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IMAGERY AND THE DEVELOPMENT OF MATHEMATICAL REASONING

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I feel somewhat constrained in attempting to focus on the topic of theories of mathematical learning, for I am unable to separate matters of learning from matters of reasoning. I suppose this is for two reasons. First, in talking about learning, we immediately bump into the question "Learn what?", and that brings us into the arena of concepts, methods, schemes, and so forth that learners always express within occasions of reasoning. Second, any cognitive theory I know of postulates that learning happens by way of action—some theories focus on habituation of overt behavior, some focus on assimilation and accommodation, others focus on compilation of propositions. These various theories' constructs all call for explicit attention to events that occasion learning; I tend to think of such events as necessarily entailing students' reasoning in the context of problems. So, in addressing the issue of mathematical learning I find it rewarding to orient the discussion toward the development of mathematical reasoning—purposeful inference, deduction, induction, and association in the areas of quantity and structure.

The idea I put forward is this: Mathematical reasoning at all levels is firmly grounded in imagery. I ought to say something about what I mean by imagery and why I take interest in imagery. This will take a while.

IMAGERY

By "image" I mean much more than a mental picture. Rather, I have in mind an image as being constituted by experiential fragments from kinesthesis, proprioception, smell, touch, taste, vision, or hearing. It seems essential also to include the possibility that images can entail fragments of past affective experiences, such as fearing, enjoying, or puzzling, and fragments of past cognitive experiences, such as judging, deciding, inferring, or imagining. In regard to this last item, imagining, Tom Kieren and Susan Pirie (Kieren, 1988; Kieren & Pirie, 1990; Kieren & Pirie, 1991; Pirie & Kieren, 1991; Pirie & Kieren, 1994) make it evident that the act of imagining can itself inform our images.

I admit that this meaning for the word *image* is too broad, but that is where my thinking is now, and it has afforded me the ability to hear much more in students' expressions of their reasoning than I used to. Nevertheless, this formulation does suggest that a person's actual images can be drawn from many sources, and hence individual's actual images are going to be highly idiosyncratic.

The roots of this overly broad characterization of image go back to Piaget's ideas of praxis (goal-directed action), operation, and scheme. I discuss this connection more fully in other papers (Thompson, 1985a, 1991, 1994a). For this chapter I focus on Piaget's idea of an image and its relationship to mental operations.

Piaget distinguished among three general types of images. The distinctions he drew were based on how dependent on the image were the actions of reasoning associated with it. The earliest images formed by children are an "internalized act of imitation … the motor response required to bring action to bear on an object … a *schema* of action" (Piaget, 1967, p. 294, italics in original). By this I take Piaget to have meant images associated with the creation of objects, whereby we internalize objects by acting upon them—we internalize them by internalizing our actions. His characterization was originally formulated to account for object permanence, but it also seems especially pertinent to the creation of mathematical objects.

A later kind of image people come to create is one having to do with primitive forms of thought experiments:

> In place of merely representing the object itself, independently of its transformations, this image expresses a phase or an outcome of the action performed on the object . . . . [but] the image cannot keep pace with the actions because, unlike operations, such actions are not coordinated one with the other" (Piaget, 1967, pp. 295–296).

It is advantageous to interpret Piaget's description broadly. If by actions we include ascription of meaning or significance, then we can speak of images as contributing to the building of understanding and comprehension, and we can speak of understandings-in-the-making as contributing to ever more stable images.

A third kind of image people come to form is one that supports thought experiments, and supports reasoning by way of quantitative relationships. An image conjured up at a particular moment is shaped by the mental operations one performs, and operations applied within the image are tested for consistency with the scheme of which the operation is part. At the same time that the image is shaped by the operations, the operations are constrained by the image, for the image contains vestiges of having operated, and hence results of operating must be consistent with the transformations of the image if one is to avoid becoming confused.[1]

> [This is an image] that is dynamic and mobile in character … entirely concerned with the transformations of the object .… [The image] is no longer a necessary aid to thought, for the actions which it represents are henceforth independent of their physical realization and consist only of transformations grouped in free, transitive and reversible combination .... In short, the image is now no more than a symbol of an operation, an imitative symbol like its precursors, but one which is constantly outpaced by the dynamics of the transformations. Its sole function is now to express certain momentary states occurring in the course of such transformations by way of references or symbolic allusions." (Piaget, 1967, p. 296)

Kosslyn (1980) characterized images as data structures that result from the processes of perception. He, along with Piaget, dismissed the idea of images as mental pictures.

> These organizational processes result in our perceptions being structured into units corresponding to objects and properties of objects. It is these larger units that may be stored and later assembled into images that are experienced as quasi-pictorial, spatial entities resembling those evoked during perception itself.... It is erroneous to equate image representations with mental photographs, since this would overlook the fact that images are composed from highly processed perceptual encodings. (Kosslyn, 1980, p. 19)

On the other hand, he took issue with Piaget's notions of early images emerging by way of internalized imitation.

> Even if it were clear what was meant [by imitation], this sort of treatment would seem closer to describing what is taking place than to explaining it. I do not want to deny the value of describing a phenomenon; rich descriptions facilitate theorizing, and there is no more astute observer than Piaget. But in my view explanations of cognitive phenomena should specify the ways in which functional capacities operate. Piaget and Inhelder's account is more on the level of intentionality, and hence is open to multiple interpretations at the level of the function of the brain. The do not specify how interiorized imitation operates, nor have they specified the format or content of the image. This level of discourse will never produce process adequacy, and hence seems of limited value. (Kosslyn, 1980, p. 411)

Kosslyn's objection seems to have three sources. First, his is a correspondence theory whereby images represent features of an objective reality. Piaget's theory assumes no correspondence; it takes objects as things constructed, not as things to be represented (von Glasersfeld, 1978). Second, Kosslyn's notion of image seems to be much more oriented to visualization than is Piaget's. Piaget was much more concerned with ensembles of action by which people assimilate objects than with visualizing an object in its absence. Third, I believe Kosslyn misunderstood Piaget by separating Piaget's notion of image from its theoretical context, it being one piece of the puzzle in describing the emergence of mental operations. Kosslyn focused on images as the *products* of acting. Piaget focused on images as the products of *acting*. So, to Kosslyn, images are data produced by perceptual processing. To Piaget, images are residues of coordinated actions, performed within a context with an intention, and only early images are concerned with physical objects. In regard to Kosslyn's criticism that Piaget's theory does not specify how the brain manages to create images, I suspect it is less severe than he thought. The strong computational metaphor within which Kosslyn frames his theory may not be as lasting as he thinks. In fact, the notion of imagery may be its undoing (Cobb, 1987).

Piaget's idea of image is remarkably consistent with Johnson's (1987) detailed argument that rationality arises from and is conditioned by the patterns of our bodily experience. Johnson took to task realist philosophy and cognitive science (which he together called "Objectivism") in his criticism of their attempts to capture meaning and understanding within a referential framework.

> Meanings, conceptual connections, inference patterns, and all other aspects of rationality are distinguished, according to Objectivism, by their universality and independence from the particularities of human embodiment. They are supposed to be that which is shared by all of us, that which transcends our various embodiments and allows us to partake of a common objective realm of meaning. The chief difference, then, between the Objectivist view of meaning and non-Objectivist "semantics of understanding" being proposed here can be summed up as follows: For the non-Objectivist, *meaning is always a matter of human understanding, which constitutes our experience of a common world that we can make some sense of. A theory of meaning is a theory of understanding. And understanding involves image schemata and their metaphorical projections, as well as propositions.* .... Grasping a meaning is an *event* of understanding. Meaning is not merely a fixed relation between sentences and objective reality, as Objectivism would have it. What we typically regard as fixed meanings are merely sedimented or stabilized structures that emerge as recurring patterns in our understanding. The idea that understanding is an event in which one has a world, or, more properly, a series of ongoing related meaning events in which one's world stands forth, has long been recognized on the Continent, especially in the work of Heidegger and Gadamer. But Anglo-American analytic philosophy has steadfastly resisted this orientation in favor of meaning as a fixed relation between words and the world. It has been mistakenly assumed that only a viewpoint that transcends human embodiment, cultural embeddedness, imaginative understanding, and location within historically evolving traditions can guarantee the possibility of objectivity. (Johnson, 1987, pp. 174–175, italics in original)

Piaget maintained throughout his career that all knowledge originates in action, both bodily and imaginative (Piaget, 1950, 1968, 1971, 1976, 1980). Although Johnson's primary purpose was to give substance to this idea in the realms of everyday life, Piaget was primarily concerned with the origins of scientific and mathematical reasoning—reasoning that is oriented to our understandings of quantity and structure. For example, although Johnson focused on the idea of balance as an image schema emerging from senses of stability and their projection to images of symmetric forces (Johnson, 1987), it requires a non-trivial reconstruction to create an image of balance as involving countervailing twisting actions—where we imagine the twisting actions themselves in such a way that it occurs to us that we might somehow measure them. It seems to involve more than a metaphorical projection of balance as countervailing pushes to have an image of balance that entails the understanding that any of a class of weight–distance pairs on one side of a fulcrum can be balanced by any of a well-determined class of weight–distance pairs on the other side of a fulcrum.

The meaning of "image" developed here is only tangentially related to the idea of concept image as developed by Vinner (Tall & Vinner, 1981; Vinner, 1987, 1989, 1991, 1992; Vinner & Dreyfus, 1989). Vinner's idea of concept image focuses on the coalescence of mental pictures into categories corresponding to conventional mathematical vocabulary, whereas the notion of image I've attempted to develop focuses on the dynamics of mental operations. The two notions of image are not inconsistent; they merely have different foci.

Vinner's distinction between concept image and concept definition arose originally in the work of Vinner, Tall, and Dreyfus (Tall & Vinner, 1981; Vinner, 1991; Vinner & Dreyfus, 1989). In their usage, a concept definition is a customary or conventional linguistic formulation that demarcates the boundaries of a word's or phrase's application. On the other hand, a concept image comprises the visual representations, mental pictures, experiences and impressions evoked by the concept name.

In lay situations, people understand words through the imagery evoked when they hear them. They operate from the basis of imagery, not from the basis of conventional constraints adopted by a community. People understand a word technically through the logical relationships evoked by the word. They operate from the basis of conventional and formal constraints entailed within their understanding of the system within which the technical term occurs. Vinner, Tall, and Dreyfus arrived at the distinction between concept image and concept definition after puzzling over students' misuse and misapplication of mathematical terms like function, limit, tangent, and derivative. For example, if in a student's mathematical experience the word *tangent* has been used only to describe a tangent to a circle, then it is quite reasonable for him to incorporate into his image of tangents the characteristic that the entire line lies to one side or the other of the curve, and that it intersects the curve only once (Vinner, 1991). Notice that this image of tangent—uniquely touching at one point—has nothing to do with the notion of a limit of secants. It is natural that a student who maintains this image of tangent is perplexed when trying to imagine a tangent to the graph of $f(x)=x^3$ at (0,0), or a tangent to the graph of $g(x)=x$ at any point on its graph.

A predominant image evoked in students by the word *function* is of two written expressions separated by an equal sign (Fig. 15.1). We might think that only neophytes hold this image of function. I suspect it is far more prevalent than we acknowledge. An example illustrates my suspicion and at the same time illustrate how Tall, Vinner, and Dreyfus envision the influence of concept images over concept definitions.

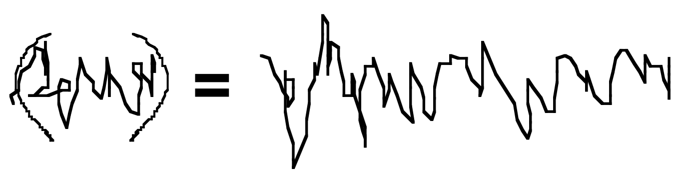


Fig. 15.1. A concept image of "function." Something written on the left is "equal to" something written on the right.

My wife, Alba Thompson, teaches a course designed to be a transition from lower to upper division undergraduate mathematics. It focuses on problem solving and proof. Students are supposed to take it after calculus and linear algebra, but a fair portion of the class typically have taken at least one term of advanced calculus or modern algebra. In the context of studying mathematical induction she asked one student to put his work on the board in regard to deriving and proving a formula for the sum $S_n = 1^2+2^2+\ldots+n^2$. The student wrote f(x) = $n(n+1)(2n+1)/6$. Not a single student thought there was anything wrong with this formulation. It turned out, after prolonged probing by Alba, that students accepted it because it fit their concept image of function, which I've presented in Fig. 15.2.

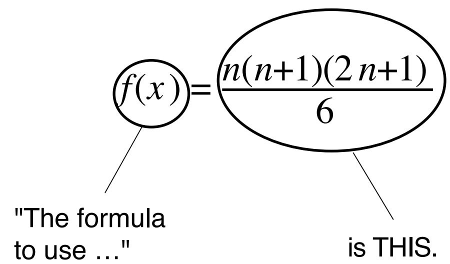


Fig. 15.2. Students' acceptance of an ill-formed function representation because it fit their concept image of function.

Why Imagery is Important

There are two aspects to imagery that I suspect have a significant influence on the development of mathematical reasoning. The first has to do with students' immediate understandings of the situations they encounter during schooling. The second has to do with more global aspects of their development of mental operations.

INFLUENCES OF IMMEDIATE IMAGERY

I am often in classrooms with children, either as observer, interviewer, or teacher. Over the past 4 years I have become ever more aware of the difficulties students experience because of insufficient attention being given to their images of the settings in which problems ostensibly occur. Some examples might make this clear.

Restocking the Shelves

A seventh-grade teacher presented this problem to his class.

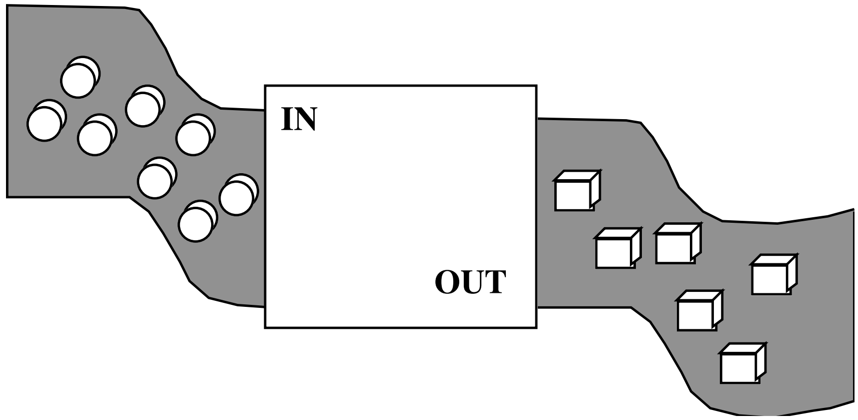
> A grocer buys Sara Lee cakes from his distributor in packages of 8 cakes per package. Each package costs $4.25. The grocer figures he needs 275 cakes for the next week. How much money should he plan on paying for cakes?

The teacher first reemphasized a theme he had made prominent through the year—that first they understand the situation. One student, Chris, said that he “didn’t understand.” The teacher, himself trying to understand Chris’ difficulty, asked Chris to put himself in the grocer’s position, and asked Chris “How will you figure how much it will cost you?” Chris’ response: “They’ll tell you when you walk out.”

It seems Chris did put himself into the situation. But his image of the situation centered around his experience of shopping, where you get what you need, take it to a cashier, and the cashier rings up the amount you owe. The teacher, an ex-businessman, had a much different image in mind—an image of a person sitting at his desk trying to set up a budget for the coming week. The teacher’s instructional objectives had to do with bringing out the issue of when it is reasonable to round a calculation up and when to round it down. He was not sensitive to the situation as Chris had constructed it, and the image Chris had made of the context made the situation non-problematic. He quite literally did not understand what the problem was about.

Fractions

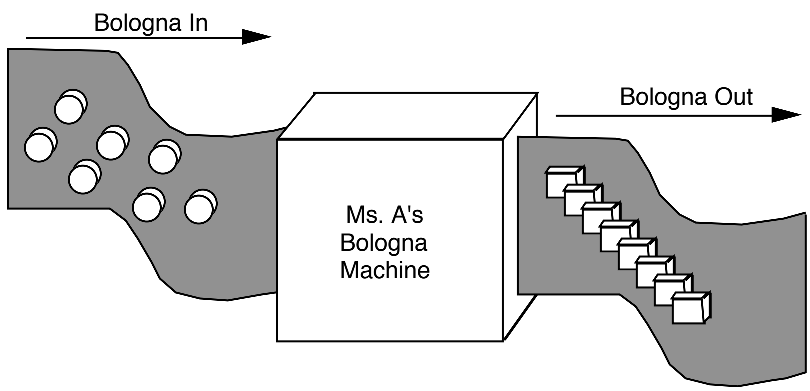
A seventh-grade teacher used the problem presented in Fig. 15.3 in a unit on fractions. Students working this problem looked for patterns in number-pairs. Their conversations were empty in regard to the machine or why the number-pairs were related in a natural way. Instead, students focused completely on filling-out the table as a type of “guess my rule” problem. It occurred to me that there was nothing about the machine that they could talk about, for there was nothing imaginative about it except that, as if by magic, things came out of it when other things came in.



Here is a machine. Eight objects are going in and 6 packs are coming out.

Fig. 15.3. Fraction as operator represented by a machine. The text gives five other in–out pairs for this machine and asks students to complete a table with eight entries, each of which gives a number of pieces going in but does not give a number of pieces going out.

I suggested to the teacher that he change the problem's setting and orientation slightly, as in Fig. 15.4.



Ms. Allowishus has a machine that turns round chunks of bologna into rectangular packets. All round chunks are the same size and all packets are the same size.

The same amount of bologna comes out of the machine as goes into it. So, if 5 pounds of bologna goes into the machine, then 5 pounds of bologna comes out of it.

Eight packets come out of the machine when seven round chunks go into the machine.

1. Which contains more bologna, a round chunk or a packet? Explain.
2. How many packets does one round chunk make? Explain.
3. How many round chunks of bologna make up one packet? Explain.
4. How many packets will come out of the machine when 10 round chunks go into the machine? Explain.

Fig. 15.4. Revised machine problem.

Following these questions was a problem of filling in a table that was similar to the one given in the original text.

Students' discussions of this situation had a completely different content than their original discussions. Whereas originally they were wondering about what numbers to add, subtract, multiply or divide, in this case they talked initially about the machine (e.g., why would anyone want to turn chunks into packets) and they soon began talking about cutting up an amount of bologna into sevenths versus cutting up the same amount of bologna into eighths, and they ended up talking about turning one-seventh of the bologna into a number of eighths of the bologna. Other problematic matters that came into their discussion were whether or not the bologna on the right was the same stuff as the bologna on the left (that is, could you talk about "filling" a packet with a chunk, or did you need to "reshape" a packet into a chunk), and whether it mattered that there might be any bologna in the machine (and if so, was it the same amount as on either side). I should also point out that this problem was more difficult for many students, in that they experienced difficulties arising from lacunae in their understanding of fractions that they had not experienced in the original problem. One source of difficulty, I claim, is that their images of the situation entailed a severe constraint—the invariance of amount of bologna whether made of a number of chunks or a number of packets.

Someone might argue that it would be better had these difficulties been avoided—that it would be preferable to use the original setting. I would argue in response that the difficulties and issues that these students encountered are among those they will encounter when finding themselves in occasions where fractional reasoning would be appropriate, and that they should encounter occasions to experience these difficulties early in their learning of fractions. Otherwise we enhance the probability that children's school mathematics is useless in their everyday lives (Lave, 1988). The general point is that if students do not become engaged imagistically in ways that relate mathematical reasoning to principled experience, then we have little reason to believe that they will come to see their world outside of school in any way mathematical. On the other hand, we should not think that just any imagistic reasoning guarantees relevance.

GLOBAL INFLUENCES OF IMAGERY

Although immediate imagery always influences direction in a local space of possibilities, present in every moment of reasoning is a cumulative influence of imagery that orients our reasoning. One way this happens is during our uses of notation. Another way this happens is through the mental operations by which we constitute the situations in which we use the mathematics we know. Both aspects of imagery provide a large part of the background for our events of reasoning.

Uses of Notation

When a person applies mathematics skillfully, her uses of notation are largely nonproblematic. It is much like a skilled writer's use of written language; difficulties reside more in deciding what to write than in the writing itself. However, the expression of an idea in notation provides her an occasion to reflect on what she said, an occasion to consider if what she said was what she intended to say and if what she intended to say is what she said. To act in this way unthinkingly is common among practicing mathematicians and mathematical scientists.

Behind such a dialectic between understanding and expression is an image, most often unarticulated and unconsciously acted, of what one does when reasoning mathematically. This image entails an orientation to negotiations with oneself about meaning, something that is outside the experience of most school students. The dialectic between understanding and expression just depicted is not the normal stuff of students' experiences in school mathematics, at least in the United States. Instead, the predominant image behind students' and teachers' notational actions seems to be more like "put the right stuff on the paper."

The robustness of the predominant image of acting mathematically—putting it right, on paper—is striking. I observed a senior mathematics major's demonstration lesson on the graphing of inequalities in two variables. The words point, pair, open sentence, and solution never appeared; his entire discussion was about numbers. I asked him where are these y's that are greater than $2x$-3. He waved his hand up and down the y-axis. I asked him, "Where are these x's so that $2x$-3 is less than y?" He waved his hand back and forth along the x-axis. I then asked, "So, if all of the y's are on the y-axis and all of the x's are along the x-axis, why did you draw a graph of $y=2x-3$ and shade the portion above the graph?" He couldn't say. Now, the fact that this student didn't relate inequalities to open sentences, truth values, or solution sets is not pertinent to my present point. What is pertinent is that he did not feel that he had done anything improper, because his students quickly caught on to his procedure for graphing linear inequalities. From his perspective he taught a successful lesson even though it hadn't made any sense.

A related example comes from students' images of the activity of solving an equation. Their image of solving equations often is of activity that ends with something like "$x=2$." So, when they end up with something like "2=2" or "$x=x$" they conclude without hesitation that they must have done something wrong.

These images of mathematical activity and successful teaching, and their concomitant uses of notation, are more like what Johnson addressed than what Piaget addressed. They have to do with normative patterns of activity and what they are supposed to produce, so in this regard one can say they are highly conditioned by social arrangements and interactions (Cobb & Yackel, 1991; Yackel, Cobb, Wood, Wheatley, & Merkel, 1990).

At the same time that images of mathematical activity are conditioned by social interactions, they are personal images that inform students' and teachers' activities. To change the social arrangements that are supported by actions emanating from these images, which thereby reinforce them further, it would seem that students and/or their teacher need to experience a different pattern of intellectual engagement surrounding their use of notation. In (Thompson, 1992) I attempted to design an instructional unit that would support the kind of notational dialectic outlined at the beginning of this section. I hoped that students' experience of a dialectic among intention, action, and expression would enrich their images of notation usage and their understanding of convention.

The unit was on decimal numeration, and it was for fourth-graders. They worked 7 days on it. At the center of the unit was a computer program that simulated base-ten blocks (see Fig. 15.5). Students represented numerical values by dragging copies of blocks into various parts of the screen (with a mouse). They could act on the display by moving blocks around or by "exploding" one block into ten of the next smaller kind or "imploding" ten blocks of one kind into one of the next larger. They implemented these actions on blocks by clicking on an appropriate digit in the value's expanded-notation representation and then clicking *Carry* or *Borrow*. *Carry* caused ten blocks to implode into one. *Borrow* caused one block to explode into ten.

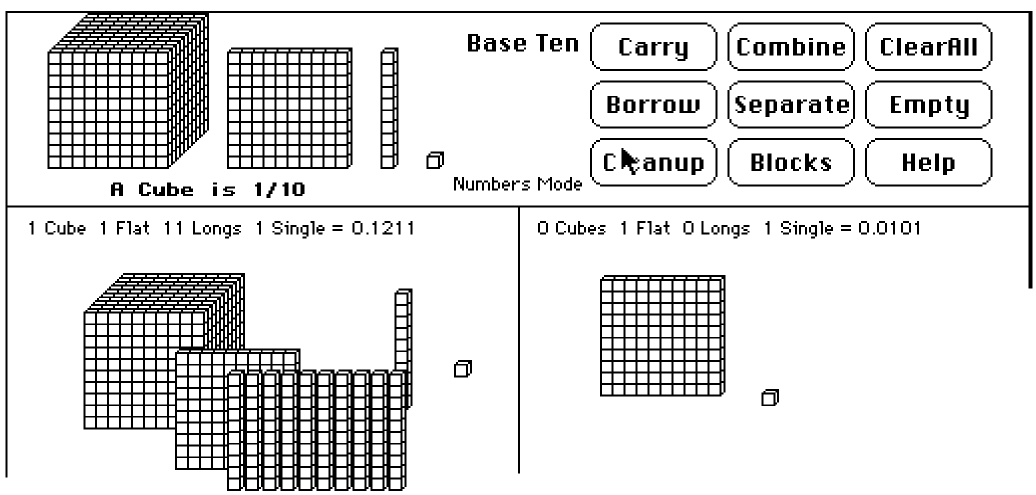


Fig. 15.5. Screen display of Blocks microworld. From Thompson (1992). Reprinted by permission.

Part of the instructional design was a continual orientation to the fact that students were free to devise blocks-based methods for addition and subtraction, with one constraint. They were required to devise notational methods by which they could reflect each action and its effect on the current state of their problem.

Instructional tasks and the program were designed to support students' back-and-forth movements between the things they were acting on and notational expressions of the effects of whatever they had done (Fig. 15.6). In fact, "back and forth" from situation to expression to situation was precisely the image of notational use I hoped they would form.

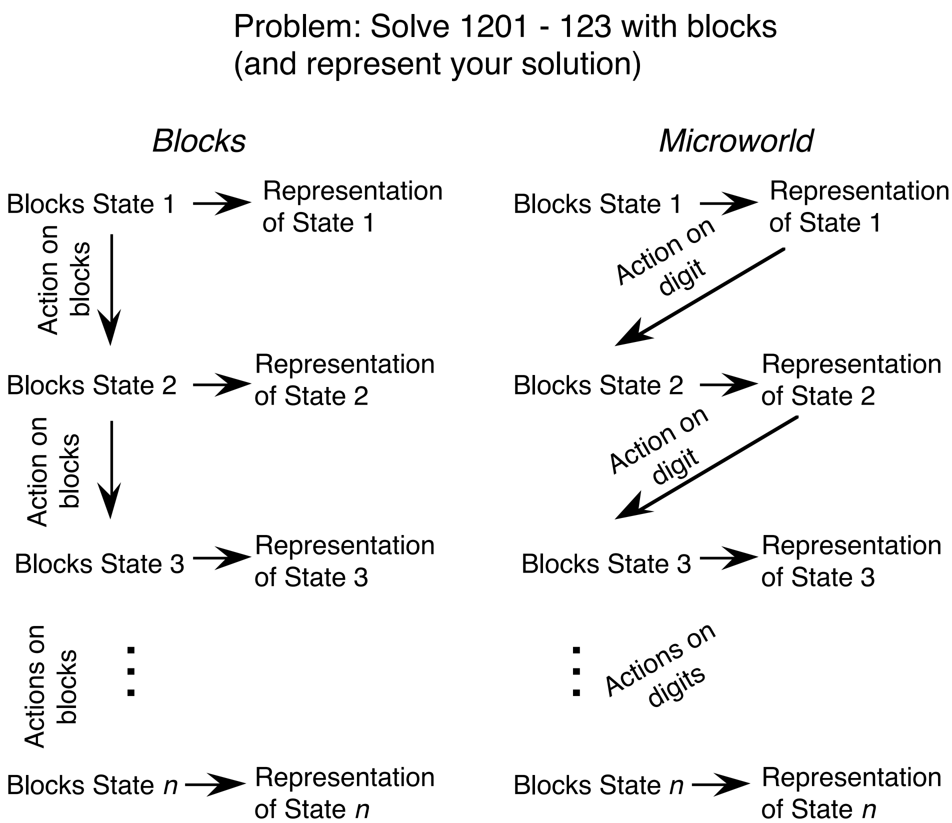


Fig. 15.6. Differences in students' engagement with notation when working with wooden base-ten blocks and with the computer program. From Thompson (1992). Reprinted by permission.

The results of this experiment are available in (Thompson, 1992). What I would like to highlight here is an observation mentioned only in passing in that article. Students at first were highly disoriented by having the freedom (and responsibility) to create their own uses of notation. They would have rather been told what to do, presumably because being told what to do would have fit their image of learning mathematics in school. The students in the control group, who experienced the pattern of engagement illustrated in the left side of Fig. 15.6, remained disoriented by their notational freedom and responsibilities. Students in the experimental group, who experienced the pattern of engagement illustrated in the right side of Fig. 15.6, developed a predilection to speak imagistically about things happening with blocks *as they spoke about notational actions*. That is, the dialectic nature of their experience when acting notationally supported their synthesis of blocks and numerals as alternative notational systems for representing numerical value.

Table 15.1 repeats one item from the posttest given in (Thompson, 1992). Students who received instruction using wooden base-ten blocks appeared to retain a prescriptive image of notational method, whereas the children receiving instruction in the experimental setting were more open to the validity of alternative, creative uses of notation. An explicit orientation to having students build dialectical images of mathematical activity and its notational entailments seems to be a promising avenue for further investigation and development. The same approach can be taken, in principle, with secondary school and university mathematics.

Statement		Response	Blocks	Microworld
This is the RIGHT way to add 8276 and 4185. Other ways might give the same answer, but they are not the right way:	$\begin{array}{r} {}^{1\,1} \\ 8276 \\ +\ 4185 \\ \hline 12461 \end{array}$	Yes No Don't Know	5 2 3	1 7 2

Note: n=10 in each group.

Table 15.1. (From Thompson, 1992).

Although this example focuses explicitly on representations of numerical operations and numerical value, the issue it addresses—teachers' and students' background images of what they are doing while engaged in notational activity—spans all levels of mathematics. For example, if students' image of multiplication is repeated addition, then it is understandable that multiplication of fractions is difficult. But, if their image of multiplication is “make copies of,” then there is no built-in incoherence for multiplication of fractions. An image can easily be formed of , say, $3\frac{2}{5} \times 4\frac{5}{8}$, by first constructing something having value $4\frac{5}{8}$, making 3 copies of it, then making $\frac{2}{5}$ copy of it. I suspect this sort of image, if stressed from the outset even with whole numbers, would ameliorate common obstacles, such as “multiplication makes bigger” (Fischbein, Deri, Nello, & Marino, 1985; Greer, 1988, 1992).

Constituted Situations

The previous section focused on students’ images of mathematical activity—what it feels like to be mathematically engaged and to use notation to create systematic expressions of that engagement. There is also a kind of imagery that is traditionally aligned with conceptual development. It has been referred to in the past as process–product duality (Gray & Tall, 1994; Sfard, 1989, 1991; Thompson, 1985b), distinctions between procedure and concept (Hiebert & Lefevre, 1985), and distinctions between operations and figurations in relation to reflective abstraction (Bamberger & Schön, 1991; Dubinsky & Lewin, 1986; Rota, Sharp, & Sokolowski, 1988; Thompson, 1985a, 1991; von Glasersfeld, 1991).

It is in describing the development of images to support operational reasoning-in-context that the power of Piaget’s notion of image emerges. In (Thompson, 1994a) I used Piaget’s notion of image and mental operation, in support of my own theory of quantitative reasoning, to give a detailed account of one child’s construction of speed as a rate. In (Thompson & Thompson, 1992) Alba Thompson and I summarize several years of research on the role of imagery in students’ constructions of ratios and rates, as typified by their construction of the mental operations of speed and acceleration. The pertinent aspect of these reports is that they give operational detail to the development of “mathematical objects,” and they highlight the important role played by imagery throughout that development. It is through imagery and the operations entailed within images that we constitute situations and act in them.

In many regards these accounts parallel Piaget's (1970) description of children's construction of speed. Children's initial image is of motion—something moving—and their awareness of displacement. They abstract from movement the image of distance traveled and duration of movement, but those images are uncoordinated, and the image of distance dominates in their constituted situations. Children's early image of speed as distance shows up in their inability to reason about completed motion in relation to duration—such as, at what speed must I travel to go 100 feet in 6 seconds? Later images of speed as quantified motion entail the coordinated images of the accrual of incremental distance and incremental time in relation to images of accumulated distance and accumulated time. Images of objects being accelerated entail an image of accumulated accruals of increments to incremental distance in relation to incremental time—the speed of the object "speeds up."

The latter images, accruals in relation to accumulations, are much like what has often been described as proportional reasoning (Hart, 1978; Kaput & West, 1994; Karplus, Pulos, & Stage, 1983; Lesh, Post, & Behr, 1988; Tourniaire & Pulos, 1985). In (Thompson & Thompson, 1992) we made the case that the kinds of images built up in understanding speed and accelerations are foundational for comprehending many areas classically thought of as proportional reasoning. In (Thompson, 1994b), I demonstrated how advanced mathematics students' impoverished images of rate obstructed their understanding of derivative, integral, and relationships between them.

My claim here is that without students having developed images such as these, images that entail both figurative and operative thought, students cannot constitute the situations that their visible mathematics is supposed to be about with sufficient richness to support their reasoning. When this happens they are reduced to forming figural associations between a teacher's notational actions and superficial characteristics of a problem statement's linguistic presentation. Such figural reasoning then orients students and teachers to patterns of activity, and hence images of mathematics, that I spoke of earlier as “get it right, on paper.”

CONCLUSION

I have raised the matter of imagery as it relates to four areas of pedagogical and psychological concern. These are:

(1) Teachers’ and students’ images of mathematical activity—the kinds of activities in which they expect to be engaged and the kinds of products these activities should end with;

(2) Students’ and teachers’ background images of situations immediately under discussion frame their understandings of what is being discussed;

(3) Students’ and teachers’ images of notational activity and what might transpire while they are engaged in it;

(4) Coordinated images can consolidate in mental operations and can come to provide the conceptual substance by which students constitute situations.

I invite others to expand or alter this list, and to expand the discussion of imagery in mathematical reasoning.

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FOOTNOTE

[1] The Latin root of "confused" is *confundere*, to mix together. Thus, one way to think of being in a state of confusion is that we create inconsistent images while operating.