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RE-THINKING COVARIATION FROM A QUANTITATIVE PERSPECTIVE:

SIMULTANEOUS CONTINUOUS VARIATION

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We hypothesize that students' engagement in tasks which require them to track two sources of information simultaneously are propitious for their envisioning graphs as composed of points, each of which record the simultaneous state of two quantities that covary continuously. We investigated this hypothesis in a teaching experiment involving one 8th-grade student. Details of the student's experience and an analysis of his development are presented.

Confrey and Smith (1994, 1995) explicate a notion of covariation that entails moving between successive values of one variable and coordinating this with moving between corresponding successive values of another variable (1994, p.33). They also explain, "in a covariation approach, a function is understood as the juxtaposition of two sequences, each of which is generated independently through a pattern of data values" (1995, p. 67). Coulombe and Berenson build on these definitions, and on ideas discussed by Thompson and Thompson (1994b, 1996), to describe a concept of covariation that entails these properties: "(a) the identification of two data sets, (b) the coordination of two data patterns to form associations between increasing, decreasing, and constant patterns, (c) the linking of two data patterns to establish specific connections between data values, and (d) the generalization of the link to predict unknown data values." (p. 88)

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Thinking of covariation as the coordination of sequences fits well with employing tables to present successive states of a variation. We find it useful to extend this idea, to consider possible imagistic foundations for someone's ability to "see" covariation. In this regard, our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value.

In our theory, images of covariation are developmental. In early development one coordinates two quantities' values — think of one, then the other, then the first, then the second, and so on. Later images of covariation entail understanding time as a continuous quantity, so that, in one's image, the two quantities' values persist. An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image (Thompson, 1994a). In the case of continuous covariation, one understands that if either quantity has different values at different times, it changed from one to another by assuming all intermediate values.

Purpose and method of inquiry

We asked the question, "What conceptual operations are involved in students coming to envision and reason about continuous covariation of quantities?" We hypothesized that students' engagement with tasks requiring them to track two sources of information simultaneously are propitious for their envisioning graphs as composed of points, each of which records the simultaneous state of two quantities that covary continuously. We elaborated this hypothesis in a teaching experiment involving one 8th-grade student, Shawn.

The teaching experiment covered three sessions. In these sessions Shawn engaged in a sequence of tasks centered around the activity of tracking and describing the behavior of the distances between a car and each of two cities as the car moves along a road (see right side of Figures 1). The bulk of this report is on the results of the first two sessions.

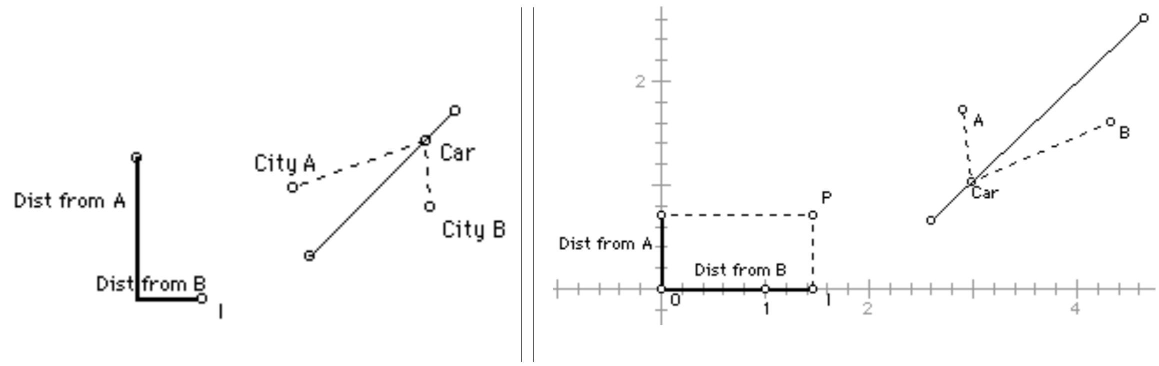

Figure 1. Two snapshots of car positions. In each snapshot, Distance from A and Distance from B are each represented by a line segment's length. In the snapshot on the right point P is displayed as the correspondence of the perpendicular segments representing AC and BC.

The activity employed a Geometer's Sketchpad sketch that allowed Shawn to simulate the car's movement by dragging a point with a computer mouse. This sketch allowed Shawn to display, or not, individually or simultaneously, the car's distances between it and each city. He could also chose to display those distances as perpendicular segments (Figure 1). Finally, the sketch allowed him to display a point of correspondence and its locus. These latter options, however, were made available to him as instruction proceeded (as were others, such as displaying axes or not). They were unavailable at the outset.

The sequence of tasks was in three phases, each focusing on successive levels of operativity in images of covariation. We call the phases *engagement*, *move to representation*, and *move to reflection*. *Engagement* tasks focused on having Shawn come to understand the setting portrayed by the sketch and the basic task of tracking distances. *Move to Representation* tasks were intended to support Shawn's internalization of the covariation. *Move to Reflection* tasks were intended to have Shawn come to imagine completed covariation and its emergent properties.

Results and analyses

Phase 1: Engagement

In the *Engagement* phase, Shawn was directed to move the car along the road while watching the distances between the cities and the car. He was also asked to describe each distance's behavior in relation to the car's position along the road. The vertical segment was visible while Shawn investigated the behavior of AC; the horizontal segment was visible while he investigated BC (Figure 1).

Shawn's observations of AC were at two levels. First, he immediately noticed the decrease in AC as he moved C (the car) away from one end of the road, and the ensuing increase in AC as the car passed the point where AC was smallest.[2] He watched the vertical "bar" closely while moving C, referring to its height (and changes in it) interchangeably with the distance between C and City A. Shawn at first focused only on the distances, not the rates at which distances changed. After being asked whether AC changed faster in some places than in others, Shawn focused on the deceleration and acceleration of AC's length with respect to the changes in C's position. Shawn built up images of this accelerated change by noticing that the variation in the bar's height is almost imperceptible for positions of C near where AC is minimum and that at points farther away (e.g., endpoints) AC changes more with the same changes in C's position. His coordinations remained between changes in the bar's height and changes in C's *position*. This is as opposed to coordinating changes in AC with changes in C's distance from its start.

[2] For brevity, we use the symbols AC and BC to denote both the segments and their lengths.

Shawn's observations of BC were at the first level; he gave an analogous description of BC's behavior in terms of the systematic variation of the horizontal segment.

Shawn then displayed both perpendicular segments simultaneously, together with a point, P, showing the correspondence of their lengths (see right side of Figure 1). He tracked the motion of point P as he moved the car along the road. His tasks were to describe the behavior of point P, and to say what P and its locus represent. Shawn said of P's motion, "It moves with the two bars … It moves along with the X and Y axis". He eventually wrote the following description:

> It shows the car getting closer to the cities as both are decreasing. At one point the bar indicating City B pauses, meaning the closest point to City B, as the bar referring to City A keeps declining, because it is not yet at the closest point to City A. As I approach City A the bar pauses, telling me that I am closest to City A, while the bar referring to City B increases because the distance is increasing and City A is increasing.

Shawn gave this description with no recourse to the GSP sketch. It was as though he mentally moved the car, watching the variation in AC and BC. This is consistent with his having internalized the experience into a coherent set of actions and images which he could re-present (von Glasersfeld, 1995; Piaget, 1970).

Shawn struggled to understand the relationship between P and its locus, however. Underlying his difficulty was some uncertainty as to what P represented. At first Shawn explained that a location of P is a graphical representation of the position of the car on the road. He eventually began to develop a proto-multiplicative view of P, seeing that it's location "combines" the distance between the car and the two cities and represents "how far or close you are to the two cities [...] 'cause you see both the bars".

Shawn's initial conception of the locus of P is revealed by his assertions: "this [the graph] is where the car's traveling [...] the road must not be straight, it must be curved", and "the car must be the correspondence point and the road must be the graph". After being asked if the graph tells the car's position on the road, Shawn eventually came to view the graph as "the path of P which marks the distance between the two cities, kind of …". He arrived at this by reflecting on the relationship between P's location and its locus, a process that involved having to explain what information was given by P's being in each of several specific locations.

Phase II: Move to representation

In this phase Shawn was presented with depictions of various road-city configurations. He was asked (a) to imagine and to describe the two distances' behaviors, (b) to draw a prediction of P's locus as the car moved along the road. He used the sketch to test a prediction; we discussed each result.

Imagining and describing the behavior of P were difficult for Shawn. He required pencil and paper to reflect on the details of how changes in AC and BC would affect changes in P's location. He drew a hypothetical starting position of P and orthogonal arrows to indicate the change in P's position according to changes in AC and BC. By coordinating the values and changes in AC with those of BC he would deduce, and plot, new positions of P. He would then decide on the graph's shape and draw a rough sketch of it. In this way he was generally able to successfully predicted the locus' monotonic portions. The concavities he predicted were always opposite those generated by the sketch. For the first road-cities arrangement, Shawn was unperturbed by the discrepancy. He downplayed it, saying, "So my prediction was pretty accurate but I forgot to leave out that little part there". When asked why he thought the graph should be arced one way instead of the other, he stated, "All I knew is it was gonna go forward and down. …. I didn't know which way is the arc". In succeeding tasks Shawn became increasingly concerned about discrepancies between predicted and actual concavity. The question of how to know the concavity remained an open one.

Phase III: Move to reflection

In this phase Shawn was presented with various graphs plotting AC versus BC. His task for each was to explore and predict possible locations of the two cities relative to the road so that the car's movement would produce that graph. Figure 2 shows a graph examined by Shawn.

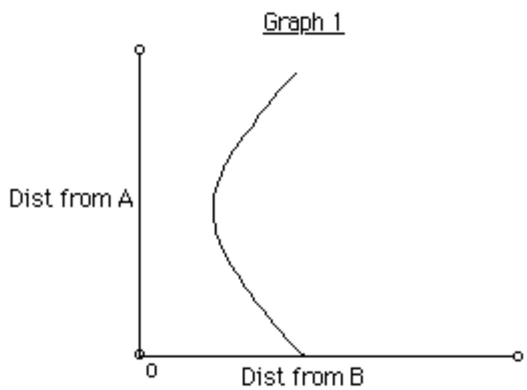

Figure 2. A graph of AC versus BC corresponding to the completed motion of the car along the road relative to fixed locations of city A and city B.

Shawn identified those points on the graph corresponding to extreme values of AC and BC and he anchored his descriptions of the variations of the two distances around these points:

> This is always going closer to A, 'cause it's always going down [...] This seems to be the closest point to B then it starts going right, back up [...] You're farthest from A and closest to A at the extremes of the graph. So figuring it's at the extremes of the road too, at the end of the road [...] The closest point to B must be in the middle of the closest point to A and the farthest one from A.

Thus, Shawn's images were of the completed variation of each of AC and BC individually. Coordinating each quantity's variation with its extreme values allowed him to deduce plausible locations for each city. In this way he was able to construct a corresponding road-cities arrangement, apparently without imagery of AC and BC explicitly covarying simultaneously.

Conclusions

The results of this study lead us to believe that understanding graphs as representing a continuum of states of covarying quantities is nontrivial and should not be taken for granted. Shawn's predominant imagery is consistent with his having developed a level of operativity at which he could intricately coordinate images of two individually varying quantities. There was also suggestion of his developing images of their sustained simultaneity, one that did not explicitly entail a conception of *tight coupling* so that one variation is not imagined without the other. Seeing graphs as intended here seems to require having tight coupling as a central feature of one's imagery.

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