Notation, Convention, and Quantity in Elementary Mathematics†

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Running Head: Notation, Convention, and Quantity

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My aim for this brief chapter is to analyze separately and then bring together three aspects of mathematical understanding, teaching, and activity that often are unmentioned when we talk about "teaching for understanding." These are the matters of notation, convention, and quantity and how teachers can productively place them in opposition so that learners synthesize them productively in their mathematical activity.

NOTATION, CONVENTION, AND MATHEMATICAL ACTIVITY

I'll begin with an anecdote. I observed a second-grade teacher use base-ten blocks in teaching addition. Children happily combined blocks and counted to find sums. One child added 59 and 23 like this: Put out blocks to make 59 and 23 (Figure 1). She then put aside the 9 ones in 59 and combined the tens in the two numbers, saying "seventy" (Figure 2). She then moved the 9 ones over next to the tens, saying "seventy-nine" (Figure 3). Finally, she moved the 3 ones, originally part of 23, next to the rest, saying "eighty, eighty-one, eighty-two" (Figure 4). The teacher said "You need to start with the ones." The child was perplexed.

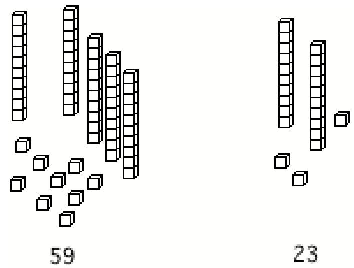

Figure 1. Make 59 and 23

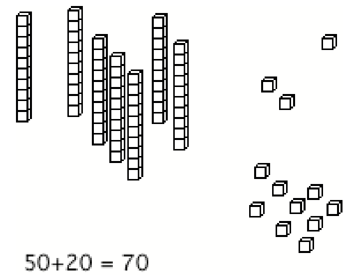

Figure 2. Add the tens.

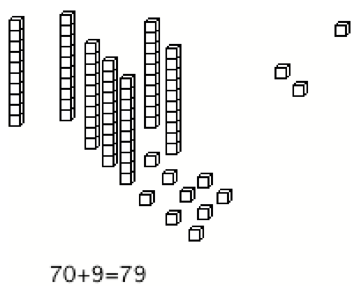

Figure 3. Add the ones of the first addend

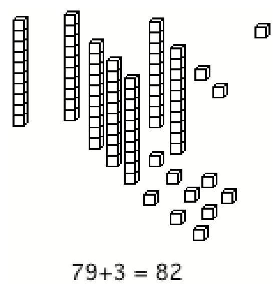

Figure 4. Add the ones of the second addend.

Soon afterward, the teacher presented a worksheet with two- and three-digit addition problems in vertical format. The child I spoke of above solved each by first adding the hundreds of both addends, then the tens of one addend followed by the tens of the other, then the ones of one addend followed by the ones of the other—all the time keeping a mental tally, writing the sum when she finished. The teacher's response was, "But you didn't show your work. You need to start with the ones and show all your trading."

What is at issue in this anecdote? Several things: (1) The ostensive purpose of using concrete materials, (2) the teacher's purpose in using concrete material (in this case, base-ten blocks), and (3) this episode's effect on the child.

Concrete materials are used productively for two purposes. First, they enable teachers and students to have grounded conversations. Their use provides something "concrete" about which they can talk. Second, concrete materials provide something on which students can act. Our pedagogical goal is that they reflect on their actions in relation to an idea the teacher has worked to establish and in relation to the constraints of the task as they have conceived it. This child evidently reflected her understandings of numeration and her actions on base-ten blocks into her way of thinking about symbolically-represented addition. That is, this child did precisely what we hoped she would do.

The teacher evidently had a different purpose for base-ten blocks—that students would act in a way that paralleled the standard paper-and-pencil addition algorithm. In other words, she had a definite activity-pattern in mind that she wanted students to perform. Perhaps she reasoned that if students could learn the activity pattern with base-ten blocks, they would find it easier to remember the steps in the paper-and-pencil algorithm.

We cannot know the long-term effect of this episode on this one child, but the child was clearly confused; it was clear that she wondered what she had done wrong. I suspect the long term effect of this episode (and continual encounters with similar ones) will be that she learns the school-math game: Mathematics is about guessing how your teacher wants you to do things. There may be conceptual effects as well. This child, in using her invented method, clearly had in mind that she was adding two numbers. If she adopted the teacher's prescription of "starting with the ones," and adapted to the constraint of performing according to a prescription that did not fit her reasoning and imagery, then she could easily come to think that one "adds" by adding a bunch of little numbers (Figure 5). That is, she could easily come to understand numerical algorithms in ways that have nothing to do with numeration and forget the conceptualizations which led to her invention.

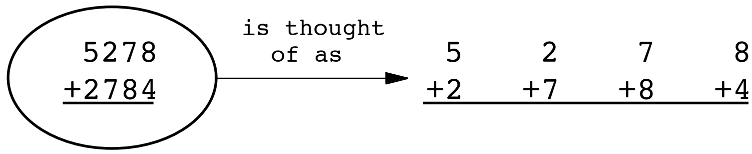

Figure 5. The child loses the sense that it is TWO numbers being added.

I offered this anecdote to draw attention to two larger issues. The first is that it is easy to confuse convention and principle unthinkingly. There is nothing principled about standard algorithms. They are standard only because they have become customary within a community. In the case of algorithms for addition, subtraction, multiplication or division of whole numbers and decimals, the only principles involved are that we adhere to the constraints that come with representing numerical value within a base-ten numeration system and that we adhere to the constraints of the task giving rise to the algorithm's use.[1] It is wrong to correct a child merely because of his or her not having performed in a prescribed manner. It is tragic when, moreover, the child's thinking is completely valid.

The second issue is practical, given that our school curriculum and the textbooks we use are aimed largely at instilling prescribed behavior. How may teachers respect students' thinking and still achieve the conceptual goals underlying the curriculum? In one sense the answer is easy: Separate what is new and conceptual from what is new and conventional, and teach for what is conceptual while postponing new conventions

[1] I should point out here that this principle is itself based on a convention—the convention of using a base-ten numeration system.

(especially those having to do with conventional notation). In another sense, this same answer is hard to put into practice. We are often unaware that what we want students to learn is no more than prescribed behavior, and we must delve into our own understanding of an idea—perhaps only to find out that we ourselves do not understand it conceptually.

Here is a practical bit of advice that almost always leads to productive engagement among teacher, student, and curriculum. Present problems to students which you think demand students to understand the idea you want them to understand, discuss the problems at length so they know what is being asked of them, and *allow students to approach these problems in any way they choose*. Allow them to construct personal methods for solving problems, and give lots of occasion for them to practice and solidify their methods. Ask students to explain their approaches, and stipulate that it is incumbent upon both speakers and listeners to work toward mutual understandings.

After students have developed stable methods for solving a class of problems, it makes sense to raise matters of efficiency and notation.[2] If students do not have stable methods, then there is nothing they can identify in their thinking which needs to be made efficient. If they do not have stable methods, then there is little for them to represent in notation. Matters of efficiency and notation will often arise naturally in their earlier work, but the teacher need not give it high priority until students have something to talk about. Efficiency and representation are important mathematical ideas, and they should be stressed often and continually, but they are the cart while conceptualization of method is the horse. I often ask teachers, "Do *you* want to be that teacher who a child understands as saying 'Don't think about it. Just do what I tell you'?" Our primary question should always be, “What do I want my students to understand?” It is too often, “What shall I have my students do.”

CONVENTION AND QUANTITY

I have often been asked, “How do you get all students to use the same method?” My response has been, “Why would you *want* them all to use the same method?” This non-answer typically is a surprise, and the ensuing conversation often uncovers an unexamined image of mathematics teaching and learning: Everything a teacher or student does, including motivational and developmental activities, is aimed at bringing students to a point where they can *do* some target, prescribed activity.

My position is that one of the most important aspects of students’ mathematics is that it should help them see the world quantitatively. Notice the words “see” and “quantitatively” in the previous sentence. Big ideas in mathematics are not to be found in algorithms; they are found in how we *see*. To *see* a mathematical idea entails having built conceptual operations which themselves incorporate the idea. To see the world *quantitatively* means to conceptualize aspects of objects as things that can be *measured*.

Here is an example that integrates these two points: *seeing* and *quantity*. The diagram in Figure 6 is often given in textbooks as an example of the fraction $3/5$. Now, if students see this diagram as “three shaded pieces out of five,” and generalize this way of seeing fractions to “a/b means *a* pieces out of *b* pieces,” then it is little wonder that they are confused by fractions like $12/5$. How do you take 12 pieces out of 5 pieces?

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[2] Here it is important to keep in mind that “class of problems” does not mean only one variation of a theme. It means *all* variations of a theme.

On the other hand, if they see Figure 6 as entailing the relationship that the whole piece is 5 times as large as each smaller piece, then the relationship between the whole and the parts is that each smaller piece is one-fifth as large as the whole, and the whole is five times as large as each piece. To understand the relationship of part to whole as a two-way multiplicative relationship is the heart of understanding fractions.

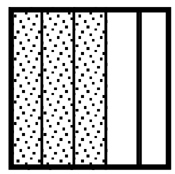

Figure 6. See in this diagram what you can.

When students can see parts and wholes in *multiplicative* relationship, then you, as teacher, can ask them to see many fractions and relationships in Figure 6. For example:

- Can you see 3/5 of something in Figure 6? *Think of the whole thing as one. Then think of the shaded part in relation to the whole.*
- Can you see 5/3 of something in Figure 6? *Think of the shaded part as one. Then think of the whole thing in relation to the shaded part.*
- Can you see 5/3 of 3/5? *Think of the whole thing as one. Then think of the shaded part in relation to the whole, which gives 3/5 of one. Then think of the shaded part as one three-fifths. Then think of the whole thing in relation to the shaded part. The whole thing (one) is 5/3 of the shaded part, which itself is 3/5 of one. The whole thing (one) can be seen to be 5/3 of (3/5 of one).*
- Can you see 2/3 of 3/5? *Think of the whole thing as one. Then think of the shaded part in relation to the whole, which gives 3/5 of one. Now think of 2/3 of the shaded*

part, which is 2/3 of (3/5 of one), which is also 2/5 of one. Two-fifths of one can be seen to make up 2/3 of (3/5 of one).

- Can you see 1 ÷ 3/5? *Think of the whole thing as one. Then ask, "How many units of 3/5 are in one?" There is one 3/5 and 2/3 of another 3/5. The one whole can be seen as containing 5/3 units of (3/5 of one).*
- Can you see 3/5 ÷ 2? *Think of the whole thing as one. Then think of two ones. Then ask, "How many twos are contained in 3/5 of one?" Three-tenths of a two is contained in 3/5 of one. Three-fifths of one can be thought of as 3/10 of* 2.
- Can you see 5/4 ÷ 3/4? *Think of the shaded part as 3/4 of one. Then one whole is four smaller pieces. Five-fourths of one would be five smaller pieces. Then ask "How many 3/4 of one are contained in 5/4 of one?" There is one 3/4 and 2/3 of another 3/4. So within 5/4 of one we can see 5/3 units of (3/4 of one).*

The reasoning depicted in the discussion of Figure 6 is called *quantitative* reasoning. Quantitative reasoning is not reasoning about numbers, it is about reasoning about objects and their measurements (i.e., *quantities*) and relationships among quantities (Thompson, 1989; Thompson, 1993; Thompson, 1994a; Thompson, in press). Here is an illustration of how quantitative reasoning might be employed to solve a sophisticated problem:[3]

> *In a country called Booglevillia, two-thirds of the men are married to three-fourths of the women. What is the ratio of men to women?*

First we need to assume that, in Booglevillia, marriages are monogamous, for otherwise we cannot conclude anything about the relationship between the number of

[3] Bob Davis of Rutgers University.

men and the number of women. If all marriages are monogamous, the number of married men is the same as the number of married women.

Imagine some amount of men. Two-thirds of them are married. Figure 7 shows the relationship between the number of married men and the number of men.

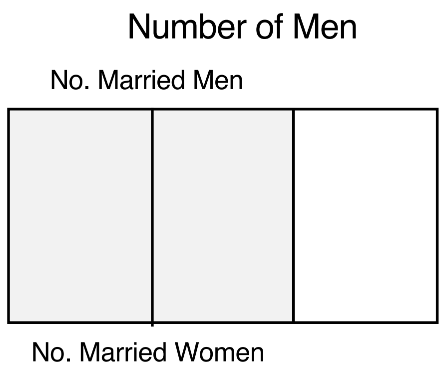

Figure 7. Two-thirds of the number of men is the same as 3/4 of the number of women.

The region which represents the number of married men also represents the number of married women, so it represents $\frac{3}{4}$ of the number of women. If we cut up this region into 3 parts (Figure 8), then each part represents $\frac{1}{3}$ of ($\frac{3}{4}$ of the number of women), or $\frac{1}{4}$ of the number of women. Each part also represents $\frac{1}{3}$ of ($\frac{2}{3}$ of the number of men), or $\frac{2}{9}$ of the number of men. So $\frac{1}{4}$ of the number of women is $\frac{2}{9}$ of the number of men. Therefore the number of women, which is four one-fourths of the number of women, is four (two-ninths of the number of men), or $\frac{8}{9}$ of the number of men. That is, the number of women make up $\frac{8}{9}$ of the number of men. The ratio of women to men is 8 to 9, so the ratio of men to women is 9 to 8.[4]

[4] You might try duplicating this chain of reasoning by first imagining some amount of women.

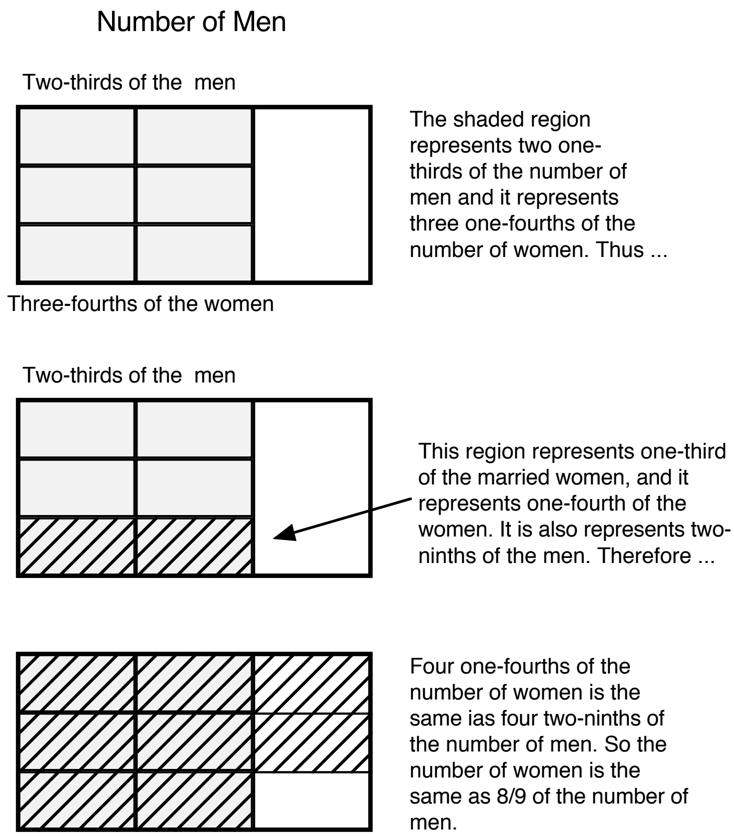

Figure 8. The shaded region represents 3/4 of the women, so cutting it into 3 parts makes each part represents 1/4 of the number of women. Each part also represents 2/9 of the men. So all the women (4 one-fourths of the women) make up 4 (2/9 of the men), or 8/9 of the men.

The discussion of Figure 6 illustrated the conceptual operations of multiple-identifications—considering some amount as constituting different fractional parts of different quantities. These same operations come into play in the quantitative reasoning portrayed for the Booglevillia problem. The identification of the same region as representing $\frac{2}{3}$ of the number of men and $\frac{3}{4}$ of the number of women was the same kind of reasoning that allowed us to see the entire region in Figure 6 as representing, simultaneously, 1 rectangle and $\frac{5}{3}$ of the shaded area. The importance of multiple-identifications of quantity is discussed by Lamon (this volume). Theoretical discussions of fractional reasoning and operations on fractions are given by Behr, Harel, Post, and

Lesh (Behr, Harel, Post, & Lesh, 1992) and by Kieren (Kieren, 1988; Kieren, 1992; Kieren, 1993). The foundation of fractional reasoning is in students' abilities to reason with complex units (Steffe, 1991a; Steffe, 1991b; Steffe, 1994).

We can capture symbolically aspects of the quantitative reasoning demonstrated in discussing the Booglevillia problem. In doing so, we are following the dictum given earlier of first developing in our thinking something to represent, then devising a way to express it in notation. We might write something like:

$$
\begin{aligned}
M &= \text{number of men} \\
W &= \text{number of women} \\
\tfrac{2}{3}M &= \text{number of married men} \\
\tfrac{3}{4}W &= \text{number of married women} \\
\tfrac{3}{4}W &= \tfrac{2}{3}M
\end{aligned}
$$

The last line, $\frac{2}{3}M = \frac{3}{4}W$, reflects the realization portrayed in Figure 7 that the shaded portion represents *both* the number of married men *and* the number of married women. There is one important difference between the informal reasoning given narratively and its symbolic expression. The first thing mentioned in the narrative, before making any consideration of fractional quantities, was that the number of married women and the number of married men are the same. We expressed this realization symbolically *after* giving fractional representations of each amount. Sometimes students will represent values of a quantity by an expression that evaluates the quantity (e.g., "$\frac{3}{4}W$" to represent "the number of married women"). Teachers should point out that to represent quantities' values with expressions sometimes forces us to write relationships in an order that differs from the order in which we realized them.

The informal, quantitative reasoning depicted in Figure 8 could also be expressed symbolically. It might be written as given below. Refer to Figure 8 while you read.

$\frac{1}{3}\left(\frac{3}{4}W\right) = \frac{1}{3}\left(\frac{2}{3}M\right)$	*One-third of (three-fourths of the number of women) is the same as one-third of (two-thirds of the number of men).*
$\frac{1}{4}W = \frac{1}{3}\left(\frac{2}{3}M\right)$	*One-third of (three-fourths of the number of women) is one-fourth of the number of women [note that only the left-hand side changed from the previous line].*
$\frac{1}{4}W = \frac{2}{9}M$	*One-third of (two-thirds of the number of men) is two-ninths of the number of men [note that only the right-hand side changed from the previous line].*
$4\left(\frac{1}{4}W\right) = 4\left(\frac{2}{9}M\right)$	*Four (one-fourths of the number of women) is the same as four (two-ninths of the number of men).*
$W = \frac{8}{9}M$	*The number of women is eight-ninths the number of married men. So, for every 8 women there are 9 men.*

There is another important difference between the informal, quantitative reasoning portrayed in Figures 7 and 8 and expressing that reasoning symbolically. When reasoning quantitatively, the spatial makeup of the diagrams implicitly supported the idea that we were always speaking about numbers of people. When we reasoned symbolically, we needed to remind ourselves continually that *W* stood for the *number* of women and that *M* stood for the *number* of men. When students fail to keep in mind that letters represent numerical values, they will think of an expression like $W = \frac{8}{9}M$ as saying "one woman is eight-ninths of a man" instead of thinking "the number of women is eight-ninths the number of men." Also, students will often read the (equivalent) equation $9W = 8M$ as "There are 9 women for every 8 men" instead of as "9 times the number of women equals 8 times the number of men." Students' thinking of letters as standing for objects is well researched, and it has pernicious consequences for students' understanding of algebra (Clement, 1982; Clement, Lochhead, & Monk, 1981; Lochhead, 1980; Lochhead & Mestre, 1988; Rosnick & Clement, 1980; Wollman, 1983).

Quantitative Operations and Quantitative Complexity

Quantitative Operations

A quantitative operation is not the same as a numerical operation like addition, subtraction, multiplication, or division. You employ quantitative operations at the first moment of thinking of a situation quantitatively. Quantitative operations are the conceptual operations one uses to *imagine* a situation and to *reason* about a situation—often independently of any numerical calculations.

The quantitative operation "form a difference" comes by imagining two quantities being compared additively—asking the question, "By how much does this quantity exceed (or fall short of) that quantity?" We customarily call the amount by which one quantity exceeds another the *difference* between the two quantities. We also, often, call the result of subtracting one number from another the "difference" between the two numbers. But we do not always subtract to find the difference between two quantities. Here is an example.

> Team 1 played a basketball game against Opponent 1.
> Team 2 played a basketball game against Opponent 2.
>
> The captains of Team 1 and Team 2 argued about which team won by more.
>
> The captain of Team 2 won the argument by 8 points.
>
> Team 1 scored 79 points.
>
> Opponent 1 scored 48 points.
>
> Team 2 scored 73 points.
>
> How many points did Opponent 2 score? (Thompson, 1993)

In calculating an answer to the question of "How many points did Opponent 2 score?" we first determine the difference between Team 1's and Opponent 1's scores

(79-48 points, or 31 points) and then calculate the difference between Team 2's and Opponent 2's scores (31+8, or 39 points). Notice that we *added* to calculate the *difference* between Team 2's and Opponent 2's scores. This shows that we do not necessarily subtract to evaluate a difference. In fact, depending on the situation and the information known about it, we might add, subtract, multiply, or divide to evaluate a difference. This demonstrates that quantitative operations and numerical operations should not be thought as being the same. In particular, teachers should take great care to help students understand that "difference" does not mean "subtract" and that "ratio" does not mean "divide."

Quantitative Complexity

It is incumbent upon us, as teachers, to help students become capable of dealing with complex ideas and complex situations. But complexity can have two sources. A situation may be complex because it involves one or more highly sophisticated ideas, and it may be complex because of a large number of relationships that must be kept in mind simultaneously. The present elementary and middle-school curriculum tends to minimize students' opportunities to deal with complex situations, yet I have always found students far more capable of working with complexity than the teachers and textbook writers appear to presume.

The problem set shown below comes from a teaching experiment on fifth-grade students' work with complex additively-structured situations (Thompson, 1993). The students were given all three problems as homework. The first situation and question were straightforward for all of them. The second situation and question were problematic for just a few. The last problem stymied all but one student—who solved it and the second with one conceptualization that entailed both. This student's reasoning illustrates how a focus on quantities and quantitative operations can orient students to reason about situations in ways that resemble algebraic reasoning.

1. Metcalf has two third grade rooms (C and D) and two fourth grade rooms (E and F). Together, rooms E and F have 46 children. Room C has 6 more children than room F. Room D has 2 fewer children than room E. Room F has 22 children.

   How many children are there altogether in rooms C and D?

2. Metcalf has two third grade rooms (C and D) and two fourth grade rooms (E and F). Together, rooms E and F have 46 children. Rooms C and D have 50 children together. Room C has 6 more children than room F. Room D has 2 fewer children than room E. There are __ children in Room E.

   What number or numbers can go in the blank so that everything works out?

3. Metcalf has two third grade rooms (C and D) and two fourth grade rooms (E and F). Together, rooms E and F have 46 children. Rooms C and D have 48 children together. Room C has 6 more children than room F. Room D has 2 fewer children than room E. There are __ children in Room E.

   What number or numbers can go in the blank so that everything works out?

The approach taken by this student was to reason about the pair-wise differences between rooms and keep track of the differences between combinations of rooms (Figure 9). His reasoning was that if there is a difference of six between Rooms C and F, and there is a difference of two between Rooms E and D, when you combine C and D you will get an amount that is four more than when you combine E and F (Figure 10). Thus, this student determined that no matter what number of students you say are in Room E, Rooms C and D together will have four more students than Rooms E and F together. You can put any number in the blank in Problem 2, and there is no number that you can put in the blank in Problem 3.

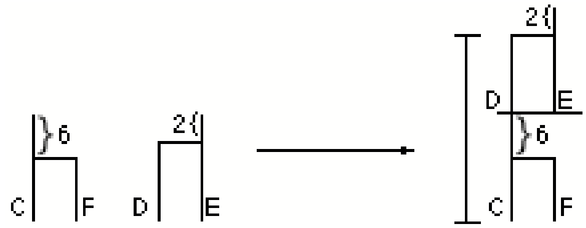

Figure 9. Combine the two comparisons.

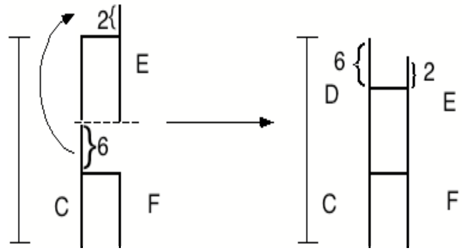

Figure 10. The difference between C and F is part of C, so it is part of the combination of C and D. Shuffle the parts of "C and D combined" and we see that "C and D combined" is four greater than "E and F combined."

The fifth-grade teaching experiment on complexity and additive structures also revealed why relationally complex situations are difficult for students. One class session included a discussion of the following situation.

> Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B.
> The two fellows argued about which one stood taller over his sister.
> It turned out that Brother A won by 17 centimeters.
> Brother A was 186 cm tall.
> Sister A was 87 cm tall.
> Brother B was ___ cm tall.
> Sister B was ___ cm tall.
> Put numbers in the blanks so that everything works out. (Thompson, 1993)

In their discussion of this setting students revealed several major difficulties. A principal one was that they would employ numerical operations without being able to identify what quantity they had evaluated.

In the following excerpt, students had been discussing the "Brothers" problem for ten minutes, but had met obstacles in a number of directions. The last of many options at one moment was that they had calculated 186 minus 87, and had just entertained calculating 99 minus 17 (getting 82). There was some confusion about what quantities they had evaluated.

Excerpt 1.

1.	PT:	Could I ask you to think about something … Try this. Think about what the 82 means.[5]
2.		*Long pause.*
3.	Jill:	[*inaudible*] … okay, so the difference between Brother A and Sister A and subtracted … uh … how much … he won … by [*to Molly*] …
4.		*Pause.*
5.	Molly:	Huh?
6.	Jill:	… We took the difference between Brother A and Sister A. Does everybody understand that?
7.	All:	Yeah.
8.	Jill:	And then we took away how much Brother A won by.
9.	All:	Yeah.
10.	Molly:	Yes … yes … yes!
11.	Peter:	[*Giggles at Molly.*] So 186 …
12.	PT:	[*To Jill*] And you got 82, right? Now, what does the 82 stand for?
13.		*Pause.*
14.	Jill:	Uhhhhh …
15.	All:	[*Giggles.*]
16.	PT:	Well, what does that 99 stand for?
17.	Molly:	How much the difference is … Oh, so the 82 could be the difference between Brother B and Sister B.
18.	PT:	Is that what it is?
19.	Don:	It could.
20.	PT:	Or is that how tall Sister B is?
21.	Molly:	It couldn't be how tall she is because then Brother B would win.
22.	Peter:	No it wouldn't. He could be 82 or more.
23.	Jill:	She has to be shorter than 90 cm … or taller …
24.	Don:	It doesn't matter.
25.	Peter:	This would work … 185 and 82.
26.	PT:	Would Brother A win by 17 cm, Peter?
27.	Peter:	… no.
28.	Molly:	He has to win by 17 cm, so you could have Sister B be 127 cm shorter than Sister A.
29.	Peter:	It could be 99. That would be 17 difference.
30.		*Pause.*
31.	PT:	Let's think again. The 99 [*186 minus 87*] stands for what?
32.	Jill: } Don: }	How much difference there is between Sister A and Brother A.

[5] I use an elispsis (…) to indicate a pause. It does not indicate omitted text.

33. Molly: I know. I know. You can subtract 82 minus 186 and then you'd get 104. And 104 is 17 more than 87.
34. *Pause*.
35. Don: Huh?
36. PT: Explain it again, Molly. But instead of talking about the arithmetic, talk about what you are thinking.
37. Molly: The first thing we did was the difference between Brother A and Sister A, and it was 99. And minus 17 is 82.
38. Jill: But what was the 82 for?
39. Molly: Huh?
40. Jill: What does the 82 stand for?
41. Molly: [*Pause*.] I don't know.

Two aspects of Excerpt 1—cognitive and pedagogical—require comment. The excerpt shows students' willingness to employ a calculation without knowing what quantity the calculation evaluated (¶s 1-12, 29-41), and it shows their willingness to use a number in other calculations even though they did not know for what that number stood (¶s 33-41).

In regard to pedagogy, I have been asked why I misled the students after one of them had offered a correct answer to my question of what 82 stood for (¶s 16-20). First, I do not agree that I misled them. Molly suggested that 82 "could" stand for the difference between Brother B's and Sister B's heights. Her suggestion sounded like an analogy; it did not sound like a deduction. My question gave Molly an opportunity to *explain* why 99 minus 17 evaluates the difference between Brother B's and Sister B's heights. Had I sanctioned Molly's answer, we would not have found out that, in fact, she was unsure of what 82 stood for.

A second pedagogical aspect of Excerpt 1, which was recurrent in the teaching experiment, is the crucial role of teacher questioning in promoting productive discussions. I continually asked the question "What does this number stand for?" or "What did you just find?" any time students calculated a number. These questions oriented students to reflect on their understanding of a situation so that their calculations remained grounded in it. Without my continual questioning, students would loose sight of the situation and speak only of numbers and operations. Their attempts to answer my questions often revealed, to me and to the students, when they had incomplete understandings—when they did not know whether a particular calculation had any relevance to the problem. I suspect that we too often let students use numbers and operations meaninglessly, to the point where meaningless use of numbers and operations becomes their habitual activity.

This teaching experiment has curricular implications, also. The discussion in Excerpt 1 could not have happened if we had been discussing a typical textbook problem. Textbook problems do not have sufficient complexity for issues of representation and deduction to arise naturally. Since textbook problems do not have sufficient complexity, teachers must generate problems of their own. Brown and Walter (Brown & Walter, 1983; Brown & Walter, 1993) provide excellent guidance for generating rich problems.

QUANTITATIVE REASONING AND CONCEPTUAL UNDERSTANDING

I mentioned that quantitative reasoning is not the same as numerical reasoning. This chapter's first section illustrated how a conceptual understanding of fractions can be a basis for solving what are often considered to be algebra problems. The second section illustrated the importance of students' learning to deal with relational complexity. In this last section, I will illustrate my claim that mathematical reasoning often is based in a conceptual understanding of quantities themselves, independently of any arithmetic that might be employed in situations regarding them.

The discussion here comes from a one-week teaching experiment on fifth-grade students' understanding of area and volume (Thompson, in preparation). In that teaching experiment, six fifth-grade students worked in various settings with the area and volume formulas that they already knew well. It became evident during the teaching experiment that their conceptions of area and of volume were what I call "one-dimensional." A conception of area as a one-dimensional object is that it is built out of items that you can rearrange in a linear order (Figure 11). Similarly, a conception of volume as a one-dimensional object is that it, too, is made by items that you can rearrange linearly (Figure 12). In both cases, the items can be arranged conveniently for rapid counting (e.g., counting by repeated addition or by skip-counting).

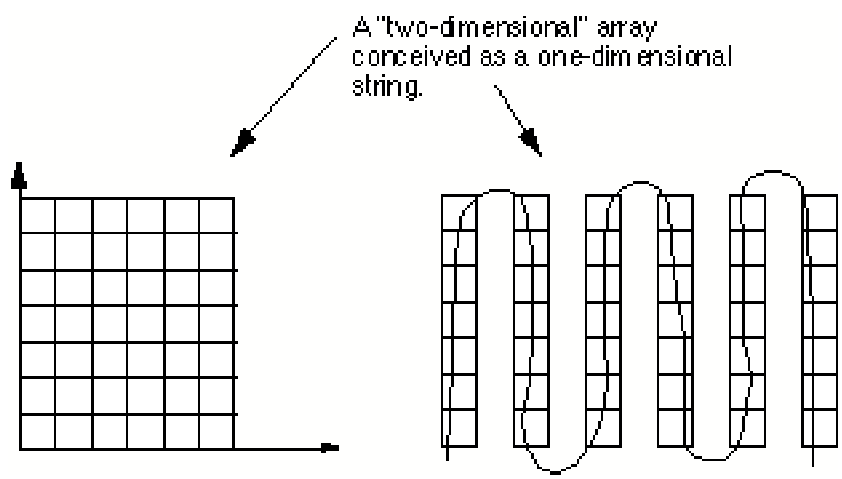

Figure 11. A one-dimensional conception of area. Area is measured by counting a string of squares, which sometimes are arranged conveniently for purposes of rapid counting (e.g., counting by sevens).

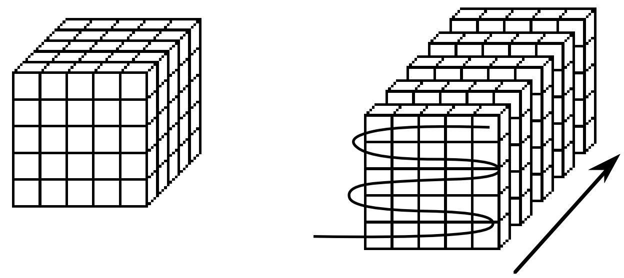

Figure 12. A one-dimensional conception of volume. Volume is measured by counting a string of cubes, which sometimes are arranged conveniently for purposes of rapid counting (e.g., counting by fives or by twenty-fives).

Students conceive of area and volume as one-dimensional objects by imagining that they measure something by repeatedly putting down a unit until you exhaust the object being measured. In the case of area, students imagine themselves counting “little squares,” which themselves have no conceptual significance. They are just objects to count and which have a nice shape for covering up rectangles. In the case of volume, students imagine themselves counting “little cubie things,” which themselves have no conceptual significance. They are just objects to count and which have a nice shape for filling empty parallelepipeds. The crucial issue is that students conceptualize *units* of area and volume as multiplicatively-structured objects.

During the teaching experiment I attempted to challenge students conceptions of area and volume. I did this initially by presenting Figure 13 to students, asking them “What is the area of this rectangle.”

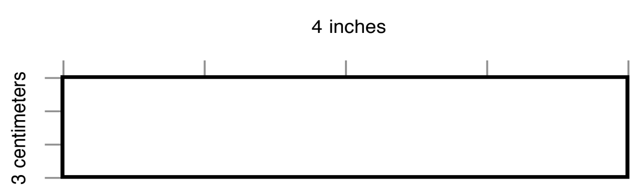

Figure 13. What is the area within this rectangle?

Excerpt 2.

1. PT: One more. What … what are the dimensions of this rectangle. Somebody...
2. Students: Three by four.
3. MH: Three by five. One , two , three , four , five...four by five.
4. JJ: Three centimeters by four inches.
5. PT: Okay, three centimeters by four inches. So, what's the area?
6. MH: It's twelve.
7. PT: BM?
8. BM: Twelve cubic inches.

MH's and BM's responses were typical of students orientations. MH said the area was 12, but did not say what things there were 12 of. BM's response was typical of students' attempts to specify derived units—they often were inappropriate for the object ostensibly being measured. It required over 10 minutes of discussion and several analogies with other multiplicative settings before JJ offered that it might be "little one-centimeter by one-inch" rectangles of which there were 12. Afterward, two days of instruction were devoted to getting students to conceptualize multiplicatively-derived units of measurement.

I interviewed each student after the teaching experiment. One interview problem is presented in Figure 14. My intention for offering it was to see if students had conceptualized the formula $L \times W \times H$ as $(L \times W)$ giving an area of a surface and $(L \times W) \times H$ giving the volume generated by sweeping a surface through space through a distance of measure H.

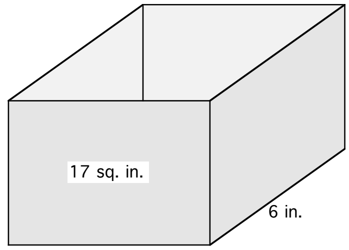

Figure 14. What is the volume of this box?

Five of the six students responded similarly to BJ, as related in Excerpt 3. BJ thought he needed numbers for the length and width of the box's front face so that he could then multiply them (¶s 1-5). Even then, he did not see that he should get 17 square inches were he to actually have them and do the multiplication (¶s 5-7). Moreover, it was clear to him that the diagram showed that the box's front face has an area of 17 square inches. It was also clear to him that multiplying length times width had nothing to do with the area of that face (¶s 8-11).

Excerpt 3.

[*PT puts problem sheet in front of BJ. BJ reads it out loud.*]
1. BJ: I can't tell. I don't have all the numbers.
2. PT: What numbers?
3. BJ: Along the other sides.
4. PT: What would you do with them if you had them?
5. BJ: Multiply.
6. PT: Any idea what you would get when you multiplied them?
7. BJ: No … I would need to know the numbers first.
[portion of text omitted]
8. PT: Do you know what "17 sq. in." stands for in this diagram?
9. BJ: Yeah. It's the area of that part of the box.
10. PT: Does 17 have anything to do with the numbers you say you need?
11. BJ: [*Pause*.] Not that I can see.

One student, JA, had conceptualized the formula $(L\times W)\times H$ quantitatively. His interview is presented in Excerpt 4. It was clear to him that whatever lengths the sides might be, when you multiply them you will get 17 square inches.

Excerpt 4.

[*PT puts problem sheet in front of JA. JA reads it to himself.*]

1. JA: Oh … somebody already did part of it.
2. PT: What do you mean?
3. JA: Somebody already multiplied for me … so it's just 17x6.
4. PT: Some kids think that they need the other numbers … the ones along the other sides. Why don't you need them?
5. JA: I'd just multiply them.
6. PT: Do you know what you would get when you multiplied them?
7. JA: Yeah … 17.

A pretest established that every student in this teaching experiment possessed the requisite calculational skills to solve the problem in Figure 14. Lack of calculational facility was not the source of the other children's difficulty. Rather, they had not conceptualized two- and three-dimensional units of measurement, and they had not conceptualized multiplication as an operation that *evaluates* a quantity.

One last issue deserves mentioning. The post-experiment interviews made it evident that two days of remedial instruction was insufficient to make up for students' having internalized area and volume formulas as calculational procedures instead of as quantity-evaluating operations. The lesson to be learned is that, as in public health, the best policy is to ensure that remediation is unneeded.

CONCLUSION

Quantitative reasoning is more than reasoning about numbers, and it is more than skilled calculating. It is about making sense of the situations to which we apply numbers and calculations. The research I reported here was intended to illustrate the central point that before students can make sense of numbers and operations, either in relationally complex or conceptually sophisticated settings, they must first make sense of the settings themselves. Students generally are not aware of this requirement. It is incumbent upon their teacher to make this a central aspect of his or her instruction.

It is not essential for teachers to perform conceptual analyses of every situation that children will deal with. Instead, it is essential that teachers adopt a *conceptual orientation* to their instruction (Thompson, Philipp, Thompson, & Boyd, 1994). This means to follow religiously a number of guidelines. These are:

- Maintain high expectations of students' reasoning and creativity. Otherwise, children will not have occasions to reveal their capabilities.
- Respect students' creativity. Do not emphasize correct performance of prescribed behavior.
- Insist that any time students speak of a number within some setting, they speak of the quantity for which the number is a value. Do not let students use numbers meaninglessly.
- Insist that any time students employ a calculation, they speak of the quantities they are operating on and the quantity their calculation evaluates. Do not allow students to employ operations casually.
- Do not use trivial settings. Students too often learn that the words in problems serve no useful purpose. They simply make it harder to find the numbers (Schoenfeld, 1992; Sowder, 1988).
- Use sophisticated and complex settings frequently. If students see such settings infrequently, they will learn that they are unimportant.
- Make performance on applications a significant part of your evaluation. Students too often learn that they can "blow off" word problems. If 10% of their assignment is word problems, they can maintain high averages and yet have no ability to *apply* their calculations.

Finally, even though my examples have been drawn from elementary mathematics, other research suggests that quantitative reasoning forms the conceptual foundation for a significant amount of secondary and college mathematics (Thompson, 1994a; Thompson, 1994b; Thompson, 1994c; Thompson, in press). In the long run, we must conceptualize our curriculum so that it reflects the long-term building of ideas and understandings rather than the accrual of ever more complex, meaningless procedures.

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