Fractions and Multiplicative Reasoning[†]

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DRAFT

In this chapter we begin with a relatively simple observation and follow its implications to end with an analysis of what it means to understand fractions well. In doing this, we touch upon related issues of curriculum, instruction, and convention that sometimes impede effective teaching and learning. We make these connections with the aim of bringing out aspects of knowing fractions that are important for considering the design of fraction curricula and instruction over short and long terms. We hope readers see our attempt to clarify learning goals for fractions as a helpful contribution of research to improving mathematics curricula and teaching.

Our observation is that how students understand a concept has important implications for what they can do and learn subsequently.[1] While this observation is neither new nor breathtaking, it is rarely taken seriously. To take it seriously means to ground the design of curricula and teaching on careful analyses of what we expect students to learn and what students do learn from instruction.

Careful analyses of what students learn means more than creating a catalog of their behaviors or strategies you hope they employ. They also entail tracing the implications that various understandings have for related or future learning. For example, many students understand "a/b" as denoting a part-whole relationship, that "3/7," for example, means "three out of seven" (Brown, 1993). This is unproblematic until they attempt to interpret "7/3". Students often will think, if not say aloud, "7/3 sort of doesn't make any sense. You can't have 7 out of 3." (Mack, 1993, p. 91; 1995). Even further, students who understand "a/b" as meaning "a things

[1] We use phrases like "understand a concept" and "concept of *x*" reluctantly. To say "understand a concept" suggests we are comparing a person's concepts and something that constitutes a correct understanding. We do not mean this at all. Rather, by "concept of *x*" we mean "conceptual structures that express themselves in ways people would conventionally associate with what they understand as *x*." But saying that is too cumbersome, so we continue to use "concept of *x*" and "understanding of *x*".

out of b things" cannot interpret "8/(3/7)". It would have to mean something like "8 things out of (3 out of 7 things)," which does not make sense to them or to us. We see a strong possibility that non-introductory lessons about fractions are largely meaningless to many students participating in them.

Students also can understand inscriptions as commands to engage in a sequence of actions. When students read "$7\overline{)3}$" as a command to act they anticipate writing things below and above it until they satisfy some criterion for stopping, such as a remainder of 0 or a remainder that has appeared before. But anyone thinking of $7\overline{)3}$ as a command to act will find it difficult to interpret the expression "$\left(7\overline{)3}\right)\overline{)8}$". It simply doesn't make sense from an action perspective. It would, however, make sense if we understood $7\overline{)3}$ as being a number. Then $\left(7\overline{)3}\right)\overline{)8}$ is a number—namely, the number that is 8 divided by the number that is $7\overline{)3}$.

To reiterate our point, the way students understand an idea can have strong implications for how, or whether, they understand other ideas. This observation is important for thinking about what students have learned or actually understand and it has implications for how instructional and curricular designers think about what they intend that students understand. Designers always intend some understanding whether or not they make it available for public scrutiny. We contend that mathematics education profits from efforts to both publicize and scrutinize those intentions. Such efforts increase the likelihood that meanings we intend students to develop actually have the potential of being consistent with and supporting meanings, understandings, and ideas we hope they develop from them.

Pedagogical Contexts

Although our intent is to describe understandings that might support sophisticated fractional reasoning, we cannot ignore contexts in which learning and teaching occur. What students learn through instruction at any moment is not just a function of the instruction; it is influenced by what they already know (including beliefs they have about mathematics, doing it, and learning it) and by instrucion in which they have participated. Reciprocally, a teacher's instructional actions at any moment are not simply a matter of executing a plan. They are influence both by what the teacher understands about what he or she is teaching and by what he or she discerns about what students know and how students might build productively upon that knowledge. We examine each consideration briefly in regard to fractions and multiplicative reasoning.

Pedagogical context of present mathematics learning

A variety of sources suggest that there is a problem with the nature of and coherence of mathematics instruction in the United States. The TIMSS report of 8^{th}-grade mathematics instruction in the United States, Germany, and Japan states this clearly.

> Finally, as part of the video study, an independent group of U.S. college mathematics teachers evaluated the quality of mathematical content in a sample of the video lessons. They based their judgments on a detailed written description of the content that was altered for each lesson to disguise the country of origin (deleting, for example, references to currency). They completed a number of in-depth analyses, the simplest of which involved making global judgments of the quality of each lesson's content on a three-point scale (Low, Medium, High). Quality was judged according to several criteria, including the coherence of the mathematical concepts across different parts of the lesson, and the degree to which deductive reasoning was included. Whereas 39 percent of the Japanese lessons and 28 percent of the German ones received the highest rating, none of the U.S. lessons received the highest rating. Eighty-nine percent of U.S. lessons received the lowest rating, compared with 11 percent of Japanese lessons. (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999, p. iv)

The TIMSS sampling technique was to draw nationally representative samples from each of its participating countries (Stigler et al., 1999), so we can expect its results to be fairly representative. That no U.S. lesson's content received the highest quality rating from these mathematicians and that 89% of the U.S. lessons' content received the lowest quality rating suggests a general lack of attention among teachers about the ideas students develop. Instead, lessons tended to focus on having students do things and remember what they had done. There was little emphasis on having students develop robust and generalizable ideas. We see the emergence of conversations about goals of instruction—understandings we intend that students develop—as being an important catalyst for changing the present situation.

One major source of personal reform is when teachers realize that what they are teaching does not support what students should be learning. Thus, discussions that entail descriptions of understandings we intend that students develop need also to address how various instructional practices might support or impede this development.

Post, Harel, Behr, and Lesh (1991) and Ma (1999) shed additional light on present contexts in which students engage ideas of fractions. Post et al. (1991) gave several versions of a ratio and rate test to 218 intermediate (grades 4-6) mathematics teachers in Illinois and Minnesota. The test reflected concepts covered in the mathematics curriculum they taught. Teachers scored between the13-year-old NAEP average and the17-year-old average on items drawn from the 1979 NAEP; overall performance among teachers varied widely across test versions, but average performance scores ranged from 60% to 69% across test versions, and more than 20% of the teachers scored less than 50%.

Post et al. found that teachers had significant difficulty with problems like "Melissa bought 0.46 of a pound of wheat flour for which she paid $0.83. How many pounds of flour could she buy for one dollar?" (Post et al., 1991, p. 193). Forty-five percent of the teachers answered this question correctly while 28% left the page blank or said "I don't know." Teachers were also asked to explain their solutions as if to a student in their class. In the case of the "Melissa" problem (given here), only 10% *of those who answered correctly* could give a sensible explanation of their solution. The authors concluded:

> Our results indicate that a multilevel problem exists. The first and primary one is the fact that many teachers simply do not know enough mathematics. The second is that only a minority of those teachers who are able to solve these problems correctly were able to explain their solutions in a pedagogically acceptable manner. (Post et al., 1991, p. 195).

Ma (1999) compared elementary schoolteachers' understandings of mathematical topics they commonly taught. She found that Chinese teachers were far more likely than U.S. teachers to exhibit richly connected and pedagogically powerful understandings of what they expected students to learn despite the U.S. teachers' more extensive educational backgrounds[2]. Thus, both the Post et al. and Ma studies point to the distinct possibility that phrases like "teach for deep understanding" and "teach with meaning" will not convey, for many teachers, a personally-meaningful message without further professional development.

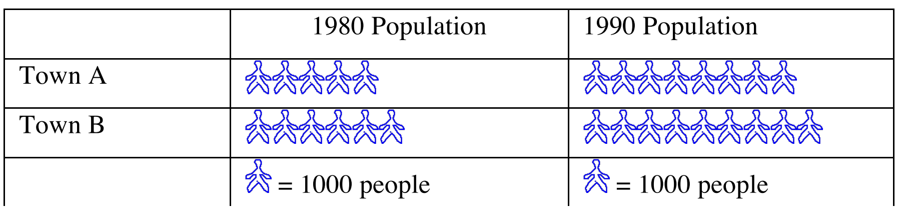
We hope no one interprets our remarks as attacking teachers or the important role that they play in students' mathematical development. We are mathematics teacher educators as well as researchers; we work daily with teachers and prospective teachers. We are mathematics teachers ourselves. However, we have a "public health" perspective on problems of mathematics teaching. Communities resolve a problem most effectively when they discusses its scope, severity, and sources openly and objectively.

[2] Chinese elementary school teachers enter normal school after ninth grade, graduating two to three years later.

Learning context of present mathematics teaching

While U.S. students often are asked to understand fractions in pedagogical contexts that provide little support, many U.S. teachers who are capable of engaging in appropriate instruction find themselves with students who are poorly prepared to participate in it. For example, the 1996 National Assessment of Educational Progress (Reese, Miller, Mazzeo, & Dossey, 1997) gave these items to 8th and 12th graders:

1. Luis mixed 6 ounces of cherry syrup with 53 ounces of water to make a cherry-flavored drink. Martin mixed 5 ounces of the same cherry syrup with 42 ounces of water. Who made the drink with the stronger cherry flavor? Give mathematical evidence to justify your answer. (NAEP M070401)



2. In 1980 the populations of Towns A and B were 5000 and 6000, respectively. In 1990 the populations of Towns A and B were 8000 and 9000, respectively.

   Brian claims that from 1980 to 1990 the two towns' populations grew by the same amount. Use mathematics to explain how Brian might have justified his answer.

   Darlene claims that from 1980 to 1990 the population of Town A had grown more. Use mathematics to explain how Darlene might have justified her answer. (NAEP M069601)

Problem 1's intent was to see the extent to which students could quantify intensity of flavor (higher ratio of cherry juice to water means more intense cherry taste). Problem 2's intent was to see whether students could compare quantities additively (by difference) as well as multiplicatively (by ratio). Compared additively, both towns grew the same amount (1000 people). Compared multiplicatively, Town A's 1990 population was 8/5 (160%) as large as its 1980 population, whereas Town B's 1990 population was 9/6 (150%) as large as its 1980 population.

While these problems might seem straightforward, they challenged 8th and 12th grade students who took part in the1996 National Assessment of Educational Progress (Reese et al., 1997). Less than half the 12th graders gave even partially acceptable answers to Problem 1; about one-fifth of 8th and one-fourth of 12th graders gave partially correct responses to Problem 2 (Table 1). NAEP students' performance was consistent with findings from a long line of studies that examined students' abilities to reason about ratios and relative amounts (Harel, Behr, Lesh, & Post, 1994; Harel, Behr, Post, & Lesh, 1992; Hart, 1978; Karplus, Pulos, & Stage, 1979, 1983; Noelting, 1980a, 1980b; Tourniaire & Pulos, 1985). U.S. students have done poorly on such items in comparison to students in other countries (Dossey, Peak, & Nelson, 1997; McKnight et al., 1987).

Table 1. Student performances in 1996 NAEP on Problems 1 and 2 (data not released)*

Grade	Problem 1	Problem 2
8	Correct:........................*	Correct: 1%
	Partially correct ....*	Partially correct........... 21%
	Incorrect ....*	Incorrect...................... 60%
	Omitted.........................*	Omitted 16%
	Off task.........................*	Off task 2%
12	Correct:....................23%	Correct: 3%
	Partially correct 26%	Partially correct........... 24%
	Incorrect 42%	Incorrect...................... 56%
	Omitted.......................9%	Omitted 16%
	Off task.......................1%	Off task 1%

Explanations of US students' poor performance on questions like these are complicated. One reason is that the understandings tapped by these questions can "go wrong" at many developmental junctures during students' schooling. As such, our analysis will therefore avoid focusing on any one path to understanding. Instead, we will analyze sophisticated understandings of fractions, leaving for other discussions of what instructional approaches might support their development.

Distinction between fractions and rational numbers

Before discussing what we mean by "to understand fractions," we will distinguish between fractions as what Kieren (1988; 1993b) calls a personally knowable system of ideas and the development of what is commonly taken as the system of rational numbers. We do this for two reasons. First, we detect a tendency among textbooks to confound fractions and rational numbers. Second, we find it profitable to point out that understanding the rational number system, where "rational numbers" is used as mathematicians use it, is so far beyond the grasp of school students that curriculum and instructional designers must be clear on what they mean by "fractions" and by "rational numbers" so that they avoid designing for incoherent learning goals.

Mathematicians rely heavily on symbol systems to aid their reasoning. Symbol systems are tools for them. Mathematicians strive to develop symbol systems (inscriptions and conventions for using them) that capture essential aspects of their intuitive understandings and means of operating, so that they need not rely explicitly on conceptual imagery and operations as they move their reasoning forward or generate further insight.

In the 18th and 19th centuries, mathematicians found that their symbol systems, used according to established conventions, led to contradictory results. Many mathematicians saw that the problem was not in the symbol systems they had devised. Rather, the problems were in their understandings of number and functional relationship that the symbol systems had intended to capture. The idea of "mathematical development of number systems" arose from the need to make commonly held, but tacit, meanings of number more precise and articulate. This development was influenced also by the emergence of new numeric understandings, such as the understanding that number systems could be created even if no one understood what they were.[3]

[3] Which was the case with complex numbers, and later with hyperreal numbers (Henle, 1986; Tall &

The mathematical construction of rational numbers is tremendously general and abstract for the same reasons that the mathematical definition of function is general and abstract. It addresses a history of paradoxes and contradictions, and the present definition of the set of rational numbers is the result of a long line of accommodations to eliminate them. For example, one motivation for the modern development of rational numbers is a spin-off from the period in which the calculus was grounded in the analysis of real-valued functions (Eves, 1976; Wilder, 1968). The original notion of derivative as a ratio of differentials (first-order changes in two quantities' values) fit with the notion of rate of change as a relationship between two varying quantities. But contradictions that traced back to notions of derivative-as-ratio led d'Alembert, for one, to wonder whether "$^{dy}\!/\!_{dx}$" should be thought of as merely a symbol that represents one number instead of as a pair of symbols representing a ratio of two numbers (Edwards, 1979). The significance of d'Alembert's question is easily missed. He suspected that *the cultural practice of considering a rate of change as being composed of two numbers was conceptually incoherent*, and that what was conventionally interpreted as a ratio of two numbers was indeed one number *that was not the result of calculating*.

Mathematicians' questions about rates-as-numbers and about continuity of functions threaded themselves into formal constructions of rational and real number systems (Eves & Newsom, 1965; Heyting, 1956; Kasner & Newman, 1940; Kneebone, 1963). It would be inappropriate to summarize that development here. Rather, we wish to emphasize that the mathematical developments of rational and real number systems interconnect many issues that typically are not treated until advanced undergraduate or introductory graduate mathematics courses. As such, we have no idea what school mathematics textbook authors or other writers

Vinner, 1981). Likewise, surreal numbers and quaternions had no immediate use except to challenge prevailing intuitions of number systems (Conway & Guy, 1997).

intend when they say they want middle-school students to "understand" rational and irrational numbers.

Conceptual Analyses of Learning Objectives

We digress momentarily to address what we mean by "to understand *x*" and to explicate our method for developing descriptions of an understanding. We do this in hopes of making our intentions precise, and thereby increasing the likelihood that you create meanings that we intend.

"Understand" has both colloquial and technical meanings. The American Heritage Dictionary (4th Ed.) lists eight senses of "understand." The first six define "understand" by reference to "comprehend" and "apprehend", themselves being defined somewhat circularly. The last two senses of "understand" operationalize it more directly:

> … (7) To accept something as an agreed fact: *It is understood that the fee will be 50 dollars*, and (8) To supply or add (words or a meaning, for example) mentally. (American Heritage, 2002)

These last two senses of "understand" underpin most colloquial uses of it and match more technical meanings as well. Skemp (1979) used Piaget's notion of assimilation (Piaget, 1950, 1971a, 1971b, 1976) when he described understanding as "assimilating to an appropriate scheme," by which he meant that a person attach appropriate meanings and imagery to the utterances or inscriptions that a person interprets. Skemp's definition of understanding coincides with Hiebert and Carpenter's notion of understanding as a rich set of meaningful connections by which a person acts flexibly with respect to problems he or she encounters (Carpenter, 1986; Hiebert & Lefevre, 1986).

We follow the tradition of Piaget, Skemp, Carpenter, and Hiebert when we speak of understanding. We choose to omit "appropriate," however, for then we can speak of a person's understanding as "assimilation to a scheme," which allows us to address understandings people *do* have even though someone else may judge them to be inappropriate. Also, we note that to describe an understanding requires addressing two sides of the assimilation—what we see as the thing a person is attempting to understand and the scheme of operations that constitutes the person's actual understanding. We see understanding as, fundamentally, what results from a person's interpreting signs, symbols, interchanges, or conversations – assigning meanings according to a web of connections the person builds over time through interactions with his or her own interpretations of settings and through interactions with other people as they attempt to do the same.

The goal in this line of work, specifying what it might mean to understand a complicated idea like fractions, is to "consider any mental content (percepts, images, concepts, thoughts, words, etc.) as a result of operations" (Cecatto, 1947 as cited in Bettoni, 1998). That is, one must describe *consapevolezza operativa*, or conceptual operations (translated literally as "operating knowledge")[4] in order to answer the question "which mental operations do we perform in order to conceive a situation in the way we conceive it?" (Bettoni, 1998). Glasersfeld (1972; 1995) combined aspects of Ceccato's operational analysis and Piaget's genetic epistemology to devise a way to talk about reasoning and communicating as imagistic processes and of knowledge as an emergent aspect of them (Glasersfeld, 1978). This produced an analytic method that Glasersfeld called *conceptual analysis*, the aim of which was to describe conceptual operations that, were people to have them, might result in them thinking the way they evidently do. Different researchers engage in varying levels of detail (Steffe, 1996; Thompson, 2000), but the aim is to describe conceptual operations in ways that are "near the surface" of the kinds of reasoning one hopes to explain.

[4] Quotations translated by M. Bettoni. Phrases translated by GO Translations, http://translator.go.com./.

When engaging in conceptual analysis, one may focus on understandings as they might exist at some level of sophistication or on how people might get to them. In regard to understanding fractions, we will focus on what might be called "mature" understandings of fractions. This is not to say that developmental issues are unimportant. They are, especially for designing curriculum and instruction to support students' developing understandings. We have chosen to describe "mature" understandings of fractions for the simple reason that later discussions can be grounded in a common image of the overall curricular and instructional goals.

Understanding Fractions

In the spirit of being explicit about the meanings and understandings we intend that students develop, we will describe one view of what "to understand" can mean with regard to the panoply of ideas and behaviors associated with the school subject called "fractions." Specifically, we will focus on fractions and multiplicative reasoning. Our use of the conjunction "and" in "fractions and multiplicative reasoning" points to the particular stance we take in our analysis: coherent fractional reasoning develops by interrelating several conceptual schemes often not associated with fractions. We use the phrase "conceptual scheme" to indicate that we are talking about stable ways of thinking that entail imagining, connecting, inferring, and understanding situations in particular ways. We emphasize that we are *not* talking about abstract formulations that reside outside every person dealing with these situations. We are talking about ways people reason when they understand fractions in the way we are attempting to convey. The schemes we characterize here are division schemes, multiplication schemes, measurement schemes, and fraction schemes.

Our discussion in this section will take a somewhat circuitous route from examples of mildly sophisticated reasoning, to discussing factors in its development, to a discussion of more general characteristics of advanced multiplicative reasoning. Our discussion draws heavily from foundational research by Gerard Vergnaud (1983; 1988; 1994), Les Steffe and Ron Tzur (Steffe, 1988, 1993; Tzur, 1999), Tom Kieren (1988; 1992; 1993b), the Rational Number Project (Behr, Harel, Post, & Lesh, 1992, 1993; Lesh, Behr, & Post, 1987; Lesh, Post, & Behr, 1988; Post, Cramer, Behr, Lesh, & Harel, 1993; Post et al., 1991), Jere Confrey (1994; Confrey & Smith, 1995) and from our own work in quantitative reasoning (Saldanha & Thompson, 1998; Thompson, 1988, 1993; P. W. Thompson, 1994; Thompson, 1995, 1996).

Kieren (1988; 1993a; 1993b) and the Rational Number Project (Behr et al., 1992, 1993; Lesh et al., 1987) have given the most extensive analyses of rational number meanings. Their approach was to break the concept of rational number into subconstructs—part-whole, quotient, ratio number, operator, and measure—and then describe rational number as an integration of those subconstructs. Our feeling is that their attempt to map systems of complementary meanings into the formal mathematical system of rational numbers will necessarily be unsatisfactory in regard to designing instruction for an integrative understanding of fractions. Each subconstruct is portrayed as a body of meanings, or interpretations, of the "big idea" of rational numbers. Mathematical motivations for developing the rational numbers as a mathematical system, however, did not emerge from meanings or subconstructs. Rather, they emerged from the larger endeavor of arithmetizing the calculus. So, to focus on subconstructs or meanings of the mathematical system of rational numbers ultimately runs the risk of asking students to develop meanings for a big idea that they do not have. Our approach will be to place fraction reasoning squarely within multiplicative reasoning as a core set of conceptual operations.

As noted already, Post et al. (1991, p. 193) gave this problem to a sample of intermediate school mathematics teachers: *Melissa bought 0.46 of a pound of wheat flour for which she paid*

\$0.83. How many pounds of flour could she buy for one dollar? A standard solution was to set up an equation, as in $\frac{0.46}{0.83}=\frac{x}{1}$, and solve for x. However, as Post et al. found, setting up this equation correctly and having a coherent understanding of it are not the same.

A solution to the *Melissa* problem that relies on reasoning instead of on equations might go like this: *If 0.46 lb costs \$0.83, then \$0.01 (being 1/83rd of \$0.83) will purchase 1/83rd of 0.46 lb. Thus \$1.00 will purchase 100/83 of 0.46 lb*. A more sophisticated expression of the same reasoning would be *\$1.00 is 100/83 as large as \$0.83 [100 times as large as 1/83rd of \$0.83], so you can buy 100/83 of 0.46 lb for \$1.00*.

What conceptual development might lead to such reasoning? A variety of sources suggest it is through the development of a web of meanings that entails conceptualizations of measurement, multiplication, division, and fractions. We emphasize *conceptualizations* of measurement, multiplication, division, and fractions. This is not the same as measuring, multiplying, and dividing. The latter are activities. The former are images of what one makes through doing them.

Measurement schemes

To conceive of a measured quantity is to imagine the measured attribute as segmented (Minskaya, 1975; Steffe, 1991b) or in terms of a coordination of segmented quantities (Piaget, 1970; Schwartz, 1988; P. W. Thompson, 1994).

The idea of ratio is at the heart of measurement. To conceive of an object as measured means to conceive of some attribute of it as segmented, and that segmentation is in comparison to some standard amount of that attribute. Suppose, for example, one wished to publicize a horse racing track's distance. How one measures its distance, however, is not straightforward. Every lane has a different length and not every horse runs in the same lane throughout. Therefore, by custom, the length of a specific lane, measured on its shortest perimeter, is taken as the "race course length."

Even when an object's attribute is clear, the matter of conceiving its measure can still remain. A mile-long race course, measured in yards, is 1760 yards because the length of one mile is understood to be 1760 times as long as the length called one yard. The same racecourse's length is 5280 feet because it is 5280 times as long as the length called one foot. "There are 5280 feet in one mile" does not mean just that a mile contains 5280 feet, in the same sense that a fielded baseball team contains 9 players. The distinction we make here is between a part-whole relationship between a set and its elements and a multiplicative comparison in which the measuring unit is imagined apart from the thing measured.

The ratio nature of measurement is trivial to people who have a quantitative scheme of measurement, but it is nontrivial to students who are building one. A conceptualized measure entails an image of a ratio relationship (A is some number of times as large as B) that is invariant across changes in measurement units. For example, Figure 1 illustrates that if m is the measure of quantity B in units of quantity A (i.e., B is m times as large as A), then nm is the measure of quantity B in units of $1/n^{th}$ of A. Conversely, if m is the measure of quantity B in units of quantity A, then m/n is the measure of B in units of nA. Put another way, the measure of a quantity is m times as large when you dilate the unit by a factor of $1/m$, and its measure is $1/m^{th}$ as large when you dilate the unit by a factor of m.

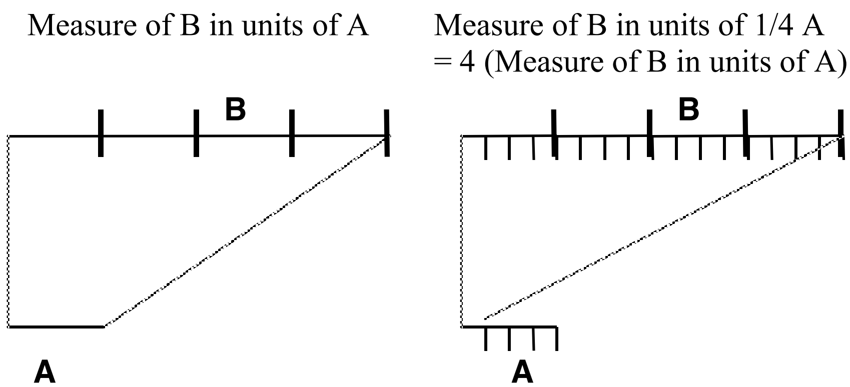


Figure 1. Change of measurement unit.

A conceptual breakthrough underlying students' understanding of unit substitutions is their realization that the magnitude of a quantity (its "amount") as determined in relation to a unit does not change even with a substitution of unit. Wildi (1991) emphasized this point by making two distinctions. The first was between a quantity's measure and its magnitude. A quantity's magnitude (it's "amount of stuff" or its "intensity of stuff") is independent of the unit in which you measure it. If we let $m(B_U)$ denote quantity B's measure relative to unit U, then $|B|$, the magnitude of B, is $m(B_U)|U|$. A change of unit does not change the quantity's magnitude—making the unit 1/4 as large makes the measure 4 times as large, leaving the quantity's magnitude unchanged.[5]

Wildi also distinguished between numerical equations and quantity equations. A numerical equation, like $W=fd$, says how to calculate a particular quantity's measure. As such, the formula's result in any particular instance depends on the particular units used. Quantity equations suggest a quantity's construction. Wildi wrote the equation [W] = [f][d] to say that accomplished work, as a quantity, is created by applying a force to an object and thereby moving

[5] The first author observed a 4th-grade lesson on the metric system in which students measured their heights. He asked one boy who had measured his height with a meter stick what he got. "140 centimeters." How tall are you? "Four feet seven." Do you know how many centimeters make 4 feet 7 inches? (Pause.) "No." This child had not realized that his height, as a magnitude, was the same in both instances and therefore did not realize that 140 cm and 4 ft 7 in were equivalent, in that both were measures of his *height*.

it some distance. The equation makes no reference to measurement units. Wildi wanted the quantity formula to say that the product quantity's magnitude remains the same regardless of the units in which you measure force or distance, as long as you measure them appropriately.[6]

Students in a 5th-grade teaching experiment on area and volume alerted us to the distinction between understanding a formula numerically and understanding it quantitatively. The first author presented the question in Figure 2. Portions of two students' interviews are given after the diagram.

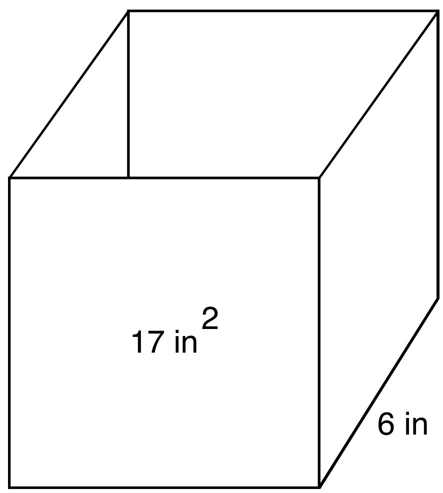


Figure 2. What is the volume of this box?

PT: (Discusses with BJ how the diagram represents a hollow box and what about it each number in the diagram indicated.)
BJ: (Reads question.) I don't know. There's not enough information.
PT: What information do you need?
BJ: I need to know how long the other sides are.
PT: What would you do if you knew those numbers?
BJ: Multiply them.
PT: Any idea what you would get when you multiply them?
BJ: No. It would depend on the numbers.
PT: Does 17 have anything to do with these numbers?
BJ: No. It's just the area of that face.

PT: (Discusses with JA how the diagram represents a hollow box and what about it each number in the diagram indicated.)

6

JA: (Reads question.) Oh. Somebody's already done part of it for us.
PT: What do you mean?
JA: All we have to do now is multiply 17 and 6.
PT: Some children think that you have to know the other two dimensions before you can answer this question. Do you need to know them?
JA: No, not really.
PT: What would you do if you knew them?
JA: I'd just multiply them.
PT: What would you get when you multiplied them?
JA: 17.

To BJ, the formula V=LWD was a numerical formula. It told him what to do with numbers once he had them. However, it had no relation to evaluating quantities' magnitudes. To the second child, JA, the formula V=LWD was a quantity formula. To him, it was V = [LW][D], where [LW] produced an area, and [LW][D] produced a volume. JA recognized that being provided one face's area was as if "somebody's done part of it for us," that part being the quantification of one face's area.

Proportionality and measurement

Proportional reasoning is important in students' conceptualizing measured quantities. Vergnaud (1983; 1988) emphasized this when he placed single and multiple proportions at the foundation of what he called the multiplicative conceptual field.[7] A single proportion is a relationship between two quantities such that if you increase the size of one by a factor *a*, then the other's measure must increase by the same factor to maintain the relationship. If you are ordering food to feed guests at a party and someone tells you that there will be three times as many guests, you will order three times as much food so that the amount of food and the number of people remain related the way you originally intended. More formally, if *x* is the measure of one quantity, and if

[7] Confrey (1994) takes a counter position, claiming that multiplicative reasoning emanates from a primitive operation called *splitting*, which does not emanate from proportionality but instead underlies it. We cannot here discuss the two in contrast, as that would take us astray of our present purpose. We will say only that we agree with Steffe (1994) that when we consider the developmental origins of splitting we see that Confrey and Vergnaud are not in opposition.

$f(x)$ is the measure of the other, and if the two are related proportionally, then $f(ax) = af(x)$.

A double proportion is a relationship among three quantities, where one is thought of as created from the other two, in which the created quantity is proportional to each of the others. More formally, if x and y are measures of two quantities, and if $f(x,y)$ is the created quantity's measure, and the created quantity is understood as being related proportionally to the other two, then:

$$\begin{aligned} f(ax,by) &= af(x,by) \\ &= bf(ax,y) \\ &= abf(x,y). \end{aligned}$$

The concept of torque, the idea that applying a force at some distance from a resistant hinge or fulcrum creates some amount of twisting force (Figure 3), provides an example of how being able to conceptualize a double proportion provides insight into devising a measuring scheme for a new quantity.

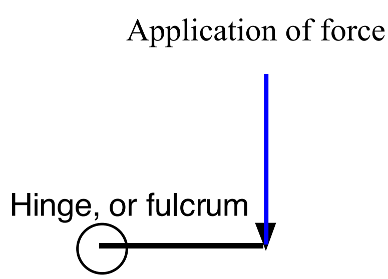


Figure 3. An amount of twisting force created by applying a force at a distance

It is easy to convey an experiential understanding of torque by having students attempt to hold weights of various amounts on a broomstick at various distances from their hands. They experience personally that the same weight placed farther out is harder to hold than when placed closer in. Unfortunately, such activities do not convey an idea of how one might *measure* the amount of twisting force created by a specific weight held a specific distance from their hands. However, if students understand *amount of twisting force* as being related to *weight* and *distance* *from fulcrum* in a double proportion, then its quantification is natural:

I. Suppose that applying a force of y force-units at a distance of x distance-units from a fulcrum produces T twist-units.

II. Increasing the applied force by a factor of u while applying it at the same distance increases the amount of twist by a factor of u. That is, applying a force of uy force-units at a distance of x distance-units produces uT twist-units.

III. Increasing the distance at which the original force is applied by a factor of v increases the amount of twist by a factor of v. That is, applying a force of y force-units at a distance of vx distance-units produces vT twist-units.

Observations I-III in combination imply that if applying a force of y force-units at a distance from a fulcrum of x distance-units produces T twist-units, then applying a force of uy force-units at a distance of vx distance-units produces a twisting force of measure uvT twist units. By convention, a force of 1 force-unit applied at a distance of 1 distance-unit produces 1 twist-unit. Thus, a force of $u \cdot 1$ force-units applied at a distance of $d \cdot 1$ distance-units produces u·v twist units. An understanding that *amount of twisting force* is proportional to each of distance-from-fulcrum and amount-of-applied-force leads directly to quantifying torque by the formula "force times distance". That is, the quantification of torque by multiplying measures of force and distance can be grounded naturally in understanding torque as entailing a double proportion. It needn't be taught as a formula to memorize.

Multiplication schemes

Contrary to most textbooks, and contrary to Fischbein's (1985) well-known treatise, multiplication is not the same as repeated addition. To be fair, we should say *conceptualized* multiplication is not the same as repeated addition.

> The conceptual foundation of multiplication of whole numbers is quite like the Biblical "multitudes"—creating many from one. That is to say, multiplication of whole numbers is the systematic creation of units of units. (Thompson, 1982, p. 316)

The difference between conceptualized multiplication and repeated addition is between envisioning the result of having multiplied and determining that result's value (Steffe, 1988). Envisioning the result of having multiplied is to anticipate a multiplicity. One may engage in repeated addition to evaluate the result of multiplying, but envisioning adding some amount repeatedly cannot support conceptualizations of multiplication.

We generalize the previous section's torque example to illustrate that the capability to conceive of multiple proportions leads generally to numerical multiplication. Suppose a double proportion relates three quantities. By convention, the standard unit of product quantity is defined as that amount made by one unit of each constituent quantity. (If we measure the three edges of a box in units of light years, centimeters, and inches respectively, then one unit of volume is one light year-centimeter-inch.) If x and y are the constituent quantities' measures, and $f(x,y)$ is the product quantity's measure expressed as a function of the other two, then the measure of the product quantity's standard unit is 1.

$$\begin{aligned} meas(StandardUnit) &= f(1,1) \\ &= 1. \end{aligned}$$

As such, the third quantity's measure will be:

$$\begin{aligned} f(x,y) &= f(x \cdot 1, y \cdot 1) \\ &= x \cdot y \cdot f(1,1) \\ &= x \cdot y \cdot meas(StandardUnit) \\ &= x \cdot y \cdot 1 \\ &= x \cdot y \end{aligned}$$

We use the statement "$f(x,y) = f(x \cdot 1, y \cdot 1)$" as a model of a particular understanding. That understanding is *not* that a student knows that x may be rewritten as $x \cdot 1$. Rather, it is that, since $f(x,y)$ stands for "the measure of the object constructed from attributes having measures x and y," our rewriting x as $x \cdot 1$ and y as $y \cdot 1$ models the explicit understanding that measurement entails a ratio comparison. For example, if a student conceives a rectangle as being made by a product of its sides, where one side is 12 inches long and the other is 2 cm long, then to say $f(12,2) = f(12 \cdot 1, 2 \cdot 1)$ reflects the move from thinking of 12 inches as a number of inches to 12 inches as a length that is 12 times as long as one inch (Figure 4.). The rectangle's (area) measure is, then, 24 in-cm, or 24 times the area of a rectangle of dimension 1 inch by 1 cm.

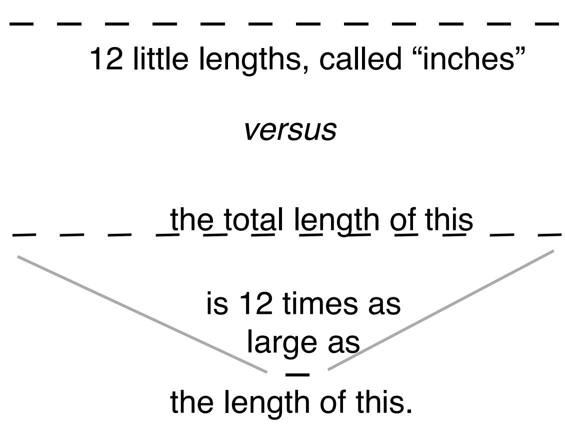


Figure 4. Measure as a number of things vs. measure as a ratio comparison.

The fact that to understand objects as entailing a single or multiple proportion relies on understanding measures as ratio comparisons highlights the special way in which multiplication must be conceived—as entailing a multiple proportion—if it is to cohere with students' understandings of fractions. Generally, most students do not see proportionality in multiplication. In fact, a large amount of curriculum and instruction has the explicit aim that students understand multiplication as a process of adding the same number repeatedly. But an extensive research literature documents how "repeated addition" conceptions become limiting and problematic for students having them (de Corte, Verschaffel, & Van Coillie, 1988; Fischbein et al., 1985; Greer, 1988b; Harel, Behr, Post, & Lesh, 1994; Luke, 1988).

As mentioned earlier, Fischbein (1985) and Burns (2000) propose repeated addition as the fundamental initial idea of multiplication. Because of students' prior, additive, numerical and quantitative experience, portraying multiplication as repeated addition makes multiplication easy to teach and, in this limited calculational sense, easy to learn. But thinking about multiplication as repeated addition leads to severe difficulties in later grades. Specifically, if 5×4 means 4+4+4+4+4, "add 4 five times", then it is difficult to say what $5\frac{2}{3} \times 4$ means. It cannot mean, "add 4 five and two-thirds times".

To introduce multiplication as repeated addition is, in principle, like introducing the quantification of work before discussing how to think of work so that it is quantifiable. Repeated addition is a quantification technique. It is not the thing being quantified. What does 4+4+4+4+4 quantify? Five fours. As described by Confrey (1994) and Steffe (1988, 1994) multiplication can be introduced to students by asking them to think about quantities and numbers in settings where they need to envision a multiplicity of identical objects. The question, "How much (many) do these make?" comes after conceiving what "these" are, and serves to orient students toward the quantification of multiplicities. The quantification itself may or may not involve repeated addition.

This approach to conceiving multiplication as about quantifying something made of identical copies of some quantity solves another problem created by students' thinking of multiplication as repeated addition. If we start with the basic meaning of 5×4 as "five fours", then $5\frac{2}{3} \times 4$ means "(five and two-thirds) fours". The principle difference between "add 4 five times" and "five fours" is that the former tells us a calculation to perform while the latter suggests something to imagine. Similarly, if students read "$5\frac{2}{3} \times 4\frac{3}{5}$" with a multiplicity meaning, they will think about some number of (four and three-fifths). They will understand

"$5\frac{2}{3} \times 4\frac{3}{5}$" as meaning 5 (four and three-fifths) and 2/3 of (four and three-fifths). When students understand to what the statement refers they can do whatever is sensible given the surrounding context.

This example carries a larger significance. It is that how students decode mathematical statements can have a large impact on the connections they form. If they read "5×4" as a command to calculate, then "$5 \times x$" is highly problematic. But if they read "5×4" as "five fours", this supports the image that we are speaking of a product – (5×4) is a *number* that is 5 times as large as four. "5×4" becomes a noun phrase and therefore points to something. Thus, understanding "$5 \times x$" as "a *number*, a number that is 5 times as large as the number x," can be a natural extension of their understanding of numerical multiplication, but only if their understanding of numerical multiplication is appropriately general in the first place.

We re-emphasize that when a curriculum starts with the idea that "__×__" means "some number of (or fractions of) some amount," it is not starting with the idea that "times" means to calculate. It is starting with the idea that "times" means to envision something in a particular way—to think of copies (including parts of copies) of some amount. This is not to suggest that multiplication should not be about calculating. Rather, calculating is just one thing one might do when thinking of a product. Non-calculational ways to think of products will be important in comprehending situations in which multiplicative calculations might be useful. The comprehension will enable students to decide on appropriate actions.

Though we will discuss fraction schemes in a later section, it is worthwhile to point out here that for someone to understand multiplication multiplicatively he or she must also understand fractions as entailing a proportion. We will propose that the phrase "quantity X is $1/n$ of a quantity Y" mean that Y is n times as large as X. From that perspective, it does not mean that

X is one of n parts of Y.[8] Thinking of $1/n$ as "one out of n parts" is to think of fractions additively—that Y is cut into two parts, one being X and the other being the rest. It is merely a way to indicate one part of a collection. When students' image of fractions is "so many out of so many", it possesses a sense of inclusion—that the first "so many" must be included in the other "so many". As a result, they will not accept the idea that we can speak of one quantity's size as being a fraction of another's size when they have nothing physically in common. They will accept "The number of boys is what fraction of the number of children?", but they will be puzzled by "The number of boys is what fraction of the number of girls?"

To think of multiplication as producing a product and to think at the same time of the product in relation to its factors entails proportional reasoning. To understand (5×4) multiplicatively, students must understand that 4 in 5×4 is not just 4 ones, as in $20 = 4 + 9 + 7$. Rather, 4 is special—it is 1/5 of the product.[9] In general, when students understand multiplication multiplicatively, they understand the product (nm) as being in multiple reciprocal relationships to n and to m:

- (nm) is n times as large as m,
- (nm) is m times as large as n,
- m is $1/n$ as large as (nm)
- n is $1/m$ as large as (nm).

An example using "ugly numbers" might clarify this point. In the expression $4\frac{3}{7} \times 7\frac{2}{3}$

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[8] This poses an interesting question. To what extent do we want to respect meanings that are commonly held but that end up hurting students who adopt it? A person with a sophisticated understanding of fractions can say "one out of n," really mean it additively at the moment of saying it (like, one of those six boys has a blue shirt), and yet flip effortlessly and unawarely to "the number of boys over there is six times as large as the number of boys with blue shirts". We do not object to the use of the words "one out of n". Our objection is that many people think that "one out of n" automatically conveys the latter. We contend that it takes a consistent, systematic effort over multiple grades to ensure that the majority of students internalize the multiplicative point of view, so that it becomes "the way they see things."

[9] We are indebted to Jose Cortina for this observation.

(read as [four and three-sevenths] seven-and-two-thirds), $7\frac{2}{3}$ is special. It is $\frac{1}{4\frac{3}{7}}$ as large as the product. What does "$\frac{1}{4\frac{3}{7}}$ as large as the product" mean? Recall that "*a* is one-n$^{\text{th}}$ of *b*" means that *b* is *n* times as large as *a*. So, to say that $7\frac{2}{3}$ is $\frac{1}{4\frac{3}{7}}$ as large as $\left(4\frac{3}{7}\times 7\frac{2}{3}\right)$ means that $\left(4\frac{3}{7}\times 7\frac{2}{3}\right)$ is $4\frac{3}{7}$ times as large as $7\frac{2}{3}$ (see Figure 5). One shaded region ($7\frac{2}{3}$) is $\frac{1}{4\frac{3}{7}}$ of the total, because the total is $4\frac{3}{7}$ times as large as $7\frac{2}{3}$. The other three reciprocal relationships can be elaborated similarly.

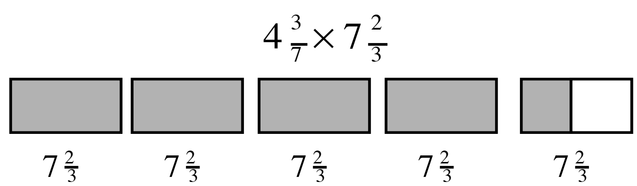

Figure 5. (Four and three-sevenths) seven and two-thirds.

We have attempted to convey the generality inherent in thinking of multiplication in a way that entails proportionality and multiplicity. When presenting this perspective in professional development settings we sometimes hear the remark, "This is too complicated. Thinking of multiplication as repeated addition is much easier." We agree that repeated addition is easier than proportionality. But understanding multiplication as repeated addition keeps it divorced conceptually from measurement, proportionality, and fractions.

Division schemes

Several research studies point to students' difficulties in conceptualizing division and its relationships with multiplication and fractions. Greer (1987; 1988a) and Harel (1994) noted the great frequency with which students will decide upon the use of multiplication or division depending on the size or type of numbers involved, not on the underlying situation. Graeber

(1988) found the same to be true among pre-service elementary teachers. Ball (1990) found that college mathematics majors had difficulty proposing situations in which it would be appropriate to divide by a fraction. Simon (1993) found that pre-service elementary teachers' knowledge of division was tightly intertwined with their whole-number schemes, which resulted in weak connections among various meanings of division and in their inability to identify the result's unit of measurement. In short, it is unquestionable that division is a problematic area of mathematics teaching and learning.

The fact that division is a problematic area does not clarify what is problematic or what constitutes an understanding of it. In addressing this issue we will begin with a widely used distinction between two "types" of division settings (sharing and segmenting), and demonstrate that, among other things, operational understanding of division entails a conceptual isomorphism between them.

Sharing (or partitioning) is the action of distributing an amount of something among a number of recipients so that each recipient receives the same amount.[10] Segmenting (or measuring) is the action of putting an amount into parts of a given size. "13 people will receive 37 pizzas. How many pizza's will each person receive?" usually is classified as a sharing situation, because of the request to distribute the pizzas into a pre-established number of groups. "You are to give 3 pizzas to each person, and you have 37 pizzas. How many people can you serve?" usually is classified as a segmenting situation, because of the request to put the pizzas into pre-established group sizes. It is well known that students often see these situations as being very different (Bell, Fischbein, & Greer, 1984; Bell, Swan, & Taylor, 1981; Greer, 1987, 1988b),

[10] We will speak as if all sharing situations are equi-sharing, and that all segmenting situations are equi-segmenting.

and they are often puzzled that both can be resolved by the numerical operation of division.[11]

How is it possible that the results of sharing and segmenting are evaluated by the same numerical operation? To see why the numerical results are the same in either situation entails the development of operative imagery—the ability to envision the result of acting prior to acting—and to suppress attention to the process by which one obtains those results.

To illustrate why the same numerical operation resolves the two situations we will speak of two scenes. In Scene 1 we will share an amount of chocolate (measured in bars) so that each of 7 people receive the same amount of chocolate (the same number of bars) and ask how many bars each person receives. In Scene 2 we will cut up the chocolate in parts of size 7 bars and ask how many parts we make. In both scenes, it is advantageous to consider the chocolate as one mass (that comprises some number of bars).

Scene 1: Share the chocolate among 7 people.

- The process of sharing ends up with each recipient having the same amount of chocolate. So, if we put the chocolate into 7 parts, each part contains 1/7 of the chocolate. So, the number of candy bars in each part is 1/7 as large as the total number of candy bars that comprise the entire amount of chocolate.
- Each person receives a number of bars that is 1/7 of the total number of bars.

Scene 2: Put the chocolate into 7-bar parts

- The process of segmenting cuts up the mass of chocolate into a number of parts of a given size, perhaps with an additional part that is a fraction of the given size.
  - We cut this mass into parts, each part the size of 7 bars, and consider any chocolate left

[11] It is also well known, by these same studies, that if you change the numbers so that in either situation each person receives 2/3 pizza, students will see these problems even differently yet. We agree with Confrey (1994) that this result is entirely an artifact of persistent experience with stereotypical problems in which products are always larger than either factor and quotients are always smaller than the minuend.

over to make some fraction of a part (i.e., a fraction of 7 bars).

- We can determine the number of parts by copying one bar from each part and a proportionate part of a bar from any fraction of a part.
- Each part (or fraction of a part) is 7 times as large as what we copied from it, so the total is 7 times as large as what we copied in all.
- The mass of chocolate is 7 times as large as what we copied, and what we copied counts the number of parts.

- The number of parts made by cutting up the chocolate into 7-bar sized parts is 1/7 as large as the number of bars that make up the entire amount of chocolate.

These two analyses show that sharing (partitioning) a number of candy bars among 7 people and segmenting (measuring) that number of bars into 7-bar-sized parts both produce a number that is 1/7 as large as the number of candy bars. The above analysis also reveals that sharing (partitioning) and segmenting (measuring) are highly related even though, at the level of activity, they appear to be very different. When students understand the numerical equivalence of measuring and partitioning they understand that any measure of a quantity induces a partition of it and that any partition of a quantity induces a measure of it.

Fraction schemes

A common approach to teaching fractions is to have students consider a collection of objects, some of which are distinct from the rest, as depicted in Figure 6.

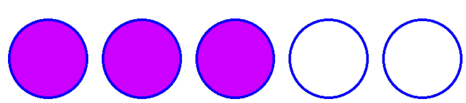

Figure 6. What does this collection illustrate?

What might Figure 6 illustrate to students? It often is portrayed as illustrating 3/5 as if it cannot illustrate anything else. A person could also see Figure 6 as illustrating that $1 \div \frac{3}{5} = 1\frac{2}{3}$—that within one whole there is one three-fifths and two-thirds of another three-fifths, or that $5 \div 3 = 1\frac{2}{3}$—that within 5 is one 3 and two-thirds of another 3. Finally, they could see Figure 6 as illustrating $\frac{5}{3} \times \frac{3}{5} = 1$—that five-thirds of (three-fifths of 1) is 1 (Figure 7).

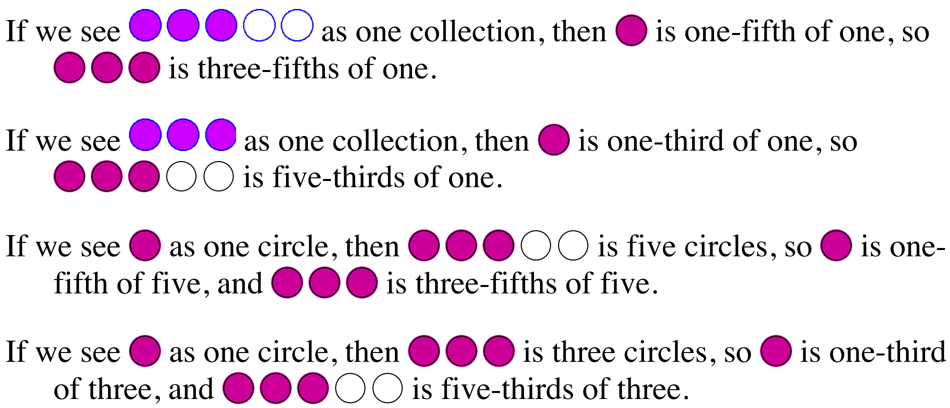

Figure 7. *Various ways to think about the circles and collections in Figure 6.*

We rarely find texts or teachers discussing the difference between thinking of 3/5 as "three out of five" and thinking of it as "three one-fifths." How a student understands Figure 6 in relation to the fraction 3/5 can have important consequences. When students think of fractions as "so many out of so many" they are puzzled by fractions like 6/5. How do you take six things out of five?[12]

The system of conceptual operations comprising a fraction scheme is based on conceiving two quantities as being in a reciprocal relationship of relative size: *Amount A is 1/n the size of amount B means that amount B is n times as large as amount A. Amount A being n times as large as amount B means that amount B is 1/n as large as amount A*. Another way to say "reciprocal relationship of relative size" is to say that the two amounts in comparison are

[12] We often hear teachers and teacher education students say "change 6/5 to $1\frac{1}{5}$ and they'll understand." This misses the point. It is problematic if a student *must* change 6/5 to $1\frac{1}{5}$, for it means that students cannot understand any situation in which they must see fractions as entailing a proportional relationship.

each measured in units of the other. Amount B being 7 times as large as amount A is to say that amount B is measured in units of A. Amount A being 1/7 as large as amount B is to say that amount A is measured in units of B.

We worded these sentences very carefully, avoiding the phrase "1/n of". This is to emphasize the slippery connection, mentioned earlier, between comparison and inclusion. Fraction sophisticates can say, "A is some fraction *of* B" and get away with it because they imagine that A and B are separate amounts even though A might be perceived as including B (see Figure 4). Students are often instructed, and therefore learn, that the fractional part is contained within the whole, so "A is some fraction *of* B" connotes a sense of inclusion to them, that A is a subset of B. As a result, statements like "A is 6/5 of B" make no sense to them. However, when thought of multiplicatively, "A is $^{m}/_{n}$ as large as B" means that A is *m* times as large as $^{1}/_{n}$ of B. So, from a multiplicative perspective, "A is 6/5 as large as B" does not imply that A and B have anything in common. Rather, it means that A is 6 times as large as (1/5 of B).

Our conceptual analysis of fraction knowledge resembles the approach taken by the Rational Number Project (Behr et al., 1992, 1993; Lesh et al., 1988). They characterized "m/n of B" as m one-n^{ths} of B. The difference between their characterization and ours is that we stress the need for a direct link to students' conceptualizing reciprocal relationships of relative size. Without our insistence on students having that link they could appear to be speaking of fractions in ways we intend while in fact be thinking of additive inclusion—that 1/n of B is just one of a collection of pieces—without grounding it in an image of relative size.

Students gain considerable mathematical power by coming to understand fractions through a scheme of operations that express themselves in reciprocal relationships of relative size. For example, "related rate" problems are notoriously difficult for U.S. algebra students.

> It took Jack 20 seconds to run as far as Bill did in 17 seconds. Jack ran at about 18 ft/sec throughout. About how fast did Bill run?

One way to reason about this problem using basic knowledge of fractions is as below. The chain of reasoning is based on Figure 8. In it we see that Jack and Bill ran the same distance but in different amounts of time. It is important to note that 1 second is 1/20 of Jack's time and is 1/17 of Bill's time.

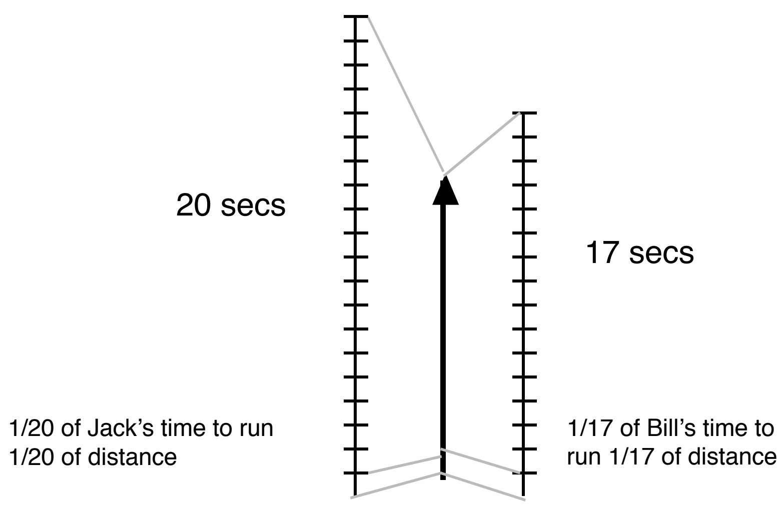

Figure 8. Jack and Bill took different amounts of time to run the same distance.

- Bill runs 1/17 of the distance in 1/17 of his time.
- Jack runs 1/20 of the distance in 1/20 of his time.
- 1/17 of Bill's time is as large as 1/20 of Jack's time.
- Were Bill to continue running 3 seconds, he and Jack will have run the same amount of time.
- Were Bill to continue running another 3 seconds, he will run an additional 1/17 of the distance in each second
- Bill runs 20/17 as far as Jack in the same amount of time
- Therefore, Bill runs 20/17 as fast as Jack.

To understand fractions as based in reciprocal relationships of relative size draws heavily on relationships among measure, multiplication, and division. For example, if we interpret "$a \div b$" as "the number of b's in a". Then whatever number $a \div {}^{m}\!/_{n}$ is (i.e., whatever number is the number of ${}^{m}\!/_{n}$ in a) it is n times as large as $a \div m$, because ${}^{m}\!/_{n}$ is ${}^{1}\!/_{n}$ as large as m. (see the discussion following Figure 1).[13] When we recall that $(a \div m)$ is ${}^{1}\!/_{m}$ as large as a we can conclude that whatever number is $a \div {}^{m}\!/_{n}$, it is n times as large as ${}^{1}\!/_{m}$ of a. Put more briefly, $a \div {}^{m}\!/_{n} = {}^{n}\!/_{m} \cdot a$. "The number of ${}^{m}\!/_{n}$'s in a is n times as large as ${}^{1}\!/_{m}$ of a." This is a conceptual derivation of the "invert and multiply" rule for division by fractions, and its interpretation is straightforward when thinking in terms of relative sizes. Similarly, we can use fraction relationships to reason about algebraic statements like $x = y/32$. "$x=y/32$" means that x is 1/32 as large as y. That means that y is 32 times as large as x. So, $y=32x$.

The evolution of students' understandings of reciprocal relationships of relative size is still being researched, especially by Steffe and his colleagues (Hunting, Davis, & Pearn, 1996; Steffe, 1991a, 1993, 1994, in press; Tzur, 1999). The fact that this understanding happens so rarely among U.S. students makes it quite hard to research its development. But the fact that these understandings of fractions exist so rarely is a significant problem for U.S. mathematics education, for there is some evidence that it is expected more routinely elsewhere (Dossey et al., 1997; Ma, 1999; McKnight et al., 1987). It should be a dominant topic of discussion that U.S. instruction fails to support its development, and we should understand the reasons for that failure in detail. At present, we can say little more than "it doesn't happen because few teachers and teacher educators expect it to happen." We should be able to say more.

[13] How you say the symbols to yourself while reading this passage can help or impede understanding it. To read "m/n" as "m slash n" or "m over n" makes it virtually impossible to make sense of the passage. To read "m/n" as "m n$^{\text{ths}}$" helps. To read it as "m one-n$^{\text{ths}}$" helps even more.

The matter of students' and teachers' inattention to what numbers are and what it means to operate on them is reminiscent of a discussion between Samuel Kutler (1998) and John Conway (1998) in which Kutler wondered about Conway and Guy's (1998) treatment of fractions. Kutler asked:

> … they [Conway and Guy] give this example: 2/3 x 1/4 = 4/6 x 1/4 = 1/6. What is the logic of this? Do [they] think that the readers have learned that the definition of the product of a/b and c/d is ac/bd and that they have forgotten it? Do they think that the reader knows or will investigate the justification for compounding the ratios of numbers, or what? (Kutler, 1998)

Conway responded:

> Guy and Conway DO think, unfortunately, that readers have learned that that is the definition of the product, which it really isn't, except in formal axiomatic contexts. Suppose one stick is three-quarters as long as another, whose length is two-and-a-half inches. Then do you really think that the reason the shorter stick's length is 15/8 inches is because the DEFINITION of a/b times c/d is ac/bd? It ISN'T! It's because what actually happens is that if you cut the longer stick into 4 equal quarters and throw one of them away, the total length of what's left will actually be 15/8 inches. (Conway, 1998)

Kutler spoke from a presumed image that the result of multiplying fractions is determined by an algorithm. Conway pointed out that the result of multiplying fractions is determined by what multiplication of fractions *means*. In his example, $3/4 \times 2\frac{1}{2}$ refers to a length that is 3 times as long as is $1/4$ of $2\frac{1}{2}$. That is the way he turned "$3/4 \times 2\frac{1}{2}$" into something to which it referred. It happens that the result is 15/8 not by definition, but by coincidence. The rule "a/b × c/d = ac/bd" is a generalization that derives from the meanings of multiplication and of fractions in conjunction with a particular notational system for expressing those meanings. We agree with Conway that it is misguided to portray multiplication of fractions as being defined by a generalization, and we support the larger implication that it is misguided to portray numbers and operations as being defined by what could be (under a more enlightened pedagogy) generalizations rooted in meaning-making activities.

Though we have chosen not to speak of developmental issues of fraction understanding, we do note that Steffe (in press) describes a form of reasoning that is quite close to a conception of fraction that, while not entailing proportionality, students having it are close to understanding two magnitudes in reciprocal relationship of relative size. Steffe describes students who, when asked to make a stick that is $1/12^{th}$ the length of a given stick, generated a length that, when iterated 12 times, produces a length as long as the given stick. These students did not have a conception of fractions that entailed proportionality, but they were fully aware that the longer stick was made of 12 of the shorter ones, and that 12 of the shorter ones made one of the longer ones. What they were lacking was the sense of reciprocity of relative size—that the shorter stick was measured in units of the larger one as well as the larger one in units of the shorter. Steffe claims that this is an intermediate state between additive and multiplicative understandings of fractions, and that through repeatedly generating such parts in relation to wholes students will solidify their image into one that entails proportionality and reciprocity of relative size.

Fractions—a synthesis

The title of this section does not point to what we will do in it. We will not synthesize what we've said about measurement, multiplication, division, and fraction. Rather, it points to what fractions are. Fractions *are* a synthesis. We do not claim this in a realist sense. We mean it in a cognitive sense. Fractions become "real" when people understand them through complementary schemes of conceptual operations that are each grounded in a deep understanding of proportionality. At the same time, a focus on developing these schemes will enrich students understanding of proportionality.

Most evidence points to the firm conclusion that it is rare for fractions to be real to U.S. students. The extent to which other countries' cultures and educational systems support students' creation of that reality is an open question, but an important one. Nevertheless, our system would be improved were it to support students' creation of fractions. We need not compare ourselves to other countries in setting goals for ourselves.

Conclusion

We have attempted to make explicit one image of what "to understand fractions" means. In doing so we described conceptual operations that might underpin those understandings. We stress that we do not intend this chapter to be definitive. Rather, we offer it as a "sacrificial example" that, we hope, contributes to productive discussions of what we intend students learn from instruction.

We have not discussed issues of symbolization, representation, symbolic skill, problem solving, and so on. The issues we've raised are largely independent of symbolism and symbolic procedure. Conceptual operations are about what one *sees*, not what one does, and about reasoning that is grounded in meaning. This is not to say that "doing" is unimportant. Rather, it expresses the common-sense adage that you should conceive of what you are trying to accomplish before trying to do it. Rules and shortcuts for operating symbolically should be generalizations from conceptual operations instead of being taught in place of them. Of course, part of building conceptual operations is the attempt to express them in symbols and diagrams. Symbolic operations can become the focus of instruction once students have developed coherent and stable meanings that they may express symbolically. But as a number of people have stressed, it is important that students have a rich web of meanings to fall back on when they become confused or cannot recall a remembered procedure (Brownell, 1935; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; I. Thompson, 1994; Thompson, 1992; Wearne & Hiebert, 1994).

There is a practical drawback to scheme-based characterizations of learning objectives. Assessment of whether students have achieved a learning objective is more complicated when expressing it in terms of schemes of conceptual operations than when expressing it in terms of behavioral skills. When learning objectives are stated in terms of skills, determining whether a student has achieved them is straightforward. When learning objectives are stated as schemes of operations, students' behavior must be interpreted to decide whether it reflects reasoning that is consistent with the objectives' achievement. This complication is unavoidable. However, coming to see reasoning in students' behavior is part of coming to understand the scheme-based approach we've used, so attempts at making teachers sensitive to students' reasoning may have the side-effect of preparing them to think of learning objectives in terms of conceptual operations.

In closing, we re-emphasize our motive for this chapter. We believe that conditions for improved instruction entail an enduring discussion of what the community intends that students learn. When teachers and teacher educators lack clarity and conceptual coherence in what they intend students learn, they introduce a systematic disconnection between instruction and learning. This becomes a major obstacle to creating instruction that empowers students to think mathematically. Please note that we said the problem is a lack of clarity and coherence in what teachers and teacher educators intend that students learn; we did not say that any one intention is correct. Nevertheless, not all intentions are equally coherent or equally powerful. We hope this chapter helps initiate prolonged discussions of what those intentions should be.

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