Constructivism, Cybernetics, and Information Processing: Implications for Technologies of Research on Learning†

Patrick W. Thompson

Center for Research in Mathematics and Science Education

and

Department of Mathematical Sciences

San Diego State University

San Diego, CA 92182

Running Head: Cybernetics and Information Processing

Thompson, P. W. (1995). Constructivism, cybernetics, and information processing: Implications for research on mathematical learning. In L. P. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 123–134). Hillsdale, NJ: Erlbaum.

† Preparation of this paper was supported by National Science Foundation Grants No. MDR 89-50311 and 90-96275, and by a grant of equipment from Apple Computer, Inc., Office of External Research. Any conclusions or recommendations stated here are those of the author and do not necessarily reflect official positions of NSF or Apple Computer.

Constructivism as a philosophical orientation has been widely accepted in mathematics and science education only since the early 1980s. As it became more broadly accepted, it also became clear that there were incongruous images of it. In 1984, Ernst von Glasersfeld (von Glasersfeld, 1984) introduced a distinction, echoed in Steier's paper at this conference, between what he called "naive" constructivism and "radical" constructivism. At the risk of oversimplification, suffice it to say that naive constructivism is the acceptance that learners construct their own knowledge, while radical constructivism is the acceptance that naive constructivism applies to everyone--researchers and philosophers included. Von Glasersfeld's distinction had a pejorative ring to it, and rightly so. Unreflective acceptance of naive constructivism easily became dogmatic ideology, which had and continues to have many unwanted consequences.[1] On the other hand, I will attempt to make a case that to do research we must spend a good part of our time *acting* as naive constructivists, even when operating within a radical constructivist or ecological constructionist framework. To make clear that the orientation I have in mind is not unreflexive, I will call it "utilitarian" constructivism, and will use Steier's and Spiro's papers as a starting point in its explication.

---

[1] One such consequence is the widespread conclusion that exposition is an unacceptable teaching method. I am continually amazed by the admonition, seen frequently in mathematics education trade journals, that teachers must not give knowledge to students ready-made since students should construct their own knowledge. This says to me that many people do not understand that there is no such thing as "ready-made" knowledge, and that students construct their knowledge *regardless* of what a teacher does--but what they construct can be influenced by the nature of the social and intellectual occasions in which the constructions take place.

Cybernetics and Reflexivity

Steier makes a strong argument that researchers of human systems need to keep in mind their contributions to the phenomena they analyze, which, in turn, contribute to their construction of the system being investigated. The system being investigated is as much an artifact of the theoretical conversations guiding the researcher's actions as it is of the participants' actions and orientations. Steier contrasts this position with early cybernetics. I have tried to capture his comparison in two diagrams (Figures 1 and 2).

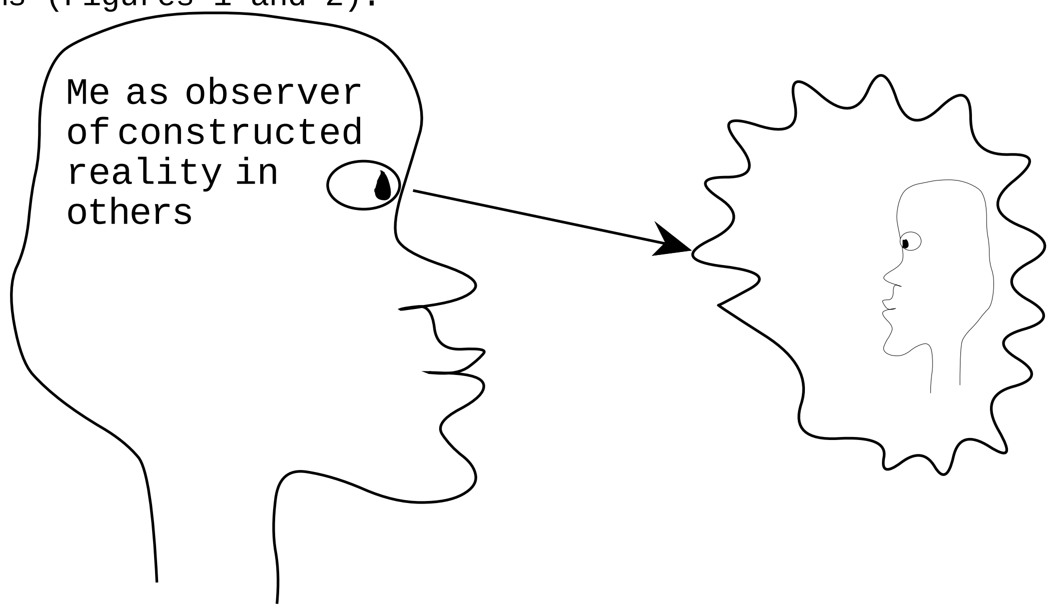

Figure 1. An early, cybernetic, image of theorizing about others' cognitions.

Figure 1 is of me, as researcher, doing research from an early cybernetic perspective on others' mathematical understanding. I study others in natural or provoked settings with the intent of constructing models of their realities. I approach the task with a sense of omnipotence — others construct their realities, and I make sense of what those realities are.

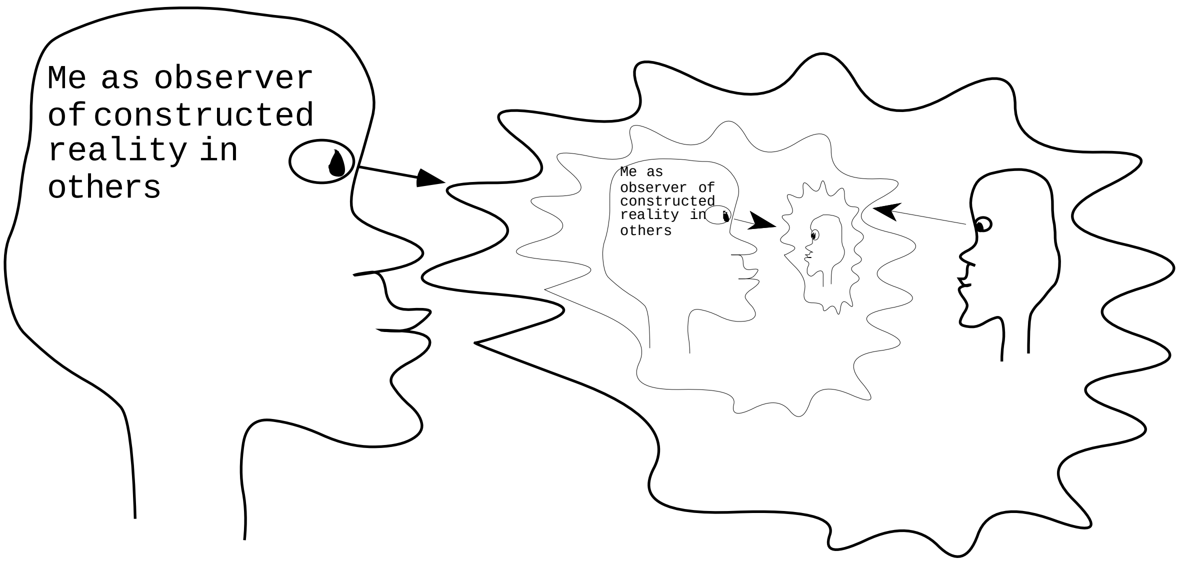

Figure 2. A contemporary, cybernetics of cybernetics, image of theorizing about others' cognitions.

Figure 2 is of me, as researcher, doing research from a later cybernetic perspective on others' mathematical understanding. From this perspective, I take into account that I, my artifacts, and the events occasioned by me are part of others' constructed realities. The images, goals, and intentions guiding my actions appear explicitly in my image of another's understanding, and I attempt to take those aspects of others' experiences into account as I try to understand their realities.

There is one thing that Figure 2 does not reveal explicitly. Reflexivity in one's research activities, explicated so well by Steier, is more than being aware that your involvement helps to create the behavior you wish to study. Reflexivity in research also entails reflecting on:

- Your sense of your research domain,
- How that sense is expressed in your researching actions,
- The contributions your actions make to the behavior you wish to study,

• and how your observations of behavior influence your sense of the research domain.

The circularity in the “cybernetics of cybernetics” perspective should be evident, but it is not the closed circularity that renders circular definitions meaningless. It resembles the circularity in “a barber is ordered to shave everyone in town who does not shave himself.”[2] The circularity is temporal, as is the circularity of dynamic, nonlinear systems. At the moment we reflect on any of the items listed above we do so at a particular moment. Those reflections express themselves in our actions, but only subsequent to that moment of reflection, and we will find occasions to reflect again on “us in the data.” The flip side of reflexive research is that we can never capture where we are (i.e., our current understandings); at best we can capture where we have been, and we *use* our current understandings in our attempts to capture where we were (MacKay, 1955; MacKay, 1965). When we reflect on “us in the data” on occasion *x*, we will get a sense of the influence of those reflections when we reflect on “us in the data” on occasion *x*+1.

The Practice of Reflexive Research

Why do I dwell on this aspect of reflexive research, the idea that we always reflect retrospectively on our contributions to the phenomena of interest? In part it is because of the experience of

[2] This example is often given to illustrate the need for logical hierarchies to exclude self-contradictory, self-referential propositional systems (Kneebone, 1963). Percy Bridgeman (1934) noted that, were the barber to make a list of the people who shaved themselves at the moment he is given this order, he would either be on the list or not. Either way, there is no contradiction when he subsequently shaves himself or not.

teaching recursive functions and recursive programming to prospective teachers. Those who try to restructure their thinking so that recursive functions and recursive procedures are an expression of their thinking invariably go through a phase where they feel trapped by recursive thinking. There are two sides to this sense of feeling trapped: When thinking of the process as a whole, they have the feeling that nothing ever gets done (closed circularity). When they are thinking of the process as a series of steps, they think through the recursion step-by-step, developing a sense of infinite regress, instead of objectifying the product of recursive thinking (Thompson, 1985). This evidently is a necessary phase; it is through reflecting on the product of thinking recursively in relation to their process of thinking recursively that they finally objectify recursive processes. I suspect that initiates to reflexive research will experience the same sense of being trapped if they are not aware that this is an ongoing process — that on every occasion in which they practice reflexive research they will be using radical constructivism instrumentally, they will be acting as instrumental constructivists. I say that they will be acting as instrumental constructivists because during those moments of retrospective reflection, they will be attempting to understand people's realities "as they are." The difference between practicing naive constructivism and using radical constructivism instrumentally is that while using constructivism instrumentally, we must do so with the awareness that our deliberations are tentative — that what we discern now will affect our future actions. Our future actions, in turn, may provide occasions for us to rethink our current deliberations and will affect the realities of the people participating in our investigations.[3]

An example

When reading Steier's discussions of reflexive research, one passage struck me as being especially relevant to my own work. In relating his research on organizational identity, Steier remarked "... my relationship to the group can be seen to co-create the very organizational identity I am trying to understand." This passage caused me to reflect on a recent series of teaching experiments that have become influential in several researchers' projects on quantitative and algebraic reasoning. The behavior of children in these teaching experiments (one on complexity, the other on concepts of rate) revealed ways of reasoning that I did not fully expect, and most people reading of them did not expect, yet when other researchers have looked more closely within their own projects, they also have found these ways of reasoning. Why were these ways of reasoning not seen before? According to Steier, it is because no one was looking for anything that would give children occasions to express themselves in ways that reflected such reasoning. Evidently, children were positioned to express this "new" type of reasoning, but researchers were not positioned to co-create it.

[3] The notion of future actions entails the possibility of creating artifacts such as problems, tests, software, and discussions surrounding them. It also entails developing sensitivities to occasions for deeper probing, and the initiation of discussions that otherwise might not occur without having so reflected.

In a marginal comment to Steier's paper, I paraphrased the quotation given on the previous page so that it pertained to my investigation of one student's concepts of rate: "... my relationship to [JJ] can be seen to co-create the very [concept of rate] I am trying to understand." My relationship to this girl, JJ, was that I brought her to the teaching experiment, designed special software that would support discussions about speed and rate, asked questions of her about situations surrounding the software, encouraged her to abandon her (unthinkingly) self-imposed constraint that she calculate all intermediate results, and prepared for the next lesson by thinking about her understandings as expressed in response to my questions. In retrospect, I can also see that I was attempting to separate my contributions to JJ's provoked responses from what she contributed to my questions.

Social Constructivism

I shall anticipate an objection. I imagine some might react that I have accommodated radical constructivism so that instead of being focused on individuals it now focuses on individuals in relation to me (or, more generally, a researcher), and that it still ignores the importance of "socially constructed" knowledge (Davydov, 1990; Goodwin & Heritage, 1990; Lave, 1988a; Lave, 1988b; Saxe, 1991; Solomon, 1989; Wertsch, 1985). First, radical constructivism has never ignored the importance of social relationships (Cobb, 1990; Cobb, Yackel, & Wood, 1992; Confrey, 1991; Thompson, 1979; von Glasersfeld, 1992). Second, this dispute amounts to one of figure versus ground. If we can agree that social relationships involve individuals and that individuals are continually involved in social relationships, then it is legitimate to take either individuals (as the things related to one another) or relationship (among individuals) as figure and the other as ground — as long as one keeps in mind which is being taken as figure and which is being taken as ground. I happen to prefer taking the individual as figure, framing discussions of social interactions within discussions of the mental operations by which individuals constitute situations, and framing social interactions within a general paradigm of mutually orienting accommodations among individuals (Bauersfeld, 1980; Bauersfeld, 1988; Bauersfeld, 1990; Cobb et al., 1992; Powers, 1978; Thompson, 1979).[4] This orientation to the individual is not opposed to social constructivism; it just puts the emphasis in a different place.[5]

Learning, Teaching, and Constructivism

Spiro et al.'s papers (hereafter, "Spiro's papers") have a very different orientation than Steier's. They orient us toward an application of constructivism in addressing the very serious problem of students' difficulties in learning advanced, complex subjects. My remarks about reflexive research of the previous

[4] One notion that I resist strongly is the notion of "social cognition," that somehow knowledge, as a socially-constructed object, is "out there," in-between the individuals interacting socially. It is only in the mind of an observer that socially-constructed knowledge is "out there" (Maturana, 1987), and it is "out there" only as a consensual domain (Maturana, 1978; Richards, 1991).

[5] This position *is* in opposition to simple-minded versions of social constructivism, wherein cognitive explanations of student's actions in interviews are dismissed with the counter-explanation that "the children had not learned how to behave in that situation" (Solomon, 1989).

section apply indirectly to Spiro's research program; I will draw connections later.

Problems of Learning Advanced Domains

Spiro's argument is that we must break away from current curricular and pedagogical practices because of their insidious effects on students' abilities to learn advanced, complex, ill-structured domains. I agree whole-heartedly. Spiro's list (Spiro, Coulson, Feltovich, & Anderson, 1988, pp. 376-377) is repeated below; I have annotated it with comments pertaining directly to mathematics education.

- Oversimplification of complex and irregular structure
- Overreliance on a single basis for mental representation

These are common characteristics of mathematics students at middle and secondary levels of public school. I suspect that they are direct reflections of the orientations common among teachers (Porter, 1989) and mathematics texts (Fuson, Stigler, & Bartsch, 1989; Stigler, Fuson, Ham, & Kim, 1986).

- Context-independent conceptual representation
- Overreliance on precompiled knowledge structures
- Rigid compartmentalization of knowledge components

The first item above is a hallmark of typical mathematics instruction. An over-emphasis on symbolic methods in elementary and secondary mathematics, sometimes with the misguided intention to first teach general forms so that they can later be widely instantiated, has little chance of producing anything other than the visually moderated sequences described by Bob Davis (Davis, Jockusch, & McKnight, 1978).[6] The latter two items remind me of research on schemas in mathematical understanding (Anderson, 1977; Mayer, 1981; Mayer, 1982; Wu & Yarbough, 1990). This research mistook market-enforced uniformity and stereotypicality in textbooks as somehow indicating the nature of competent mathematical understanding. What they failed to realize was that their research told us more about textbook authors' predispositions and mathematics teachers' overreliance on textbooks than about students' mathematical understandings.

- Passive transmission of knowledge
- Overreliance on “top down” processing

By “passive transmission of knowledge” I understood Spiro as referring to teachers' predilection to talk at instead of with their students, and students' learned disposition to expect such instruction. This type of instruction fits the common view that to teach is to tell (McDiarmid, Ball, & Anderson, 1989).

Spiro explained that students' overreliance on top down processing means that they approach problems thinking that the problem should be solved by applying a general rule. In mathematics and science, this shows up in students' beliefs that all problems are solved by a formula or their attempts to categorize problems by superficial characteristics (e.g., “river” problems in algebra (Mayer, 1982)).

I believe there is a common theme underneath all the problems of learning advanced ideas listed by Spiro: public school

[6] I should note that this characterization too often applies at college levels, too.

education. When we combine a mathematics curriculum that spirals around ever-increasing complexity of procedures instead of ever-increasing sophistication of ideas (McKnight, Crosswhite, Dossey, Kifer, Swafford, Travers, & Cooney, 1987), teachers whose image of the curriculum fits with what is contained in textbooks and whose image of understanding is to remember (Thompson, in press), and a tradition of instruction delivered in incoherent chunks (Porter, 1989; Stigler & Barnes, 1988), we get our present situation.

I will give an example from a current research project being conducted with sixth graders by Alba Thompson and me. It is a whole-class teaching experiment on middle-school students' development of quantitative and algebraic reasoning. The kind of instruction used in the teaching experiment relies heavily on students' participation in conversations about the ideas being taught. Early in the experiment we noticed that most students were largely disengaged from the discussions, especially if it was another student who was speaking. We began to realize that their lack of engagement was more than lack of interest; even when they listened to a discussion they commonly did not hear what was said.

We found a plausible explanation for the resilience of their disengagement after inspecting the instruction they had received in previous years. Here is a typical pattern of engagement: The teacher explains a procedure and works several examples. The students need not pay attention to what the teacher says, they need only watch what he or she does with the examples. Then a worksheet appears in front of them. If they recognize the items on it as being like what they just observed, then they proceed to mimic what they recall the teacher doing. If they do not recognize the items, or if they cannot recall what the teacher did, they raise their hands. When the teacher approaches their desk, they say “I don’t understand.” The teacher understands that they really mean “I don’t know what to do,” and proceeds by saying “Here is how to do it” as he or she works another example.[7] At no time in this interchange do students need to listen and reflect on what they hear, or express a difficulty they are experiencing in terms of underlying ideas. They need only pay passing attention to the procedure they are supposed to mimic. Many students carry this image of classroom engagement from elementary and middle school into secondary school, and find little in their experiences in secondary school to keep them from carrying this image into college.

Advanced Learning of Introductory Ideas

Spiro says problems of conceptual complexity and flexible knowledge acquisition occur “only later, when students reach increasingly more advanced treatments of subject matter.” My research suggests that, if this is true, its truth is an artifact of our present curriculum. In three teaching experiments, one on area and volume (in preparation), one on additive structures (Thompson, in pressb), and one on speed and rate (Thompson, in pressa), two things stands out: (1) Even at the introduction of an

[7] I suspect that one reason that teachers feel this approach is successful is that the mathematics curriculum is populated by essentially trivial problems. On trivial problems, it is possible to experience local success by demonstrating a procedure. Were the curriculum populated with more complex problems, teachers might feel lees satisfied with this approach and students might not experience a satisfactory level of success through mimicry.

idea, if the idea is trivialized (made “bite sized”) it is difficult for students to go beyond their initial images of it, and (2) it is far more productive to have students deal with an idea’s complexity as part of their introduction to it. The difficult problem for us is to provide support for students as they grapple with novelty and complexity simultaneously. Spiro’s notion of cases as the focus of study seems promising.

We need to remove ourselves form the shackles of the subjects *we* know (e.g., mathematics) and our idea of the way it fits together. A rigorous treatment of, say, geometry, to be rigorous, need not follow an axiomatic or even a *logical* development. The only criteria we need consider is that we follow a *conceptual* development, which may turn some things on their customary heads. For example, in our Saturday Math Club[8] we began geometry by focusing on the idea of invariance of relation and the dependence of relation on the construction leading to it. We used Geometer’s Sketchpad (Jackiw, 1991) to have students construct figures, and then focused discussions on why a diagram changed as it did when some part of it was transformed. Discussion frequently culminated with our making “dependency diagrams”--networks of relationships among parts of a diagram--and the use of dependency diagrams to explain why the “same” figure behaved differently when made by different constructions.

The aim of focusing on invariance was that Math Clubbers come to understand that relationships remain the same under

[8] The Saturday Math Club is a group of neighborhood children with whom I and Alba Thompson meet (yes, on Saturdays).

transformations of an initial (given) diagram, and that equivalence of diagrams is determined by correspondence of relationships. The aim of focusing on dependence was that Math Clubbers become skilled at distinguishing contingent relations from given ones, and that they become skilled at identifying relationships that are crucial to constructions based on them.

For example, after developing constructions for inscribed and circumscribed triangles, Math Clubbers identified four relationships they had used implicitly that were central to their constructions. These were: perpendicular bisectors of a triangle are concurrent, angle bisectors of a triangle are concurrent, every point on a perpendicular bisector to a segment is equidistant from the endpoints of the segment, and every point on an angle bisector is equidistant from the sides of the angle. Their realization that their constructions for inscribed and circumscribed triangles might not always work unless these relationships are true under any circumstance made them want to establish their truth. They had a stake in it. This gave us (as instructors) a natural occasion to raise the ideas of congruent triangles--ideas upon which each of these relationships rest. By focusing on a conceptual development of geometry, we ended up following neither a logical nor an axiomatic approach.

The idea of focusing on conceptual development of a subject is consistent with Spiro’s insistence that we “revisit the same material, at different times, in rearranged contexts, for different purposes, and from different conceptual perspectives” to help learners construct knowledge that is useful in complex, ill-structured situations. But it is not easy to take this approach, for we cannot begin with the intent to structure the subject according to how we know it. We must structure it with the goal that it be learnable, which may make its development appear different than the subject we or our colleagues know.[9]

Software Solutions and Constructivism

Spiro’s discussion of KANE, a hypertext environment for flexible access to varying perspectives of literary artifacts, reveals a thoughtfully and powerfully designed use of information technology to support students’ development of integrated, thematic perspectives on literary pieces. It is a major advance, both technically and instructionally, over the single-minded design of software that we typically see. I tried imagining a mathematics version of KANE; here is my stab at it.

The medium would have a videodisk of a “real” situation. The situation that came to mind was “Galloping Girdy,” a long, narrow suspension bridge that spanned the Tacoma Narrows at the south end of Puget Sound in Washington State. A prevailing wind through the Narrows caused sympathetic vibrations in the bridge, to the point that the bridge bucked, rolled, buckled, and finally collapsed. There is famous film footage of the final minutes before the bridge fell. Now, to understand what happened, we need to view the

[9] At the 1991 AERA Meeting, Andy diSessa presented some work he had done with 6th-graders on representations of speed (diSessa, in press). He had taken a conceptual approach to getting the students engaged with the ideas; after his presentation, an audience member remarked that he seriously doubted that what these children were doing was physics. The representations these children developed resembled nothing in the physics he knew, and the development of the ideas was “all out of order.”

bridge from multiple perspectives. From one perspective, the bridge was an air foil, like a wing. The prevailing wind lifted the bridge; turbulence made it lift more in some places than others. From another perspective, the bridge was like a vibrating string, where waves sometimes passed through it with sympathetic frequencies. The two perspectives are compounded by their interaction--the wave forms in the bridge changed its air foil characteristics, which then changed the forces acting to make it vibrate, which changed its wave forms. That is, from yet a third perspective the bridge constituted a nonlinear (chaotic) system. Now, suppose that a student could choose to have any of these perspectives dominate as an overlay on the video of the actual, galloping, bridge. Suppose also that the overlays were structured so that the more or less emphasis would be placed on visual models and correspondingly less or more emphasis would be placed on mathematical models in geometry, algebra, differential equations, and nonlinear systems. I don’t have a clear image of how this might be done, but it seems like an interesting project for someone to try.

There is one aspect of KANE, and of Spiro’s approach, that needs to be discussed. This is the question of whether it falls within the framework of constructivism. On one hand, it seems the themes embodied in KANE are as “ready-made” as any that Spiro criticizes. KANE provides a much richer environment of ready-made themes, and provides multiple overlays of perspectives, but “wealth corrupts” as a theme in KANE is still not a student’s construction. On the other hand, it *is* a marvelously rich environment for students to explore. If KANE is designed so that students can create their own editions--such as by building sequences of scenes, giving each scene a list of characteristics, and then summarizing these sequences according to some thematization--then it is clearly designed to support students' constructions.

I suspect that Spiro's work is in a phase where it would be productive to allow students using KANE to do some co-creating of their (Spiro et al's) theory, in the sense that Steier uses the phrase. Perhaps Spiro has already done this since writing the papers appearing before this conference. I would be delighted to hear about it.

References

Anderson, R. C. (1977). The notion of schemata and the educational enterprise. In R. Spiro & W. Montague (Eds.), Schooling and the acquisition of knowledge (pp. 415-431). Hillsdale, NJ: Erlbaum.

Bauersfeld, H. (1980). Hidden dimensions in the so-called reality of a mathematics classroom. Educational Studies in Mathematics, 11(1), 23-42.

Bauersfeld, H. (1988). Interaction, construction, and knowledge: Alternative perspectives for mathematics education. In T. J. Cooney & D. Grouws (Eds.), Effective mathematics teaching . Reston, VA: National Council of Teachers of Mathematics.

Bauersfeld, H. (1990). Activity theory and radical constructivism: What do they have in common, and how do they differ? In F. Steir (Ed.), Ecological understanding . Oslo, Norway: American Society of for Cybernetics.

Bridgman, P. W. (1934). A physicist's second reaction to Mengenlehre. Scripta Mathematica, 2, 101-117; 224-234.

Cobb, P. (1990). Multiple perspectives. In L. P. Steffe & T. Wood (Eds.), Transforming children’s mathematics education (pp. 200-215). Hillsdale, NJ: Erlbaum.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. Journal for Research in Mathematics Education, 23(1), 2-33.

Confrey, J. (1991, November). Steering a course between Vygotsky and Piaget. [Review of Soviet Studies in mathematics education: Volume 2. Types of Generalizaton in instruction.]. Educational Researcher, 20(8), 28-32.

Davis, R. B., Jockusch, E., & McKnight, C. (1978). Cognitive processes involved in learning algebra. Journal of Children’s Mathematical Behavior, 2(1), 10-320.

Davydov, V. V. (1990). Types of generalization in instruction (Joan Teller, Trans.). Reston, VA: National Council of Teachers of Mathematics.

diSessa, A. (in press). Knowledge in pieces. In G. Forman & P. Pufall (Eds.), Constructivism in the computer age . Hillsdale, NJ: Erlbaum.

Fuson, K., Stigler, J., & Bartsch, K. (1989). Grade placement of addition and subtraction topics in Japan, Mainland China, the Soviet Union, and the United States. Journal for Research in Mathematics, ??, ??-??

Goodwin, C., & Heritage, J. (1990). Conversation analysis. Annual Review of Anthropology, 19, 283-307.

Jackiw, N. (1991). Geometer's Sketchpad 1.0. Computer Program for Macintosh. San Francisco, Key Curriculum Press.

Kneebone, G. T. (1963). Mathematical logic and the foundations of mathematics. London: D. Van Nostrand.

Lave, J. (1988a). Cognition in practice. Cambridge, UK: Cambridge University Press.

Lave, J. (1988b, April). Word problems: A microcosm of theories of learning. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco.

MacKay, D. M. (1955). The place of 'meaning' in the theory of information. In E. C. Cherry (Ed.), Information theory (pp. 215-225). London: Buttersworth.

MacKay, D. M. (1965). Cerebral organization and the conscious control of action. In J. C. Eccles (Ed.), Brain and conscious experience (pp. 422-445). New York: Springer.

Maturana, H. (1978). Biology of language: The epistemology of reality. In G. A. Miller & E. Lenneberg (Eds.), Psychology and Biology of Language and Thought (pp. 27-63). New York: Academic Press.

Maturana, H. (1987). Everything is said by an observer. In W. I. Thompson (Ed.), Gaia: A way of knowing . Great Barrington, MA: Lindisfarne Press.

Mayer, R. (1981). Frequency norms and structural analysis of algebra story problems into families, categories, and templates. Instructional Science, 10, 135-175.

Mayer, R. (1982). Memory for algebra story problems. Journal of Educational Psychology, 74(2), 199-216.

McDiarmid, G. W., Ball, D. L., & Anderson, C. W. (1989). Why staying one chapter ahead doesn't really work: Subject-specific pedagogy. In M. C. Clinton (Ed.), Knowledge base for beginning teachers (pp. 193-206). New York: Pergamon Press.

McKnight, C., Crosswhite, J., Dossey, J., Kifer, L., Swafford, J., Travers, K., & Cooney, T. (1987). The underachieving curriculum: Assessing U. S. school mathematics from an international perspective. Urbana: Stipes.

Porter, A. (1989). A curriculum out of balance: The case of elementary mathematics. Educational Researcher, 18(5), 9-15.

Powers, W. (1978). Quantitative analysis of purposive systems: Some spadework at the foundations of scientific psychology. Psychological Review, 85(5), 417-435.

Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), Radical constructivism in mathematics education . The Netherlands: Kluwer.

Saxe, G. (1991). Culture and cognitive development: Studies in mathematical understanding. Hillsdale, NJ: Erlbaum.

Solomon, Y. (1989). The practice of mathematics. London: Routledge & Kegan Paul.

Spiro, R. J., Coulson, R. L., Feltovich, P. J., & Anderson, D. K. (1988). Cognitive flexibility theory: Advanced knowledge-acquisition in ill-structured domains. In Proceedings of the Tenth Annual Conference of the Cognitive Science Society (pp. 375-383). Hillsdale, NJ: Erlbaum.

Stigler, J., & Barnes, R. (1988). Culture and mathematics learning. Review of Research in Education, 15, 253-306.

Stigler, J., Fuson, K., Ham, M., & Kim, M. (1986). An analysis of addition and subtraction word problems in American and Soviet elementary mathematics textbooks. Cognition and Instruction, 3(3), 153-171.

Thompson, A. G. (in press). Teachers' beliefs and conceptions: A synthesis of research. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning . New York: Macmillan.

Thompson, P. W. (1979, April). The teaching experiment in mathematics education research. Paper presented at the NCTM Research Presession, Boston, MA.

Thompson, P. W. (1985). Understanding recursion: Process ≈ Object. In S. Damarin (Ed.), Proceedings of the 7th Annual Meeting of the North American Group for the Psychology of Mathematics Education (pp. 357-362). Columbus, OH: Ohio State University.

Thompson, P. W. (in pressa). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), The development of multiplicative reasoning in the learning of mathematics . Albany, NY: SUNY Press.

Thompson, P. W. (in pressb). Quantitative reasoning, complexity, and additive structures. Educational Studies in Mathematics.

von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), The invented reality (pp. 17-40). New York: W. W. Norton.

von Glasersfeld, E. (1992, February). A constructivist approach to teaching. Paper presented at the International Symposium on Alternative Epistemologies in Education, Athens, GA.

Wertsch, J. V. (1985). Vygotsky and the social formation of mind. Cambridge, MA: Harvard University Press.

Wu, Z., & Yarbough, D. (1990). Schema-based instruction and assessment of math word problem solving. Paper presented at the Annual Meeting of the American Educational Research Association Meeting, Boston, MA.