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Chapter 18

Radical Constructivism: Reflections and Directions[†]

Patrick W. Thompson

I would like to bring the book full circle, returning to two points raised by Ernst von Glasersfeld in his opening chapter. These are the fact that radical constructivism is misinterpreted so persistently by its critics, and the need for radical constructivism to provide a clear model of social interaction. I return to these points not only to give the book a particular rhetorical structure, but because they penetrate many of the controversies both internal to mathematics and science education and at the boundaries of radical constructivism. At the same time, I'll point out the importance of conceptual analysis in Glasersfeld's method and urge more people to use it in mathematics and science education.

Misinterpretations of Constructivism

Glasersfeld (Cha. 1, p. **) noted that some misinterpretations of radical constructivism may be inevitable by-products of readers' conceptual operations.

> Insofar as these misunderstandings are honest, they seem to be caused by conceptual blinders the traditional epistemology has placed on the readers. As with panicky horses, the blinders shut out perturbing sights and insights.

Statements like this might seem a bit defensive to non-constructivists. Who are constructivists to be speaking of others' blinders? However, if you know Glasersfeld personally, you know his remark emanates from frustration at a persistent lack of communication instead of from his attempts to defend radical constructivism or from arrogance. As he points out, critics of radical constructivism often base their criticisms on assumptions that are not made by radical constructivists.

Lerman's (1994, 1996) criticisms of radical constructivism is a case in point. Lerman criticized radical constructivism for taking an incoherent stance in regard to the idea of intersubjectivity.

> Taking constructivism's view of the autonomy of the individual in the construction of her or his knowing, given her or his particular conceptual system and its particular filter, leads to a consistent, albeit very restricted, view. To argue for an integrated view is to argue that sometimes the filter has very large holes and what is occurring beyond the

> individual can somehow enter without constraint. I argue, then, that it makes no sense to strengthen the functioning of the "social" into a social constructivism. (Lerman, 1996, p. 140)

A close look at Lerman's criticism reveals some of the conceptual blinders Glasersfeld noted. The constructivism he criticizes deserves criticism. But it is not the constructivism I know.

Lerman said	*My comment*
Taking constructivism's view of the autonomy of the individual in the construction of her or his knowing …	Constructivists evidently believe that people have explicit control over what knowledge they end up constructing.
given her or his particular conceptual system and its particular filter, …	Lerman believes that conceptual systems act as filters, meaning that there are "things to filter" and that what is not filtered is registered as such within the person's conceptual system
To argue for an integrated view is to argue that sometimes the filter has very large holes and what is occurring beyond the individual can somehow enter without constraint.	Lerman has set up constructivism for his criticisms by portraying it as taking stances to which few constructivists would agree.

Lerman's understanding of assimilation is very different from mine. In Piaget's genetic epistemology, assimilation is *constitutive* – a scheme's activation *constitutes* the subject's experience of the "things" assimilated to it. It makes no sense when explaining persons' experiences to speak of their schemes acting as filters. It is only when we speak as observers of someone living his or her experiences, when we have identified that part of *our* experience that is not the observed person, that the idea of something outside the person passing through a filter becomes intelligible. However, this account describes *our* conception of what we have observed. It does not describe the observed person's experience. This is an epistemological stance. It is not an ontological stance. As observers of living organisms that have no metaphysical access to (what we see as) their environments, whose "sensing" and "knowing" entail nothing more than the chemical and electrical activity generated by their nervous systems, we understand that all knowledge boils down to a dynamic autoregulation of nervous functioning.[1] As Glasersfeld (Cha. 1) notes, "I consider metaphysical assumptions vacuous as long as they do not specify a functional model of how ontology might determine the experiences from which we generate our knowledge. To say that something exists does not explain how we come to know it" (p. **).

Lerman's criticism reflects what I suspect is a major source of miscommunication between constructivists and non-constructivists. Lerman seems to have in mind a standard observer, one who watches all that happens, and for whom he can speak unproblematically. This image of a standard observer for whom anyone can speak suggests, to those having it, that human experience is *caused* by something penetrating a boundary between environment and individual. If this is the conceptual background within which we frame discussions of human experience, then our discussions of "knowledge" will point to experience of those things that somehow penetrate the imagined boundary. Hence, discussions of "construction" from this perspective will be about experiencers' constructions of those things that penetrate. This, indeed, is an incoherent model of knowing, and anyone who understands constructivism this way *should* question it. One strategy by which constructivists could counter objections that emanate from a "standard observer" frame is to acknowledge the objections' legitimacy, within that frame, and then point out that this is not the model we have in mind.

(Mis)Interpretations Are Sometimes Well-Founded

Not all misinterpretations of constructivism are due to others' conceptual blinders. Some, in fact, originate in honest attempts to apply constructivism directly to mathematics education. For example, Funderstanding, (Funderstanding, 1999), defines constructivism as follows.

> Constructivism is a philosophy of learning founded on the premise that, by reflecting on our experiences, we construct our own understanding of the world we live in. Each of us generates our own "rules" and "mental models," which we use to make sense of our experiences. Learning, therefore, is simply the process of adjusting our mental models to accommodate new experiences.

They state four principles of constructivism: Learning is a search for meaning, learning processes must focus on primary concepts, we must understand students' mental models if we are to teach them well, and the purpose of learning is for each individual to "construct personal meaning instead of regurgitating others' meanings". From this they derive several conclusions, which include:

> Constructivism calls for the elimination of a standardized curriculum. Instead, it promotes using curricula customized to the students' prior knowledge. Also, it emphasizes hands-on problem solving. Under the theory of constructivism, educators focus on making connections between facts and fostering new understanding in students. Instructors tailor their teaching strategies to student responses and encourage students to analyze, interpret, and predict information. Teachers also rely heavily on open-ended questions and promote extensive dialogue among students.
>
> Constructivism calls for the elimination of grades and standardized testing. Instead, assessment becomes part of

> the learning process so that students play a larger role in judging their own progress. (Funderstanding, 1999)

Funderstanding's definition of constructivism and its principles and the conclusions they derive therefrom are not incompatible with the constructivism of Glasersfeld and Piaget. However, they are not implied by it either. First, constructivism is not a philosophy of learning. It is a model of knowing. Piaget's genetic epistemology addresses, to some extent, the learning of scientific concepts broadly, but it does not address learning per se. Theories of learning based on constructivism (Steffe *et al.*, 1983), which claim to characterize knowledge people come to have and ways they might come to have it, might support such conclusions. But constructivism per se does not. Second, as an intellectual position, constructivism does not call for eliminating anything. The authors may have derived their position from other considerations, but tenets of constructivism, by themselves, do not warrant it.

Nevertheless, when non-constructivists see constructivism presented this way they are not positioned to understand that it is a mischaracterization. It would be interesting to know what percent of people's contacts with constructivism are with such ideological portrayals. I suspect the rate is high enough to support a popular understanding of constructivism as being about discovery learning, cooperation, and a ban on lecturing.

One might think that the above applies only to "popularized" constructivism. But even well known constructivists sometimes write as if applying constructivism directly to mathematics education. For example, Dubinsky and Schwingendorf appeal directly to constructivism in justifying their student-centered approaches to instruction and curriculum development.

> The emphasis of the C^4L program [Calculus, Concepts, Computers and Cooperative Learning] is a pedagogical approach based on a constructivist theoretical perspective of how mathematics is learned. According to this emerging theory, students need to construct their own understanding of each mathematical concept. Hence, we believe that the primary role of teaching is not to lecture, explain, or otherwise attempt to "transfer" mathematical knowledge, but to create situations for students that will foster their making the necessary mental constructions. (Dubinsky & Schwingendorf, 1997)

Constructivism, by itself, cannot sanction any particular pedagogical approach. Though Dubinsky and Schwingendorf do not say *not* to lecture, many people interpret them to say so, and it seems clear that they do not value lecturing highly. However, it may be that lecturing is an appropriate pedagogical move when used with learners who are in a position to listen actively to a speaker and engage in a private debate with the points they understand the speaker to make. It could also be that, at a particular moment in a student's conceptualization of a particular idea, it is counterproductive to have him or her remain focused on concrete activity. This is not to say that lecturing is always appropriate or that focus on concrete activity is always inappropriate. Rather, our pedagogical decisions must be sensitive to context and goals and must be informed by theories that tie directly into issues of mathematics learning and teaching. As Simon (Cha. 14) notes, "Whereas constructivism provides a foundation for reconceptualizing mathematics teaching, it does not provide a particular model of mathematics teaching." Our audiences often understand that constructivism does not imply a particular model of teaching, so when they hear statements that it does, they wonder rightfully whether constructivism is just ideology or dogma. It may be, to quote Walt Kelly's Pogo, that "We have met the enemy and he is us."

Non-constructivists rightfully see hypocrisy in someone saying, on one hand, that *all* knowers' knowledge is constructed because it has proven viable within their experience, and on the other hand saying that realism and empiricism are untenable positions. Are realists and empiricists not knowers? Do they, in fact, not know what they claim to know? Perhaps it is productive to qualify our comments on realism or empiricism by saying that they are not useful if, as Glaserfeld (Cha. 1) notes, we require our models of learning to suggest how someone comes to know whatever they know and when we disallow metaphysical accounts of learning.

While I point out the need for constructivists to show signs of humility and to avoid projecting a sense of self-righteousness, I also am compelled to give a warning. It is that some people will employ others' attitudes of openness, humility, and tolerance to the others' disadvantage. Chomsky and Fodor did this in their debate with Piaget on whether language is largely innate or emerges through an interaction between sensori-motor intelligence and the general semiotic function (Piattelli-Palmarini, 1980). Piaget was polite and open to criticisms of his position; they were not. They expressed an attitude similar to, "You are uncertain, we are certain, therefore we are right."

Similar stances have happened in the "math wars". Bishop (1999) gives one of the more blatant admissions that his group's goal is power, not insight. O'Brien (1999) observed that such public posturing can be as much a personality trait as an academic position.

- Parrot math reflects a deep-seated longing to control children through external rewards and punishments, rather than to harness children's urge to make sense of things. We've all seen controlling parents at poolside: "Jonathan, get out of the pool. You're grounded for five minutes." "Gee, Dad, what did I do?" "Out! Now you're grounded for 10 minutes!" (p. 435)

O'Brien (1999) also considered the motives of people advocating procedure- and memorization-oriented approaches to mathematics teaching. He observed that advocates of "basic" mathematics (what he called "parrot math") are interested more in garnering power to decide children's fates and seem interested more in controlling school instruction and curriculum than in children's education.

- [the debate over child-centered approaches to mathematics education] is a useful battering ram for advancing political candidates at the expense of teachers and educational leaders. Ironically, the critics' arguments seem to have a special appeal to parents and editors who themselves learned to hate and fear math because of the same parrot math approaches that are being recommended today.

- Publishers who specialize in back-to-basics approaches stand to gain from the rejection of other methods. For example, in December 1997 California's politically appointed state board rejected middle-of-the-road, "balanced" mathematics standards that had been developed by a state-mandated academic standards commission and replaced them with a set of parrot math standards that were developed under highly controversial circumstances, a process and a result that were greeted by nationwide protests from mathematicians, math educators, teachers, and parents. In papers shared via the Internet, some from as far away as Hungary, [ostensive] leaders in math education reached the consensus view that the new standards were "a disaster." Several months later,
- California Gov. Pete Wilson called for some $250 million to be earmarked for new math textbooks that would be tied to the standards. (O'Brien, 1999, p. 435)

To put it sharply, O'Brien illustrates the practical knowledge that one cannot argue from a position of tolerance and openness with someone who is bent on achieving his own agenda by hanging you with your own words.

The Role of Radical Constructivism in Theories of Mathematics Education

Radical constructivism can be thought of as a background theory, or, as Noddings (1991) said, a post-epistemological stance. Background theories cannot be used to explain phenomena or to prescribe particular actions. Rather, their function is to constrain the types of explanations we give and to frame our descriptions of what needs explaining. The primary importance, to me, of radical constructivism is that it provides a continual reminder that humans are biological organisms whose only way to exert mutual influence, aside from physical harm or pleasure, is through mutual interpretation. This constraint on how we think of human knowing is always present – regardless of where we cast our net for theoretical problems. It is present when we consider how individual students come to understand a mathematical idea and it is present when we consider how newcomers are initiated to what we take as cultural practices.

As a background theory, constructivism orients us to formulating descriptions, problems, explanations, and theories in specific ways. First, constructivism orients us to matters of what people know and how they might have come to know it.[2]. It forces us to speak in the active voice. The passive voice is a strong ally of people speaking for the universal observer. By speaking in the passive voice, by writing subject-less sentences, a writer can proffer universal truths when actually expressing one viewpoint. However, within a constructivist framework, knowledge is knowledge because someone knows it, and when writers speak in subject-less sentences they should be criticized for being vague. You should insist that they say for whom they speak – for themselves, their subjects, or, perhaps, for a universal observer. To be specific, statements like

> Signs are incomplete, fundamentally context-dependent and possess imminently multiple meanings. Context, or what is brought to the communicational situation, inumbrates the sign, and is shaped by equivocality and ambiguity in messages. (Cullum-Swan & Manning, 1995)

are very problematic from a constructivist view. Within constructivism, it makes no sense for anyone to talk about a sign possessing a meaning if he or she intends anything other than that people customarily bring to mind a particular meaning when observing what the speaker thinks is a sign. It also makes no sense, within constructivism, to speak of context being shaped by properties of messages if he or she intends anything other than that how a person contextualizes some event is influenced by the meanings he or she attributes to what a speaker has identified as worthy of discussion.[3]

Second, constructivism orients us to look for explanations and descriptions of interaction that are grounded in some equivalent of information theory (MacKay, 1951, 1955, 1964) and symbolic interactionism. More accurately, it orients us toward looking at social interaction as would someone from the school of symbolic interactionism that highlights the interplay of pragmatics and semantics instead of reifying symbols and interactions.

In summary, to pronounce constructivism as a background theory is not to announce a commitment to a particular theory of learning or pedagogy. Instead, it is to announce a set of commitments and constraints on the kinds of explanations we may accept and on the ways we frame problems and phenomena. A commitment to constructivism may have ramifications for the instructional actions we anticipate will be effective regarding students with certain characteristics coming to have particular understandings in the context of certain environments, but it does not, of itself, prescribe or exclude any particular action as being possibly effective.

A Conundrum

After having already said that constructivism cannot be applied prescriptively, I must confess that constructivism *does* seem to compel openness to alternative conceptualizations and tolerance of competing perspectives. It says that we cannot claim that any one perspective is "right" – in the sense that all others are somehow "wrong." Thus, on the one hand constructivism seems to suggest a pedagogy based on openness, empathy, and tolerance. On the other hand, it suggests that even this suggestion cannot be proffered or accepted dogmatically. These are ethical issues, in a sense, because they address matters of "right" behavior in specific contexts and a system of beliefs that is more acceptable than are others. On the one hand, constructivism implies a correct behavior. On the other hand, it cannot do this. This is the issue Larochelle (Cha. 5) raises when she warns against the siren song of ethics. "If constructivism is indeed a reflexive theory (i.e., one that practices what it preaches) it cannot then present itself as a meta-perspective dictating what ethics should be without simultaneously running the risk of lapsing into the very thing it denounces" (p. **). That is, it seems, on the surface, counter to constructivism that Lewin (Cha. 4) urges that we develop a constructivist ethics.

In addressing this conundrum I have found it useful to think of constructivism operating at different levels for different types of observers. The first is as if we are Martians, orbiting the earth, watching animals called humans scurrying about and apparently interacting by mutual influence and adaptation. We do not attribute any special sensory powers to them except that they each seem to send signals into and register signals from (what we see as) their environments. They also seem predisposed to attribute the same sensory and interpretive powers they experience to what we see as things in their environment, to the extent that this attribution proves viable within their experience. As Martians, it never occurs to us that any one or group of humans has "correct" knowledge of their environment. It is a non-issue. It seems evident that none of them does. We do notice, however, that patterns of interaction and interpretation seem to stabilize over time for some groups, and the groups in fact seem to be defined reflexively by these patterns. That is, it seems that patterns of "right" interaction emerge among various groups as members of them engage over time in mutual adaptation. But, as Martians, the idea that any group develops *the* correct pattern of interaction (i.e., ethics) never occurs to us. Rather, we conclude that ethical issues emerge within groups as patterns of interaction stabilize, and we observe that there are a great variety of stable patterns of interaction.

A second type of observer is one who empathizes with the observed. The earth-orbiter is an earthling like us, who can imagine being a member of the groups being observed. The observer can imagine experiencing what the observed experiences because of having been a member of one or several groups. This observer's observations are similar to the Martian's, but are colored by the projection of personal experience into the process of interpreting the interactions. That is, this observer also sees that there is no "right" pattern of interactions, but instead sees that any stable pattern provides its own constraints on the kinds of interactions that will allow the pattern to persist.[4] But this observer also feels a sense of identification with one of the groups. As such, he or she must act according to the constraints that come with having a set of values that support his or her participation in that particular pattern of interaction. However, this observer must also act according to the added constraint that comes with the knowledge that these constraints are not the only possible ones.

A model of ethics consistent with constructivism would explain, in principle, how *any* ethics comes about. However, our model of ethics would also have to allow the possibility of competing, incompatible ethics. It is a fact that the ethics of New Guinea headhunters differs from the ethics of Polynesian traders. As neutral observers interested in the emergence of ethical systems, our model of ethics must encompass both.

In short, constructivism, as an epistemological or post-epistemological stance, predicts that ethical considerations will emerge as groups of people exert mutual influence among themselves and come to value certain effects of having interacted in particular ways. It neither predicts nor dictates what any group's ethics will or should be. However, *if* one adopts a constructivist stance, *and* one considers ethical systems *consistent with having adopted a constructivist stance*, then these ethical systems will require its adherents to be open to competing viewpoints, tolerant of differences, and tentative in the claims they make. This is not to say that adherents of constructivism never may act as if their knowledge is certain. Rather, it says they must be willing to reflect on their certainties – they must be ready to call into question anything they know.

This seems to resolve the conundrum. Constructivism does not dictate that anyone have a particular ethics. Rather, it predicts that ethical considerations will emerge as humans interact and form expectations about others' actions. At the same time, constructivism does impose constraints on anyone who consciously adopts it and who realizes, as Larochelle (Cha. 5) puts it, that people who adopt constructivism must practice what they preach.

It is important that adherents not convey constructivism as dogma. The fact that many people do think of it as dogma is, I believe, a result of having heard others who claim to be constructivists but at the same time seem to argue that constructivism is correct (and that other viewpoints are incorrect). Glasersfeld never implies that constructivism is correct. Rather, he says only that constructivism is *useful*. At the risk of sounding amoral, I think we are wise to remove ethics from our educational decision making and replace it with matters of practical reason. For example, it is commonplace for educators to state that students should feel a sense of responsibility for their own learning and decision making. How might we justify this to teaches and parents?[5] On the one hand, we could make it an ethical argument about why it is the right thing to do. On the other hand, we could make it a practical issue by comparing the ramifications of students by and large *not* doing this with the ramifications of them doing it routinely.

In closing, I should point out that being open to competing viewpoints does not mean that all viewpoints are necessarily equal. If, in a particular area of inquiry, one model of knowing solves the problems a second model solves, and also solves problems the second problem does not solve, then the first model is more useful than the second is. It is in this sense that I believe constructivism is more useful in education than is realism or empiricism. It supports insight into miscommunication as well as successful communication, whereas the others only support accounts of successful communication.

A Disinterested View of Miscommunication

We must wonder about the extent to which it is unreasonable for us to expect others to understand questions of knowing in the sense that radical constructivists understand issues of knowing. After all, from our point of view, their knowledge that knowledge is a direct copy of reality must be viable in their experience – it must somehow work for them. My questions, then, are "What is the work that a representational view of mind does for the person having it?" "What is the work that a radical constructivist view of knowledge does for the person having it?" I suspect that in answering these questions we might find ways to communicate more effectively across world views.[6]

A representational view of mind, the understanding that the representations people form of their worlds somehow mirror reality, seems to work for those holding it in the same way that Platonism works for practicing mathematicians and scientists. Platonism allows mathematicians and scientists to deal with their mental worlds with the same metaphors as when dealing with the physical world. They can take their constructs as objects of study unproblematically, so that their understandings of those constructs can be refined and tested in much the same way that they refine and test their understandings of physical phenomena. They can deal with the task at hand – "discovering" connections among ideas – without having to constantly question the existence of the ideas they are connecting. They can believe that there *is* a solution to their problem if only they are clever enough to discover it.

A representational view of mind in educational research is like Platonism in mathematics and science. It relieves researchers of the need to take into account that subjects' perspectives on events of interest may not coincide at all with theirs. I am reminded of a public conversation I had in 1990 with a psychologist who is well known for his stance that all cognition is situated. The group was discussing young children's surprising answers to questions in another researcher's study of their thinking about ratios. I observed that it would be very helpful if we knew what questions the children had answered. The psychologist's response was immediate – the children had answered the questions that were asked. To him, the question was what the researcher said, or what was written on paper. To me, the question varied among children; the question any child answered was what he or she formulated *from* what the researcher said. In other words, what was unproblematic to the psychologist (the question children answered was the question the researcher asked) was problematic to me (the question children answered was what they formulated from what the researcher asked, and their surprising answers suggested their formulations were quite different from ours). Though this psychologist was convinced that all cognition is situated, it was he, in that case, who determined what the situation was in which the children's cognition happened. It is in this way that a representational view of mind simplifies theoretical life – researchers do not have to wonder about the experience of their studies' subjects. From my point of view, this omits from study the very thing that we hope to influence – our students' mathematical and scientific experiences.

The work that radical constructivism does for me is that it serves as a constant reminder that I must question my interpretations of how others understand what I take as a common setting. This is important to me because of what I do. I try to affect others' understandings of situations in ways that support the emergence of mathematical reasoning. I sometimes do this at close proximity to others (e.g., as a teacher or tutor) and I sometimes do this at great distance (e.g., as a policy maker or instructional designer). At close proximity, I try to think of ways my students might be thinking and adjust my actions accordingly. At greater distances I imagine groups of people interacting and try to think about ways in which we might intrude on those interactions to perturb them in productive ways. But at all times I am reminded that my point of view is not privileged, and I must constantly keep in mind that other people will act and interact in consonance with the inertia of their personal histories as solidified in their current conceptual operations. So, constructivism works for me. It does not guarantee success, but it reminds me that there are potential problems to avoid.

There are times when constructivists must suspend their constructivism. When designing instruction or working with students, at some point we must *act with certainty* that what we suspect is the case is actually the case. We must *act with certainty* that students actually understand an idea the way that we suspect they do; we must *act with certainty* that they interpret a diagram or a computer screen in the way we have predicted. In order to *act*, we must believe, to some extent, that we are right and the students are as they see them.[7]

In acting with certainty, we become actors rather than observers. By an actor, I do not mean a theatrical performer. Rather, I mean that one becomes fused with the situation as he or she has constituted it (MacKay, 1969). Constructivists must function, at their moment of acting, as actors who take their experiential and conceptual worlds as real, just as any living organism becomes one with their constructed reality. To remain constructivists, they must interrupt their experiential realism to consider the consequences of their presumed certainty. Here I should point out once again the reflexivity of constructivism. Theoreticians, including us, are among those beings to which constructivism applies, so theoreticians' knowledge, too, is a functional adaptation to experience.

We should take care not to convey the impression that we want mathematics and science students to become epistemologists. We want them to construct a coherent and powerful mathematics, but we want that mathematics to be real for them. We want them to *see* structure, to *see* relationships, to *see* quantity. It is unnecessary to consider epistemological questions while doing mathematics or science. Constructivism is better suited for addressing questions of what people might *know* as they have a mathematical or scientific idea, what they might know as they participate in a mathematical or scientific discussion, or what they might know as their thinking crystallizes while solving a complex problem. However, it is imperative that mathematics and science educators be aware of the epistemological claims inherent in any stance they take. It is imperative that mathematics and science teachers be sensitive to conceptions of situations that differ fundamentally from theirs, as if being seen from a completely different world view.

In using constructivism we sometimes are pushed to act as if we are non-constructivists. To communicate with students we must at times say things that point to a reality of ideas that we would, if speaking as epistemologists, deny. For example, I was working with high school math students on the idea of sampling distributions. We were discussing opinion polling, and they were having difficulty distinguishing between the ideas of population parameter and sample statistic, and I began to suspect that their main problem was that they were unable to conceive a population parameter. I found myself saying this:

> Suppose we were like Mork[8] and could stop time for everyone but ourselves. Imagine freezing everyone in our target population. At that moment, each person in the population has an answer (yes, no, or no opinion) to the question we will ask, even if we happen not to ask him or her the question. So, the population as a whole, at that moment in time, has a percent of it who *would* say "yes" to our question were they to be asked.

In other words, in order to talk about population parameters, students needed to think of populations as having characteristics whose measures have specific values at each moment in time. This is not to say that this example's population *really* had a characteristic whose measure had specific values at each moment in time. For the purpose of building a concept of sampling distribution, it is merely useful to think that it does. However, this was my realization – that it was merely *useful* to think of a population having a particular measurable characteristic. Students needed to *believe* that populations *can* have measurable characteristics, or else they would have been unable to conceive of sampling distributions as arising from repeatedly drawing samples of a given size from that population. They also would have been unable to consider how the set of sample statistics clusters around the population parameter. To coordinate all these aspects of sampling distributions, population parameters needed to be real to them.

The preceding example illustrated that the principles of constructivism, by themselves, do not provide guidance for acting in specific contexts – that one cannot simply turn responsibility to understand important ideas over to students. It does illustrate how one can employ constructivism in conjunction with an image of mathematical understanding to produce a model of knowing that *is* pedagogically useful.

I find distinctions between first- and second-order observers and between first- and second-order models of knowing (Steffe, 1995; Steffe et al., 1983; Steffe & Thompson, 2000) very useful in explicating conditions that enable people to employ constructivism in pedagogically powerful ways. To understand these distinctions, one must first understand Maturana's famous position that "Everything said is said by an observer" (Maturana, 1987) who in fact may be the observed. Maturana meant that any description of affairs must be done at a level of monitoring that is above what is being described. We can think of this monitoring happening in different ways. The person saying, "I feel great" is not just feeling great. She is also monitoring how she feels. The researcher who says about children with whom she interacts, "It seems they think of fractions as an additive part-whole relationship" is not just acting from an image of how students think. She is monitoring how she understands their thinking, and she has connected it with ways of thinking about whole numbers. It is also helpful to keep in mind that when Steffe et al. distinguished between levels of observers and levels of models they imagined *themselves* observing someone else in interaction with a third person (e.g., a student).

Observers act within the reality of their experience. As actors, they impute significance reflectively or unreflectively to aspects of what they take to be a situation of interaction. If they are interacting with another person, they are immediately aware of many aspects of that person. If they interact unreflectively, they are essentially actors in interaction. If they act reflectively as they interact with the other person, they are acting as an observer. To reflect means that one "steps out of the stream of direct experience, to re-present chunks of it, and to look at it as though it were direct experience, while remaining aware of the fact that it is not"(Glasersfeld, 1991, p. 47). If teachers do not reflect on aspects of an interaction that are contributed by students, if they are fused with the situation as they have constituted it, then they are actors in the interaction, not observers. To be an observer of another while involved in interaction one necessarily moves to a level of reflection. When this happens it adds a dimension to the social interaction – the observer is now acting purposefully and thoughtfully, *using* the social interaction instrumentally. Reflection also opens the possibility that the teacher or researcher learns a mathematics consistent with the mathematical knowledge of those with whom he or she interacts.

The distinction between a first- and second-order observer is that a first-order observer addresses what someone understands, while a second-order observer addresses what *they* understand about what the other person *could* understand. As a first-order observer, a teacher would be oriented to understand that students might think differently than he or she and be oriented to formulate a particular image of the students' current thinking. As a second-order observer, the teacher would be oriented to also think about ramifications of positing alternative ways of thinking that might prove more profitable were students to think in those ways[9]. But the teacher could not draw directly on constructivism's principles to generate either those images of the students' current ways of thinking or images of alternative ways of thinking. Rather, those images would emerge by teachers conjoining a theory of mathematical understanding, which addresses explicitly the composition of individuals' understandings, and constructivism, which constrains and orients the descriptions they could formulate.

The ideas of first- and second-order models are even more important in their implications for mathematics education than the ideas of first- and second-order observer. First-order models are "models the observed subject constructs to order, comprehend, and control his or her experience (i.e., the subject's knowledge)" (Steffe et al., 1983, p. xvi). Second-order models are "models observers may construct of the subject's knowledge in order to explain their observations (i.e., their experience) of the subject's states and activities" (Steffe et al., 1983, p. xvi).

The distinctions between first- and second-order models and first-and second-order observers point to reasons why constructivism is so easily misused in teaching, curricula development, and research. As necessary as they are, second-order models of knowing made by a first-order observer provide only weak guidance to a teacher, developer, or researcher. The only thing they can draw from them is that what students end up knowing comes out of the sense they make of teaching and bears no necessary relationship with what the teacher, developer, or researcher intended. Second-order models of knowing made by a second-order observer, however, can provide strong guidance for the teacher, developer, or researcher who have developed a vocabulary and system of constructs to describe students' conceptual schemes together with transformations, reorganization, or other modifications in them. Thus, we should not be overly optimistic about the ease with which constructivism can be used appropriately in teaching or research. I shall discuss this again in the section on conceptual analysis.

The elusive social dimension

Another perceived shortcoming of research conducted within a radical constructivist perspective is that it ignores the social dimension of human cognition. However, it is worth noting that von Glasersfeld's elaboration of Piaget's genetic epistemology into what he eventually called radical constructivism (Glasersfeld, 1978) grew in large part out of his keen interest in understanding human communication and language (Glasersfeld, 1970, 1975, 1977, 1990; von Foerster, 1979). So, from its very beginning, the core problems of radical constructivism entailed the question of how physically-disconnected self-regulating organisms could influence each other to end in a state where each presumes there is essential common agreement on what is their shared environment (Maturana, 1978; Richards, 1991). That is, from its very outset an image of cognizing individual's in social contexts was central to radical constructivism. This is one reason for its early appeal within mathematics education (Steffe & Kieren, 1994).

There is an essential difference between radical constructivism and social constructivism. It is that the former takes social interaction as a phenomenon needing explanation, whereas the latter takes it as a constitutive element of human activity. This difference expresses itself most vividly in the types of explanations coming from radical constructivists and social constructivists. The former tend to focus on human discourse as emanating from interactions among self-organizing, autonomous individuals. The latter tend to focus on the collective activity in which individuals participate. That is, from a radical constructivist perspective, what we take as collective activity is constituted by interactions among individuals each having schemes by which they generate their activity and by which they make sense of other's actions. From a social constructivist perspective, collective activity and social interaction are given, predating any individual's participation in it. The individual accommodates to social meaning and practice.

Characteristics of group activity, from a radical constructivist perspective, emerge by way of individuals' mutual interpretation of what each perceives as other-oriented action. Collective activity is constituted by these interactions. If we also assume that individuals reflect on their actions, then it follows that each individual's participation changes as she becomes aware of, elaborates, and interiorizes[10] her activity and her understanding of its repercussions. A radical constructivist perspective on the constitution of collective activity is similar to points of view originating in complexity theory and chaos (Mainzer, 1994; Sandefur, 1993). In complexity theory, the intent is to model complex phenomena by attempting to identify elementary process that, through large numbers of interactions over sufficient amounts of time, regenerate the phenomena. The elementary processes, from a radical constructivist perspective, to account for collective activity are intersubjective operations within individuals[11] and interactions among overlapping groups of individuals. This, combined with the facts that people have memories and use them, and that interactions often produce artifacts that people use both informationally and practically, engender social activity from a complexity theory point of view.

The elementary processes of social interaction in radical constructivism are similar in many respects to ideas from symbolic interactionism (Cobb *et al.*, 1996; Miller *et al.*, 1997; Prus, 1996), at least that part that treats the fact of communication problematically. From this symbolic interactionist perspective, people do not communicate meanings *per se*. Rather, the phenomenon of one person "communicating a meaning" to another is accomplished by listeners attributing meanings to the utterances they hear that are compatible with their own understandings and that are compatible with the image they've built of, or impute to, a speaker (see Figure 1).

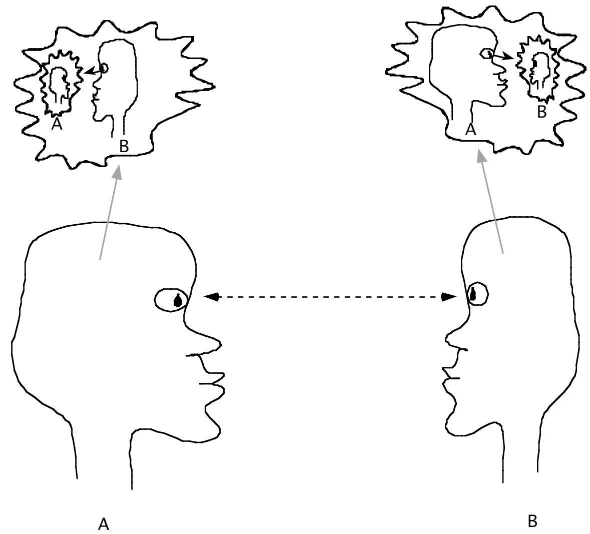

Figure 1. A symbolic interactionist view of one moment in human communication

Figure 1 portrays a symbolic interactionist understanding of two people interacting socially. We see the two of them interacting, but we understand both to be predicating their actions on an image they have of the other and implications that image has for their actions. Each person acts self-reflexively, and yet acts with consideration of others, much as Baursfeld (1980) has described. Figure 1 makes this point a bit too strongly, in that it portrays interactors as thinking this all through carefully, when that is not my intention at all. Instead, I mean that each person imagines and acts, sometimes reflectively but always through imagery built from past interactions. This is precisely the model of communication that allowed us to locate and describe the reasons for severely dysfunctional communication between a teacher and his student (P. W. Thompson & Thompson, 1994) and to hypothesize a principled course of action that successfully remediated that miscommunication (A. G. Thompson & Thompson, 1996).

As Glasersfeld (1995) notes, to say two people communicate successfully means no more than that they have arrived at a point where their mutual interpretations, each expressed in action interpretable by the other, are compatible – they work for the time being. Intersubjectivity is the state where each participant in a socially-ongoing interaction feels assured that others involved in the interaction think pretty much as he or she anticipates they do. That is, intersubjectivity is *not* a claim of identical thinking. Rather, it is a claim that no one sees a reason to believe others think differently than he or she presumes they do.

The importance of these considerations for educational research from a radical constructivist perspective is threefold. First, they point to some of the tacit assumptions behind the observation that individual's cognitions are, at once, psychological and social (Cobb, 1994, Ch. 11). Individual cognitions are simultaneously psychological and social *in the eyes of an observer who sees cognition happening in typical settings over prolonged periods*. From an observer's perspective, cognitions become *other-oriented*.[12]

Second, the processes by which we imagine conceptual development happening will be reflected in the theories we develop. If we imagine children becoming more mathematically developed by internalizing collective activity in which they participate ever more centrally (Forman, 1996; Lave, 1991), then that is what we will try to describe, explain, and affect. If we see children becoming more scientifically literate by participating in discussions in which participants contribute and takes away what is consistent with their individual schemes, then that is what we will try to describe, explain, and affect.

Third, if educational research from a constructivist perspective is to carry weight with interpreters of it, then methodologies must be grounded in an epistemology that is compatible with the notion of intervention. "Teaching" as the attempt to affect what children know is an oxymoron without a coherent base for thinking about intervention. Any epistemology that is solipsist – in the sense that personal reality can be made without constraints and can be conjured independently of surroundings – cannot support the idea of intervention. While radical constructivism does not entail ontology, it does not deny a reality. It says only that it is essentially unknowable in any way that can be labeled "correct".[13] People can affect others' cognitions intentionally by placing themselves in a position to be interpreted by the persons they intend to affect. An "informed" intervention is one in which the interventionist is guided by a model of those whom he wishes to affect. The effect may be circuitous, in that they (as teachers) might start by saying something they know will be interpreted by students in ways that differ predictably from what they intend students understand, but which will provide springboards for moving the discussions in directions not possible without the initial (mis)interpretations. This is the model of teaching in radical constructivism – informed interventionists (i.e., folks with models of what they hope learners will learn) place themselves in positions to be interpreted in ways they intend by the persons they wish to affect.

Glasersfeld's method of conceptual analysis

Most discussions of radical constructivism are about what it is and what it implies. However, radical constructivism is not just a set of principles. It is also an instrument for thinking about knowing. As such, when *doing* radical constructivism – as an earthling educator, not as a Martian observer – it is useful to describe ways of knowing that operationalize what it is students might understand when they know a particular idea in various ways. Glasersfeld (1995) calls his method for doing this *conceptual analysis*. As Steffe (1996) notes, the main goal of conceptual analysis is to propose answers to this question: "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (Glasersfeld, 1995, p. 78).

Glasersfeld first introduced me to conceptual analysis when he wondered how to convey the concept of triangle to a person who is congenitally blind and does not know the word already. His example went like this (if you are sighted, close your eyes)

Imagine that you:

- are in some location, facing in some direction;
- walk, straight, for some distance.
- Stop. Turn some amount.
- walk straight for another distance.

- Stop. Turn to face your starting position.
- Walk straight to it.

  Your path is a triangle.[14] (Glasersfeld & Czerny, 1979)

Glasersfeld employed conceptual analysis in two ways. The first was to generate models of knowing that help us think about how others might know particular ideas. Glasersfeld's meaning of model is very much like Maturana's (1978) notion of scientific explanation.

> As scientists, we want to provide explanations for the phenomena we observe. That is, we want to propose conceptual or concrete systems that can be deemed intentionally isomorphic to the systems that generate the observed phenomena. (p. 29)

Glasersfeld's operationalization of "triangle" was more than a way to define it to a blind person. It was also an attempt to develop one hypothesis about the operational aspects of imagining a triangle. I find this approach especially powerful for research on mathematics learning. For example, in research on students' emerging concepts of rate it has been extremely useful to think of students' early understanding of speed as, to them, speed is a distance and time is a ratio (P. W. Thompson, 1994; P. W. Thompson & Thompson, 1992, 1994). That is, speed is a distance you must travel to endure one time unit; the time required to travel some distance at some speed is the number of speed-lengths that compose that distance. Upper-elementary school children bound to this way of thinking about speed will often use division to determine how much time it will take to travel a given distance at a given speed, but use guess-and-test to determine the speed required to travel a given distance in a given amount of time. Their employment of guess-and-test is not a change of strategy. Rather, it is an attempt to assimilate the new situation into their way of thinking about speed – that it is a distance. Guess-and-test is their search for a speed-length that will produce the desired amount of time when the given distance is actually traveled.

There is a second way to employ Glaserseld's method of conceptual analysis. It is to devise ways of understanding an idea that, if students had them, might be propitious for building more powerful ways to deal mathematically with their environments than they would build otherwise. Steffe and Tzur (Steffe, 1993; Tzur, 1999) have employed this use of conceptual analysis to guide their instruction in teaching experiments on rational numbers of arithmetic. Confrey and her colleagues have employed conceptual analysis in similar ways to convey how one might think about multiplication so that it will simultaneously support thinking about exponential growth (Confrey, 1994; Confrey & Smith, 1994, 1995). Thompson (1998) employed conceptual analysis to show how a person's understandings of multiplication, division, measurement, and fraction could each be expressions of a core scheme of conceptual operations, all entailed by multiplicative reasoning. As Steffe (1996) has noted, conceptual analysis (the conjoining of radical constructivism as an epistemology and a theory of mathematical understanding) emphasizes the positive aspect of radical constructivism – that knowledge persists because it has proved viable in the experience of the knower. Knowledge persists because it *works*.

Conceptual analysis can also provide a technique for making operational hypotheses about why students have difficulties understanding specific situations as presented in specific ways. For example, standard fractions instruction often proposes fractions as “so many out of so many” (e.g., 3/5 of 10 apples is “three parts out of five equal-sized parts of the 10 apples”. When students understand fractions, in principle, as “so many out of so many”, they understand fractions as an additive part-whole relationship. Fractional relationships like “7/5 of 10” apples make no sense whatsoever to students who understand fractions additively, because they would have to understand it as specifying “seven parts out of five equally-sized parts of 10 apples”.[15] Another example of the utility of conceptual analysis: It has been known for a long time that children have greater difficulty understanding quantitative comparisons specified as “less” than comparisons specified as “more”. Conceptual analysis can reveal possible reasons for it, and specify them in ways understandable in terms of the conceptual operations involved in understanding actual statements.

Figure 2 depicts an unfolding image of person X comprehending the statement, “Joe has $15 more than Fred.” It shows X as first understanding that this story is about Joe having $15, then the phrase “more than” brings to mind that the $15 they originally imagined as being what Joe had is actually an excess in comparison to some other amount of money (which happens to belong to someone named Fred). Figure 3, on the other hand, shows an unfolding image of person X comprehending the statement “Fred has $15 less than Joe.” It shows X as first understanding that Fred has $15, as before, but then person X had to backtrack when he realized that the $15 he imagined is *not* in Fred’s possession. It actually belongs in another amount of money, and is the excess of that amount in comparison to whatever amount of money Fred has. Also, the fact that Fred actually has an amount of money must be imagined independently of the text, and in opposition to the realization that he did not have the $15 person X originally imagined him having.

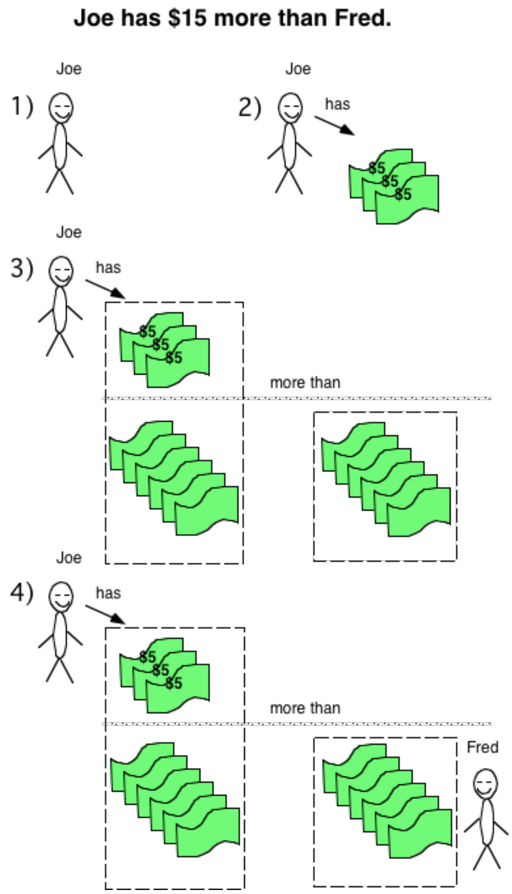

Figure 2

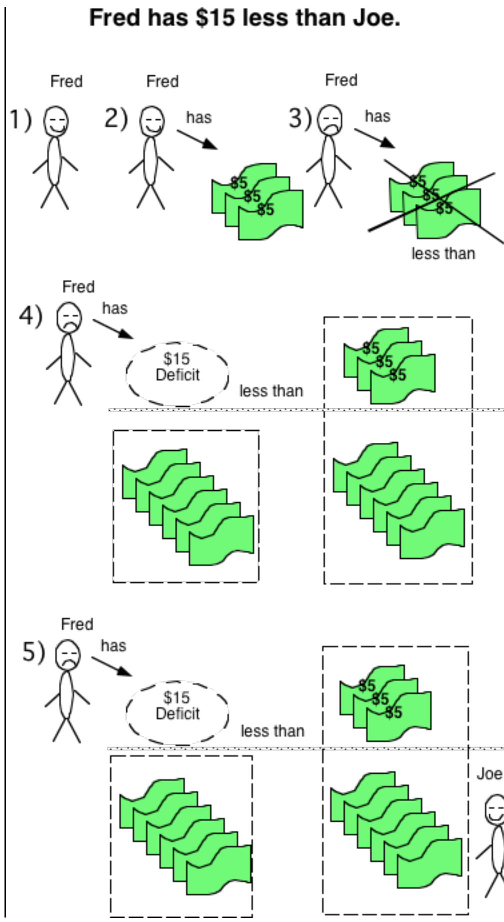

Figure 3

In summary, conceptual analysis can be used in three ways:

(1) in building models of what students actually know at some specific time and what they comprehend in specific situations,

(2) in describing ways of knowing that might be propitious for students' mathematical empowerment, and

(3) in describing ways of knowing that might be deleterious to students' understanding of important ideas or their understandings that might be problematic in specific situations.

I find that conceptual analysis, as exemplified here and practiced by Glasersfeld, provides mathematics educators an extremely powerful tool. It orients us to providing imagistically-grounded descriptions of mathematical cognition that capture the dynamic aspects of knowing and comprehending without committing us to the epistemological quagmire that comes with low-level information processing models of cognition (Cobb, 1987; P. W. Thompson, 1989). Conceptual analysis provides a technique for making concrete examples, potentially understandable by teachers, of the learning trajectories that Simon (1995) calls for in his re-conceptualization of teaching from a constructivist perspective, and which Cobb (Cha. 11) and his colleagues (Gravemeijer, 1994;

Gravemeijer *et al.*, 2000) employ in their studies of emerging classroom mathematical practices. In addition, when conceptual analysis is employed by a teacher who is skilled at it, we obtain important examples of how mathematically substantive, conceptually-grounded conversations can be held with students (Bowers & Nickerson, in press). Teachers in the U. S. rarely experience these kinds of conversations, and hence they have no personal image of them. Having positive examples of such conversations will be very important for mathematics teacher education.

Coming full circle

Ernst von Glasersfeld opened his chapter by thanking the conference organizers and the people who wrote chapters for this book, noting the large debt he owed them for contributing to his point of view. I believe I speak for all those who were present at the conference, and for the mathematics and science education communities at large, when I say to Ernst, "Thank *you* for contributing so much to *our* points of view."

Allow me to end on a personal note. I have grown immensely through the inspiration of Ernst's work. However, I have grown even more by having had the pleasure and honor of seeing Ernst make radical constructivism real through his living example.

[†] I thank Les Steffe for his helpful comments on earlier drafts of this chapter. Preparation of this chapter was supported by National Science Foundation Grant No. REC-9811879. Any conclusions or recommendations stated here are those of the author and do not necessarily reflect official positions of NSF.

[1] It is through Glasersfeld's and von Foerster's contributions that we are aware that such statements must apply to the observer as well as the observed. That is, you must read every passage in this book as the authors' best attempts at finding coherence in their respective physical, intellectual, and social experiences, even as they propose theories to account for others' social, intellectual, and physical experiences.

[2] I am using "know" synonymously with "understand" and "comprehend". So, "to know a concept" does not point to a binary relationship between a person's knowledge and a concept that they have knowledge of. Rather, "to know a concept", in my usage, is slightly redundant. It would be better expressed as, "What this person knows can be called a concept."

[3] This itself may be problematic if the observed person does not make distinctions compatible with the speaker's distinctions

[4] We must be careful to remember that patterns are stable in our observation because of the interplay of interpretations made by actors within the group, so "provides its own constraints" does not imbue properties to the pattern, rather it speaks about the interpretations people make in interacting.

[5] Assuming we can explain what we mean by "take responsibility for his or her learning" well enough to distinguish between when a student is doing it and when not.

[6] See Bickard (Bickhard, 1991) and Cobb (Cobb *et al.*, 1992), and Steffe (1991) for discussions of the deep issues involved in thinking of the mind as somehow repesenting reality and how constructivism offers a non-representational view of mind.

[7] This was my point in (P. W. Thompson, 1995) when I spoke of the need to act as a realist or a trivial constructivist when engaging in mathematics education research. In order to act, we must presume that what we think is the case actually is the case, and then after having acted, and sometimes while acting,

we again put on the hat of the radical constructivist.

[8] Of Mork and Mindy, a television program of the 1970's about an alien living on earth, starring Robin Williams. I know from other conversations that many of them had watched this program.

[9] These alternative ways of thinking must be such that they are in the range of possibility for students as modifications of their current ways of thinking.

[10] I use *interiorize* in a Piagetian sense. For an individual to intereriorize actions means that they construct schemes of mental operations they may carry out in thought and by which they may anticipate outcomes of particular courses of action.

[11] I address a constructivist meaning of intersubjectivity in later paragraphs.

[12] This, in essence, was Piaget's opinion when he emphasized the importance of socialization for the development of semiotic and formal operations (Piaget, 1950, 1977, 1995).

[13] This is not to say that there *cannot* be a "correct" knowledge of reality. Even if there were such a thing, however, we could not know whether anyone has it.

[14] For readers who recall Logo, the similarity between Glasersfeld's operationalization of a triangle and a turtle-procedure for drawing one is striking. However, his example predates the general availibility of Logo, and neither of us had heard of it anyway.

[15] This is not to say students couldn't give *some* meaning to "7/5 of 10 apples". Children often change the meaning from the intended fractional relationship to a procedure for determing some number of apples, such as "5/5 is 10, 2/5 is 4, so the answer is 14." But that is not the same as understanding 7/5 of 10 as the multiplicative structure that indicates a number that is 7 times as large as 1/5 of 10, where 1/5 of 10 refers to a number to which 10 is 5 times as large. This latter understanding of 7/5 of 10 generalizes easily to 7/5 of 9 apples. The former reasoning does not generalize so easily.

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