A general discussion of learning theory and its relationship to instructional design and instructional practices is provided. This is followed by the presentation of two individual cognitive frameworks. A covariational reasoning framework that characterizes five mental actions of covariational reasoning is presented. A case is made for why the reasoning abilities of the covariation framework are foundational for understanding major concepts of first semester calculus (e.g., limit, derivative, accumulation, the FTC). A brief discussion of how this framework influenced the design and development of the accumulation curriculum is included. An overview of the multi-dimensional problem solving framework is also provided and includes a description of specific reasoning patterns and their interaction with metacognitive behaviors, conceptual knowledge, affective responses, and heuristics in problem solving. We discuss how the findings of this research influenced Carlson’s thinking about instructional approaches for calculus instruction, proposing that instructional design from a radical constructivist perspective is made compatible with more situative perspectives through the creation of epistemic students and by the understanding that the situations students live in are of their making. It is through the act of conceptual modeling that researchers can address students’ situated cognitions in the service of designing for individual students’ learning.

INTRODUCTION

Learning theory has often seen a tug-of-war between perspectives that focus on the individual student and those that focus on the social context in which the student learns. Many researchers in mathematics education have investigated the process and products of learning mathematics from the perspective of the cognition of the individual learner (e.g., Dubinsky, 1992; Carlson, 1998; Thompson, 1994). Others have advocated that learning is a social phenomenon with learning being defined by the individual’s ability to participate in a community of practice (Cobb, Gravemeijer, Yackel, McClarin & Whitenack). This situated cognitive theory of learning maintains that situations co-produce knowledge and that the activity and context in which learning takes place is fundamentally linked to the nature of the learning that occurs (Brown, Collins & Duguid, 1989). Both approaches have their critics. Cognitive models have been criticized for focusing exclusively on individual learners, while ignoring the role of social practices on the individual’s psychological processes (Solomon, 1989). Situated cognitive models, in turn, have been criticized for placing too much emphasis on evaluating learning based on what has become known as the taken as shared meaning and practices of the community, with little emphasis on attending to and accounting for the cognitive development of the individual.

More recent positions advocate that these two perspectives need not be in conflict. We maintain that the theory of radical constructivism, properly understood, allows the theorist to account for both the individual-cognitive and the social-cognitive aspects of learning. In the next section of this paper we revisit the philosophy of constructivism and explain why we maintain that it does not preclude accepting that social influences play a part in students’ learning. We then look at how, working within a radical
constructivist/RME perspective, Carlson leveraged cognitive research on understanding functions and rate of change to design curricular modules for a first-semester calculus course and to test and refine those modules over three semesters. Finally, after thoroughly examining Carlson’s research-based interventions, we circle back to emphasize why radical constructivism is a logical philosophical home for this RME-style of instructional theory.

THE CONSTRUCTIVIST THEORY OF KNOWING

Constructivism is a philosophy of knowing that can be traced to the 18th century and the work of philosopher Giambattista Vico, who maintained that humans can understand only what they have constructed themselves. The more contemporary notions of constructivism are rooted in the work of Jean Piaget and John Dewey, who imbued constructivism with a theory of learning that is grounded in the idea that learning happens through a dialectic between processes of assimilation and processes of accommodation. To them, learning results in individual mental constructions by matching new information with old, and by making meaningful connections and order. Learning is judged to be successful when a student is able to demonstrate conceptual understanding, including the ability to use newly acquired knowledge as a tool.

Constructivism has been characterized as a background theory; that is, it is a theory that does not explain phenomena or prescribe actions. Hence there can be no constructivist pedagogical theory (Simon, 1995; Thompson, 2002). The notion of constructivism as a background theory does not dictate instructional practices. It posits that whatever sense a person makes of an experience, that sense is constructed by the individual. Many researchers interpret this stance to be solipsist, and therefore anathema to instructional design. However, if one keeps in mind that constructivism provides a model of knowing, not a theory of learning, it is not in conflict with the tenets for Realistic Mathematics Education (RME). It supports the notion that genetic decomposition of a concept or mathematical idea provides useful insights for defining a hypothetical learning trajectory for a lesson. Information of both the process of learning and the products of learning should provide useful knowledge for creating situations in which the learning experiences shift to promote construction by individual students.

Clarifying the Tenets of Radical Constructivism

A perceived shortcoming of research conducted within a radical constructivist perspective is that it ignores the social dimension of human cognition. However, von Glasersfeld's elaboration of Piaget's genetic epistemology into what he eventually called radical constructivism (Glasersfeld, 1978) grew in large part out of his interest in understanding human communication and language (Glasersfeld, 1970, 1975, 1977, 1990; von Foerster, 1979). So, from its very beginning, the core problems of radical constructivism entailed the question of how physically disconnected self-regulating organisms could influence each other to end in a state where each presumes there is essential common agreement on what is their shared environment (Maturana, 1978; Richards, 1991). That is, from its very outset an image of cognizing individuals in social
contexts was central to radical constructivism. This is one reason for its early appeal within mathematics education (Steffe & Kieren, 1994).

There is, however, an essential difference between radical constructivism and social constructivism. The former takes social interaction as a phenomenon needing explanation, whereas the latter takes it as a constitutive element of human activity. This difference expresses itself most vividly in the types of explanations coming from radical constructivists and social constructivists. The former tend to focus on human discourse as emanating from interactions among self-organizing, autonomous individuals. The latter tends to focus on the collective activity in which individuals participate. That is, from a radical constructivist perspective, what we take as collective activity is constituted by interactions among individuals each having schemes by which they generate their activity and by which they make sense of other’s actions. In contrast, from the social constructivist perspective, collective activity and social interaction are given, predating any individual’s participation in it. The individual accommodates to social meaning and practice.

Characteristics of group activity, from a radical constructivist perspective, emerge by way of individuals’ mutual interpretation of what each perceives as other-oriented action. Collective activity is constituted by these interactions. If we also assume that individuals reflect on their actions, then it follows that each individual’s participation changes as she becomes aware of, elaborates, and interiorizes her activity and her understanding of its repercussions. From a radical constructivist perspective, the elementary processes that account for collective activity are intersubjective operations within individuals and interactions among overlapping groups of individuals. As two people interact we understand both to be predicating their actions on an image they have of the other and implications that image has for their actions. Each person acts self-reflexively; each person imagines and acts, sometimes reflectively but always through imagery built from past interactions. As Glasersfeld (1995) notes, to say two people communicate successfully means no more than that they have arrived at a point where their mutual interpretations, each expressed in action interpretable by the other, are compatible—they work for the time being. Intersubjectivity is the state in which each participant in a socially ongoing interaction feels assured that others involved in the interaction think pretty much as he or she anticipates they do. That is, intersubjectivity is not a claim of identical thinking. Rather, it is a claim that no one sees a reason to believe others think differently than he or she assumes they do.

The importance of these considerations for educational research from a radical constructivist perspective is threefold. First, they point to some of the tacit assumptions behind the observation that an individual’s cognitions are, at once, psychological and social (Cobb, 1994, 2002). Individual cognitions are simultaneously psychological and social in the eyes of an observer who sees cognition happening in typical settings over prolonged periods. From an observer’s perspective, cognitions become other-oriented.

Second, the processes by which we imagine conceptual development happening will be reflected in the theories we develop. If we imagine children becoming more mathematically developed by internalizing collective activity in which they participate ever more centrally (Forman, 1996; Lave, 1991), then that is what we will try to describe, explain, and affect. If we see children becoming more scientifically literate by participating in discussions in which participants contribute and take away what is
consistent with their individual schemes, then that is what we will try to describe, explain, and affect.

Third, if educational research from a constructivist perspective is to carry weight with interpreters of it, then methodologies must be grounded in an epistemology that is compatible with the notion of intervention. “Teaching” as the attempt to affect what children know is an oxymoron without a coherent base for thinking about intervention. While radical constructivism does not entail ontology, it does not deny a reality. It says only that it is essentially unknowable in any way that can be labeled “correct.” People can affect others’ cognitions intentionally by placing themselves in a position to be interpreted by the persons they intend to affect. An “informed” intervention is one in which the interventionist is guided by a model of those whom he wishes to affect. The effect may be circuitous, in that they (as teachers) might start by saying something they know will be interpreted by students in ways that differ predictably from what they intend students understand, but which will provide springboards for moving the discussions in directions not possible without the initial (mis)interpretations. This is the model of teaching in radical constructivism — informed interventionists (i.e., folks with models of what they hope learners will learn) place themselves in positions to be interpreted in ways they intend by the persons they wish to affect.

The more recently developed theory of RME fits comfortably into the broad philosophical tradition of radical constructivism, and it gives us a way to develop a local instruction theory that respects both the individual-cognitive and the social-cognitive perspectives of learning. Recent theories of instructional design call for increased attention on the developmental nature of learning, including calls for instruction to help students develop their current ways of reasoning into more sophisticated ways of reasoning (Gravemeijer, 2004). The theory for RME can be used to develop a local instruction theory, i.e., a description of and rationale for the envisioned learning route. By describing the path by which learning occurs one is able to devise a hypothetical learning trajectory (Simon, 1995) of the daily instructional activities of a class. Various forms of constructivism have provided the catalyst for instructional shifts that attend to both the development and inventions of the student. Current discussions about mathematics teaching often reveal tensions between the product and process of learning. In fact, the development of a hypothetical learning trajectory benefits from having a careful characterization of what is to be learned, and the process by which it emerges in students. It is in this spirit that one can envision the critical role that conceptual analysis of a concept might play in determining instructional practices within a classroom.

**GENERATING LOCAL, RME-BASED INSTRUCTIONAL THEORY**

The next section of this manuscript describes how research into individual cognition influenced the development of curricular modules used in Carlson’s instruction of first semester calculus over three semesters. The research first resulted in the emergence of a framework that characterizes the cognitive processes involved in learning to reason about rate-of-change. The research also produced a multidimensional problem solving framework that emerged from examining the problem solving behaviors of mathematicians.
Carlson’s research into individual cognition uses *conceptual analysis* (Glasersfeld, 1995), a method that operationalizes what students might understand when they know and can use a particular idea. As Steffe (1996) notes, the main goal of conceptual analysis is to propose answers to the question, “What mental operations must be carried out to see the presented situation in the particular way one is seeing it?” (Glasersfeld, 1995, p. 78). Conceptual analyses of mathematical ideas cannot be carried out abstractly. Instead, doing conceptual analyses entails imagining students thinking about *something* in the context of *discussing* it or *using* it. The covariation framework (Carlson, Jacobs, Coe, Larsen & Hsu, 2000), described in more detail in the next section of this manuscript, emerged from a conceptual analysis of students’ thinking about and using rate-of-change in a variety of contexts. It articulates the reasoning abilities identified in students as they were observed responding to a broad collection of rate-of-change tasks, over several different studies (e.g., Thompson, 1994; Carlson, 1998; Carlson et al., 2002). It characterizes the mental actions involved in *covariational reasoning*, the coordination of two quantities’ values as they vary simultaneously, while attending to how the quantities change in relation to each other (Thompson, 1994).

The multi-dimensional problem solving framework, also described in more detail later in this manuscript, characterized the interaction of specific reasoning patterns and problem solving attributes (e.g., conceptual knowledge, affective responses, heuristics) of problem solving (Carlson & Bloom, in press). This framework emerged from investigating the problem solving behaviors of 12 mathematicians.

The presentation of these frameworks is followed by a discussion of their influence in guiding both the development of calculus modules and the classroom practices in calculus instruction.

**Covariational Reasoning**

According to several studies, calculus students are slow to develop an ability to interpret varying *rates of change* over intervals of a function’s domain. (Carlson, 1998; Kaput, 1992; Monk, 1992; Monk & Nemirovsky, 1994; Nemirovsky, 1996; Tall, 1992; Thompson, 1994a). According to Thompson (1994a), once students are adept at imagining expressions being evaluated continually as they “run rapidly” over a continuum, the groundwork has been laid for them to reflect on a set of possible inputs in relation to the set of corresponding outputs (p. 27). Such a *covariation view* of function has also been found to be essential for understanding central concepts of calculus (Cottrill et al., 1996; Kaput, 1992; Thompson, 1994b; Zandieh, 2000) and for reasoning about average and instantaneous rates of change, concavity, inflection points, and their real-world interpretations (Carlson, 1998; Monk, 1992).

Our work to characterize the thinking involved in reasoning flexibly about dynamically changing events has led to our decomposing *covariational reasoning* into five distinct mental actions. This decomposition has been useful for guiding the development of curricular modules to promote covariational reasoning in students. These five categories
of mental actions (Table 2) describe the reasoning abilities involved in meaningful representation and interpretation of a graphical model of a dynamic function situation.

The initial image described in the framework for covariational reasoning is one of two variables changing simultaneously. This loose association undergoes multiple refinements as the student moves toward an image of increasing and decreasing rate over the entire domain of the function (Table 2).

**Table 1. Mental Actions of the Covariation Framework**

<table>
<thead>
<tr>
<th>Mental Action</th>
<th>Description of Mental Action</th>
<th>Behaviors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mental Action 1 (MA1)</td>
<td>Coordinating the dependence of one variable on another variable</td>
<td>• Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)</td>
</tr>
<tr>
<td>Mental Action 2 (MA2)</td>
<td>Coordinating the direction of change of one variable with changes in the other variable</td>
<td>• Constructing a monotonic straight line</td>
</tr>
<tr>
<td>Mental Action 3 (MA3)</td>
<td>Coordinating the amount of change of one variable with changes in the other variable</td>
<td>• Plotting points/constructing secant lines</td>
</tr>
<tr>
<td>Mental Action 4 (MA4)</td>
<td>Coordinating the average rate-of-change of the function with uniform increments of change in the input variable</td>
<td>• Constructing secant lines for contiguous intervals in the domain</td>
</tr>
<tr>
<td>Mental Action 5 (MA5)</td>
<td>Coordinating the instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function</td>
<td>• Constructing a smooth curve with clear indications of concavity changes</td>
</tr>
</tbody>
</table>

In our work to study and promote students’ emerging covariational reasoning abilities, we have found that the ability to move flexibly between mental actions 3, 4 and 5 is not trivial for students. We have also observed that many precalculus level students only employ Mental Action 1 and Mental Action 2 when asked to construct the graph of a dynamic function situation.

**Covariational Reasoning in the Context of the Bottle Problem**

When prompting students to construct the graph of the height as a function of the amount of water in a bottle (Figure 1), we found that many precalculus students appropriately labeled the axes (MA1) and then constructed an increasing straight line (MA2). When prompted to explain their reasoning, they frequently indicated that “as more water is put into the bottle, the height of the water rises (MA2).” These students were clearly not attending to the amount of change of the height of the water level or the rate at which the
water was rising; nor were they exhibiting reasoning patterns about how the rate of change of the water was changing over the interval of the function’s domain.

Imagine this bottle filling with water. Sketch a graph of the height as a function of the amount of water that’s in the bottle.

**Figure 1.** The Bottle Problem.

We have observed that calculus students frequently provided a strictly concave up graph in response to this question (Carlson, 1998). When probed to explain their reasoning, a common justification was, “as the water is poured in it gets higher and higher on the bottle (MA2).” In contrast, other students who were starting to be able to construct an appropriate graph began coordinating the magnitude of changes in the height with changes in the volume (MA3). This is exemplified in the strategy of imagining pouring in one cup of water at a time and coordinating the resulting change in height based on how “spread out” that layer of water is.

Other students have demonstrated the ability to speak about the average rate of change locally for a specific interval of a function’s domain (MA4), but were unable to explain how the rate changes over the domain of the function. Even when calculus students produced a graph that was correct, they commonly had difficulty explaining what was conveyed by the inflection point and why the graph was “smooth” (i.e., $C^1$ rather than piecewise linear). Students frequently exhibited behaviors that gave the appearance of engaging in Mental Action 5 (e.g., construction of a smooth curve with the correct shape), however when prompted to explain their reasoning, they expressed that they had relied on memorized facts to guide their constructions. They were relying on facts such as faster means steeper and slower means less steep, but they were unable to explain why this was true.

**The role of covariational reasoning in understanding major concepts of first semester calculus**

Research suggests that the predominant approach to calculus instruction is not achieving the foundational understandings and problem solving behaviors that are needed for students’ continued mathematical development and course taking. It is our view that the mathematics community is ready for a careful rethinking of the precalculus and calculus curriculum—one that is driven by the broad body of research on knowing and learning function, including calls for a covariation approach to precalculus and calculus instruction.

As students move through their school and undergraduate mathematics curricula, they are frequently asked to manipulate algebraic equations and compute answers to specific types of questions. This strong procedural emphasis has not been effective for
building foundational function conceptions—ones that allow for meaningful interpretation and use of function in various representational and novel settings. Even understanding functions in terms of input and output can be a major challenge for many students (Carlson, 1998).

When interpreting graphs such as the one in Figure 2, students often confuse velocity for position (Monk, 1992) since the curves are laid out spatially, and position refers to a spatial property. This confusion leads to erroneous claims such as: the two cars collide at \( t = 1 \) hour or that Car B is catching up to Car A between \( t = .75 \) hour and \( t = 1 \) hours. In one study, 88% of students who had earned a high-A in college algebra made such mistakes, as did 63% of students earning an A in second semester calculus, and 42% of students earning an A in their first graduate mathematics course (Carlson, 1998).

In this example, students are thinking of the graph of a function as a picture of a physical situation rather than as a mapping from a set of input values to a set of output values. They show no evidence of engaging in covariational reasoning; they are not imagining the coordination of changes in time with changes in accumulated distance; nor are they forming a coordinated image of how speed (output of the function) is changing while imagining changes in time. Our research suggests that these covariational reasoning abilities are needed for the conceptual developments we desire in first semester calculus students. We also believe that these reasoning patterns are an important bridge for success in advanced mathematics.

**Covariational Reasoning Patterns Needed for Understanding Accumulation**

In our work to develop and validate the Precalculus Concept Assessment Instrument\(^1\) (Carlson, Oehrtman, & Engelke, submitted), we found that students’ ability to respond correctly to a diverse set of function-focused tasks is tightly linked to two types of dynamic reasoning abilities. First, as mentioned above, students must develop an understanding of functions as general processes that accept input and produce output. Second, they must be able to attend to the changing value of output and rate of change as the independent variable is varied through an interval in the domain.

Development of an intuitive understanding of the Fundamental Theorem of Calculus in the context of a graph requires the formation of an image of accumulating area, while

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\(^1\) The Precalculus Concept Assessment Instrument is a 25-item multiple choice instrument for assessing students’ understanding of the major aspects of the function concept that are foundational for success in beginning calculus. The answer choices include the correct answer and the common misconceptions that have been expressed by students in research studies (e.g., interviews that have probed students’ thinking when providing specific responses to conceptually based tasks).
imagining changes in the independent variable in the function. As an example, 
explaining why the derivative of the volume of a sphere, \( v = \frac{4}{3} \pi r^3 \), is its surface area 
requires students to coordinate mental images of changes in the “radius” with the 
corresponding changes in the volume over a range of small variations. For such 
variations, students must then be able to imagine the computation of rate of change of 
volume and see its connection to surface area.

To understand the relationship between average and instantaneous rates and the graphical 
analog between secant and tangent lines, a student must first conceive of an image as in 
Figure 3a, below (Monk, 1987). By employing covariational reasoning (e.g., 
coordinating an image of two varying quantities and attending to how they change in 
relation to each other), the student is able to transform the image and reason about values 
of various parameters as the configuration changes. Being able to answer questions that 
require such variation as “When point \( Q \) moves toward \( P \), does the slope of \( S \) increase or 
decrease?” is significantly more difficult than being able to answer questions about the 
value of a function at a single point.

![Figure 3. Foundational images for the definitions of a) the derivative and b) the 
definite integral](image)

The standard calculus curriculum presents accumulation in terms of methods of 
determining static quantities such as the area of an irregular region of the plane or the 
total distance traveled given a changing velocity (but as a completed motion). Equally 
important, however, is a dynamic view in which an accumulated total is changing 
through continual accruals (Kaput, 1994; Thompson, 1994). For example, in a typical 
“area so far” function as in Figure 3b, this involves being able to mentally imagine the 
point \( P \) moving to the right by adding slices of area at a rate proportional to the height of 
the graph. This requires students to engage in covariational reasoning (Carlson, Smith & 
Persson, 2003) and is significantly more difficult for students than evaluating and even 
comparing areas at given points (Monk, 1987).

Newton first determined the rate-of-change of an area and determined that the total area 
could be determined by multiplying the rate-of-change with the accumulation of the 
independent variable, thus noticing that a relationship existed between the accumulation 
of a quantity and the change in accumulation. This led to his observation that the rate of 
change of the accumulated quantity is equal to the immediate accrual.
This line of thinking also emphasizes the importance of understanding that the accrual is a multiplicative relationship and that the total accumulation is made of infinitesimal accruals of quantity (e.g., accruals of lines compose area and accruals of area compose volume). It is these understandings that enable the relationship expressed by

\[ \frac{d}{dx} \left[ \int_a^x f(t)dt \right] = f(x) \]

to be appreciated and understood. A more careful articulation of the reasoning abilities and understandings related to accumulation and the Fundamental Theorem of Calculus are provided in the theoretical framework that emerged from investigating calculus students learning while completing the accumulation module. An FTC framework contains four dimensions that describe the foundational reasoning abilities and understandings of the Fundamental Theorem of Calculus.

Table 2. A Framework for Accumulation and the FTC

**Part A: Foundational understandings and reasoning abilities**

(FR1) Ability to view a function as an entity that accepts input and produces output.
(FR2) Ability to coordinate the instantaneous rate-of-change of a function with continuous changes in the input variable (Level V—Covariation Framework).
(FU1) Understand that the average change of a function (on an interval) = the average rate-of-change (multiplied by) the amount of change in the independent variable.
(FU3) Understand that the multiplicative relationship that represents the accrual of change on an interval can be represented by area.

**Part B: Covariational reasoning with accumulating quantities**

The mental actions of the Fundamental Theorem of Calculus:

(MA1) Coordinating the accumulation of discrete changes in a function’s input variable with the accumulation of the average rate-of-change of the function on fixed intervals of the function’s domain.
(MA 2) Coordinating the accumulation of smaller and smaller intervals of a function’s input variable with the accumulation of the average rate-of-change on each interval.
(MA 3) Coordinating the accumulation of area from some fixed starting value to some specified value.

**Part C: Notational aspects of accumulation**

<table>
<thead>
<tr>
<th>Notation</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(x) = \int f(x)dx )</td>
<td>i) the antiderivative of ( f ) is ( F )</td>
</tr>
<tr>
<td></td>
<td>ii) ( f ) is the function that describes the rate-of-change of ( F ).</td>
</tr>
<tr>
<td>( F(x) = \int_a^x f(t)dt )</td>
<td>i) The value of ( F(x) ) represents the accumulated area under the curve of ( f ) from ( a ) to ( x );</td>
</tr>
<tr>
<td></td>
<td>ii) The value of ( F(x) ) represents the total change in ( F ) from ( a ) to ( x ).</td>
</tr>
</tbody>
</table>
Part D: The statements and relationships of the FTC

\[ \int_a^b f(x)dx = F(b) - F(a) \]

i) The accumulated area under the curve of \( f \) from \( a \) to \( b \) is equal to the total change in \( F \) from \( a \) to \( b \).

\[ \frac{d}{dx} \left[ \int_a^x f(t)dt \right] = f(x) \]

i) the instantaneous rate-of-change of the accrual function at \( x \) is equal to the value of the rate-of-change function at \( x \).

Designing instruction with the intent of producing instructional conversations of a particular type and grounding that design in an analysis of understandings that would support them, that conceptual analysis overlaps with Gravemeijer’s and Cobb’s notion of emergent models in instructional design (Cobb et al., 1997b; Gravemeijer, 1994; Gravemeijer et al., 2000). As Gravemeijer et al. (2000) note,

The term model as it used here can refer to a task setting or to a verbal description as well as to ways of symbolizing and notating. Thus, although we will speak of models and symbolizations interchangeably, there is a slight difference. In Realistic Mathematics Education, the term model is understood in a dynamic, holistic sense. As a consequence, the symbolizations that are embedded in the process of modeling and that constitute the model can change over time …. This approach of proactively supporting the emergence of taken-as-shared models involves both the judicious selection of instructional activities and the negotiation of the ways of symbolizing that students create as they participate in communal classroom practices (pp. 240-241).

Thus, an emergent model in RME is something to be aimed for as an outcome of instruction in the same way as is an instructional conversation within conceptual analysis.

The genetic decomposition of the reasoning abilities, understandings, and notational aspects of understanding accumulation and the FTC guided the curriculum development for the accumulation module (See Table III for one activity within this module). It also served as a lens for analyzing students’ thinking while completing the module, with successive refinements of both the framework and curricular module resulting from research on students’ cognitive development relative to this concept. This emergent theory of reasoning about accumulation enabled us to describe the understandings we desire in students. It also provided a framing for the instructional design of this module, i.e., the sequencing and design of specific activities, including specific prompts and requests of students. This framework should continue to provide a means for characterizing the emerging reasoning abilities, understandings, and notational fluency of individual students.
Issues of Instructional Design

An emergent model in RME is intended to be something of which students become more or less aware and which has a tool-like character to them. An intended instructional conversation is something of which the teacher is aware, but students needn’t be aware of it in the same way or even aware of at all. As such, when viewed as a method by which to anticipate the conceptual operations that underlie a particular way of thinking and therefore the design of conversations that might support students’ developing them, conceptual analysis can be viewed not just as a method of producing psychological models of individual's understanding, but as an important activity in instructional design.

The methods of designing instruction for producing instructional conversations and of emergent models differ in another respect. In Realistic Mathematics Education one attends to tools that will influence students’ activity. In the method of conceptual analysis, one thinks less of influencing students’ activity and more of describing things students might re-perceive and things about which a teacher might hold fruitful discussions with them. It is in this respect that we see the need to attend to what we take as the “something” we imagine teachers and students discussing and to the nature of the discussions we imagine surrounding it. Put another way, conceptual analysis supports the design of “things to talk about” that, were the larger objective held in mind by a teacher as he or she manages classroom discussions, could engender reflective discourse around a desired theme, issue, or way of thinking (Cobb et al., 1997a; A. G. Thompson et al., 1994).

It often is useful to coin a phrase for an idea still germinating. Thompson (2002) has proposed the phrase didactic object to refer to “a thing to talk about” that is designed with the intention of supporting reflective mathematical discourse. I hasten to point out that objects cannot be didactic in and of themselves. Rather, they are didactic because of the conversations that are enabled by someone having conceptualized them as such. In one sense, a didactic object is a tool, but one designed for a teacher who can conceive of ways he or she might use it to produce desirable student engagements and classroom conversations.

In this sense the activities within a module (Table 3) can also serve as didactic objects within a classroom setting. The scaffolding of the activity includes prompts that draw from conceptual analysis of what is involved in knowing and learning accumulation and the FTC. The purpose of these prompts is to assure that all individuals within the classroom are afforded an opportunity to ponder the central and relevant issues relative to learning these concepts, as articulated in the accumulation and FTC framework (Table 2). After students complete and discuss their responses to these within a smaller group of 5-6 students who are seated at the same table, the instructor leads a discussion in which the student responses become “a thing to talk about.” The products of student thinking are used to produce the desired student learning, as defined in the FTC framework.
### Table 3. A Sample Activity From One Module

**Accumulation Project: Activity 1**

1. While running at the gym, I typically set the treadmill speed at 6.8 miles per hour. For the month of May, I plan to continue setting the speed at 6.8 m.p.h. and run as far as I can, depending on how I feel that day. Set up an integral that I can use to compute the distance of my run for any day of the month. (Let \( g(x) \) be defined as the total distance run for varying amounts of time, \( x \).)

   a) \( g(x) = \ldots \)

   b) Using any method you like, compute the distance covered in a 1 hour and 24 minute run.

   c) Determine the algebraic function that represents distance as a function of time for this situation.

   d) Now graph both the speed function and the distance function (determined in c above) on the below axes. Explain the relationship between these two functions.

2. Let \( f(t) \) represent the rate at which the amount of water in Chandler’s water reservoir changed in (100’s of gallons per hour) in a 7 hour period from 10 a.m. to 5 p.m. last Saturday (Assume that the tank had 5700 gallons in the tank at 10 a.m. \( (t = 0) \)). Use the graph of \( f \), given below, to answer the following.

<table>
<thead>
<tr>
<th>hours</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>100’s of gallons per hour</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>-1</td>
<td>-2</td>
<td>-3</td>
<td>-1</td>
</tr>
</tbody>
</table>

   a. How much water was in the tank at noon?

   b. What is the meaning of \( g(x) = \int_0^x f(t) dt \)?

   c. What is the value of \( g(6) \)?

   d. During what intervals of time was the water level decreasing?

   e. At what time was the tank the fullest?

   f. Using the graph of \( f \), given above, construct a rough sketch of the graph of \( g \).
MATHEMATICAL REASONING PATTERNS AND ATTRIBUTES OF EFFECTIVE PROBLEM SOLVERS

A formal investigation of the reasoning patterns of 12 mathematicians when completing four mathematics tasks has revealed consistent reasoning patterns among these mathematicians as they engaged in problem solving activity. In this investigation we were interested in knowing why some individuals emerge as highly effective problem solvers, while others do not. We chose to work with mathematicians because we hypothesized that observing individuals with a broad, deep knowledge base and extensive problem solving experience would afford the clearest possible view of the problem solving process, the interactions of various problem solving attributes (e.g., thought processes, monitoring and reflecting behaviors, emotional responses), and the subtle reasoning patterns and attributes that contribute to problem solving success. We further believed that this knowledge could have practical impact, providing valuable information for informing course design, curriculum development, and classroom instruction.

The 12 mathematicians in our study were asked to complete four problems in a clinical interview setting. The problems required knowledge of secondary mathematics concepts, but were complex enough to elicit multiple solution paths and strong affective responses, even from mathematicians. For more details about the methods for collecting and analyzing our data refer to Sections 4.1 and 4.2 of Carlson and Bloom’s 2005 article.

A Sample Problem Solving Task: A Mathematician Solves the Paper-Folding Problem

Paul, an active research mathematician, was one of 12 mathematicians whose problem solving behavior we investigated in this paper. When presented with the paper-folding problem (Table 4), he first drew a sketch of the problem situation (Figure 4). Paul first sat silently, reading the text while repeatedly clicking his pen. He then lifted his head and gazed out the window with his eyes fixed ahead and his face expressionless. After about three minutes of silence, he redirected his eyes toward the interviewer—he grinned slightly and exclaimed, “The distance of A’ from the fold line is 1.”

Table 4. The Paper-Folding Problem

| A square piece of paper ABCD is white on the front side and black on the back side and has an area of 3 square inches. Corner A is folded over to point A' which lies on the diagonal AC such that the total visible area is ½ white and ½ black. How far is A' from the fold line? |

When prompted to verbalize his thinking, Paul explained, “I quickly deduced that after unfolding the paper, the new fold line would divide the paper into three equal areas. Thus each of the pieces has an area of 1.” When asked to provide more detail, he said, “Transitivity.” The interviewer waited for an explanation, and he followed with, “…let the three areas be x, y and z; then since x = y, and y = z, then x = z, and since the total area is 3, each area would be 1.” When asked to explain how he determined the value of 1 for the distance of A’ from the fold line, he responded that he had used his knowledge of the relationships among the sides of a 45-45-90 triangle to arrive at the answer.
This prompting provided some insights into the reasoning patterns that Paul employed to arrive at the answer; however, it does not explain how Paul was able to quickly observe that all three areas were equal; nor does it explain why Paul was so efficient in computing the exact value of the distance of $A'$ from the fold line. What sequence of mental actions led Paul to his simple and elegant solution? Were all of his initial ideas fruitful, or did he reject some on the way toward a correct response? Why did Paul click his pen repeatedly when reading the problem—was it a nervous habit, an expression of frustration, or something else? Why did Paul grin when he looked up to tell the interviewer his answer—was this grin an expression of pride?

Paul’s approach to the paper-folding problem illustrates the powerful reasoning abilities in which he was engaged. Contrasting Paul’s solution with those of the other mathematicians also illuminates the complex nature of problem solving, including the important role of conceptual knowledge, facts and procedures in moving efficiently toward a solution attempt. A detailed description of methods, including our coding and analyses for other solutions to this problem, can be found in Carlson and Bloom (in press).

**An Overview of the Problem Solving Findings**

We noticed that the subjects in our study engaged in cyclic reasoning patterns that were highly effective in their solution attempts. We also observed that four distinct phases emerged in the subjects’ problem solving process: orienting, planning, executing, and checking. Our observations illustrate that these phases are linked in a cycle that is executed repeatedly until a solution is obtained or the problem is abandoned (Figure 5). Moreover, we also noticed another distinct cycle embedded in the planning phase of the larger cycle. We have labeled this the conjecture—imagine—evaluate sub-cycle. This sub-cycle usually was signified subtly; the mathematicians might pause and contemplate briefly, for example, before actively pursuing a solution approach. Other mental actions may well occur during these cyclical phases, but we believe that the labels we have chosen provide an accurate characterization of the primary and most general form of cognitive activity that our subjects exhibited as they cycled through these phases during their solution attempt.
The mathematicians in our study consistently exhibited patterns of reflection on and regulation of their thought processes and products. The effectiveness of their monitoring was observed to be highly dependent on their fluency in accessing both conceptual and procedural knowledge. Strong emotional responses (both positive and negative) emerged regularly in their solution attempt. Even their small successes were followed by exhibitions of joy or pride, while exhibitions of frustration were followed by coping mechanisms that included their diverting attention from the problem. What emerged from our analysis was a more structured, coherent, and descriptive characterization of the interplay between the problem solving phases and various problem solving attributes (resources, heuristics, affect, and monitoring) than had been previously reported in the literature. We illustrate this characterization in the form of a multidimensional problem solving framework (Table 4).
Table 4. A Multi-Dimensional Problem Solving Framework

<table>
<thead>
<tr>
<th>PHASE</th>
<th>Resources</th>
<th>Heuristics</th>
<th>Affect</th>
<th>Monitoring</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orienting</td>
<td>Mathematical concepts and algorithms are accessed when attempting to make sense of the problem. The solver also scans her/his knowledge base to categorize the problem.</td>
<td>The solver often draws pictures, labels unknowns and classifies the problem. (Solvers were sometimes observed saying, “this is an X kind of problem.”)</td>
<td>The curiosity and interest level of the solver affects the solver’s motivation to make sense of the problem. If the solver is not interested, he/she may lack motivation and stall before starting.</td>
<td>Self-talk and reflective behaviors serve to keep the mind engaged. The solvers were observed asking “What does this mean”; “How should I represent this?”; “What does that look like?”</td>
</tr>
<tr>
<td>Planning</td>
<td>Conceptual knowledge and facts are needed to construct conjectures and make informed decisions about strategies.</td>
<td>Specific heuristics were accessed and considered while considering and choosing a solution approach.</td>
<td>Beliefs about the methods of mathematics and one’s abilities influence conjectures and decisions. Signs of intimacy, anxiety, and frustration are also displayed.</td>
<td>Solvers monitor their strategies and plans. They ask themselves “Will this take me where I want to go?”, “How efficient will approach x be?”</td>
</tr>
<tr>
<td>Executing</td>
<td>Conceptual knowledge, facts and algorithms are essential for executing, computing and constructing. Without conceptual knowledge, monitoring of constructions is misguided.</td>
<td>Fluency with a wide repertoire of heuristics, algorithms and computational approaches are needed for the efficient execution of a solution.</td>
<td>Intimacy with the problem, integrity in constructions, frustration, joy, defense mechanisms and concern for aesthetic solutions emerge in the context of constructing and computing.</td>
<td>Conceptual understandings and numerical intuitions are employed to monitor both the solution progress and products while constructing statements</td>
</tr>
<tr>
<td>Checking</td>
<td>Resources, including well-connected conceptual knowledge informs the solver as to the reasonableness or correctness of the solution attained.</td>
<td>Computational and algorithmic shortcuts are used to verify the correctness of their answers and to ascertain the reasonableness of their computations.</td>
<td>As with the other phases, there were a number of affective behaviors displayed. It is often at this phase that frustration overwhelmed the solver, causing the solver to abandon the task.</td>
<td>Reflections on the efficiency, correctness and aesthetic quality of the solution provide useful feedback to the solver.</td>
</tr>
</tbody>
</table>

The Problem Solving Process

Each problem solving phase and the predominant behaviors exhibited during that phase are listed in the cells on the far left column of Table I. For example, in the orienting phase the subjects consistently engaged in behaviors of sense making, organizing, and constructing. The primary role of each problem solving attribute (e.g., resources, heuristics) during that
phase is illustrated in the cells to the right. When orienting themselves to the problem, the experts predominantly exhibited behaviors of sense making, organizing, and constructing, and the accessing of resources that included concepts, facts, and algorithms. During this phase, the mathematicians in our study also applied a variety of heuristics such as drawing pictures, labeling unknowns, and classifying the problem in a specific category of problems. The primary affective behaviors demonstrated in this phase were curiosity and interest—the intensity of the subjects’ efforts to make sense of the problem was typically influenced by their interest in the problem type and curiosity to find a solution. Various monitoring behaviors such as self-talk and reflections about the productiveness of their orienting behaviors were also observed to be influential in keeping their thinking moving in productive directions. Reviewing the decomposition of the collection of behaviors revealed a diverse classification of behaviors during each phase of the problem solving process.

In addition to their global decisions about their solution approach—deciding to cycle forward or cycle back, or to act on a conjectured approach—the mathematicians were making decisions about their moment-to-moment constructions and actions. They were continually asking themselves: Is this approach getting me anywhere? What does this tell me? Does that calculation make sense? The effectiveness of this monitoring, like their sense-making, was highly dependent on their knowledge base and ability to access helpful mathematical information as it was needed. Effective monitoring played a powerful role in moving our subjects in positive directions.

Other attributes of problem solving (e.g., resources, heuristics, affect, and monitoring) also were evident during each problem solving phase. Consistent with the findings of DeBellis (1998) and Hannula (1999), we observed that local affective pathways play a powerful role in the problem solving process. We found that frustration occurred frequently during the solution attempts of these mathematicians; however, unlike what has been reported when observing students, these mathematicians effectively employed a variety of coping mechanisms to manage the frustration and anxiety during the solution attempt. Other expressions of affect included mathematical integrity and intimacy. When we reviewed the interviews, it became apparent that all decisions were based on a logical foundation. We refer to this as mathematics integrity. Our experts displayed moments of joy and satisfaction when they were successful and high levels of frustration when they were stumped. This suggests the existence of a strong bonding with the problem that has been previously observed in children and described as mathematical intimacy (DeBellis, 1998). During a general interview after solving the problems the subjects each described other instances of mathematical intimacy. Some subjects reported that they sometimes become so obsessed with a problem that it occupied much of their waking hours. Others told of being unable to “let go” of a problem for years, even though rationally they had concluded that the problem was beyond their abilities.

The multiple dimensions and diverse components of the MPS Framework suggest that learning to become an effective problem solver requires the development and coordination of a large reservoir of reasoning patterns, knowledge, and behaviors. It also requires the effective management of emotional responses, as well as a great deal of practice and experience. The many years of problem solving experience of these mathematicians clearly contributed to their confidence and effective access and use of mathematical concepts, facts, and heuristic strategies. However, we do believe that explicit instructional and curricular effort to promote the reasoning patterns and attributes
of effective problem solving, as described in the *Multidimensional Problem Solving Framework*, may facilitate the acquisition of these behaviors in students.

**THEORY DRIVES CURRICULAR INTERVENTIONS**

The instructional setting for the three instructional interventions was small class with less than 35 students. The classroom contained 7 tables that seated 5 to 6 students; thus facilitating group discussion among the students in the class. Stewart (fourth edition) calculus was the required text for the course. Students were expected to read at least one section prior to each class session, with exercises and problems from the text being assigned twice per week. All homework was collected, with select problems graded and returned to the students. In most cases the homework was returned at the beginning of the subsequent class meeting.

The primary instructional goals for the course included the promotion of a deep understanding of the major concepts of first semester calculus and improved problem solving behaviors as was discussed and elaborated in the multidimensional problem solving framework and previous section. In order to facilitate the deep understandings and the foundational reasoning abilities that were desired, curricular modules were developed for six major concepts of the course (function, limit, derivative, mean value theorem, accumulation and The Fundamental Theorem of Calculus). The design of this curriculum was guided by the broad body of research related to knowing and learning calculus, but was primarily driven by our past research on function, covariation and accumulation (Thompson and Thomson, 1994; Thompson, 1994a; Thompson, 1994b; Carlson, 1998, Carlson et al., 2002). The six conceptual modules included three to five in class activities that contained prompts that were designed to assist students in building the reasoning abilities and to overcome the common obstacles that had been reported in the literature. The in-class activities were primarily completed by students in groups, with one individual from each group producing a solution product on a whiteboard, and one person from each group being chosen to explain that group’s solution to the class. (The students did not know in advance who would be selected to explain the group’s solution.)

The modules also included two to four take home assignments designed to promote students’ use of and extension of the conceptual tools central to that module. The final assignment for each module utilized the design principles of Lesh’s (2004) Model Eliciting Activities (See Appendix A for a sample accumulation activity).

Each day of instruction began with a *question of the day*, with the expectation that the students at each table would discuss and negotiate a response for their group. The question was designed to prompt students’ sense making and to promote curiosity and sense making relative to the central topic for the day. The daily expectation that students begin thinking about the question as soon as they were seated appeared to encourage their regular execution of cycles of *conjecture, imagine and verify*, as they made multiple attempts to construct a reasonable response to the question. The questions were also intended to transfer greater authority for learning to the students. As the students at each table reported their responses to the question, the instructor posed questions along a trajectory to promote the emergence of interaction patterns in which the knowledge and concepts for that lesson emerged from the ideas that were offered during the discussion.
The Intended Role of the Instructor

The stated goals of the instructor were to provide students’ experiences that would facilitate the acquisition of effective problem solving behaviors and to promote students’ understanding of and ability to use the major concepts of the course. The instructor used questioning to promote student conjecturing, imagining and verifying, with explicit efforts made to elicit understanding and application of knowledge in students. The development of skills and algorithms were shared through short lectures, followed by reinforcement tasks given in class. The expectation was set that minimal in class instruction was needed to prepare students for procedurally oriented homework; instead students were expected to rely on their text and each other for assistance as needed.

Intended Classroom Norms

The instructor created classroom norms that promoted student discourse, with students encouraged to make and test conjectures. Explicit efforts were made to make students feel safe in sharing their thinking and reasoning; however, constructions that were based on a logical foundation were acknowledged as having greater value. Students were encouraged to verbalize their thinking. They were also expected to listen and attempt to make sense of others’ thinking and solutions. Explicit efforts were also made to encourage students to provide coherent and clear explanations of their thinking, both when expressing their ideas to one another at their tables and when addressing the class. This was achieved by the instructor making efforts to gently move individuals in this direction, by attempting to read the confidence level of the individual speaking, and making suggestions for improved clarity as she believed it would be useful. Occasionally humor was used to help students through awkward moments that arose in class, in attempt to help them acquire coping mechanisms that were so powerful for mathematicians. The instructor tried to appear competent, but did not hesitate to reveal her imperfections as they surfaced; she attempted to leverage them as opportunities to transmit beliefs that learning mathematics is a process that requires refinements of one's thinking, persistence is a key element in problem solving success, and it is only through making logical conjectures and attempts that one makes progress in arriving at a solution to a complex problem. Ongoing efforts were made to make class enjoyable through honest intellectual engagement.

CLOSING REMARKS

From a radical constructivist perspective, teaching draws heavily on ideas of intersubjectivity, meaning that teaching is the teacher's purposeful positioning of herself in relation to her epistemic students so that students interpret her instructional actions in ways she intends. Use of individual cognitive theories of learning or knowing, such as the covariation framework and the multidimensional problem solving framework, increase the purposefulness of instructional materials and instructional actions. These frameworks provide increased power for assessing and acting on the sense students have made of instruction. "Materials" are things selected (or designed) that will support the emergence
of conversations (often shaped by teacher's instructional actions) that will support the emergence of a conceptual space ("space" meaning ways of thinking among individuals and attempts that students make to interpret one another) in which students' thinking is moved forward by their attempts to express and refine their own thinking and make sense of others' thinking. This view of teaching is not in conflict with the realistic mathematics education instructional theory. In fact, in this manuscript, we have attempted to make a case that, in a radical constructivist perspective, there exists a useful reflexive relationship between individual cognitive theories and the social engineering of classroom instruction. The covariation, accumulation, and problem solving frameworks can inform the hypothetical learning trajectory for the class; they can also provide useful insights for the module developers in such things as devising meaningful prompts to both support individual cognition and emergence of mathematical practices. They can also provide useful knowledge for the instructor in deciding on such things as promoting and managing meaningful classroom discourse, setting useful classroom norms and determining optimal instructional moves.

REFERENCES


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i We use *interiorize* in a Piagetian sense. For an individual to interiorize actions means that they construct schemes of mental operations they may carry out in thought and by which they may anticipate outcomes of particular courses of action.

ii I address a constructivist meaning of intersubjectivity in later paragraphs.

iii This, in essence, was Piaget’s opinion when he emphasized the importance of socialization for the development of semiotic and formal operations (Piaget, 1950, 1977, 1995).

iv This is not to say that there cannot be a “correct” knowledge of reality. Even if there were such a thing, however, we could not know whether anyone has it.