Technological Implementation Models

In considering future needs, we suggest that there may be less need for new technological tools than for successful implementation models for technological tools in general. Most technological tools, in and of themselves, do not determine what kinds of algebraic concepts students learn, or even encounter. The ways in which the tools are used can have much more influence than the software designer's original intent. Teacher preparation, involvement, and what accompanies the software are sometimes most important of all. Many uses of computers or calculators for algebra curriculum reform need the support of carefully crafted, guided exploration activities. Such activities can be critical to the learning outcomes of students interacting with computer-based exploratory mathematical tools. For students to engage consistently in productive explorations, it is necessary for software tools to be accompanied by activities which are easy for both students and teachers to use.

Activities need to incorporate varied examples that highlight the mathematical objective of an activity. For example, an activity on factoring polynomials should include at least one example of a double root (two equal factors) and an example that is unfactorable. A few carefully selected examples can be more effective than the long list of similar problems often found in textbooks.

In order to serve both teachers and students effectively, these activities need to:

- focus student attention by suggesting questions, possible avenues of inquiry, and work strategies,
- build students' inquiry skills,
- suggest provocative examples that illuminate important cases to explore,
- enable students to apply, generalize, integrate and extend their knowledge, and
- be relatively easy to use.

More attention needs to be directed toward creating model curricula for use with existing software tools, and toward doing the necessary research to provide guidance to new curriculum designers. This research must not only examine students' cognitive gains, but must look at appropriate mechanisms for teacher support as the roles of both teacher and student change.

Technological Tools

Although we view the development of new technological tools as less crucial than attention to appropriate implementation models for technological tools in general, we offer the following examples of possible directions that merit exploration.

Generalizing functions as mappings. Software tools can be designed to help students understand function as a kind of mapping from one set of objects to another. These objects need not be numbers—they can even be geometric objects. Iterated mappings of various geometric objects can produce fractals: a fertile domain for mathematical investigation and intrigue.

Even when we are dealing with mappings of numbers, we often impose unnecessary constraints. For example, we usually introduce functions as R^1 to R^1 mappings (the real numbers to the real numbers) which can be most easily represented as a graph in R^2 (the coordinate plane), and we usually draw graphs by plotting points from left to right. Other plotting modes could be used—for example, graphs could appear instantaneously, or they could slowly appear as random values in the domain are chosen and plotted. The effects of such differences on students' understanding of function could be investigated. Further, we could explore R^2 to R^2 mappings. Much can be learned by exploring why these are hard to represent graphically (in general R^2 to R^2 mappings would be graphed in R^4 space), how we might otherwise visualize them (select a limited domain of R^2—say a circle—and see how it gets transformed, in R^2, by the mapping), and how these approaches all relate to R^1 to R^1 mappings and graphs in R^2.

How we represent these mappings, how we talk about the function that defines them, and especially, how we use computers to help us see and explore them, all have substantial effects on students' understanding of function and its usefulness.

New notational systems. Interactive electronic media may allow students not only to use new dynamic notational systems, but also to create their own notational systems and experience the real *doing* of mathematics. Traditional algebraic and arithmetical notations were created long before current information technologies existed, and they are therefore constrained by the technologies that existed at the time. They needed to be concise and abstract so that their use imposed few constraints in comparison to the constraints inherent in problematic situations and contexts. (For example, "50" is far more practical in most situations than a series of fifty unit markers.) However, as notations became more concise and more abstract, ideas represented in notational expressions became harder to construct for persons new to both the ideas and the systems for representing them.

With the advent of computers (and, in particular, graphical interfaces) there is potential for a true paradigm shift in how we think about algebraic notation. The ability to make multiple representations readily available and easily manipulable has highlighted the importance of representation to mathematical understanding. It is much clearer now that we cannot access mathematical ideas independent of some representation of them, and that different ways of representing something promote different aspects of the thing.

It is now possible, with the use of linked notational systems, to provide students with occasions to build an experiential basis for making these realizations in practically powerful settings. Graphic interfaces can amount to notational systems that incorporate the activities of students as part of the system. Graphs and tables can be notations; semantic networks can be notations; phase-state diagrams can be notations. Even graphic depictions of events, event structures, or situations can be rendered as forms of notations. There is reason to believe that students who work in the context of these sorts of notational systems engage in cognitive, strategic and affective acts that are more apparently mathematical than with traditional paper-and-pencil notations (Lesh, 1987; Thompson, 1992, in press; Thompson & Thompson, 1987).

Recursive definitions of functions. Closed form definitions of functions, for example $f(x) = 2^x$, are clearly useful in the study of algebra. However, computers facilitate other function definition approaches, such as recursion. The function, $f(x) = 2^x$, x an integer, can also be defined recursively as, $f(x) = 2*f(x - 1)$, $f(0) = 1$. The recursive definition makes some properties of this function more apparent, for example that the function is multiplicative, rather than additive, in nature (Confrey, 1990). Environments designed for algebraic exploration need to allow recursively defined functions to be created, modified, viewed, and explored in order to broaden students' understanding of the function concept. Computer spreadsheets offer a promising tool in this regard. (See, e.g., Bannard, 1991; Cornell & Siegfried, 1991; Dugdale, 1994; Fey, 1989; Levin & Abramovich, 1992; Maxim & Verhey, 1991; Pinter-Lucke, 1992.)

Research Questions

Algebra curricula, and curricula aimed at students' transitions from arithmetic to algebra, have been influenced greatly by Gagne-like analyses of skills. That is, scope-and-sequence has been determined largely by reductionist analyses of procedures. Thus, when skilled performance of specific answer-getting procedures is not an issue, when the issue is schematic knowledge that supports broader patterns of reasoning, we are largely without historical or psychological precedent upon which to base long-term curricular recommendations in regard to uses of computers and calculators.

There is not a body of systematic research that provides a firm foundation for the recommendations we give. The positions expressed herein are influenced by our experiences in development projects, isolated empirical studies, and personal use. Moreover, the pace of technological innovation has been such that existence proofs of feasibility have been the primary area of research. To keep pace with technological and curricular innovations, which will continue unabated and will refuse to "wait" to be researched, we suggest a mid-level course, between tightly controlled experimental studies and unstructured action research. Research in this vein can be carried out through case studies (Stake, 1988), ethnographies (Eisenhart, 1988), or teaching experiments (Steffe, 1991; Steffe & Richards, 1980; Thompson, 1979). The aim of such research is to enrich our understanding of students' and teachers' subjective experiences while they use powerful technological aids in the context of conceptually-oriented instruction.

Before discussing potential areas of research it is appropriate that we raise a matter of epistemology. Many changes in mathematics instruction and content have been proposed at all levels of schooling, many of which have arisen because of implications drawn from novel uses of computers and calculators. However, while considering major reforms in the content and orientation of algebra instruction, we should keep in mind these questions:

- To what extent do our positive feelings toward "intuition-building" software arise because this genre of software expresses or enriches an algebra *that we already know*, but which students (and perhaps teachers) do not know? Can we turn the edifice of our understanding on its head, expecting students to build rich understandings of something that is understood by us via a set of cognitions that our students will never have?

- If technological advances are woven into curricular reforms, students will know a mathematics that is much different from today's, because the very same software we design will engender dramatically different instructional experiences and cognitive objectives than those we experienced ourselves. What space of possibilities is opened by the introduction of powerful technological aids?

These are unsettling questions. They make it clear to us that we cannot claim objective insight into the consequences of major changes in the content and structure of mathematics curriculum and instruction. We consider that mathematics education research will be an essential component of reform efforts; it is essential that we evolve deep understanding of the potential and actual consequences of changes we propose or implement. We also wish to point out that the areas which we feel require systematic research and theory building focus on students' conceptualizations and not on technological tools. This reflects our position that technological tools are subservient to issues of mathematics learning and teaching, and it reflects our position that, as tools, technological aids must serve purposes larger than the use of technology.

Transitions from arithmetic to algebra. In 1989 NCTM published its Research Agenda monograph on algebra (Wagner & Kieran, 1989). A common theme among chapters in this monograph is that students' abilities to reason algebraically are highly influenced by the kind of arithmetical reasoning they develop in elementary and middle school. This raises many questions that need clarification and investigation: What conceptual foundations and arithmetic skills are propitious for students to begin to reason algebraically? How might students be exposed to algebra in the elementary grades before they have become fully competent with arithmetic? What do other countries, such as Japan and Russia, do in the elementary and middle grades to engender algebraic reasoning in their early study of mathematics?

When arithmetic instruction in the elementary curriculum emphasizes numerical and quantitative relationships, students have occasions to develop "algebra-like" mental operations of representing and denoting (Lesh, 1987; Noss, 1988; Sutherland, 1989; Thompson, 1992; Thompson & Thompson, 1987). We strongly concur with the position that arithmetic instruction needs to de-emphasize the memorization of answer-getting rituals, and instead needs to give more emphasis to reasoning about quantities, quantitative operations, and quantitative and numerical relationships.

Many technological tools now exist to support the latter emphases. Their implications for curriculum reform at elementary and secondary levels are unclear, however, because the cognitions they aim to promote are not developed within traditional K-9 curricula. It is not at all clear what advantage might be taken of the knowledge students construct in these new contexts. Longitudinal studies of students' learning are essential for informed technology-enriched curriculum reform in middle school and secondary school mathematics.

Connections between formal and applied algebra. We find it useful to draw a simple-minded distinction between applied algebra and formal algebra. By "applied algebra" we mean the use of standard algebraic notation to represent quantitative or numerical relationships. Expressions comprised of letters and operations signs have reference in a particular situation, and it is the situation that is prominent in one's reasoning. By "formal algebra" we mean that it is the expressions themselves that are the objects of reasoning, and matters of equivalence are predominant in one's awareness.[2] Writing an expression to represent how fast one is traveling after accelerating for some amount of time is applied algebra. Applying an equivalence-preserving transformation to an expression or equation, or replacing a letter in one expression by an equivalent expression, is formal algebra. Competence appears to entail moving fluidly from doing applied algebra to doing formal algebra, and vice versa, at all phases of solving a problem.

We find ourselves unable to articulate a conceptual relationship between these two kinds of algebra. Significant theoretical analyses are needed so that we may make informed curricular recommendations for promoting students' articulation of formal and applied algebra. One approach that seems worth pursuing is to think of all algebraic reasoning as being applied, but that the objects represented become more and more conceptual while retaining genetic links to the situations from which they were abstracted (Harel & Dubinsky, 1991; Harel & Kaput, 1991; Kaput, in press).[3] We urge researchers to investigate students' establishment of formal and applied algebra through the use of software that supports both.

Concepts of variable and function. It has become widely acknowledged that the ideas of variable and function are foundational in mathematical understanding in middle school and secondary school mathematics, and that they are poorly understood by students in the United States (Leinhardt, Zaslavsky, & Stein, 1990). In the recent past it has been customary to equate the idea of variable with the use of letters in expressions (Usiskin, 1988). This practice is giving way to a "new" view of variable, originating with Oresme and continued by Newton and Galileo. To them "variable" was not a noun. It was an adjective. A variable quantity is a quantity whose value varies; we can use letters to stand in place of a quantity's value, but we do so with the image that we may "run through" the possible values of this quantity by imagining the quantity itself changing (Kaput, in press). The notion of letter as a placeholder is much more abstract than the idea of variable quantity, and, genetically, comes much later than the idea of variable quantity (Piaget & Garcia, 1989).

When thinking of "variable" as variable quantity, the conceptual connection between variable quantities and functions becomes much clearer. Values of quantities can always be thought to vary in relation to variations in some other quantity, and the relation, being unaffected by changes in values, is the functional connection between the quantities. When viewed from this perspective, the notions of functional relationship and variable quantity can be seen to be inextricably intertwined. It has also been well established that students now do not learn these foundational ways of reasoning about functional relationships (Bell & Janvier, 1981; Clement, 1989; Harel & Dubinsky, 1991; Leinhardt, et al., 1990; Piaget et al., 1977; Vinner & Dreyfus, 1989).

Students' development of the ability to imagine situations comprised of variable quantities and functional relationships among quantities needs careful and systematic investigation. With tools to create situations that have invariant relationships among varying quantities, such as *Geometer's Sketchpad* (Jackiw, 1991), students can be given tasks that explicitly require them to create or represent quantities, and functional relationships among them, so that the value of one quantity varies according to that particular relationship.[4] The effect of such experiences may be propitious for students' understanding of mathematical models as capturing dynamic aspects of functioning systems. Nevertheless, it remains to be investigated how an instructional orientation to envisioning dynamic systems that change according to invariant relationships may be incorporated into a curriculum, and what effect such an orientation may have on students' understanding of variable quantities and functional relationships remains to be established. One thing seems clear: Without powerful technological aids the question would be moot, for it is with technological aids that we are able to make dynamic images and structured imagination something to discuss publicly.

Mathematical modeling. One of the more powerful uses of mathematics is to create complex systems of functional relationships to model real-world phenomena. Equation-solving programs and symbolic algebra systems provide natural tools for building such models. However, if equation-solving programs (that do not place an emphasis on intermediate, symbol-manipulating states) are made an integral part of students' experiences in school algebra, then algebra instruction will focus on uses of notation to model quantitative and mathematical situations, an activity that is typically de-emphasized in current school mathematics. But, is there a danger that we might shoot ourselves in the foot by going too far in de-emphasizing formal skills? Could it be the case that in order for symbolic algebra programs to be used productively as conceptual tools, we must give prominence to formal reasoning that might otherwise be de-emphasized because symbolic algebra programs do formal manipulations for the student? These questions need empirical investigation and theoretical clarification.

Graphs and functions. Graphing software that presents multiple windows on a single graph can show a graph in different scales simultaneously. Functions can be shown in various representations (equations, tables, graphs), and a modification in one representation can be reflected instantly in the other representations. The variety of perspectives subsumed by graphic approaches to functions makes it essential that we understand the nature of experts' and students' understandings of functions, graphs, tables, expressions, and their relationships. (See, e.g., Goldenberg, 1988.)

It is also imperative that we have deep understandings of how one develops the ability to read situational information from graphs. Tools that present graphs of functional relationships among quantities simultaneously with the actual situation being modeled may give us insight into such insights (Kaput, 1993, in press; Nemirovsky & Rubin, 1991; Rubin & Nemirovsky, 1991).

One of the more radical avenues now afforded by graphing software is the portrayal of graphs and tables as notations, to be manipulated within syntactical constraints inherited from operations on functions (including composition). Curricular implications of this view of graphs and tables need to be explored thoroughly and systematically. For example, what kinds of understandings might be developed through graphical approaches without first developing a significant understanding of more traditional algebraic notations? What role does a concept of formal variable play in understandings of graphs as graphs? In what ways can graphing software inform the development of concepts of variable? For example, consider a program that has two settings: The first graphs $f(x)$ as x varies from left-to-right along the x-axis. A second setting has the computer plot graphs from "random" selections of x. The former supports the notion of "variable as varying." The latter supports the notion of formal variable—a placeholder for a domain of possibilities—and it supports the notion of a graph as an image of an "instantaneous" map from domain to range.

New notational systems. We caution against hasty introduction of new notational systems. Notations are meant to be used as matters of convenience, and hence they are typically meant to be used outside of the settings from which they draw meaning. Current notations are products of long evolutions, and they are conventional because they have proved useful. Transitional notations (or "ramping" notations, as Kaput, 1986, calls them) can be beneficial, but we must be judicious about the number of such systems introduced, the intellectual overhead of coming "up to speed" with them, and the cost-benefit ratio of using them. Rapid proliferation of unconventional notations can easily produce confusion among both students and teachers. This caveat is not meant to be read as an argument against new notations, but is intended to emphasize the need for serious research as to pedagogical and cognitive benefits of particular notations before promoting widespread adoption.

Teachers' beliefs about mathematics and the practice of teaching. The introduction of technological aids in algebra instruction allows for substantial changes to the content of instruction and motivates a change in pedagogy itself. It is evident that implementation of curriculum reform depends extensively on how successful teachers are in making the intended curriculum part of their own thinking. Successful implementations of reform will be influenced by how well the reform movement understands teachers' conceptions of mathematics (algebra in particular), their conceptions of what constitutes mathematical understanding, and their conceptions of learning mathematics. *The Role of the Computer in Education IX*

Assessment and Evaluation Needs

We need new ways of evaluating student learning which has been accomplished with calculators and computers. For example, it makes little sense to use calculators and computers to express algebraic concepts graphically, then examine the students with tests that do not include graphical representations.

National curricular groups have advocated the use of calculators and computers in instruction. Although those recommendations have become part of the goals and objectives of schooling for many states in the U.S., few states permit the use of technology on required state tests. A common argument against the use of calculators on national and state tests is that some students cannot afford calculators. However, the cost of a scientific calculator is now only about one-fourth of the cost of an algebra textbook, and there is at least one type of *graphing* calculator currently available for approximately the cost of a textbook. If schools can purchase texts for their students, why not calculators? Advocates of technology in schools must become adept at convincing legislators and school boards that mathematics is a dynamic subject which has been transformed by technology. Using calculators and computers in testing is not making tests easier for students. Rather, these devices permit the assessment of higher level learning not tapped in traditional pencil and paper tests.

Admissions and placement tests for higher education carry great weight in what teachers perceive as essential elements in instruction. Many mathematics tests related to college explicitly forbid use of calculators and computers. Schools and teachers will be able to embrace existing technologies more readily as colleges, universities, and testing firms begin to permit students to use technology in solving problems on important tests.

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Notes

1. A quirk of *Eureka* is that you cannot maximize functions; you must instead declare that you want to maximize a variable.
2. When someone is reasoning about marks on paper, trying to remember "what to do" with those marks, we do not consider them to be engaged in formal algebra. Rather, we consider them to be thinking about marks on paper, trying to remember what rules might be used with them.
3. A concrete example of this would be to represent, say, the total torque of a system as a sum of constituent torques, and to ask oneself, "Can I say this more simply while still saying the same thing?" The focus moves from representing "total torque" to writing a simpler, more economical expression, but writing it within the constraint that the new expression still be a valid representation of the original situation.
4. For example, to create the locus of a point that moves in such a way that it always is the vertex of a 30° angle with a given line segment, one must come to envision the point as being the value of a function, where the "function" is a construction that yields a point having the desired relationship with the given line.

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