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Interaction or Intersubjectivity?

A Reply to Lerman[1, 2]

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Lerman, challenged constructivism by presenting Vygotsky as an irreconcilable opponent to Piaget's genetic epistemology and thus to von Glasersfeld's radical constructivism. We argue that Lerman's stance does not reflect von Glasersfeld's opinion of Vygotsky's work, nor does it reflect Vygotsky's opinion of Piaget's work. We question Lerman's interpretation of radical constructivism and explain how the ideas of interaction, intersubjectivity, and social goals make sense in it. We then establish compatibility between the analytic units in Vygotsky and von Glasersfeld's models and contrast them with Lerman's analytic unit. Consequently, we question Lerman's interpretation of Vygotsky. Finally, we question Lerman's use of Vygotsky's work in mathematics education, and we contrast that with how we use von Glasersfeld's radical constructivism.

Lerman argued in two different papers (Lerman, 1994, 1996) that an attempt to incorporate a social view of knowledge into radical constructivism leads to an incoherent model of knowing. He explained the alleged incoherence as follows:

> Taking constructivism's view of the autonomy of the individual in the construction of her or his knowing, given her or his particular conceptual system and its particular filter, leads to a consistent, albeit very restricted, view. To argue for an integrated view is to argue that sometimes the filter has very large holes and what is occurring beyond the individual can somehow enter without constraint. I argue, then, that it makes no sense to strengthen the functioning of the "social" into a social constructivism. (Lerman, 1996, p. 140)

Lerman (1996) also believes that "[A]ttempts to incorporate intersubjectivity into radical constructivism make it an incoherent theory" (p. 148). Further, he places the work of Vygotsky and von Glasersfeld in strong opposition, and concludes that mathematics education would benefit from abandoning constructivism as a view of how people learn (p.133).

In our reply, we sketch a portrait of radical constructivism in which interaction rather than intersubjectivity is a core hypothesis. In the first part of the paper, we develop several concepts within radical constructivism and contrast these concepts with what Lerman says about them. In the article's second part, we argue that Lerman's interpretation of Vygotsky differs from ours, and these differences are manifested in our differing conceptions of how general models should be used in mathematics education. In this context, we compare and contrast Lerman's ideas on mathematics education with some of our own and close the paper by questioning Lerman's interpretation of Vygotsky.

Piagetian Interactionism

Lerman's claim that social interaction has no "force" in radical constructivism repeats the Epistemic Objection, "[W]hich states that Piaget's theory is a failure because it treats cognition, in any of its forms whether physical or social, as an individual and not a co-operative enterprise” (Smith, 1982).

Radical constructivists do refer to assimilation, accommodation, and reflective abstraction as processes involved in subjective cognitive construction. Although interaction is often not mentioned, interaction of some kind is always assumed. In fact, interaction enters into radical constructivism at its very core. It is not incorporated as an afterthought or as an auxiliary hypothesis to patch up the model. Piattelli-Palmarini (1980) realized the centrality of interaction in Piaget’s genetic epistemology.

> These presuppositions lead to a core hypothesis, out of which the entire program of genetic epistemology has been developed. We read the following, italicized in the original text: "Cognitive processes seem, then, to be at one and the same time the outcome of organic autoregulation, reflecting its essential mechanisms, and the most highly differentiated organs of this regulation at the core of interactions with the environment". (p. 4).

Socio-cultural critics of constructivism often interpret the phrase “interactions with the environment” as excluding interaction with others. We do not understand how anyone can hold this interpretation and be aware of Piaget’s (1964) insistence that social transmission was one of four factors that contribute to development[3]. In particular, Piaget (1964) said:

> The third factor is social transmission--linguistic transmission or educational transmission. This factor, once again, is fundamental. I do not deny the role of any one of these factors; they all play a part (p. 13).

Piaget did not mean “social transmission” as transmission from an impersonal, objective social reality to the individual. That would be inconsistent with his reaction to the claim that knowledge and language are preformed in society and do not reside in individuals:

> The preformation [of social characteristics] is, as in other contexts, nothing but a common sense illusion consolidated by Aristotelian philosophy of potentiality and action. (Piaget, 1995, p. 340)

Instead, Piaget thought of social transmission interactively. Individuals establish equilibrium among personal schemes of action and anticipation as they interact in mutual adaptation--as constrained by the local limitations imposed by their abilities to accommodate those very schemes (Piaget, 1995).

Radical constructivists’ understanding of interaction is compatible with the idea that:

> We are all unavoidably social individuals. Right from the moment of our conception we are in company, and from birth we begin to be “socialized”, i.e. adapted to the pre-fabricated world that is the consensual framework of everybody’s permanent participant realization of the respective society or group (Köck, 1980, p. 107).

Köck, in this quote, was speaking in the context of Maturana’s contributions to theories of communication. Piaget’s interactionism is highly compatible with Maturana’s explanation of interaction. According to Maturana (1980a), an interaction occurs “Whenever two or more entities change their relative positions in their space of existence as the result of the interplay of their properties” (p. 27). The emphasis on the interplay of properties suggests that Piagetian interactionism should not be understood as direct interaction between an individual’s knowledge and the environment it knows. Rather, Maturana implies a second kind of interaction--the interaction of personal constructs in re-presenting or otherwise operating on previously constructed items. It is in this second kind of interaction that we find subjective cognitive construction.

Individual-environment interactions and interactions within an individual constitute nonintersecting domains (Steffe, 1995, pp. 509-514). As an individual interacts in its environment, interactions among constructs within the individual occur as part of the regulation of assimilation and accommodation. In other words, subject-environment interactions can engender interactions within the individual that lead to modifications of the interacting constructs or of relationships among them. These modifications, in turn, can influence subsequent subject-environment interactions, which can engender further modifications of the individual's interacting constructs. And so on.

Reflexivity between the two domains of interaction is fundamental to what radical constructivists mean by "intersubjective construction of knowledge in social interaction". When two individuals are in social interaction, intersubjective knowledge is established whenever (1) each individual reciprocally assimilates the language and actions of the other, (2) the reciprocal process of assimilation continues until no accommodations of the conceptual structures involved in the reciprocal assimilations are necessary for "successful assimilation"[4], and (3) the two individuals reach a state of mutual agreement about the meaning of the results of their interactions. By "reaching mutual agreement" we do not mean that the interacting individuals end up with identical conceptual structures. Rather, we mean only that their conceptual structures are sufficiently compatible for successful reciprocal assimilation. Lerman (1996) seems to have overlooked reciprocal interactions in his contention that "for radical constructivists it is difficult to see how there could be such a thing as the intersubjective construction of knowledge" (p. 134).

Intersubjectivity

Lerman interprets radical constructivism as a model of learning that applies only to what he calls the "autonomous individual"[5]. However, intersubjectivity was an integral part of radical constructivism from its very outset. People often forget that von Glasersfeld's elaboration of Piaget's genetic epistemology into what he eventually called radical constructivism grew in large part out of his keen interest in understanding the nature of human communication and language (von Glasersfeld, 1987/1976; Schmidt, in press). That is, from its very outset, an image of individuals in social contexts was central to radical constructivism.

It should be no surprise, then, that von Glasersfeld (1995) addressed the role of social interaction in the construction of intersubjective knowledge. He addressed it in a discussion of what he calls *second-order viability*. When another person does what we predicted, we may say that "the piece of knowledge was found to be viable not only in our own sphere of actions but also in that of the other. This bestows a second order of viability to the knowledge and the reasoning we assumed the other to have and act on" (p. 120). Of second-order viability, von Glasersfeld says:

> It is obvious that this second-order viability, of which we can say with some justification that it reaches beyond the field of our individual experience into that of others, must play an important part in the stabilization and solidification of our experiential reality. It helps create that intersubjective level on which one is led to believe that concepts, schemes of action, goals, and ultimately feelings and emotions are shared by others and, therefore, more real than anything experienced only by oneself. It is the level on which one feels justified in speaking of 'confirmed facts', of 'society', 'social interaction', and 'common knowledge'" (p. 120).

In a review of von Glasersfeld's model of knowing, Smith (1995), commented that: "A central claim of [radical constructivism] is that social construction is as essential as abstraction and that both have a basis in Piaget's work (social construction in Piaget, 1995; abstraction in Piaget, in preparation)" (p. 509). Everything that we might call language, mathematics, society, or knowledge follows on from what individuals make from interaction. This is fundamental in radical constructivism, and any assertion otherwise indicates an understanding of the model of knowing that is not shared among radical constructivists.

Even with these clarifications, Lerman would likely argue that it is also necessary for radical constructivists to demonstrate a role for intersubjectivity in subjective cognitive construction that goes “beyond the interpretation of social interactions as setting up perturbations for the individual, which is the Piagetian view” (p. 135). Indeed, in his charge of incoherence, Lerman seems concerned with subjective cognitive construction that “appears to be involuntary, through acculturation” (p. 135). However, given that Lerman attributes primacy to intersubjectivity, his arguments against the possibility of intersubjective knowledge contributing to subjective cognitive construction in radical constructivism proceeds from his idea that intersubjectivity as he understands it couldn’t be a part of radical constructivism, and from his rejection of the subjective construction of knowledge. So, his arguments are specious. Nevertheless, it is important to respond to Lerman’s contentions about intersubjectivity and radical constructivism because readers might not distinguish between Lerman’s and radical constructivism’s concepts of intersubjectivity. Even if they did make this distinction, the way in which Lerman portrays “perturbation” leaves little room for the construction of intersubjective knowledge to contribute to the construction of subjective knowledge.

> Whether the individual modifies her or his ideas or not is dependent on the challenge to the viability of her or his existing conceptual structures when faced with the new experiences. However, whether a new experience constitutes a challenge to a person’s existing schemata or not is also dependent on the individual’s schemata… . Thus, what happens in the classroom may or may not generate a cognitive conflict for an individual; social interactions have no force, the individual construes. (Lerman, 1996, p. 139)

It is particularly unfortunate that Lerman interprets von Glasersfeld (1980) notion of perturbation as mere cognitive conflict because this interpretation of “challenge” paints the concept of viability in very Spartan colors. To offset the idea that perturbation can be equated with cognitive conflict and that adaptation serves only in eliminating these conflicts, we discuss perturbation and distinguish between goals and perturbation.

Collaboration and Goals.

Lerman’s (1996) believes that radical constructivists see the function of social interaction as setting up perturbations for the individual, where perturbation is restricted to cognitive conflict. This is quite different from von Glasersfeld’s (1995) statement that “The notion of collaboration and the concerted efforts to reach a goal is probably the most powerful principle” in social construction (p. 121).

The idea that generating social goals is a basic effect of social interaction is quite different than the idea that its basic effect is setting up perturbations, even when perturbation is understood more generally than cognitive conflict. Social goals and perturbations are not the same. To infer a social goal, it must be possible for the observer to infer that a participant in an interaction is *aware that others have intentions*.[6] In making this inference, it must be also possible for the observer to infer that each individual participant establishes a goal which is used in intersubjective agreement. Even though the first inference assumes that the second can be made, it is not the same inference.

We can distinguish, from an observer’s perspective, between two types of disturbances within an individual,. Our purpose for doing so is to counter Lerman’s claim that radical constructivism has not gone “beyond the interpretation of social interactions as setting up perturbations for the individual.” Lerman collapses two senses of disturbances--corresponding to internal and social intrusions in an individual’s experience-- into the first.

A technical discussion of perturbations would address issues of negative feedback, effectors, reference values, and so on. (Glasersfeld, 1978; Powers, 1973; von Foerster, 1987; Wiener, 1948). Engaging in this technical discussion would require a separate article. So, we use the personal experience of the subject as a heuristical device to make a distinction between the two kinds of disturbances. The essential distinction between the two, which we call perturbations and experiential provocations, lies in where the experiencing person locates its source. When people experience a toothache, they typically attribute its source to a bad tooth. When they hear a voice, they attribute its source to an other. The toothache is an example of a disturbance generated wholly within the person and could be regarded as a perturbation if a physiological reaction was set in motion to heal the tooth. The latter is an example of an experiential provocation.. In other words, the person’s *attribution* of the source of the disturbance can be use heuristically to classify a disturbance as a perturbation or an experiential provocation. Interactions among individuals that generate experiential provocations in them can lead to their establishing social goals – discrepancies in their current states which point to operations that entail operations or actions to be carried out by others.[7]

. Smith (1982) clarified this for any goal, individual or social, which involve intentionality: “An intentional action is always one that occurs in the light of a subject’s beliefs together with desires and values, since the presence of factors of both types is a necessary condition of an action’s being intentional” (p. 175).

To sustain the social construction of mathematical knowledge that might follow from children’s establishing social goals is a basic problem in constructivism as it pertains to mathematics education. Each goal-directed child may proceed independently of the others for any one of several reasons, both cognitive and social, and therefore use their individual schemes without collaboration. It is no accident that Piaget considered decentering--the attempt to imagine one’s experience from another perspective--so central to both socialization and to reflective abstraction. Egocentric thought cannot support the construction of intersubjective knowledge. Therefore, learning how to engender collaboration in children’s mathematical interactions, and to sustain and modify the collaborations in such a way that we can predict in broad how they are individually affected by the interaction, is essential. In radical constructivism, intersubjectivity cannot be taken as given – as an explanatory concept in the construction of knowledge – as Lerman wishes to do. Rather, intersubjectivity must itself be explained.

The Manifold and Experiential Reality

Lerman’s claim that integrating a social view of knowledge into radical constructivism produces an incoherent theory carries with it an assumption that “information” can travel from one person to another by speech sounds, by radio waves, or by letter. This assumption is thoroughly misleading, as can be seen by considering that subject’s construct experience out of their manifold.

> The 'manifold', then, is the raw material, the stuff on which constructive perception and reason can operate. . . . In present-day neurophysiology one would say, it is the totality of electrochemical impulses continuously generated by the sensory organs of the system. Even if one assumes that these impulses are caused by differences of an ontic substrate they cannot convey qualitative information, because qualitatively they are all the same. *Experience, thus, is what the thinking*

> *subject coordinates (constructs) out of elements of the manifold* (emphasis added) (von Glasersfeld, 1995, pp. 40-41).

MacKay (1951; 1955) made a related point – that information cannot be transferred between two acting systems. Rather, signals can traffic between two systems, but information is generated internally by a receiving system when its "readiness to act" is thereby affected. If we tell you that a student is outside your door when you already know this, you may conclude that we are unaware that you already know it, but you have not gained any information from hearing us say it.

Experience is the subject's construction. The knowledge constituting the subject's experiential reality is also what the subject constructs out of its experience, which, by definition, is an internal affair. This is the case for the knowledge constituting the subject's mathematical reality as well. Mathematical reality is what the thinking subject constructs out of the experiential reality it constructs. This does not mean that mathematical reality is only one step removed from experiential reality, as the model of mathematical understanding of Pirie & Kieren (1994) shows. In any event, we want to point out that mathematical reality cannot be said to be in experiential reality any more than experience can be said to be in elements of the manifold. But mathematical knowledge can contribute to and modify experiential reality.

Lerman's portrayal of constructivism as having "filters" with large holes that allow what is occurring beyond the individual to enter without constraint assumes that the sensory signals of the manifold contain information from the subject's environment that he or she can "read" literally. However, at the level of the manifold, there are no distinctions in the raw material on which constructive perception and reasoning operate (von Foerster, 1984), which is to say that there is no information in these elements. As noted by Wright (In Press), the world is sensed before it is known.

Social and Physical Interactions.

Lerman (1996) also insists that, in the constructivist model of knowing, social interactions are on the same plane as physical interactions.

> When the source of knowledge and of meaning is the individual, social interactions are on the same plane as physical interactions; they are filtered, or refracted, through the perceptions of the receiver. (p. 45)

This repeats the Social Objection (Smith, 1982), which

> [S]tates that Piaget's theory fails because it treats social cognition as if it were physical cognition, because it takes a subject's knowledge of the social world as being not essentially different from a subject's knowledge of the physical world. Thus it is claimed … it takes the physical environment to be constitutive of a subject's environment … (p. 174).

Although at the level of the manifold there can be no distinctions in the raw material on which constructive perception and reason can operate, at the levels of experiential reality (which already involves several abstractions), the situation changes.

> The subject creates not only objects to which independent existence is attributed but also *others* to whom the subject imputes such status and capabilities as are conceivable, given his or her own experience" (von Glasersfeld, 1995, p. 128).

The issue is not that radical constructivists fail to distinguish among the nature, quality, purpose, or function of physical and social interaction. Rather, the issue is that physical interactions have no currency for Lerman in the development of what Vygotsky called spontaneous concepts. Lerman (1996) argues that "taking Luria's and Leont'ev's elaborations of Vygotsky's interpretation of the development of human consciousness indicates that a nonsocial origin to spontaneous concepts makes no sense" (p. 142).

We find Lerman's interpretation that there cannot be a nonsocial origin to spontaneous concepts puzzling because it ignores one of the four sources of development cited by Piaget, and because Vygotsky differentiated, as did Piaget, "between representations that develop primarily through the operation of the child's own thought and those that arise under the decisive and determining influence of knowledge the child acquires from those around him" (Vygotsky, 1987, p. 173). The former "representations" certainly do not exclude the contributions of physical interactions to physical experiential reality. Even if one argues that knowledge presupposes consciousness awareness and that a subject may acquire knowledge of any sort only if other subjects enter into social relationships with that subject (Smith, 1982), that does not mean that nonsocial origins to spontaneous concepts make no sense.

In spontaneous concepts, "attention is always directed toward the object that the spontaneous concept represents …" (Vygotsky, 1987, p. 191). Vygotsky (1987) made it very clear in his general law of development that: "It [consciousness awareness] arises comparative late and must be preceded by a stage where conscious awareness is absent, a stage where there is no volition in the application of a given form of conscious awareness. If we are to master something, we must have at our disposal what is to be subordinated to our will" (Vygotsky, 1987, p. 189). Consequently, in Vygotsky's system, there is a time when a child was not aware of the object of the spontaneous concept. So, if Vygotsky had lived to read the 1937 publication *La construction du réel chez l'enfant* by Piaget, we believe that he would have found that the first 80 pages, devoted to explaining the development of object concepts during the first two years of life, corroborated his understanding of the origins of spontaneous concepts. Piaget demonstrated that a child can recognize an object long before it is aware of the object after the object leaves the child's visual field, an awareness which is an indication of conscious awareness. The construction of a recognition template necessarily involves the child interacting with items an observer regards as physical objects external to the child (von Glasersfeld, 1995, pp. 160ff).

Davydov's use of operative structures

Further confirmation that Vygotsky did not reject nonsocial origins of spontaneous concepts is contained in Davydov's (1975) work on quantity. His use of Piaget's operative schemes as a foundation for his curriculum on quantity is compatible with Vygotsky's idea of spontaneous concepts as emerging from the child's own thought. In contrast to Davydov's use of Piagetian operative schemes, Lerman (1996) rejects them: "Thus the semiotic function becomes the focus of study, rather than the reversible mental structures that, for Piaget (1964), "constitute the basis of knowledge, the natural psychological reality, in terms of which we must understand the development of knowledge" (p. 9)" (p. 137). His rejection of the significance of physical interactions to the development of spontaneous concepts is echoed in Lerman's (1994) rejection of the concepts of unit, plurality, and number, as explained by von Glasersfeld, as the foundation of mathematical understanding[8].

Units of Analysis in Vygotsky's, von Glasersfeld's, and Lerman's Work

Lerman's interpretation of Vygotsky substantially influences his rejection of radical constructivism. We already indicated our disagreement with Lerman concerning origins of spontaneous concepts and his rejection of the idea of reversible mental structures. In this section, we pursue the issue of Lerman's interpretation of Vygotsky by comparing the unit of analysis in the latest period of Vygotsky's work and the unit of analysis in children's construction of counting schemes in the project IRON (Interdisciplinary Research on Number) in which von Glasersfeld participated.

The Child's Experience

Vygotsky's emphasis on the child's experience as an analytic unit representing the individual in individual/environment relationship is compatible with the idea that the child's experience is the result of an assimilation (Piaget, 1964, p. 18). This compatibility is especially striking in Minick's (1987) discussion of Vygotsky's analytic unit. Minick first noted that the practice of describing the environment in terms of "absolute indices" rather than in terms of "what it means for the child" is a fundamental inadequacy of most attempts to study the environment's influence on children's development, and that, second, this inadequacy is removed by taking the individual in the individual/environment relationship as the analytic unit. Minick (1987) then quoted Vygotsky as follows:

> The child's experience is the kind of simple unit of which it is impossible to say that it is the influence of the environment on the child or a characteristic of the child himself… . Experience must be understood as the internal relationship of the child as an individual to a given aspect of reality…". (p. 32)

Vygotsky's emphasis of "what it means for the child" implies he would have been interested in the child's mental structures in relation to the world in which, from an observer's point of view, the child lives. The child's mental structures in relation to his or her world – that is, the mathematical knowledge he or she builds from experience – is the analytic unit in the studies of children's mathematical learning in the IRON (Interdisciplinary Research on Number) project in which von Glasersfeld is a principle investigator and collaborator.

Our approach to establishing the individual/environment relationship and then studying the individual's experience within that relationship is to investigate those critical events where the observed child meets constraints. It is a study of where the child's current mathematical knowledge proves insufficient for successful action or interaction, and thus a study of learning in the context of adaptation. We find, however, that the operations involved in adaptation cannot be accounted for by the knowledge of those with whom the child interacts nor by the interaction per se. Rather, we find that the operations involved in adaptation emerge from within the child in the context of interaction.

Unit of Analysis in Lerman's Work

In contrast to the compatibility between Vygotsky and von Glasersfeld concerning the child's experience as an analytical unit in an analysis of learning, Lerman (1996) commented that, "the gaze of the psychologist must be on the social and cultural practices in which meanings and purposes function so that people act in the world" (p. 136). Moreover, "Individual mental structures are not the fundamental unit of cognition; meanings, which are first on the social plane, perform this function" (p. 148). That Lerman eliminates the subjective element of word meanings is made clear in his interpretation of Wittgenstein that private language makes no sense. In this regard, von Glasersfeld (1995) commented that: "Wittgenstein, who had undoubtedly one of the sharpest intellects in our century, struggled until his death to convert his notion of meaning and truth into a logical certainty, but the final pages of his last notebook (1969) show that he did not succeed in eliminating the subjective element" (p. 134).

So, we wonder whether Lerman's comments are more compatible with Vygotsky's earlier views than they are with the his later views. Of Vygotsky's earlier views, Minick (1987) commented that: "Vygotsky saw the higher mental functions as "social " in two senses.

> First, like other aspects of culture, their development is part of the development of the socio-cultural system and their existence is dependent on transmission from one generation to the next through learning. Second, they are nothing other than the organization and means of actual social behavior that has been taken over by the individual and internalized". (p. 21)

In the latest period, Vygotsky "quite consciously moved beyond an explanatory framework in which speech and social interaction were seen as the sole motive force underlying psychological development" (Minick, 1987, p. 30). Thus, it seems that Vygotsky would have accepted an approach that focuses on individual mental structures *in interacting children* during the last period of his research program.

Lerman (1996) made a comment concerning meanings of the word "half" which corroborates our interpretation that his ideas for mathematics education are in line with the earlier Vygotsky.

> There are many meanings of the word "half," some of which ("your half is bigger than mine") are in the domain of everyday discourses and only one of which (two halves are only "halves" if they are identical and together make the whole) is in the mathematical discourse. It is not wrong to hold both; it is important to learn which is appropriate in which discourse. (p. 146)

Vygotsky (1987), in elaborating what he meant by "The scientific concept grows downward through the everyday concept and the everyday concept moves upward through the scientific." (p. 220), commented that:

> Everyday concepts, however, move quickly along the upper section of the path which was blazed by scientific concepts. In this process, they are restructured in accordance with the structures prepared by the scientific concept. (p. 220)

So, it would seem to be a goal of mathematics teachers working within a Vygotskian framework to help a child to restructure the everyday meanings of the word "half" using the higher order mathematical meaning, and then perhaps reflect on the everyday use and possible reasons for it., We believe, as does Lerman, that a child who says "your half is bigger than mine" is not necessarily wrong. But unlike Lerman, the phrase "not wrong" is based on our ability as observers to construct a scheme that fits into our experience of the child's language and action and that allows us to understand the child's meaning as a necessary consequence of this scheme's functioning.

Where is the Observer?

Lerman makes no mention of the observer in his challenge to radical constructivism. In fact, he interprets a comment the first author made in an earlier paper concerning internalization being an observer's concept as a rejection of internalization: "Steffe (1993) is consistent and quite specific when he recognizes that, in the radical constructivist view, the process of internalization makes no sense" (Lerman, 1996, p. 136). This is unfortunate, because clarifying the role of the observer can establish points of contact in what seems to be conflicting positions as well as reveal crucial differences.

For example, when we focus on analyzing children's schemes, we work as first-order observers. Although a first-order observer makes a concerted attempt to assume the position of the child and think like he or she does, the observer's ways and means of operating are left implicit and the observer does not intentionally analyze the mental structures of the child relative to his or her own mental structures. However, the first-order observer does interpret the interactions of the child, and by this means tests the interpretations for their viability.

When we focus on analyzing the mathematical learning of a child whom we are teaching, we work from the point of view of a second-order observer, which refers to "the observer's ability through second-order consensuality to operate as external to the situation in which he or she is, and thus be observer of his or her circumstance as an observer" (Maturana, 1978, p. 61). When working from this perspective in both actual interaction and retrospectively, we focus specifically on explaining the child's learning relative to our own purposes, intentions, and contributions to mathematical interaction. Finally, as Maturana (1980b) explained, when we analyze the mental structures of two interacting individuals along with possible changes in these structures as a result of the interaction, we assume a meta-position with respect to them: "[A]ny comparison of two entities, whatever their kind, necessarily entails the adoption of a meta-position with respect to them, and the choice, by the comparativist, of a perspective that defines a domain of similarities that makes the comparison possible" (p. 71)[9]. It is from the position of a second-order observer that the researcher can analyze changes in the mathematical meanings of the individuals in interaction, and hence, formulate a model of "internalization". We say "internalization" with scare quotes because we do not mean to imply that there is something external to the individual of which we are aware as observers that becomes inside the individual's thinking or understanding. The individual must construct its meanings using elements of its manifold, which is a position that constrains any model of internalization. Von Glasersfeld said as much when commenting upon Shannon's distinction between signals and information.

> Claude Shannon's (1948) work on communication was revolutionary because it established incontrovertibly that the physical signals that pass between persons in communication … do not carry what is ordinarily considered as meaning. Instead, they carry instructions to select particular meanings from a list, which, together with the list of convened signals, constitutes the communication code. (von Glasersfeld, 1995, p. 138)

Lerman may regard this constraint as a rejection of his concept of internalization. But, in our domain of explanation, we do not reject "internalization" for that would be tantamount to rejecting the concept of communication and any notion of mathematics teaching.

If Lerman's failure to mention his role as an observer is simply an omission, it would be very helpful if he could make his stance as an observer explicit, because that would make it possible to interpret what explanatory path he takes. His basic unit of analysis suggests that as an observer he takes an explanatory path that Maturana called objectivity-without-parentheses. Of this path, Maturana (1988) commented, "[T]his explanatory path necessarily leads the observer to require a single domain of reality--a universe, a transcendental referent--as the ultimate source of validation for the explanations that he or she accepts" (p. 29). This seems to fit well with Lerman's unit of analysis, because Lerman, as observer, considers that words gain their meaning and significance for the individual through their use and that meaning is socially and culturally determined. He posits internalization as the mechanism by means of which inter-individual meanings become intra-individual meanings without feeling any need to explain this process. The alternative, objectivity-with-parentheses (which is von Glasersfeld's as well as Maturana's view), implies that knowledge is a process in the domain of explanation.

Models in the Domain of Explanation

Lerman's failure make any mention of an observer fits well with how he uses Vygotsky's work in mathematics education and also with how he interprets radical constructivism in mathematics education. For example, we don't find ourselves in Lerman's (1996) comment that: "Constructivism, after all, just attempts to describe how people learn, and that process will take place even in a formal lecture hall of 300 students. There is an independence of the two discussions, radical constructivism and teaching" (pp. 146-147).

Although learning in the way we have explained it could take place in a formal lecture hall of 300 students, it could also take place as a result of interactions outside of school as well. Acknowledging that learning can occur even when it is unintended should not be interpreted as meaning that there is an independence of radical constructivism and teaching in the sense that radical constructivism could not under any circumstances be used in building models of mathematics teaching. However, to say that there is an independence of the two discussions, radical constructivism and teaching, has meaning for us because we make a distinction between radical constructivism as a model of knowing as von Glasersfeld (1995) presented it in his latest book, and how it might be used by a mathematics educator. One wouldn't expect to read a model of mathematical teaching nor a model of mathematics learning in von Glasersfeld's book in the same way that one would not expect to read about such models in Vygotsky's work. A general model of knowing can be used in building models of children's mathematical learning, or models of teachers' mathematics teaching, but these models are not to be confused with the general model used in building them.

Researchers should not apply a general models like von Glasersfeld's or Vygotsky's directly to the practice of mathematics education. That is not their purpose. Rather, researchers should construct explanatory models that are grounded in their experience with children's mathematics, while at the same time being guided and constrained by their general models. That is, neither constructivism nor socio-culturalism provides a theory of mathematics education.

Although we agree that one can teach mathematics in a way that is not consistent with radical constructivism, we emphatically disagree with the idea that a researcher who sets out to investigate mathematics teaching should reject the core principles of radical constructivism. Researchers will find these core principles especially useful when the phenomenon to be explain is their personal experience of using whatever general model they hold The relationship between the researcher's theory and the researcher's practice may seem circular, but we regard it as a strength even in a practical sense. Researchers in mathematics education are obliged to produce ways of teaching mathematics that are consistent with their "general model" of knowing. But those researchers who wish to abstract models of teaching mathematics are obliged to explain the phenomena that are regarded as mathematics teaching using their general model. Teaching and model building are recursive processes.

First- and Second-Order Models

If the principles of radical constructivism were simply applied to teaching children mathematics without the intervening process of model building, the principles would be used as heuristical guides such as understanding the learning of mathematics as an active rather than as a passive process. Lerman (1996) characterized this spirit of application of radical constructivism beautifully in his comment that: "[T]he issue of pedagogic practices that follow from constructivist principles can only be couched in a discussion of the ways of encouraging rich constructions (or correct constructions mathematically?) on the part of students" (p. 146). We find Lerman's comment about applying radical constructivism consistent with his recommendations when applying Vygotskian theory to mathematics education: "[T]he aim of the mathematics teacher might be seen as assisting pupils to appropriate the culture of the community of mathematicians as a further cultural practice" (p. 146).

Lerman's view of the aim of a mathematics teacher emphasizes the mathematical knowledge of the teacher, but it makes no mention of the mathematical knowledge that the teacher can learn by interacting with children. The phrase "everyday knowledge" does not respect children's knowledge as legitimate and rational *mathematical* knowledge in the way that the mathematical knowledge of the teacher is legitimate and rational. If we do respect children's mathematical knowledge, we are obliged to explain it in a way that goes beyond the idea of "everyday knowledge."

We use the ideas of first- and second-order models in learning children's mathematical knowledge. Second-order models are models an observer may construct of an observed subject's knowledge in order to explain his or her observations (i.e., his or her experience) of the observed subject's states and activities (Steffe, von Glasersfeld, Richards, & Cobb, 1983, p. xvi). They are necessarily constructed through interaction with the subject because they are the observer's models of the observed. Models of mathematical learning are also second-order models, but they are the observer's models of the observed that include the circumstances of observation. Second-order models are necessarily constructed by an observer using his or her own ways and means of operating. To the extent that second-order models are constructed as a result of attempts to explain the mathematical knowledge and learning of children as understood by the observer, they can be legitimately called *social* [10] (cf. Steffe, 1995).

Classroom Culture vs. Mathematical Culture of Mathematicians.

We call second-order mathematical knowledge as it pertains to children *the mathematics of children* (Steffe, 1994). This kind if mathematical knowledge impacts dramatically on the goals of a mathematics teacher. For example, in radical constructivism, one goal of a mathematics teacher might be seen as learning how to bring forth the mathematical knowledge of children, and how to sustain and engender modifications in it, which is quite different than what Lerman deduced in his application of radical constructivism to the practice of teaching mathematics. It is different in several respects, one of which is that a radical constructivist does not take social interaction as a given. Rather, one of the main problems is to learn how to engage in purposeful mathematical interactions with children. Implicit in this problem, of course, is to learn how to bring purposeful mathematical interactions among children forth. In this learning, the teacher should also aim to learn the mathematical culture that is produced by interactive mathematical communication with and among children, and the evolution of that culture.

It is profitable to regard such a mathematical culture using Maturana's (1980a) idea of a social system as a network of interactions of a collection of living systems. According to Maturana, a living system is self-producing (autopoeitic). In a social system, the network of interactions is the realization of the living systems' autopoiesis as they constitute the social system. This recursive property of social systems is central to understanding how a teacher might regard the relation between the mathematics of children and classroom culture, and it supersedes the idea of children's appropriation of the culture of the community of mathematicians as a further social practice. Educators must remind themselves that the teacher is the most important part of a recursive social system. The teacher is the only adult mathematician in the classroom. But rather than embody only adult mathematical practices, the teacher is also obliged to be the leading mathematician with respect to the mathematics of children. In fact, how a teacher comes to understand the relation between his or her first- and second-order mathematical knowledge is another of the basic problems of mathematics education opened by radical constructivism. This problem is complicated by the fact that constructs from outside of mathematics must be used to explain the production of "mathematics"[11]. In line with Gödel's incompleteness theorem, mathematics cannot be used to explain its own construction.

In closing, we would like to say that we regard the mathematics of children as superseding Vygotsky's notion of spontaneous and scientific concepts[12]. A separate paper would be needed to elaborate on this point. Our reason for making it is twofold. First, our claim is made possible by to the compatibility we find between the analytic units in Vygotsky's and von Glasersfeld's work. Second, the claim illustrates that we do not wish to abandon Vygotsky's work as a view of how people learn. The principles of radical constructivism were developed primarily after Vygotsky's death and were not available to him. So, rather than abandon his work, we instead choose to construct models of leaning that supersede it.

Final Comments

When reading and interpreting Vygotsky, we have always found substantial compatibilities with the work of von Glasersfeld. Finding these compatibilities has always been somewhat disturbing, because interpreters of Vygotsky often reject the work of Piaget and von Glasersfeld. Not having had the privilege of working directly with Vygotsky, we do not have the confidence in our interpretations of his work that we have in our interpretations of von Glasersfeld's work. Nevertheless, we have always interpreted Vygotsky as being primarily concerned with explaining individual cognitive development, and we were surprised to found this interpretation countermanded in almost everything Lerman said in the two papers cited here. But we do find our interpretation corroborated in van der Veer's and Valsiner's (1994) insightful comment that:

> A number of blind spots can be detected in contemporary uses of Vygotsky's ideas… . Secondly, the focus on the individual developing person which Vygotsky clearly had (as did most European psychologists of the time) has been persistently overlooked. Thus, Vygotsky has been presented as an irreconcilable opponent to Piaget, with whom he differed in the evaluation of egocentric speech, but not in the focus on the developing personal--cognitive (and affective) structures. The actual closeness of the basic personalistic standpoints of both … has gone without attention. Our contemporary child and educational psychology seems to be in its socially orientated mode, within which the simple primacy of the individual's personal experiencing is yet to find its prominent place (again). (p. 6)

The basic issue seems to be how one interprets the ideas of Vygotsky, Piaget, or von Glasersfeld, for what purpose, and how viable those interpretations are. We explained why we find both Lerman's interpretations of radical constructivism and how it might be used in mathematics education untenable. We also explained how both his interpretation of Vygotsky and how he uses Vygotsky's general model in mathematics education are fundamentally different from ours. We then sketched what we regard as a much more appropriate way to use general models in mathematics education, and contrasted this use with how Lerman used Vygotsky's work. Is it the case that an attempt to incorporate a social view of knowledge into radical constructivism leads to an incoherent model of knowing? For us, this question is inappropriate. Neither von Glasersfeld's nor Vygotsky's work point to a conceptual contradiction between the idea of subjective cognitive construction and the experiential reality of social phenomena.

References

Davydov, V. V. (1975). Logical and psychological problems of elementary mathematics as an academic subject. In L. P. Steffe (Ed.), Children's capacity for learning mathematics (pp. 55-108). Chicago: University of Chicago Press.

Köck, W. K. (1980). Autopoiesis and communication. In F. Benseler, P.M. Hejl, & W.K. Köch (Eds.), *Autopoiesis, communication, and society: The theory of autopoietic systems in the social sciences* (pp. 87-112). New York: Campus-Verlag.

Lerman, S. (1994). Articulating theories of mathematical learning. In Ernest, P. (Ed.) Constructing mathematical knowledge: Epistemology and mathematics education (pp. 41-49). Washington, DC: The Falmer Press.

Lerman, S. (1996). Intersubjectivity in mathematics learning: A challenge to the radical constructivist paradigm? Journal for Research in Mathematics Education. Vol. 27, No. 2, 133-150.

MacKay, D. M. (1951). The nomenclature of information theory. In H. v. Foerster (Ed.). Proceedings of the Eighth Conference on Cybernetics (pp. 156–196). : Josiah Macy, Jr. Foundation.

MacKay, D. M. (1955). The place of 'meaning' in the theory of information. In E. C. Cherry (Ed.), Information theory (pp. 215–225). London: Buttersworth.

Maturana, H. (1978). Biology of language: The epistemology of reality. In G. A. Miller & E. Lenneberg (Eds.), Psychology and biology of language and thought: Essays in honor of Eric Lenneberg (pp. 27-63). New York: Academic Press.

Maturana, H. (1980a). Man and society. In F. Benseler, P. M. Hejl, and W. K. Köch (Eds.), Autopoiesis, communication, and society: The theory of autopoietic systems in the social sciences (pp. 11-32). New York: Campus-Verlag.

Maturana, H. (1980b). Postscriptum. In F. Benseler, P. M. Hejl, and W. K. Köch (Eds.), Autopoiesis, communication, and society: The theory of autopoietic systems in the social sciences (pp. 71-72). New York: Campus-Verlag.

Maturana, H. (1988). Reality: The search for objectivity or the quest for a compelling argument (pp. 25-82). Irish Journal of Psychology, 9, 1.

Minick, N. (1987). The development of Vygotsky's thought: An introduction. In R. W. Rieber, & A. S. Carton (Eds.), The collected works of L. S. Vygotsky. Volume 1: Problems of general psychology (pp. 17-36). New York: Plenum Press

Piaget, J. (1964). Development and learning. In R. E. Ripple & V. N. Rockcastle (Eds.), Piaget rediscovered: Report of conference on cognitive studies and curriculum development (pp. 7-20). Ithaca, Cornell University Press.

Piaget, J. (1995). *Sociological studies*. London: Routledge and Kegan Paul.

Piattelli-Palmarini, M. (1980). How hard is the 'hard core' of a scientific program? In M. Piattelli-Palmarini (Ed.), Language and learning: The debate between Jean Piaget and Noam Chomsky (pp. 1-20). Cambridge, MA: Harvard University Press.

Pirie, S. & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? In P. Cobb (Ed.), Learning mathematics: Constructivist and interactionist theories of mathematical development (pp. 61-86). Dordrecht: The Netherlands.

Powers, W. (1973). Behavior: The control of perception. Chicago: Aldine.

Smith, L. (1995). Radical Constructivism: A Way of Knowing and Learning. By Ernst von Glasersfeld. British Journal of Behavioural Research, 65(4), 508-509.

Smith, L. (1982). Piaget and the solitary knower. Philosophy of Social Science, 12, 173-182.

Steffe, L. P. (1994). The constructivist teachin experiment: Illustrations and implications. In E. von Glasersfeld, (Ed.), Radical constructivism in mathematics education (pp. 177-194). Boston: Kluwer Academic Publishers.

Steffe, L. P. (1995). Alternative epistemlogies: An educator's perspective. In L. P. Steffe & J. Gale (Eds.), Constructivism in education (489-523). Hillsdale: Lawrence Erlbaum Associates.

Steffe, L. P. (1999). Individual constructive activity: An experimental analysis. Cybernetics & Human Knowing, 6, 1.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). Construction of arithmetical meanings and strategies. New York: Springer-Verlag.

van der Veer, R., & J. Valsiner (Eds.). (1994). Introduction. In R. Van der Veer, & J. Valsiner (Eds.), The Vygotsky reader (pp. 1-9). Cambridge: Blackwell Publishers.

von Foerster, H. (1984). On constructing a reality. In P. Watzlawick (Ed.), The invented reality (pp. 41-62). New York: W. W. Norton & Company.

von Glasersfeld, E. (1978). Cybernetics, experience, and the concept of self. In M. N. Ozer (Ed.), Toward the more human use of human beings (pp. ??–??). Boulder, CO: Westview Press.

von Glasersfeld, E. (1980). The concept of equilibration in a constructivist theory of knowledge. In F. Benseler, P.M. Hejl, & W.K. Köch (Eds.), Autopoiesis, communication, and society: The theory of autopoietic systems in the social sciences (pp. 75-85). New York: Campus-Verlag.

von Glasersfeld, E. (1987/1976). The development of language as purposive behavior. In The construcion of reality (pp. 37-64). Seaside, CA: Intersystems Publications.

von Glasersfeld, E. (1995). Radical constructivism: A way of knowing and learning. Washington, D. C.: The Falmer Press.

Vygotsky, L. S. (1987). The development of scientific concepts in childhood. In R. W. Rieber & A. S. Carton (Eds.), The collected works of L. S. Vygotsky (pp. 167-241). New York: Plenum Press.

Wiener, N. (1948). Cybernetics: Or control and communication in the animal and the machine. Cambridge, MA: M.I.T. Press.

Wright, E. (In Press). The topic of entity as it relates to Ernst von Glasersfeld's constructivism. In L. P. Steffe, & P. Thompson (Eds.), Radical constructivism in action: Building on the pioneering work of Ernst von Glasersfeld. London: Falmer Press

Footnotes

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[1] We gratefully acknowledge the comments of Arne Engström, Tom Cooney, and the anonymous reviewers on an earlier version of this paper. But we assume all responsibility for what is said.

[2] This paper was written as part of the activities of NSF Projects REC-9814853 and REC-9811879. All opinions are those of the authors.

[3] The other three factors are maturation, physical experience, and self-regulation.

[4] We use quotes to indicate that when no constraints are met in assimilation, the assimilation is successful.

[5] For a constructivist, being autonomous in an environment refers to Piaget's self-regulation or to Maturana'a autopoiesis. It does not imply that an individual's biological or intellectual functions can be sustained without environmental interactions or that those functions are not influenced by those interactions.

[6] It is entirely possible that, in an interaction of two or more individuals, only one has a social goal. That is, a social goal is a type of individual goal. Two people may hold compatible social goals, but that is different from saying that two people share one goal.

[7] It is important that we understand "interactions" to include remembered and anticipated interactions and not restrict them to physical, contemporaneous interacting.

[8] von Glasersfeld's explanation of unit, plurality, and number should be regarded as first-order concepts; that is, as von Glasersfeld's concepts. Using them in explanation of children's language and actions leads to second-order concepts (concepts imputed to the child) like perceptual unit items, figurative unit items, numerical composite, initial number sequence, and etc.

[9] The concept of second-order observer opens the possibility of studying teacher learning as well as student learning. Moreover, it opens the way to studying teacher learning relative to classroom learning, where the classroom is considered as an interacting self-organizing social system in the way Maturana explains social systems (Maturana, 1980, 1988).

[10] First-order models concern the knowledge the individual constructs to organize, comprehend, and control his or her experience; that is, the subject's knowledge (Steffe, et al., 1983, p. xvi).

[11] Quotation marks are used to indicate that we include the mathematics of children in mathematics.

[12] One model supersedes another if it solves all of the problems the preceding model solved but solves them better, and solves problems the preceding model didn't solve.