REPRESENTATION, VISION AND VISUALIZATION: COGNITIVE FUNCTIONS IN MATHEMATICAL THINKING. BASIC ISSUES FOR LEARNING

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Mathematics education has been very sensitive to change needs over the last fifty years. Researches in developmental psychology, new technologies, new requirements in assessment have supported them. But their impact has been more effective on mathematics curriculum and on means of teaching than on the explanations of the deep processes of understanding and learning in mathematics. Difficulties of such research stem from the necessity to define a framework within the epistemological constraints specific to mathematical activity and the cognitive functions of thought which it involves are not separated. That requires going beyond local studies of concept acquiring at each level of the curriculum and beyond mere reference to very general theories of learning and even beyond global description of student's activity in classroom.

Representation and visualization are at the core of understanding in mathematics. But in which framework can their role in mathematical thinking and in learning of mathematics be analyzed? Already in 1961, Piaget admitted the difficulty to understand what mathematicians call "intuition", a way of understanding which has close links with representation and visualization: “rien n'est plus difficile à comprendre pour un psychologue que ce que les mathematiciens entendent par intuition". He distinguished "many forms of mathematical intuition" (1961, pp.223-241) from the empirical ones to the symbolizing ones. From a cognitive viewpoint, the question is not easier. Representation refers to a large range of meaning activities: steady and holistic beliefs about something, various ways to evoke and to denote objects, how information is coded... On the contrary, visualization seems to emphasize images and empirical intuition of physical objects and actions. Which ones are relevant to analyze the understanding in mathematics in order to bring out conditions of learning?

Our purpose in this panel is to focus on some main distinctions which are necessary to analyze the mathematical knowledge from a learning point of view and to explain how many students come up against difficulties at each level of curriculum and very often cannot go beyond. Studies about reasoning, proving, using geometrical figures in problem solving, reading of graphs... have made these distinctions necessary. They lead not only to emphasize semiotic representations as an intrinsic process of thinking but also to relativize some other ones as the distinction between internal and external representations. They lead also to point out the gap between vision and visualization. And from a learning point of view, visualization, the only relevant cognitive modality in mathematics, cannot be used as an immediate and obvious support for understanding. All these distinctions find accurate expression in different sets of cognitive variables. Within the compass of this panel we shall confine ourselves to sketching the complex cognitive architecture that any subject must develop because it underlies the use of representations and visualization in mathematics.

I. Three key ideas to define a framework to analyze the conditions of learning

1. The first one is the paradoxical character of mathematical knowledge

On the one hand, the use of systems of semiotic representation for mathematical thinking is essential because, unlike the other fields of knowledge (botany, geology, astronomy, physics...), there is no other ways of gaining access to the **mathematical objects** but to produce **some semiotic representations**. In the other fields of knowledge, semiotic representations are images or descriptions about some phenomena of the real external world, to which we can gain a perceptual and instrumental access without these representations. In mathematics it is not the case.

On the other hand, the understanding of mathematics requires not confusing the mathematical objects with the used representations. This begins early with numbers, which have not to be identified with digits, numeral systems (roman, binary, decimal...). And figures in geometry, even when they are constructed with accuracy, are just representations with particular values that are not relevant. And they cannot be taken as proofs.

2. The second one is the ambiguous meaning of the term "representation"

This term is often used to refer to mental entities: image, something away or missing that is evoked and, finally, what subjects understand. In this context, "mental" representation is considered as the opposite of signs which should be only "material" or "external" signs. Semiotic, and therefore external representations, would be at first necessary for the communication between the subjects. But this is a misleading division (Duval, 1995b pp. 24-32) which brings about two very damaging confusions.

When it is applied to the representations, the distinction mental/external refers to their mode of production and not to their nature or to their form. In that sense, signs are neither mental nor physical or external entities. More specifically, there is not a term to term correspondence between the distinction mental/material and the distinction signified/signifiant, because the signifiant of any sign is not determined by its material realization but only by its opposite relations to the other signs: it is the number of possible choices what matters, as Saussure explained it. The binary system and decimal systems are very trivial examples of this semiotic determination of significance: the significance of any digit depends not only on its position but also on the number of possible choices per position. And, as for language, any use of a semiotic system can be mental or written (that is external). Thus, mental arithmetic uses the same decimal system like written calculation but not the same strategies because of the cognitive cost.

There are two kinds of cognitive representations. Those that are intentionally produced by using any semiotic system: sentences, graphs, diagrams, drawings... Their production can be either mental or external. And there are those which are causally and automatically produced either by an organic system (dream or memory visual images) or by a physical device (reflections, photographs...). In one case, the content of the representations denotes the represented object: it is an explicit selection because each significant unit results from a choice. In the other case, the content of representations is the outcome of a physical action of the represented object on some organic system or on some physical device (Duval *et al*., 1999, pp. 32-46). In other words, the basic division is not the one between mental representation and external representation, which is often used in cognitive sciences as though it was evident and primary, but the other one between semiotic representation and physical/organic representation. We cannot deal anyway with a representation without taking into account the system in which it is produced.

3. The third one is about the need of various semiotic systems for mathematical thinking

History shows that progress in mathematics has been linked to the development of several semiotic systems from the primitive duality of cognitive modes which are based on different sensory systems: language and image. For example, symbolic notations stemmed from written language have led to the algebraic writing and, since the nineteenth century, to the creation of formal languages. For imagery, there was the construction of plane figures with tools, then that in perspective, then the graphs in order to «translate» curves into equations... Each new semiotic system provided specific means of representation and processing for mathematical thinking. For that reason, we have called them «register of representation» (Duval, 1995b). Thus, we have several registers for discursive representation and several systems for visualization. That entails a complex cognitive interplay underlying any mathematical activity.

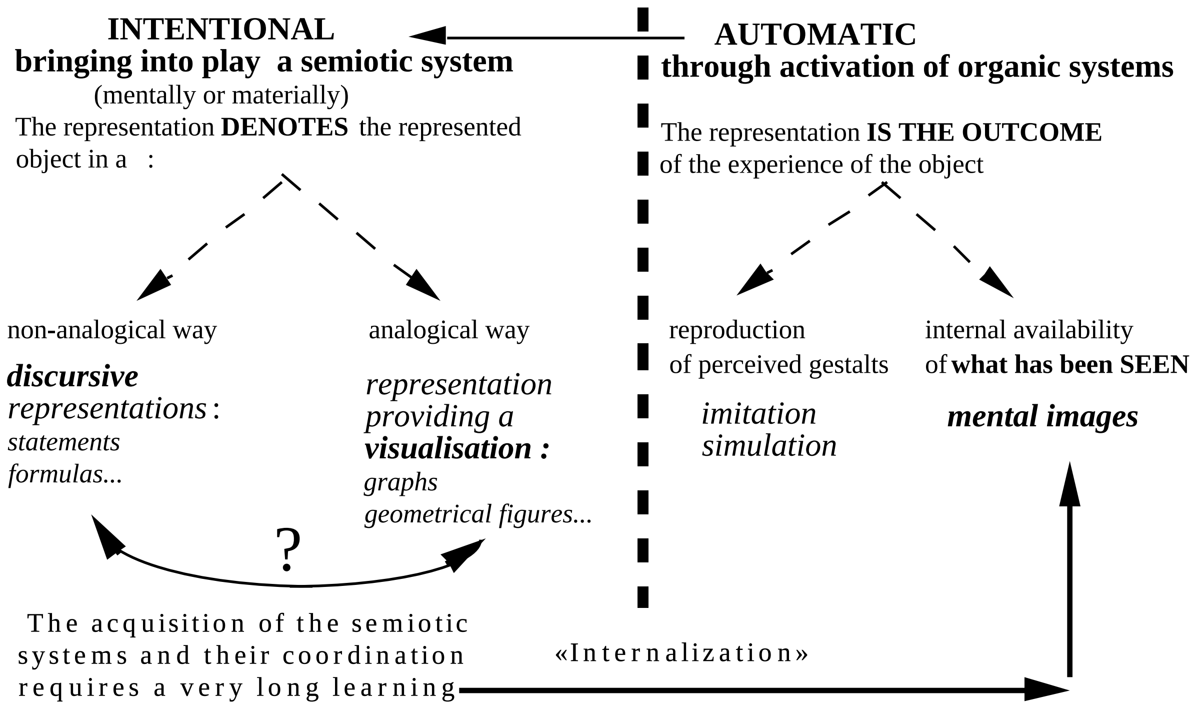


Figure 1. Cognitive classification of conscious representations. This classification can be expanded more and includes all kinds of representations. We can notice the existence of two heterogeneous kinds of "mental images": the «quasi-percepts» which are an extension of perception (on the right) and the internalized semiotic visualizations (on the left). Actions like the physical ones (rotation, displacement, separation...) can still be performed on some quasi-percepts and their time cost can be measured by reaction times to comparison tasks.

Firstly, as well as for discourse (description, explanation, reasoning, computation...) as for visualization, we have two kinds of registers: the registers with a triadic structure of significance (natural language, 2D or 3D shapes representation) and registers with a dyadic structure of significance (symbolic notations, formal languages, diagrams...) (Duval, 1995b, pp. 63-64). Within a dyadic structure any meaning is reduced to an explicitly defined denotation of objects. Within a triadic structure, we have meanings playing independently of any explicit denotation of objects and one must take into account their interplay. We can even fall into a cognitive conflict between the meaning game, which is proper to the register, and the denotation set for the representation. For example, the complexity of geometrical figures stems from their triadic structure of significance. Secondly, mathematical thinking often requires to activate in parallel two or three registers, even when only one is externally used, or seems sufficient, from a mathematical point of view.

This need of various registers of representation gives rise to several questions that are important in order to understand the real conditions of learning mathematics. First of all, there is a question about the specific way of working in each register: what operations are favored, or are only possible, within each register? This question is not trivial, because there are several registers for visualization and because they cannot be the same. Then, there are questions about the change from one register to another one. Are these changes very frequent or necessary? Are they always easy or evident to make? ... At last, is there a register more convenient or more intrinsically suitable for the mathematical thinking than others? It is obvious that registers with dyadic structure are technically more useful and more powerful than registers with triadic structure. But natural language remains essential for a cognitive control and for understanding within any mathematical activity. These questions may appear unimportant from a mathematical point of view. Even more, very often a mathematician cannot see why these questions arise. But from a didactical point of view, they are those questions that the difficulties of learning pose.

II. How the problems of mathematics learning come to light in this framework

1. No learning in mathematics can progress without understanding how the registers work

Cartesian graphs are very common examples because they look visually easy to grasp. But many observations have shown that most 15-17 year old students cannot discriminate the equations $y = x + 2$ and $y = 2x$ when looking at these two graphs:

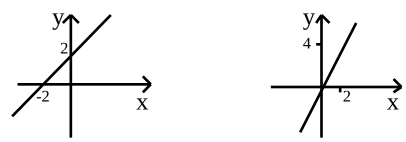


Figure 2. Visual discrimination of two elementary linear functions. This kind of discrimination presupposes that the qualitative values of two visual variables be distinguished: comparison between the angle with x-axis and the one obtained by bisection of xy angle, and position of crossing point with y-axis. Most often students confine themselves to the visual variable which is not relevant: how far some points are above x-axis (Duval, 1988).

Notwithstanding this kind of failure, students succeed in the standard tasks such as constructing the graph from a given equation or reading the coordinates of a point! This kind of failure means that graphs cannot be useful representations neither to control intuitively some calculations nor to organize and to interpret data in other fields. And we have similar observations for each register of representation, even those which look more natural, like geometrical figures, or which are very utilized, like the decimal system in which the position of digits determines the operative meaning (French National Assessment, 1992, 1996).

All these repeated observations show that semiotic representations constitute an irreducible aspect of mathematical knowledge and that wanting to subordinate them to concepts leads to false issues in learning. That amounts to forget the paradox of mathematical knowledge: mathematics objects, even the more elementary objects in arithmetic and geometry, are not directly accessible like the physical objects. Each semiotic register of representation has a specific way of working, of which students must become aware.

2. We must distinguish two kinds of cognitive operations in mathematics thinking: "processing" and conversion

Mathematical processes are composed of two kinds of transformations of representations. There are transformations that are made within the same register of representation, like arithmetical or algebraic computation. The semiotic possibilities of generating a new representation from a given representation are exploited. With the dyadic structure, these possibilities depend both on the semiotic system and on mathematical rules. The geometrical figures give also rise to the intrinsic gestalt transformations of configurations apart from any previous consideration of mathematical properties. These gestalt transformations are like the visual transformations that anamorphoses or jigsaws lead to bring into play. We have called "processing" this kind of transformation.

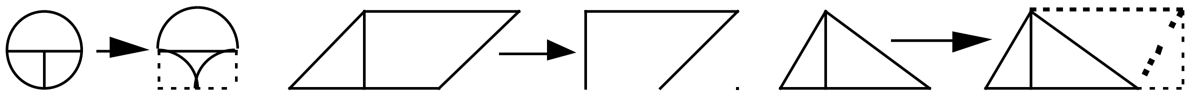


Figure 3. Visual change of configuration. The figurative units of any figure can be "reconfigured", mentally or materially, in another figure. For this kind of merely figurative transformation, neither hypotheses nor mathematical justification are required. Very often problem solving or explanations meant to convince students to resort to such transformations as if they were immediate and obvious for every student. Many observations show that this is not the case. There are factors that inhibit or trigger the «visibility» of such transformations. We can study them experimentally (Duval, 1995a).

And there are transformations that lay on a change of register: the representation of an object is «translated» into a different representation of the same object in another register. For example, when we go from a statement in native language to a literal expression. The transformation of equations into Cartesian graphs is another example. We have called "conversion" this kind of transformation.

One does not pay very close attention to the gap between these two kinds of cognitive operations that underlie mathematical processes. Nevertheless, if most students can learn some processing, very few of them can really convert representations. Much misunderstanding stems from this inability. But, very often, teachers attach more importance to the mathematical processes than to their application to daily life problems or to physical, or economic problems.

3. Conversion of representations is crucial problem in the learning of mathematics

Mathematical activity, in problem solving situations, requires the ability to change of register, either because another presentation of data, which fits better an already known model, is required, or because two registers must be brought together into play, like figures and natural language or symbolic notations in geometry. From a didactical point of view, only students who can perform register change do not confuse a mathematical object with its representation and they can transfer their mathematical knowledge to other contexts different from the one of learning. Two facts show the great complexity of conversion operation.

— Any conversion can be congruent or non-congruent. When a conversion is congruent the representation of the starting register is transparent to the representation of the target register. In other words, conversion can be seen like an easy translation unit to unit. Very accurate analyses of the congruent or non-congruent character of the conversion of a representation into another one can be systematically done. And they explain in a very accurate way many errors, failures, misunderstandings or mental bloks (Duval, 1995b, pp. 45-59; 1996, pp. 366-367).

— The congruence or the non-congruence of any conversion depends on its direction. A conversion can be congruent in one way and non-congruent in the opposite way. That leads to striking contrasts in the performances of students, such as those summarized in Figure 4.

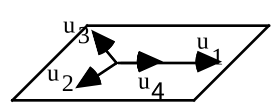
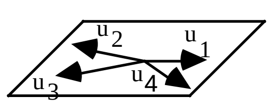
	Starting register	**Target Register**	144 students
(2D Rep.) **T ∅ G**	1 0 k p 0 1 m 0		**.83**
G ∅ T		1 0 a c 0 1 b d	**.34**

Figure 4. Elementary task of conversion (Pavlopoulou, 1993, p. 84)

Of course, the contrasts caused by the non congruence can be observed in a systematic way at all stages of the curriculum, from the more elementary verbal problems at primary school (Damm, 1992), to the university level.

It is surprising to see that this wide-ranging phenomenon is always ignored in the teaching of mathematics. Most teachers, mathematicians and even psychologists pay little attention to the difference of nature between processing and conversion. These two kinds of cognitive operations are grouped together in the unity of mathematical processes to solve a problem. And when a change of register must be introduced in the learning, one generally chooses one direction and the cases that are congruent. From a cognitive point of view, it is frequently a one-sided activity, which is proposed to students! There is something like an instinct to avoid the non-congruence situations that lead to real difficulties. But they are impossible to avoid especially when transfer of knowledge is required. Then failures and blocks are explained as conceptual misunderstanding, what is not a right explanation, since we have a contrast of successes and failures for the same mathematical objects in very similar situations. In reality the fact that students don't recognize anymore, when direction of conversion is changed, reveals a lack of co-ordination between the registers that have to bring into play together. The coordination of registers is not the consequence of understanding mathematics; on the contrary, it is an essential condition.

4. The learning of mathematics and the progressive coordination between registers

All these various and continual observations point out to a basic requirement that is specific for any progress in the learning of mathematics: the coordination between the registers of representation. This basic requirement is not fulfilled for most students, what is noticed in a global way often at the end of learning. For example, many teachers have, in one way or another, experienced what Schoenfeld (1986) described after a one yearlong study:

> [S]tudents may make virtually no connections between reference domains and symbols systems that we would expect them to think of as being nearly identical... the interplay occurs far more rarely than one would like (pp.239-242)

> [T]he students did not see any connection between the deductive mathematics of theorem proving and the inductive mathematics of doing constructions... they fail to see the connections or dismiss the proofs as being irrelevant (pp.243-244)... If students fail to see such obvious connections, they are missing what lies at the core of mathematics... (p.260)

Schoenfeld characterized this splitting rightly like an "inappropriate compartmentalization" (p. 226). But, unlike Schoenfeld's analysis, the kind of operative connections we expect to be made when learning is not between deductive and empirical mathematics, proofs and constructions, nor between mathematical structures and symbol structures, but between the different registers of semiotic representation. These connections between registers make up the cognitive architecture by which the students can recognize the same object through different representations and can make objective connections between deductive and empirical mathematics. Learning mathematics implies the construction of this cognitive architecture. It always begins with the coordination of a register providing visualization and a register performing one of the four discursive functions (Duval, 1995b, pp. 88-94).

III. Vision and visualization

From a psychological point of view, "vision" refers to visual perception and, by extension, to visual imagery. As perception, vision involves two essential cognitive functions.

— The first one consists in giving **direct access** to any physical object "in person". That is the reason why visual perception is always taken as a model for the epistemological notion of intuition. Nothing is more convincing than what is seen. In that sense, vision is the opposite of representation, even of the "mental images", because representation is something which stands instead of something else (Peirce). We shall call this function the ***epistemological function***.

— The second one is quite different. Vision consists of **apprehending simultaneously several objects** or a whole field. In other words, vision seems to give immediately a complete apprehension of any object or situation. In that sense, vision is the opposite of discourse, of deduction... which requires a sequence of focusing acts on a string of statements. We shall call it the ***synoptic function.***

In fact, visual perception performs in a very imperfect way the synoptic function. Firstly, because we are inside a three dimensional world: just one side of things can be seen, and complete apprehension requires movement, either of the one who is looking at it or of what is seen. In any case, this movement is a transformation of the perceived content: we have just a juxtaposition of successive sights which can be full-face, in profile, from above...Secondly, because visual perception always focalizes on a particular part of the field and can jump from one part to another one. There is no visual perception without such an exploration.

Now we can ask the following question that is decisive in the perspective of learning: are there cognitive structures that can perform both the epistemological and the synoptic function for the mathematical knowledge? The previous remarks lead us to answer this question negatively. More precisely, they lead to distinguish visualization from vision. Unlike vision, which provides a direct access to the object, visualization is based on the production of a semiotic representation. As Piaget, who has highlighted the synthetic inability of 3-5 year old children for the drawing of geometrical gestalts, explained it:

Le dessin est une représentation, c'est-à-dire qu**'il suppose la construction d'une image bien distincte de la perception elle-même**, et rien ne prouve que les rapports spatiaux dont cette image est faite soient du même niveau que ceux dont témoigne la perception correspondante (1972, p.65)

We have here the breaking point between visual perception and visualization. A semiotic representation does not show things as they are in the 3D environment or as they can be physically projected on a small 2D material support. That is the matter of visual perception. **A semiotic representation shows relations or, better, organization of relations between representational units.** These representational units can be 1D or 2D shapes (geometrical figures), coordinates (Cartesian graphs), propositions (propositional deductive graphs or "proof graph"), or words (semantic networks)... And these units must be **bi-dimensionally connected**, **because any organization requires at least two dimensions to become obvious**. In a string of discrete units (words, symbols, propositions) not any organization can be displayed. Thus, inasmuch as text or reasoning, understanding involves grasping their whole structure, there is no understanding without visualization. And that is why visualization should not be reduced to vision, that is to say: visualization makes visible all that is not accessible to vision. We can see now the gap between visual perception and visualization. Visual perception needs exploration through physical movements because it never gives a complete apprehension of the object. On the contrary, visualization can get at once a complete apprehension of any organization of relations. We say "can get" and "cannot get" because visualization requires a long training, as we shall prove it below. However, what visualization apprehends can be the start of a series of transformations, that makes its inventive power.

This difference between visual perception and visualization entails two consequences for the learning of mathematics.

Visualization refers to a cognitive activity that is intrinsically semiotic, that is, neither mental nor physical. Also such expressions as "mental image", "mental representation", '"mental imagery", are equivocal. They can only be the extension of visual perception. Accordingly, Neisser wrote:

"[V]isual image" is a partly undefined term for something *seen* somewhat in the way real objects are seen when little or nothing in the immediate or very sensory input appears to justify it. Imagery ranges from the extremely vivid and externally localized images of the eidetiker to the relatively hazy and unlocalized images of visual memory.

(Neisser, 1967; p.146)

Experiments on mental rotation of three-dimensional objects, since Shepard and Meztler (1971), are in the line of this conception of mental image as an extension of visual perception. But "mental imagery" can also be a mere visualization, that is, the mental production of semiotic representations as in mental calculation. Thus in "mind", we find the split into two kinds of representation back (*Figure*1). By resorting to mental images one does not avoid the difficulties arising from the paradoxical character of mathematics.

The way of watching is not the same in vision than in visualization. Two phenomena are confusing this issue. First, when they are graphically produced, semiotic representations are subject to visual perceptive apprehension. In that sense, visualization is always displayed within visual perception or within its mental extension. Second, some semiotic representations, like drawings, aim at being "iconic" representations: there is a relating likeness between the representation content and the represented object, so that one recognizes it (a tree, a car, a house...) at once, without further information. Iconic representations refer to a previous perception of the represented object, from which to their concrete character. In mathematics, visualization does not work with such iconic representations: to look at them is not enough to see, that is, to notice and understand what is really represented.

The use of visualization requires a specific training, specific to visualize each register. Geometrical figures or Cartesian graphs are not directly available as iconic representations can be. And their learning cannot be reduced to training to construct them. This is due to the simple reason that construction makes attention to focus successively on some units and properties, whereas visualization consists in grasping directly the whole configuration of relations and in discriminating what is relevant in it. Most frequently, students go no further than to a local apprehension and do not see the relevant global organization but an iconic representation.

To sum up, visualization, which performs only the synoptic function, is not intuition but representation. In that sense, there are several possible geometrical registers for visualization. Visualization in mathematics is needed because it displays organization of relations, but it is not primitive, because it is not mere visual perception. In this respect, there is learning from the geometrical registers. Is there any vision that could perform the epistemological function? That is a philosophical question. From a cognitive view, the essential fact is the paradoxical character of the mathematical knowledge, which excludes any resort to mental representations as direct grasping of mathematical objects, at least in the didactical context.

IV. How visualization works toward understanding

We have characterized visualization as a bi-dimensional organization of relations between some kinds of units. Through visualization, any organization can be synoptically grasped as a configuration. In this way, we have as many kinds of visualization as kinds of units: geometrical configurations where units are 1D or 2D shapes or Gestalts, Cartesian graphs where units are couples {point, coordinates}, propositional graphs where units are statements... For the visualization of each register of visualization there are some rules or some intrinsic constraints to produce units and to form their relations. Thus, geometrical configurations can be constructed with tools and according to mathematical properties of the represented objects. One does not draw a pentagon as an oak-leaf or as a flower. There lies the point where visualization leads away from any iconic representation of a material object. In the perspective of learning, three problems have to be taken into account about visualization: the problem of discrimination, the problem of processing and the problem of coordination with a discursive register.

1. How can the relevant visual features be discriminated?

Unlike iconic representations, visualizations used in mathematics are not sufficient to know what are the denoted objects. Very early, young children learn quickly to recognize by themselves images of physical objects, perhaps because schematizations of frequently perceived outlines are automatically developed. But learning visualization in mathematics is not quite so easy and successful as it is for physical objects and real environment.

In front of simple Cartesian graphs, most students only have a local apprehension confined to the associations of points with coordinates. They do not get a global apprehension of all visual variables, which enables them to discriminate visually between the different graphs of functions such as

$$y = 2 - x, \quad y = 2x, \quad y = x + 2.$$

In other words, Cartesian graphs do not work visually for most students except for giving the naïve holistic information: the line goes up or down, like a mountain road. But that can be misleading when they have to compare the graphs of two series of observations. And Cartesian graphs can perform anyway a checking or a heuristic function in the tasks of formulae computation or interpretation. No connections can be made between the different graphs and the definitions, descriptions or explanations that are displayed in other registers.

Some simple 2D geometrical figures are taught at the primary level: triangle, circle, different quadrilateral polygons... But all these geometrical figures are equivocal representations. They can be hard iconic representations and they are nothing further than an herbarium of mathematical Gestalts. Or they can work as representations of geometrical objects and, in this case, they must appear as 2D organizations of 1D figural units. In other words, there are quite different apprehensions of the most elementary geometrical figures; the one which is according only to the spontaneous perceptive work, and the other which is «discursive» or anchored in some statements (definitions, theorems...), (Duval, 1998; pp.39-40). Thus, with the discursive apprehension, we can have several figures for the same geometrical object: for example, there are two typical figures to represent a parallelogram (I and II in Figure 5).

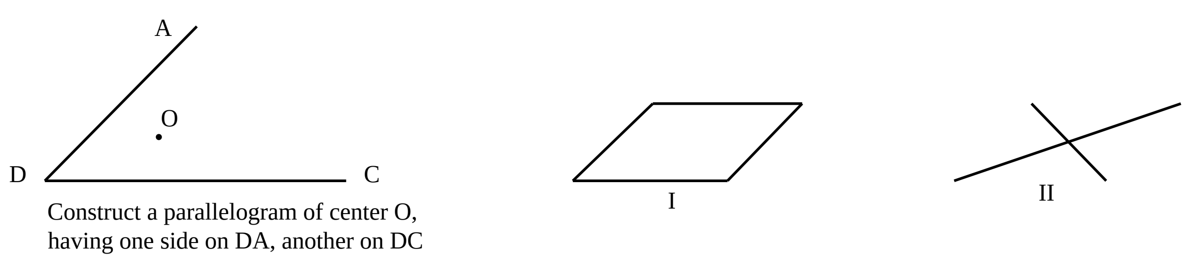


Figure 5. Which of the two figures, I or II, can be useful to solve the problem? With the visual help of *Figure* I, one can only roughly make the drawing by successive attempts of measurements on DA and DC. With the visual help of *Figure* II, one easily succeeds by drawing the diagonal DOD'. Although they knew all the properties of parallelogram, most students failed as if they were confined themselves to visualization I (Dupuis, 1978, pp.79-81). In fact, I and II give a visual help only when one works with configurations of 1D figural units.

Such observations have been made many times for very simple problems (Schoenfeld, 1986, pp. 243-244). And these phenomena are all the stronger the geometrical figure appears as a joint of several Gestalts (triangles, parallelograms, circles, straight lines...). For most students, there is like a heuristic deficiency of geometrical interpretation to visualization. But the equivocal character of geometrical figures appears also when a figure is directly taken for proof and leads to reject any resort to deductive reasoning. In that case, the figure works as a true iconic representation which makes discursive apprehension meaningless.

All these observations, which can be made anytime and anywhere in curricula, reveal the intrinsic difficulties of mathematical visualization. The intricacy of mathematical visualization does not consist in its visual units — they are fewer and more homogeneous than for the images

— but in the implicit selection of which visual contrast values within the configurations of units are relevant and which are not. Here is the representation barrier specific to learn visualization in mathematics. Is it really taken into account in teaching?

Very often one believes that to learn how to construct graphs or geometrical figures is enough to learn visualization in mathematics. Moreover, in this kind of task students get satisfactory results. But any such a task of construction requires only a succession of local apprehensions: one needs to focus on units and not on the final configuration. In other words, a student can succeed in constructing a graph or a geometrical figure and being unable to look at the final configurations other than as iconic representations. That is easy to observe and to explain.

Constructing a graph requires only to compute some coordinates and to plot a straight line, a curve: one goes ever from data tables, or from equations, to graduated axis. But visualization requires the opposite change: one must go from the whole graph to some visual values that point to the characteristic features of the represented phenomenon or that correspond to a kind of equation and to some characteristic values within the equation. Therefore, visualization causes the anticipation of the kind of equation to find out. And this gap between local apprehension and global apprehension that can exist to the end of the construction is more important for geometrical figures than for graphs. The reason is that from a geometrical figure we have not one but many possible configurations or subconfigurations. And the relevant configurations or subconfigurations in the context of a problem are not always those highlighted at first glance. What we called above a heuristic deficiency is like an inability to go further from this first glance. What reason is it due to? Teaching or some cognitive way of working?

2. Visualization and figural processing

In order to analyze any form of visualization there is a key idea: the existence of several registers of representation provides specific ways to process each register. Thus, if geometrical figures depend on a register, that is, on a system of representation, we must obtain specific visual operations that are peculiar to this register and that allow to change any initial geometrical figure into another one, while keeping the properties of the initial figure. What are these visual operations?

Three kinds of operations can be distinguished according to the way of modifying a given figure (Duval, 1988, pp. 61-63; 1995a, p.147):

— The mereologic way: you can divide the whole given figure into parts of various shapes (bands, rectangles...) and you can combine these parts in another whole figure or you can make appear new subfigures. In this way, you change the shapes that appeared at the first glance: a parallelogram is changed into a rectangle, or a parallelogram can appear by combining triangles... We call «reconfiguration» the most typical operation.

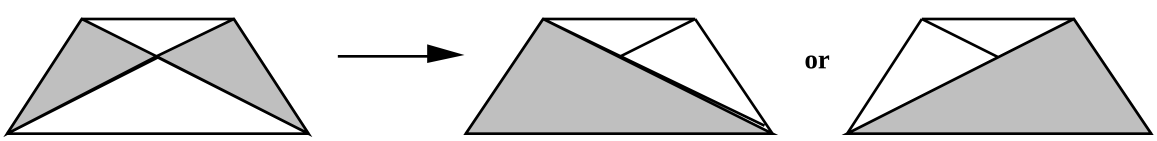


Figure 6. Figural processing by reconfiguration. Apprehension of this transformation within the only starting figure can be inhibited by the visual difficulty of double use of one sub-figure. But the starting figural frame is not changed like in the examples in Figure 3.

— The optic way: you can make a shape larger or narrower, or slant, as if you would use lenses. In this way, without any change, the shapes can appear differently. Plane figures are seen as if they were located in a 3D space. The typical operation is to make two similar figures overlap in depth (Duval, 1995 b, p.187): the smaller one is seen as it was the bigger one at the distance.

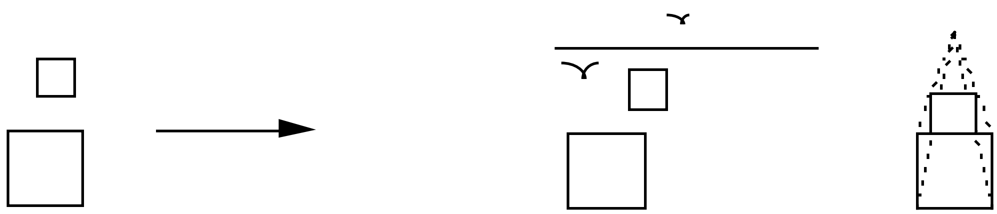


FIGURE 7. Figural processing by overlapping in depth of two similar figures

— The place way: you can change its orientation in the picture plane. It is the weakest change. It affects mainly the recognition of right angles, which visually are made up of vertical and horizontal lines.

These various operations constitute a specific figural processing which provides figures with a heuristic function. One of these operations can give an insight to the solution of a problem. We call it the operative apprehension of a given figure. It is different both from perceptual and discursive operation.

Operative apprehension is different from perceptual apprehension because perception fixes at the first glance the vision of some shapes and this evidence makes them steady.

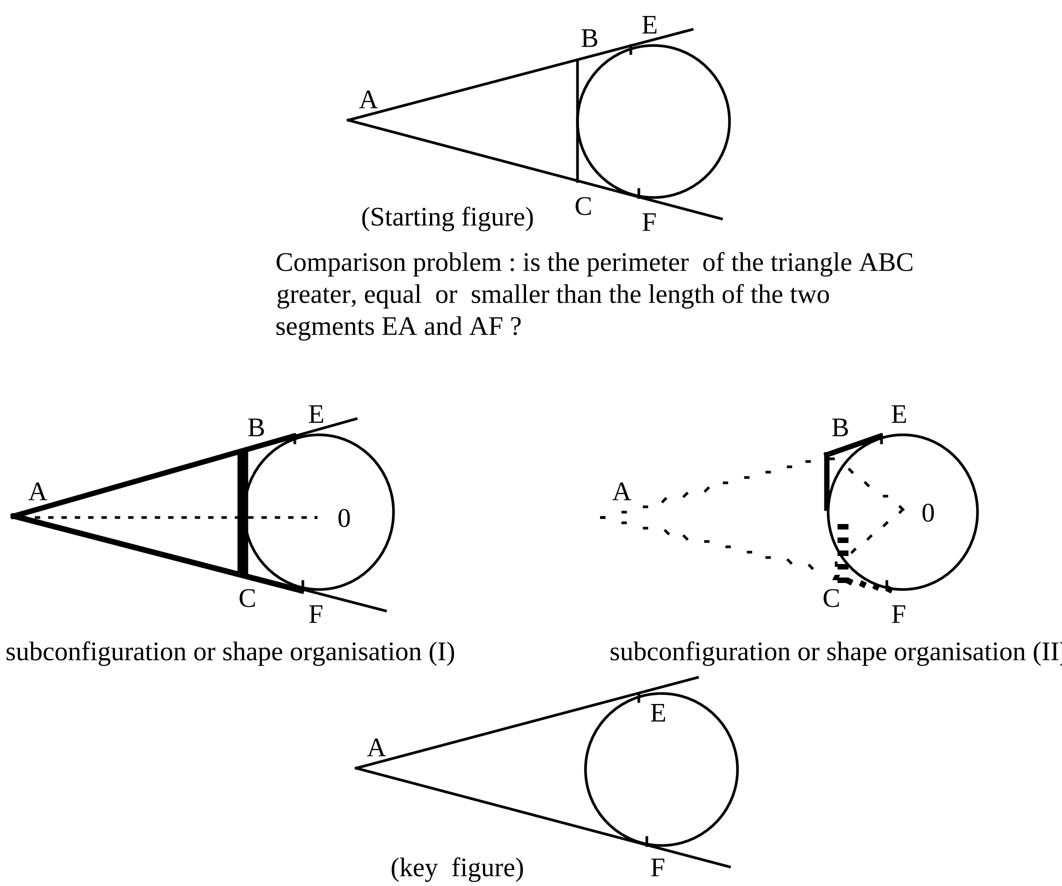


Figure 8. The perception of the starting figure highlights the shape organization (I) and makes it steady. But solving the problem requires the apprehension of the shape organization (II). Changing the perceptual apprehension of (I) into a perceptual apprehension of (II) constitutes a non-natural jump, because the symmetry axis AO sets forward the triangle within which the side BC is like an indivisible visual unit. Changing (I) into (II) requires looking at BC as a configuration of two segments! Moreover, the starting figure can be constructed without having to take into account the shape organization (II) with BO and BC as symmetry axis. Less than fifty per cent of 14-year-old students succeeded such a jump. And key figure does not help them for that (Mesquita, 1989, pp. 40, 68-69; Pluvinage, 1990, p. 27). However, by changing just a little the problem statement, and therefore the starting figure, all students can succeed: by naming I the point of intersection between AO and BC and by asking them to compare BI and IC, students are led to look at BC as a configuration of two segments. In that case, the statement of the problem becomes a congruent description of the subconfiguration (II), and geometrical visualization is reduced to an illustration function (Duval, 1999). But the learning problem is bypassed. A true didactical approach requires to embrace the whole range of variations of the conditions of a problem and to bring out the various factors that make them clear. It is only on the basis of students' knowledge that teachers can organize learning sequences.

In operative apprehension, the given figure becomes a starting point in order to investigate others configurations that can be obtained by one of these visual operations. In this respect, operative apprehension can develop several strings of figures from a given figure. According to the stated problem, one string shows an insight to the solution. The ability to think of drawing some units more on the given figure is one of the outward sign of operative apprehension. Now we can pose well the problem of heuristic deficiency: why perceptual apprehension does not ever lead to operative apprehension? For each operation, we were able to identify visual variables that trigger or inhibit the visibility of the relevant subfigure and operation within a given figure. And we were able to define the conditions of their influence on operative apprehension. Even the use of key figures in problem solving depends on these visual variables. Therefore, it would be naïve to believe that providing students with key figures would help them in problem solving. At the least change in the starting figure, most students do not recognize the correspondence with the key figure anymore. The visual variables must be taken into account in teaching. Their study opens an important field of research in order to understand the way cognition works for visualization in geometry (Duval, 1995a, pp.148-154; 1998, pp. 41-46).

Operative apprehension is independent of discursive apprehension. Vision does not start from hypotheses and does not follow from mathematical deduction. Otherwise, geometrical figures would not perform a heuristic function but only an illustrative function (Duval, 1999). That is the blind spot of many didactical studies. They do not differentiate between visualization and hypotheses, which depend on two heterogeneous registers of representation, and they subordinate the way of working of visualization to the way of working of deduction or of computation. In fact, shape recognition is independent of shape size and of perimeter magnitude. For example, when hypotheses include numbers as measures of sides or segments, operative apprehension is neutralized and the figure fulfils only an illustrative or support function. We can have even a conflict between the figure and the measures leading to a paradox. The most well known case is the reconfiguration of an 8 x 8 square into a 5 x 13 rectangle, within which a parallelogram is perceptively reduced to a diagonal.

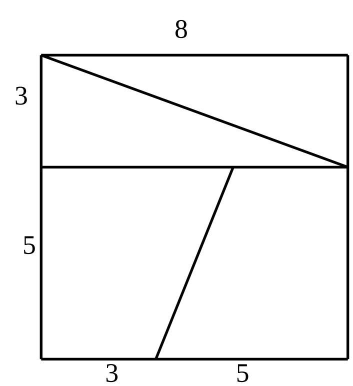


Figure 9.

Visualization consists only of operative apprehension. Measures are a matter of discursive apprehension, and they put an obstacle in the way not only for reasoning but also for visualization. Usually, the introduction of «geometrical figures» runs against this fact. Mathematical tasks are conceived as if the perceptual, discursive and operative apprehensions were inseparable! And the general outcome for most students is the inhibition of operative apprehension and a lack of interplay between perceptual and discursive apprehension.

3. Transitional visualization and development of the coronation of registers of representation

There is an introspective illusion that often distorts the analysis of mathematics learning processes. What is simple from a mathematical point of view appears also simple from a cognitive point of view when we are becoming experts. In fact, more often than not, what is taken as mathematically simple becomes cognitively evident only at the end of learning (Duval, 1998c). That is why assuming these simple-evident conditions cannot be taken as a starting point for learning and teaching. As I said above (II.4), learning mathematics implies the construction of this cognitive architecture that includes several registers of representation and their coordination. Thus geometrical figures used to solve problems involves some ability in operative apprehension and awareness of how deductive reasoning works. Students do not come into such apprehension and awareness by themselves. Moreover, some coordination is required between operative apprehension, discursive apprehension and deductive reasoning. In other words, geometrical activity requires continual shifts between visualization and discourse. In order to achieve such coordination another kind of visualization is required.

The introduction of graphs in proof learning is well known since their use with computer tutor (Anderson *et al.*, 1987). This example is interesting because it shows the hidden cognitive complexity of any visualization. In front of that use we must ask two questions:

- Firstly, what can be visualize from any propositional graph?

- Secondly, what kind of task makes the students able to understand proof by means of visualization?

The answer to the first question seems easy. «Proof graphs» display the whole deductive organization of propositions like a tree structure. But from that, one does not visualize how such organization works. The essential point is not visible on a graph: each connection is only based on the status of the connected propositions, and we have three kinds of deductive status. And in order to be able to become aware of this point, one must succeed at least once in constructing a whole proof graph. That concers to the second question, we find two kinds of task: to construct oneself the whole graph or to find out forward and backward paths from hypotheses to the to-be-proven statement, which are already given at the top and at the bottom of screen.

In Anderson's research, proof graph was used to provide heuristic help «during problem-solving». Hence the second kind of task was chosen. As to what graph is expected to visualize, it is mainly «a hint in the form of suggesting the best nodes from which to infer» (Anderson *et al.*, 1987, p.116). In other words, proof graph must focus attention on the new step to find out in order to progress. This way of using a graph turned out to be disappointing. And it is easy to know why. On the one hand, a graph cannot perform a heuristic function in geometry problem solving: that depends on figural processing. On the other hand, if the goal is to understand how deductive organization of propositions works, the task has the crucial point bypassed. In fact, proof graph becomes a helpful visualization for the students only when they have to construct it by themselves according to rules explaining how to shift the status of propositions into visual values. Then proof graph can visualize **not a particular proof of the to-be-proven statement, but how any proof works** (Duval, 1989, 1991). To understand how a mathematical proof works and why it does not work, as other language reasoning is the necessary condition for being convinced by a mathematical proof. We are there on the crucial threshold of learning in mathematics. Under very specific conditions, proof graph is a kind of visualization that allows one to explore and to check our own understanding of deductive reasoning. Once this threshold is crossed, proof graph becomes useless and interplay can start between deductive reasoning and geometrical figures. Proof graph is a transitional visualization that furthers register coordination.

It may more evident for proving than for any other mathematical activity, that what is mathematically simple is cognitively complex and can be understood only at the end of learning. Heterogeneous ways of working, specific to each register, must be first learnt in parallel. Is it possible to lead frontally all the training that this requires? For experimental reasons, our researches have aimed separately at each register and we have identified some conversion problems. But, recently, an attempt to join all the aspects involved in proof activity has been made within a computational environment (Luengo, 1997). And this attempt seems to be promising.

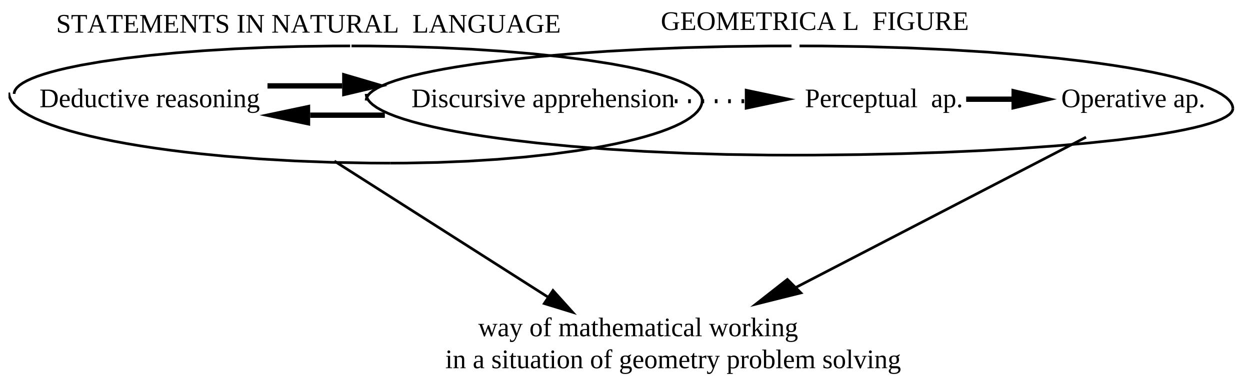


Figure 10. Skills and coordinations to be developed in mathematics education. Most often students confine to perceptual apprehension and reduce discursive apprehension to simple denomination of recognized shapes.

Conclusion

There is no direct access to mathematical objects but only to their representations. We cannot compare any mathematical object to its representations, as we can compare a model with its photo or its drawn image. This comparison remains attached to epistemological patterns to analyze knowledge (Platon, *Res Publica* VI 510 a-e, X 596 a-e), and it cannot be relevant in mathematics and in teaching of mathematics. We can only work on and from semiotic representations, because they are the means of processing. At the same time we have to be able to activate in parallel two or three registers of representations. That determines the three specific requirements in learning of mathematics: to compare similar representations within the same register in order to discriminate relevant values within a mathematical understanding, to convert a representation from a register to another one; and to discriminate the specific way of working in order to understand the mathematical processing that is performed in this register. This is not the familiar way of thinking. And it is the reason why an anchorage in concrete manipulations or in applications to real situations is often pursued in order to make sense of the activity proposed. But that comes often to a sudden end, because it does not provide means of transfer to other contexts. Besides, representation becomes usable in mathematics only when it involves physical things or concrete situations. We find the same problems with visualization use, whatever the register be, it focuses on a synoptic way, organization of particular units and it does not show objects as any iconic representation. One does not look at mathematical visualization as one does at images.

Mathematical activity has two sides. The visible or conclusive side is the one of mathematical objects and valid processes used to solve a given problem. The hidden and crucial side is the one of cognitive operations by which anyone can perform the valid processes and gain access to a mathematical object. Registers of semiotic representation and their coordination set up the cognitive architecture which anyone can perform the cognitive operations underlying mathematical processes. That means that any cognitive operation, such as processing within a register or conversion of representation between two registers, depends on several variables. To find out what these variables are and how they interact is an important field of research for learning mathematics. Indeed, from a mathematical point of view only one side matters, from a didactical point of view the two sides are equally essential. In concrete terms, any task or any problem that the students are asked to solve requires a double analysis, mathematical and cognitive: the cognitive variables must be taken into account in the same way as the mathematical structure for "concept construction" (Duval, 1996). But for that, teachers must know themselves these variables and take them into account as didactical variables. They must be able to analyze the function that each visualization can perform in the context of a determined activity (Duval *et al.*, 1999). We are here in front of an important field of research. But it seems still often neglected because most didactical studies are mainly centered on one side of the mathematical activity, as if mathematical processes were natural and cognitively transparent. There is no true understanding in mathematics for students who do not "incorporate" into their "cognitive architecture" the various registers of semiotic representations used to do mathematics, even those of visualization.

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