Teachers' Understandings of Probability

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Abstract

Probability is an important idea with a remarkably wide range of applications. However, psychological and instructional studies conducted in the last two decades have consistently documented poor understanding of probability among different populations across different settings. The purpose of this study is to develop a theoretical framework for describing teachers' understandings of probability. To this end, we conducted an eight-day seminar with eight high school statistics teachers in the summer of 2001. The data we collected include videotaped sessions and interviews, teachers' written work, and researchers' field notes. Our analysis of the data revealed that there was a complex mix of conceptions and understandings of probability, both within and across the teachers, which were: situationally triggered, often incoherent when the teachers tried to reflect on them, and which did not support their attempts to develop coherent pedagogical strategies regarding probability and statistical inference.

Teachers' Understandings of Probability

Probability is one of the most important ideas that we expect students to understand in high school. Since its origin in the mid-seventeen century, probability theory has had an enormous impact on scientific and cultural developments in society. In the 18^{th} century, it was applied to problems in gambling, jurisprudence, data analysis, inductive inference, and insurance; in the 19^{th} century it found applications in sociology, physics, biology and psychology; in the 20^{th} century it was applied to problems in agronomy, polling, medical testing, baseball and innumerable other practical matters (Gigerenzer, Swijtink, Porter, Daston, Beatty, & Kruger, 1989). Along with this expansion of applications, probability has shaped modern science, transformed our ideas of nature, mind, and society, and altered our values and assumptions about matters as diverse as legal fairness to human intelligence. Given the extraordinary range and significance of these transformations and their influence on the structure of knowledge and power, and on issues of opportunity and equity in our society, the question of how to support people's development of a coherent understanding of probability assumes increased importance.

However, the historical development of theories of probability has been riddled with controversy, which in turn, has shaped the way the curriculum and pedagogy of probability are designed and implemented today in school. In classical Laplacian theory of probability, probability is "the conversion of either complete ignorance or partial symmetric knowledge concerning which of a set of alternatives is true, into a uniform probability distribution over the alternatives" (Fine, 1973, p. 167). The core of this approach is the "principle of indifference"—alternatives are considered to be equally probable in the absence of known reasons to the contrary, or when there is a balance of evidence in favor of each alternative. However, classical probability builds on a number of troubling bases. It assumes an equal likelihood of alternative outcomes. Yet, "equal likelihood" is exactly synonymous with "equal probability." It is in this sense von Mises (1957) argued that, "unless we consider the classical definition of probability to be a vicious circle, this definition means the reduction of all distribution to the simpler case of uniform distribution" (p. 68).

Over time, "probability" has had two meanings: *frequency-type probability* "concerning itself with stochastic laws of chance processes," and *belief-type probability* "dedicated to assessing reasonable degrees of belief in propositions quite devoid of statistical background" (Hacking, 1975 p. 12; Hacking, 2001 pp. 132-133). Since 1654, there was an explosion of conceptions in the mathematical community that are compatible with either concept of probability: frequentist probability, Bayesian probability, axiomatic probability, and, probability as propensity (cf. Gillies, 2000; Von Plato, 1994). Yet, even today, mathematicians and scientists continue to debate and negotiate meanings of probability both for its theoretical implication, and for its application in scientific research. There were frequentists, such as von Mises (1957), who regarded probability as "a *scientific theory* of the same kind as any other branch of the exact *natural science,*" which applies to long sequences of repeating occurrences or of mass phenomena (p. 7). There were subjectivists, such as de Finetti, (1970), who defined probability as "the *degree of belief* in the occurrence of an event attributed by a given person at a given instant and with a given set of information" (p. 3), and who said that frequentist or objective probability can be made sense of only through personal probability. According to Hacking (1975), though most people who use probability do not pay attention to such distinctions, extremists of these schools of theories “argue vigorously that the distinction is a sham, for there is only one kind of probability” (p. 15).

The controversy surrounding the theories of probability presents a difficult question for educators: “What do we teach?” (Nilsson, 2003). Current instructional practices have sidestepped this question by simultaneously introducing multiple definitions of probability while ignoring the differences and conflicts among their theoretical underpinnings (e.g., Triola, 1997; Watkins, Scheaffer, & Cobb, 2004). These practices ignore the consequence of incompatible meanings on students’ learning over the long run, and do not support efforts to make connections between the idea of probability and that of statistical inference.

Many research studies have investigated ways people understand probability. Psychological and instructional studies have documented consistently poor understanding or misconceptions of probability among different population across different settings (Fischbein & Schnarch, 1997; Kahneman & Tversky, 1973; Konold, 1989; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993; Nisbett, Krantz, Jepson, & Kunda, 1983). The research of Kahneman and Tversky (Kahneman, Slovic, & Tversky, 1982; Kahneman & Tversky, 1972, 1973) suggested that many people used non-standard strategies to reason probabilistically. Later research, such as by Konold and his colleagues (Konold, 1989; Konold *et al.*, 1993) and by Goldstein and Hogarth (1997), suggested that Kahneman's and Tversky might have over-interpreted their data. Kahneman and Tversky seemed to assume that their subjects had a relative frequency meaning for "probability" when it may have been that many of them had a deterministic understanding of events, and that numerical probability simply reflected their degree of belief in the outcome. Konold also discovered that many students were not only reasoning deterministically; they were also reasoning about the events that researchers thought were being portrayed probabilistically as if they were never to be repeated again. As such, subjects assigned probabilities between zero and one to events whose probabilities were (since they were single trials) either zero or one. Later studies supported Konold's, and Goldstein's and Hogarth's, suspicions that a single-trial conception of situations is widespread among people asked to reason about events probabilistically and that Kahneman's and Tversky's subjects thought likewise (e.g., Fischbein & Gazit, 1984).

Even with the rich insights generated from these studies, understandings of individuals' probabilistic reasoning remain fragmented and incomplete. In particular, there is no theoretical framework for modeling individuals' probabilistic understandings and the reasoning that is grounded in them. We therefore designed and conducted this study with the goal of adding to our understanding of important conceptualizations entailed within individuals' probabilistic understandings and the ways they reason with them.

To explicate our research purposes, let us explain what we mean by "understanding" and the method we use in developing descriptions of an understanding. By "understanding" we mean that which "results from a person's interpreting signs, symbols, interchanges, or conversation—assigning meanings according to a web of connections the person builds over time through interactions with his or her own interpretations of settings and through interactions with other people as they attempt to do the same" (Thompson & Saldanha, 2003, p. 99). Building on earlier definitions of understanding based on Piaget's notion of assimilation, e.g. "assimilating to an appropriate scheme" (Skemp, 1979), Thompson & Saldanha (2003) extended its meaning to "assimilation to a scheme," which allowed for addressing understandings people have even though they could be judged as inappropriate or wrong. As a result, a description of understanding requires "addressing two sides of the assimilation—what we see as the thing a person is attempting to understand and the scheme of operations that constitutes the person's actual understanding" (*ibid.*, p. 99).

Methodology

In conducting this study, we used a modified constructivist teaching experiment (Cobb & Steffe, 1983; Hunting, 1983; Steffe, 1991; Steffe & Richards, 1980; Steffe & Thompson, 2000). The constructivist teaching experiment methodology was adapted from the Soviet-style teaching experiment (Kantowski, 1977) to serve the purpose of developing conceptual models of students' mathematical knowledge in the context of mathematics instruction.

Radical constructivism entails the stance that any cognizing organism builds its reality out of its experience. As such, it is necessary to attribute mathematical realities to subjects that are independent of the researchers' mathematical realities. While acknowledging the inaccessibility of the learners' environment as seen from their points of view, constructivists also believe that the roots of mathematical knowledge can be found in general coordination of the actor's actions (Piaget, 1971). These assumptions then frame the specific research goals of a teaching experiment as being to build models of subjects' mathematical realities. To create models of subjects' mathematical realities, we must attempt to perturb them so as to reveal both their composition and boundaries.

To construct a description of a person's understanding, we adopted an analytical method that Glasersfeld (1995) called conceptual analysis, the aim of which is to describe conceptual operations that, were people to have them, might result in their thinking the way they evidently do. Engaging in conceptual analysis of a person's understanding means trying to think as the person does, to construct a conceptual structure that is "intentionally isomorphic" (Maturana, 1978, p. 29) to that of the person. In conducting a conceptual analysis, a researcher builds models of a person's understanding by observing the person's actions in natural or designed contexts and asking herself, "What can this person be thinking so that his actions make sense from his perspective?" (Thompson, 1982b, pp. 160-161). In other words, "the researcher puts himself into the position of the observed and attempts to examine the operations that he (the researcher) would need or the constraints he would have to operate under in order to (logically) behave as the observed did" (Thompson, 1982b, p. 161).

As a researcher engages in the activity of constructing description of her subjects' understanding, she should also examine her very activity of conceptual analysis with the aim of reflectively abstracting the concepts and operations that she applies in constructing explanations of her subjects' behaviors. The former activity is called "constructing an understanding" (or constructing a model) of observed individuals. The latter activity, reflecting on the activity of coming to understand individuals' understandings, is called "constructing a theoretical framework" (Thompson, 1982b). When the researcher becomes aware of the concepts and operations upon which she draws to understand individuals' understandings, and can describe them as a coherent system of ideas, she has a theoretical framework. Having a theoretical framework then opens new possibilities for the researcher who turns to using it for new purposes (Steffe & Thompson, 2000).

There is a dialectical relationship between these two kinds of activity—constructing models of a person's understanding and creating a theoretical framework for constructing such models. Constructing a theoretical framework and constructing individual models often happen contemporaneously, and they often exert a reciprocal influence upon each other. A theoretical framework is used to explain students' behavior in terms of what they understand about the situation in which they act. As one refines the models, the framework changes; as one refines the framework, one's understandings of individuals may change (Thompson, 1982a).

Design and Implementation

With the broad purposes of constructing models of teachers' understandings of probability and statistical inference, we designed a summer seminar for high school teachers. The seminar was advertised as "an opportunity to learn about issues involved in teaching and learning probability and statistics with understanding and about what constitutes a profound understanding of probability and statistics." From 12 applicants we selected eight who met our criteria—having taken coursework in statistics and probability and currently teaching, having taught, or preparing to teach high school statistics either as a stand alone course or as a unit within another course. Participating teachers received a stipend equivalent to one-half month salary. Table 1 presents the demographic information on the eight selected teachers. None of the teachers had extensive coursework in statistics. All had at least a BA in mathematics or mathematics education. Statistics backgrounds varied between self-study (statistics and probability through regression analysis) to an undergraduate sequence in mathematical statistics. Two teachers (Linda and Betty) had experience in statistics applications. Linda taught operations research at a Navy Nuclear Power school, and Betty was trained in the Ford Academy of Manufacturing Sciences (FAMS) and taught the FAMS statistical quality control high school curriculum.

[Insert Table 1]

We prepared for the seminar by meeting weekly for eight months to devise a set of issues that would be addressed in it, selecting video segments and student work from prior teaching experiments (Saldanha, 2003) on students' stochastic reasoning to use in seminar discussions, and preparing teacher activities. The seminar lasted two weeks in June 2001, with the last day of each week devoted to individual interviews. Each session began at 9:00a.m.and ended at 3:00p.m., with 60 minutes for lunch. An overview of topics is given in Table 2. All sessions were led by a high school AP statistics teacher (Terry) who had collaborated in the seminar design throughout the planning period.

[Insert Table 2]

The seminars were conducted in a guided-discussion format. Terry began each session with pre-planned activities and a “guide” for discussions that we hoped would unfold, but the discussions often strayed from the central point, and most of the time those digressions were important enough that Terry would see where they might go. Terry would then nudge the discussions back to the current main point. As seen from Table 2, the seminar’s first week was devoted to issues of understanding and teaching statistical inference. It might seem odd that we covered inference before probability. We did this for two reasons. First, our focus on inference was highly informal, never drawing on technical understandings of probability, and emphasizing the idea of distribution. Second, the idea of distribution would be central to our sessions on probability too, and we hoped to avoid any carry-over effect brought about by covering probability before inference.

We interviewed each teacher three times: Prior to the seminar about his or her understandings of sampling, variability, and the law of large numbers; at the end of the first week on statistical inference; and at the end of week 2 on probability and stochastic reasoning. Although ideas of probability occurred in all parts of seminar, this paper will focus on week 2 where ideas of probability were prominent.

The data for analysis included video recordings of all seminar sessions made with two cameras, videotapes of individual interviews, teachers' written work, and documents made during the planning of the seminar. The analytical approach we employed in generating descriptions and explanations was consistent with Cobb and Whitenack's (1996) method for conducting longitudinal analyses of qualitative data and Glaser and Strauss' (1967) grounded theory, which highlights an iterative process of generating and modifying hypotheses in light of the data. Analyses generated through this iterative process were designed to develop increasingly stable and viable hypotheses and models of teachers' understanding.

We began by first reviewing the entire collection of videotaped seminar sessions and interviews. Our primary goal in this process was to develop a rough description of what had transpired in the seminar, and a sense of ways of organizing the data. In reviewing the tapes, we partitioned the sessions into video segments: a segment of video is defined by: a) a chronological beginning and end, b) a task/artifact, c) the rationale of task design, or the issues that the task was designed to raise, and d) discussion/verbal exchange around the task and the issues that arise. For each session, we wrote a summary of all video segments chronologically. Descriptions of video segments consisted of the four components identified above. The interview data were summarized by question. In general, our interaction with data in this phase can be characterized by openness in documenting the data. In other words, instead of looking for answers to specific questions from the data, we consistently documented what happened without differentiating certain segments from the others in terms of how well they might reveal teachers' understanding. Our purpose was to develop an overall sense of what had transpired in the seminar sessions and interviews, and to create an organizational structure and brief summaries that would facilitate further analysis of the data.

In the second phase, we reviewed the videotaped seminar sessions and interviews again, this time with the purpose of identifying video segments that seemed potentially useful to gaining insight into one or more teachers' understandings of probability and statistical inference. Segments containing direct evidence of teachers' thinking, miscommunication in discussions of problems or ideas, or controversy about mathematical meanings or pedagogical practices were transcribed. We then enriched the first level summary by providing thick descriptions of these segments. Along with these descriptions, we also made observations and developed initial hypotheses regarding what these segments seemed to reveal about teachers' understanding.

Conversations around an activity formed the primary units of analysis. For each activity, we captured its global structure by parsing the conversation into hierarchies of episodes. The first level episode was defined as conversations around the organizing questions in the activity, and thus can be conveniently named as the organizing question. The second level episodes depict the significant themes of the conversation in the first level episodes. We then took each of these second level episodes as the primary unit of annotation, where annotations contained hypotheses about teachers' individual meanings, commitments, and beliefs. Our primary goal in the annotation was to clarify the meanings of teachers' utterances,(i.e., discern from their utterances what they had in mind). For any utterance *x*, the questions we addressed included:

- *What motivated a teacher to say x? What was the point the teacher tried to make?*

- *How did it build on this teacher's interpretation of conversation preceding x?*
- *How did the other teachers interpret x?*

The guiding question for making sense of a teacher's utterance is: *What might he have been thinking (or seeing in the situation, or interpreting in the previous conversation) so that what he said made sense to himself?* An interpretation of teachers' thinking is a conjecture that can be subject to further confirmation, refutation, or modification in light of further evidence. Thus, the viability of the interpretations is achieved by working toward compatibility among interpretations across the data.

After annotating an episode, we synthesized the conjectures developed during the annotation. These conjectures were then subject to constant comparison and modification with those of the rest of the data. More substantiated conjectures would emerge from this process and serve as potential answers to the research questions. In this phase, the hierarchical structure of conjectures made during transcript analyses became data that were further analyzed (Cobb & Whitenack, 1996) and re-organized along three categories of themes: (a) teachers' conceptions of probability, (b) conceptions of hypothesis testing, and (c) understandings of variability and margin of error. In this paper we discuss only aspects of each that pertain to teachers' understandings of probability. Within each of these categories, there are sub categories of theoretical constructs that describe/capture the teachers' different conceptions or understanding.

Results

We report the results in three parts. First, we elaborate upon a normative conception of probability. We place this discussion in the results section because, while we embedded conceptions of stochastic processes as an instructional goal in our design of the seminar activities, we found the intricacy of those conceptions, as well as the difficulties teachers had in developing these conceptions, only during - and as a result of - our post-seminar retrospective analyses. Second, we present the theoretical framework that we developed as a result of analyzing teachers' conceptions of probability. Growing from, yet extending beyond, the teacher seminar, this theoretical framework provides a way to model possible ways people understand probability. Finally, we describe the teachers' understandings of probability in the context of the seminar activities and interview questions. Since teachers' emerging ideas and their responses to tasks were coupled tightly with the instruction/conversation that provoked them, we present descriptions of seminar activities and interview questions along with the results.

Stochastic Conception of Probability

Statistical inference builds on a stochastic conception of probability. Statistical inference is about inferring a population parameter by taking one sample. In hypothesis testing, an inference is made on the basis of a probabilistic statement about *the relative frequency* of the observed sample over *a large number of repetitions*. In parameter estimation, the probability of a confidence interval containing the true population parameter is the *proportion* of all confidence intervals that contains the true population parameter. Hence, understanding statistical inference entails a stochastic conception of probability.

In a stochastic conception, an outcome A's probability being x means "an expectation that the long run repetition of the process that produced the outcome A will end with an outcome like A $100x$ percent of the time." To say an outcome has a probability of *.015* is to say that we *expect* the outcome to occur 1.5 percent of the time as we perform the process repeatedly a large number of times.

A person having a stochastic conception of an event conceives of an observed outcome as but one expression of an underlying repeatable process (what Horvath and Lehrer (1998) called a "trial"), which over the long run will produce a stable distribution of outcomes. The conceptual operations entailed in a stochastic conception of an event's probability are:

1. Conceiving of a probability situation as the expression of a stochastic process;
2. Taking for granted that the process could be repeated under essentially similar conditions;
3. Taking for granted that the conditions and implementation of the process would differ among repetitions in small, yet perhaps important, ways;
4. Anticipating that repeating the process would produce a collection of outcomes;
5. Anticipating that the relative frequency of outcomes will have a stable distribution in the long run.

For example,

Suppose that 30 people are selected at random and are asked, "Which do you prefer, Coke or Pepsi?" What is the probability that 18 out of 30 people favor Pepsi over Coca Cola?

To conceive of the underlying situation stochastically entails

1. Conceiving of a random sampling process: selecting 30 people from a population, and asking each person whether he or she favors Pepsi or Coke;
2. Imagining repeatedly taking samples of size 30, and recording the number of people in each sample that favor Pepsi;
3. Understanding that this repeated process will produce a collection of outcomes;

4. Understanding that because of the random selection process there exists variability in the collection of outcomes, but over the long run, the distribution of outcomes will become stable.

Thus, "Probability of 18 out of 30 people favor Pepsi" is the percent of samples having 18 of 30 people preferring Pepsi within the distribution of counts gotten from the collection of 30-person samples.

The Theoretical Framework

A stochastic conception of processes underlies a coherent and powerful conception of probability. Using a stochastic conception of probability as an overarching organizing construct, we developed a theoretical framework (Table 3 and Figure 1) that, when applied to teachers, would result in descriptions of teachers' actual understandings of probability in the situations in which their conceptions occurred.

Table 3 is a list of the theoretical constructs that we used to describe teachers' understandings of probability. It consisted of 3 questions and 7 conceptions or interpretations of probability. Constructs 1 to 3 are chains of reasoning, or ways of thinking, about a probability situation.[1] Constructs 4 to 10 are interpretations of (observable behaviors about) probability. Figure 1 depicts how chains of reasoning could lead to different observable behaviors.

[Insert Table 3 and Figure 1]

Interpretation 4 (outcome approach) is a subjective conception of probability. One who holds an outcome approach thinks that probability is a subjective judgment based on personal experiences (Konold, 1989). Interpretations 5 and 7 both reflect an approach that reduces the sample space for event A to [A, not A]. Interpretation 7 further applies the principle of indifference—the probability of each outcome is 1/n where n is the number of outcomes. People whose meanings of probability are equivalent to the principle of indifference will not be able to make sense of common probabilistic statements, such as, the chance of rain tonight is 40%. They will think instead that the chance of rain is either 1 or 0, or the chance of rain is always 50%.

Interpretation 6 employs what we call proportionality heuristic—evaluating the likelihood of a sample statistic by comparing it against the population proportion, or statistic of a larger sample. For instance, in Example II (Table 4), "*What is the probability that 18 out of 30 people favor Pepsi?,"* a person employing the proportionality heuristic could say, “If you were to sample a larger number of people, would you get the same result?” It meant that in this scenario, 18 out of 30 people favor Pepsi; if we take a larger sample, say 90 people, and around 54 out of 90 favor Pepsi, then we would deem the result of “18 out of 30 favor Pepsi” as highly likely. The logic behind this reasoning could be that: A larger sample is more representative of the population. So if you get the same result in a larger sample, that confirms that the result of the small sample is very likely. The problem of proportionality heuristic lies in the question: What does it mean for two results to be *the same*? Would the two results be the same only when the larger sample has precisely 60% of people preferring Pepsi? What if there is 61% preferring Pepsi? Would that count as the same? If 61% could be counted as same/similar, where does one draw the line? In other words, the difference between proportionality heuristic and a stochastic reasoning is that a stochastic conception of likelihood is quantifiable, but likelihood in a proportionality heuristic is not. The answers to the question “how likely is this result?” must be expressed in qualitative terms, such as, “very likely” or “not likely.”

[Insert Table 4]

Interpretations 8 and 9 reveal a relative proportion conception of probability. Within this conception, the probability of an event is the relative proportion of the number of outcomes that lead to this event to the number of possible outcomes. As we can see from the Example I of Table 4, the results of the interpretations 8 and 9 conflict with each other. This is because interpretation 8 confounds values of a random variable (sum of the dots on the uppermost faces) with the process’ sample space (the set of all possible states in which the process can terminate, in this case the number of dots on each die). Notice that interpretation 9 comes from a standard method of computing probability, namely to divide outcomes leading to event A by the number of outcomes in a sample space. The limitation of this method is that it can be applied only to experiments that have a sample space in which all outcomes are equally likely.

Figure 2 shows the ways observable behavior could be judged were one to know the reasoning behind it. The paths indicated by the arrows in the framework indicate different conceptions of probability.

Non-stochastic conceptions: 1→4, 1→5, 1→6, 1→7, 1→8, 1→9, 1→2→6, 1→2→7, 1→2→8, 1→2→3→9

Stochastic conceptions: 1→2→3→10

Arriving at box 10 by way of 1, 2, and 3 is the only way to understand probability as entailing a stochastic process. Note that boxes 7, 8, and 9 represent behavior that is taught in schools and that is commonly taken as evidence that the person understands probability.

[Insert Figure 2]

As shown in Figure 2, people could have different conceptions of probability and interpretations of probability situations. We believe that it is important for teachers to not only develop a coherent and powerful understanding of probability, but also to have an understanding of how probability statements and situations might be interpreted differently. In other words, we wanted the teachers to understand the idea that *a situation is not stochastic in and of itself. It is how one conceives of a situation that makes it stochastic or non-stochastic*. We argue that this idea is essential for teachers to become sensitive to alternative understandings their students might have. Furthermore, as we have shown, just as a non-stochastic conception can have many different expressions, a stochastic conception, too, could lead to different interpretations of a probability situation. In other words, *a particular event could be seen as an outcome from different stochastic processes, and thus the probability one assigns to this event differs depending on how one conceives of the stochastic process underlying it*. For example, suppose we have urns A, B, and C, each containing a number of red and white marbles (see Table 5). Question: What is the chance that if you draw a red marble, it is from urn C?

[Insert Table 5]

There are two ways of conceiving this situation stochastically. The first is to imagine that all the marbles are dumped into one container and each is labeled by the urn from which it came. Of all 10 red marbles, 3 came from Urn C, thus we would expect that, over the long run, 30% of the time that we select a red marble, it would have come from Urn C. A second way is to imagine a repeated process that we will first pick an urn at random, and then select a marble from that urn. In the long run, each urn will be chosen 1/3 of the *n* times we repeat the process. We will select a red marble from Urn A 2/7 of the time Urn A is chosen, or $\frac{1}{3}\times\frac{2}{7}$ of the *n* times we repeat this process. By the same token, we will draw a red marble from Urn B $\frac{1}{3}\times\frac{5}{9}$ of the *n* times we repeat this process, and we will draw a red marble from Urn C $\frac{1}{3}\times\frac{3}{12}$ of the *n* times we repeat this process. Therefore, we select a red marble $\frac{1}{3}\left(\frac{2}{7}+\frac{5}{9}+\frac{3}{12}\right)$ of the *n* times we repeat the process, and we will have selected a red marble from Urn C $\frac{1}{3}\times\frac{3}{12}$ of the *n* times that we repeat this process. Therefore, we will select a red marble from Urn C $\frac{\frac{1}{3}\times\frac{3}{12}}{\frac{1}{3}\left(\frac{2}{7}+\frac{5}{9}+\frac{3}{12}\right)}$ of the times we select a red marble (about 23% of the time).

This example illustrates that a probability situation can be conceived of from different stochastic perspectives, and that an answer to a probability question is valid as long as it is consistent with the underlying situation as one has conceived it.

Teachers' Understandings of Probability

In this section we describe two sets of activities and interview questions in which we investigated teachers' conceptions of probability. Table 6 provides an overview of the activities and interviews, the day on which it was conducted, and the duration of conversation around each activity.

[Insert Table 6]

The first three activities and the follow-up interviews focused on teachers' interpretations of probability situations, particularly as to whether they conceived of the situations stochastically or non-stochastically. The last activity introduced a probability situation that, if conceived stochastically, may be subject to multiple interpretations. We were interested in knowing the ways with which the teachers responded to this type of situation.

Activity 1: Chance and likelihood.

Question 1: What does "a 45% chance of rain" mean?

Question 2: A pollster asked 30 people about which they liked better, Pepsi or Coca-Cola.

18 said Pepsi.

What does it mean to ask, "How likely is this result?"

We raised these two questions in the first day of the seminar. The discussion around the first question lasted for 17 minutes. The discussion around the second question lasted for 12 minutes in two separate segments: 7 minutes in the afternoon of Day 1, and 5 minutes in the morning of Day 2.

The key to interpreting chance and likelihood stochastically is to conceive of a stochastic process that generates a collection of outcomes of which the particular phenomenon in question is but one. For example, "a 45% chance of rain" means, "It rained on 45% of all those days having conditions like today." The stochastic process is to examine the weather conditions of each past day having similar conditions as today. Interpreting the Pepsi situation stochastically entails conceiving of: (a) an underlying population having a relatively stable proportion of people who favor Pepsi, and (b) a process of taking random samples of 30 people out of this population. Seeing the situation as such allows one to think of the result "18 people out of 30 favor Pepsi" as one of the possible outcomes of the sampling process. The likelihood of this outcome can then be quantified as the relative frequency of outcomes like this one against the total number of times the sampling process is repeated.

Table 7 summarizes the teachers' interpretations and ways of thinking about these two situations over the entire discussion of them.

[Insert Table 7]

As shown from Table 7, three out of eight teachers interpreted the first situation non-stochastically. They were concerned, essentially, about *today's* chance of rain. For example,

Excerpt 1

Betty [It] means you'll probably wanna take your umbrella.

Excerpt 2

Linda I would do it based on what you were going to do that day. If you had an outdoor wedding planned, I'd say there's a good chance… you need to plan something else. But if I was planning for some outdoor sports activity, then I'd probably go ahead and do it. It is a relative—I don't know, It is not a number that really means 45 of anything. It is just…forty-five percent of anything, like if it were a hundred percent chance of rain then you know it is going to rain for sure, if it is eighty-five percent then you're almost sure it is going to rain, ten percent chance, well it probably won't rain so go ahead and do whatever you want that day.

Excerpt 3

John There's always a 50% chance of rain, it just so happens that on that day there's a little bit less than a fifty percent chance. It is either it will rain or it will not.

For Linda, a 45% probability of rain conveys the strength of belief about the likelihood of rain that lies somewhere between "definitely going to rain" to "no rain." John's conception of probability closely resembled the principle of indifference: the probability of rain on any given day is 50% if we remain completely ignorant of the possible conditions or information that might provide evidence for or against any one of the two outcomes [rain] and [no rain].

Henry and Nicole interpreted the chance of rain stochastically:

Excerpt 4

Nicole If you had 100 days just like today, that in forty-five days it is going to rain in the immediate area.

Excerpt 5

Henry I think today they run a lot of models, meteorological models…if you had a hundred days just like today, the model says it is going to rain forty-five of those a hundred days.

They understood that the probability of rain on one particular day was calculated on the basis of a collection of days having similar conditions.

The discussion around the second scenario revealed that none of the teachers interpreted the situation stochastically (Table 7). John, Betty, and Linda evaluated the likelihood of "18 out of 30 people favor Pepsi" based on their beliefs of how the population was distributed with respect to their preference. Betty, a Coke drinker, believed more of the population favored Coke.

Excerpt 6

Betty As a Coke drinker I would think more people would choose Coke, because I like Coke better and I can't see how anybody can drink Pepsi.

Linda and John assumed that the population was evenly split.

Excerpt 7

John Yes, it is highly likely that it is 18. I would expect this number to be between 10 and 20. I doubt that it would be 0 and I doubt that it would be 29. But, I'm not trying to be funny. I'm being serious, it is either, either you like Pepsi or you don't like Pepsi…. So the other choice would be "not Pepsi", which would be Coca Cola. So it is either Pepsi or it is Coca Cola. And uh…how-- what does that tell us? Eighteen out of thirty, what does that tell me? Well, I can't say that 60% of the nation likes Pepsi because that's not enough data. That's just one sample of 30. But I would expect to see one sample of 30, say, 10 people liking Pepsi, or 20 people liking Pepsi, I would expect to see a lot of things, but I wouldn't expect to see 0 and I wouldn't expect to see 30.

Excerpt 8

Linda It is about half way, so I would say it is very likely. Because 15 is half, so … I mean Pepsi and Coca Cola are, are always competitive with each other.

John's justification was, "either you like Pepsi or you don't like Pepsi… it is either Pepsi or it is Coca Cola." The logic of this reasoning is: "Because any particular person either likes Pepsi or doesn't like Pepsi, about half of the population like Pepsi and half don't." The problems with this logic can be easily seen by an analogy: "Because you either have AIDS or you don't, about half of the population have AIDS." The fault of this reasoning lies in the fact that one is thinking about individual cases while making a statement about a collection. It is a coerced application of the principle of indifference, always acting in a state of ignorance. The defining characteristic of these three teachers' reasoning is that they tried to answer the question "how likely is this result?" on the basis of: (a) a particular result, and (b) their beliefs about how the population was distributed—"I believe the population is distributed in this way. Given what I believe, the likelihood that 18 out of 30 people favor

Pepsi is…" This way of reasoning is what we called proportionality heuristic—evaluating the likelihood of a sample statistic by comparing it against the population proportion.

Lucy applied a variation of this heuristic by saying, "If you were to sample a larger number of people, would you get the same result?" We interpret her as meaning something like this: "In this scenario, 18 out of 30 people favor Pepsi. If we take a larger sample, say 90 people, and around 54 out of 90 favor Pepsi, then we would deem the result of "18 out of 30 favor Pepsi" as highly likely.

Sarah's answer "would you get the result over and over" implies that she had conceived of a repeatable process and alludes to the idea of likelihood as a long-run expectation. However, since she did not elaborate on her thinking, we could not know what she had in mind about the process and how it related to the likelihood of the result.

A later discussion revealed that the teachers' interpretations of the word "likelihood" might have contributed to their interpretation of the situation. They argued that the word *likelihood* did not call for a quantitative interpretation.

Excerpt 9

Nicole	But "likely" feels like a weaker word than "what percent." "What percent" means to me that I need to do some floundering around to figure out a "mathie"-type answer and um, that's what "what percent" means. "How likely" just was (shrugs)..I mean in that case I wasn't making the assumption that it was 50-50 so it seemed quite likely 'cause I'd been basing it on my um…on a general knowledge… that's, you know, they're sort of equally distra-dist-distributing in Kroger, but I've never stood there that long and counted them up. So, sure!

Sarah	"Likely" is less definitive than percent=[1]
Nicole	=That's right! That's how I felt and so=
Sarah	=and I don't disagree with that, but I'm like Henry, I couldn't come up with a better phrase! So..
Terry	When you think of the word "likely", what question do you think, when you talk about likelihood, just in general=
Nicole	=what could've happened!?
Terry	And how would you measure "Could've happened"?
Nicole	Well, I wasn't measuring it because I thought that that's what the slide, you know=
Terry	=Ok=
Nicole	=I mean if, if you would ask me, you know, "three percent favor Pepsi, how likely is that result?" Just sort of based on my common knowledge I would've said, "I don't think it is likely, no."
Terry	OK.
Lucy	"Likely" sounds more like it is asking for a question like, for an answer like "fairly-likely" or "not very likely" or something.

This discussion revealed that teachers were using the colloquial meaning of the word "likelihood." They did not equate likelihood to the technical meaning of probability. *Likelihood* to *probability* is like, for example, *warmth* to *temperature*. When we ask a question, "How warm is the water?" we expect answers such as, "Yes, very warm," "Not very," or "It is not." When we ask, "What is the temperature of the water?" we expect a

[1] We used "=" sign to denote instances where teachers interrupted each other's utterances.

measurement of the temperature expressed in quantitative terms, such as "70 degrees." By the same token, teachers thought that the question of "how likely" called for answers such as "fairly likely" or "not very likely." This non-quantitative conception of likelihood is consistent with the teachers' non-stochastic interpretations of the Pepsi situation, and it suggests that "probability" was something that you compute.

In summary, at the beginning of the seminar most teachers had a non-stochastic conception of chance and likelihood. Their interpretations of chance and likelihood situations were subjective, expressed either as purely personal beliefs or as results of a coerced application of the "principle of indifference." They held a non-quantitative meaning of likelihood, in which case likelihood expresses strengths of beliefs about the possibility of chance occurrence, as opposed to a mathematical expectation.

Activity 2: Movie theatre scenario.

> Ephram works at a theater, taking tickets for one movie per night at a theater that holds 250 people. The town has 30 000 people. He estimates that he knows 300 of them by name. Ephram noticed that he often saw at least two people he knew. Is it in fact unusual that at least two people Ephram knows attend the movie he shows, or could people be coming because he is there? (The theater holds 250 people.)

This activity was adapted from Konold (1994, pp. 16-18) It centered on investigating the question: Is it *unusual* that at least two people Ephram knows attend the movie he shows? Ways of thinking about this question that indicate a stochastic conception of usualness would be to conceive of repeated observations of Ephram's theater. The statement, "Ephram sees *x* people he knows" would then be understood as pointing to a random process that produces a distribution of outcomes when applied repeatedly over a large number of nights. That is:

1. Assume that people go to the theater randomly (i.e., people do not go to the theater because Ephram is there);
2. Think of a collection of nights, when random samples of 250 people from the population of 30000 go to the theatre;
3. Record the number of people Ephram knows each night;
4. Plot a distribution of these numbers, and calculate the density of "at least 2": the chance of at least two people Ephram knows attend the movie he shows;
5. If the proportion is smaller than 5% (a conventional significance level), then conclude that it would be unusual that at least two people Ephram knows attend the movie.

The discussion around this activity lasted about 106 minutes. Table 8 showed that, of the six teachers whose conceptions of unusualness were revealed in the discussion, all but one teacher had non-stochastic conceptions of unusualness. John applied a proportionality heuristic to "measure" the unusualness, while Sarah, Betty, and Linda had an outcome approach to unusualness. Only two teachers, Alice and Henry, conceived of unusualness as a statistical value.

[Insert Table 8]

John's method of measuring unusualness was to compare two relative proportions: (a) the proportion of people Ephram knows by name out of the entire population in town, and (b) the proportion of people Ephram knows out of all the people coming to the theatre (i.e. 2/250).

Excerpt 10

Terry	How would you figure that out?
Henry	Develop a proportion.
Terry	Okay, and how would you develop a proportion?
John	Well, he knows 300 and there's 30,000 people so there he developed a proportion from that. Out of the proportion of 30,000, how many does he know? So basically he knows 1 out of every 100 people.
Terry	Okay
John	So, that's what I did.
John	I set up a proportion and basically got that he knows 1 out of every 100 people=
Terry	Okay and then what did you do with that?
John	So therefore I said he'd know approximately 2.5 people at the movie theater because if 250 come, out of the first 100, he should know 1 person out the next 100 he should know one and of the last 50 he should know half=

John's reasoning was this:

> Since Ephram knows 300 people out of 30,000 people in his town, it means for every 100 people, he knows 1 person. On any given night he should know 2.5 people out of 250 people who come to the theatre, given that this 250 people is a random sample of 30,000 in his town. Therefore, it is not unusual that he saw in the theatre at least 2 people he knows.

This method employed the proportionality heuristic. It suggested that John's conception of unusualness was about one particular event: how likely is it that on one particular night

Ephram sees at least two people he knows? This way of conceptualizing unusualness does not quantify unusualness. Rather, it leads to a subjective judgment on the likelihood of the outcome (of Ephram seeing at least two people he knows) in non-quantitative terms—“Yes, it is unusual,” or “No, it is not.”

Sarah claimed that the event “Ephram saw *a whole herd of* people he knew” was unusual.

Excerpt 11

Sarah I thought if he had a night where he saw 50 people that he knew, that would be unusual.

…

Terry … I’m hearing people saying ‘one night, just by proportions, 2-3 people that he knows are going to be there and what I’m saying is that that, to me, is not talking about whether that was unusual or not.

Sarah I think that would be your expectation. Something unusual would have to be something different than that, like a whole herd of people comes in that he knows.

John We’d have to go back to the assumption=

Terry Okay, okay stop! Let’s not even talk about this situation. Just tell me what your definition of unusual is.

Sarah Something that does not meet the expected.

She did not see “Ephram sees x people he knows” as a random repeatable event. While John was focusing on one particular night when x=2; Sarah focused on one particular night when x=*a big number*. Neither one of them had conceived of a collection of nights with varying outcomes. Sarah's conception of unusualness was entirely subjective—something is unusual if it is unexpected, and expectations are made on the basis of personal experience.

The following excerpts showed that Linda and Betty's conceptions of unusualness were also subjective.

Excerpt 12

Linda	Can I draw a different example? …Okay, in a college you expect that for every college algebra class, 30 people are going to sign up. But if one instructor has a 60 person waiting list, that's unusual. Okay? Because you don't see that. There's something going on there. That's unusual.

Excerpt 13

Terry	What is my criterion for me saying that [something] is unusual?

Betty	It is out of the ordinary. It is out of normal circumstances.

These excerpts suggested that they both had a subjective conception of unusualness. More specifically, Betty exhibited a typical case of someone employing the outcome approach (i.e. an event/outcome is unusual if it is unanticipated to occur).

Only two teachers, Alice and Henry, had a quantitative conception of unusualness.

Excerpt 14

Terry	Is it unusual that people get struck by lightning?

Alice	[Yes, because] It happens to very few people.

Terry	Okay when you say it doesn't happen very often, what do you mean?

Alice	Think about the whole population. Compared to the number of times it happens within the whole population.

Excerpt 15

Henry	There may be a lot of people at the table who have a mathematical definition of unusual, if you want to talk about numbers and problems and so forth, and then you may have a personal definition of unusual. For instance, myself, I tend to think 'well, something's unusual if I'm doing it less that 50% of the time'. That's just sort of my rule of thumb for me as a person. If I'm not doing it 50% of the time or more, then it is unusual. It doesn't occur for the majority. It is less than the majority. It is the minority.

Both Alice and Henry's interpretations of unusualness involved some underlying repeatable process. To Alice, the unusualness of "people get struck by lightning" was measured by the low frequency of its occurrence over a large number of times. To Henry, an event was usual if it was repeated less than 50% of the time.

Activity 3: PowerPoint presentation. We started the second week by showing a PowerPoint presentation and asking the teachers to interpret the probability situations in slides 1 to 4.

1. Risky Rides

Your risk of being killed on an amusement park ride? One in 250 million. What does that mean?

Your risk of being killed driving home from an amusement park ride? One in 7,000. How do you suppose they determined these values?

2. Vitamin Use

   In a study of over 88,000 women, total folate (vitamin Bc) intake was not associated with the overall risk of breast cancer. However, higher folate intake (or multivitamin use) was associated with a lower risk of breast cancer among women who regularly consumed alcohol. What does this mean to you? How do you suppose they determined this.

3. Gustav's Bad Luck

   Gustav read in *Newsweek* magazine that drivers with three or more speeding tickets are twice as likely to be in a fatal accident as are drivers with fewer than three tickets. What does this mean?

   The very next day he received his third speeding ticket. What does this imply?

4. Car color & Temperature

   What is the probability that my car is red?

   What's the probability that the temperature tonight is below 40? What does that mean?

5. Rashad's situation

   Rashad's sister, Betty, rolled ten sixes in a row while playing a board game.

   Rashad: "That is impossible!" (response 1)

   Rashad: "If a billion people rolled a die 10 times, what fraction of them would roll all sixes?" (response 2)

Table 9 summarizes teachers' conceptions of probability revealed in this activity.

[Insert Table 9]

We will provide details of the discussion around slide 4 as this discussion hinged on the big idea that "The situation per se is not probabilistic. It is how you conceive the situation that makes it probabilistic." Linda started by stating that the probability that "my car is red" is either 1 or 0, "because there's only two outcomes. It either is or it isn't." In her thinking, there was one particular event (color of my car) with two possible outcomes (red and not red). Thus, the probability that "my car is red" is 1 when the outcome is red, and 0 when the outcome is not red. Sarah commented that "probability of buying a red car have to do with how many red cars out of how many cars were on the lot." Terry, the seminar host, dismissed this interpretation on the ground that the situation under discussion was about *my car*. She emphasized the phrase *my car* and insisted that the car situation was about a single event/object, and argued that it did not make sense to talk about probability in this situation.

Excerpt 16

Terry	Does it even many sense to talk about that as a probability?
John	No. It doesn't make sense.
Terry	No, it just doesn't even really make sense to talk about it because think about probability as repeating a process, and one of the characteristics being that we don't necessarily know what's going to happen, but once you go out there and see what color my car is, you know. It is not until I buy, you know, if I don't buy a new car, it is going to be the same color. There's no unknown there.

The second question, "What's the probability that the low temperature tonight will be below 40 degrees?" was very similar to the probabilistic statement the teachers had discussed earlier, "Today there is a 40% chance of rain." Nicole, who had interpreted that statement stochastically, had a similar interpretation to the weather situation. However, as we will see in the following excerpt, when Sarah proposed a different stochastic interpretation (different in the sense that Sarah specified a particular collection where Nicole vaguely expressed as "over a long period of time"), Nicole questioned her own interpretation.

Excerpt 17

Terry	What's the probability that the low temperature tonight will be below 40 degrees? What does that mean?
Nicole	Well, don't you think that it means that if we had conditions just like today, and the weather conditions generally around here over a long period of time, what's the probability that we'll get a temperature below 40 degrees?
Sarah	Or does it mean that on June the 18th, of all the June 18ths how many have we had that were below 40 degrees?
Nicole	That's right, is it tonight or is it all of these nights?
…	
Nicole	Anyway, it is zero for tonight, folks.

Sarah's mention of "June the 18th"(the day of the discussion), despite the fact that she was thinking of many years of June 18th, made Nicole realize that there could be two ways of interpreting this question. One is non-stochastic (i.e. thinking of the situation as being about one particular event): "The temperature *tonight* is below 40." The probability in this case is, as she said, "It is zero for tonight." The other way of interpreting the question is stochastic, that is, imagining looking at the night temperature over an extended number of June 18ths.

Henry observed that the weather situation could be interpreted in two ways. One way was to say: tonight's temperature is or isn't going to be below 40 degrees; another way was to look at historical data about temperatures of days like today. But the only acceptable interpretation for the car situation was "it is either red or isn't red." Henry raised the question, "If there were two ways of interpreting the weather situation (stochastically and non-stochastically), why couldn't the car situation be interpreted stochastically?"

Excerpt 18

Linda One has a large number of nights to base your---your statement about tonight, you can base it on a large number of previous nights, but if you talk about your car—if I said something about my car, I wouldn't base it on anything that I've ever owned before. I mean…

Henry I've got a friend who's owned 50 cars. What's the probability=

John I see. You've made a good point, Henry. Those are exactly the same question. Because, if the weatherman had to make a forecast about what kind of color your car would be, the weatherman would have the model go and look at the previous cars, just like he goes and looks at previous weather situations, so predict what tonight is going to be like.

Linda But there's a question about tonight. There's no question about what kind of car she has.

John I think those are the exact same question=

Linda There is no, no question. She either has a red car, or she doesn't=

Excerpt 19

Linda Maybe the question should be is it determined now. Whether we know it or not is really not the issue, it [the issue] is, "Is it determined?"

Terry Right. Is it determined what the low temp—that's a good way to put it. It is determined what color my car is. There's no chance there. My car is a certain color. It is not going to change color when I walk out there. Whereas, it is not predetermined, at this moment, what the low temperature's going to be. So there is some chance involved there. So there's an element of—I can't predict what the low temperature is going to be.

Discussion around Henry's question revealed that, while one group of teachers (Henry, John, and Nicole) realized that a probability situation could be interpreted in both ways, another group (Terry and Linda) insisted that the two situations in slide 4 must be interpreted differently. The difference in the two groups' opinions hinged on the question: What was the outcome? To Terry and Linda, the outcomes for the weather situation were the possible temperatures for either tonight, or a collection of nights, but the outcomes for the car situation were "red" or "not red." The epistemological difference, it seems, is that in one case they anticipate variability of outcome (temperature) and in the other case they do not (color of car). In the former, they would be asking the same question about *different nights (or different time tonight).* In the latter, they would be asking the same question about *the same car.* The other group, Henry and John, had a different perspective. The outcomes of the weather situation could be either "below 40" or "above 40" when the situation was seen non-stochastically, which was parallel to the non-stochastic interpretation of the car situation. By the same token, the car situation could be seen from a stochastic perspective, in which case, the outcomes were colors of a collection of cars (previously owned by a person), or alternatively, seeing the presented car owner as one of a population of car owners having similar known characteristics, and then considering the random variable to be the color of each such person's car.

In the follow-up interview at the end of the seminar we attempted to see the extent to which teachers were able to control ways of interpreting probability situations. Table 10 summarized the teachers' interpretations of the following probability situations in Question 1.

1. What are the chances that it will snow in Billings, Montana on April 23, 2002?
2. What's the probability that the length of Dean Benbow's driveway is between 29' and 30'?
3. Your risk of being struck by lightening is 1 in 400,000.
4. How likely is it to be dealt one pair in a 5-card hand from a standard deck?
5. What's the probability that you are off by no more than 2" when you measure the length of your driveway?

[Insert Table 10]

Table 10 shows that the teachers consistently interpreted Statement 4, the situation of "dealing a 5-card hand" stochastically. Majority of the teachers also interpreted Statement 1 and 3 stochastically. Most of the teachers conceived of statement 2 non-stochastically. Only two teachers, Lucy and Henry, interpreted several situations in both ways (Lucy 2 out of 5, Henry 3 out of 4). Looking at interpretations across situations given by each teacher, we found that Nicole, Linda, and Alice seemed to have an orientation to conceive of probability situations stochastically. John, Sarah, Lucy, Betty, each conceived of two situations non-stochastically. Most teachers made statements about whether the situation was or was not probabilistic, as opposed to whether they conceived of the situation as probabilistic. Henry was the only one who consistently tried to interpret all situations in two ways. He acknowledged that he had difficulty seeing the third statement (i.e., risk of being struck by lightning) from a non-stochastic perspective. Perhaps it was because had he done so, then the probability would be either 0 or 1, which was in conflict with the given measure of probability1/400000.

The second interview question resembled slide 6, Rashad's situation, in the PowerPoint presentation, in that the distinction between non-stochastic and stochastic interpretations was made clear in the question.

You <u>must</u> make a choice between:

a) definitely receiving $225
b) a 25 percent chance of winning $1,000 and a 75 percent chance of winning nothing

1) Suppose this is a one-time choice. That is, you are presented with these options once and you will never be presented with these options again. Would you choose (a) or (b)? Why?

2) Suppose you are a gambler who will be presented with these same options many, many times. Would you choose (a) or (b)? Why?

Teachers' answers to both questions (Table 11) revealed that they understood the distinction between making a one-time choice and making repeated choices, and they understood that consistently choosing *b* would yield more benefit over the long run. Four teachers made their choices based on this understanding. The other four teachers, while understanding the distinction, nevertheless made their choices based on their personal preference of avoiding risk. It is worthwhile noting that Henry chose to accept risk even in the one-time situation. Perhaps he thought of this one-time situation as one of many one-time situations.

[Insert Table 11]

Activity 4 Clown & Cards situation.

At the Cobb County fair a clown is sitting at a table with three cards in front of her. She shows you that the first card is red on both sides, the second is white on both sides, and the third is red on one side and white on the other. She picks them up, shuffles, hides them in a hat, then draws out a card at random and lays it on the table, in a manner such that both of you can see only one side of the card. She says: "This card is red on the side we see. So it is either the red/red card or the red/white card. I'll bet you one dollar that the other side is red."

1. What is the probability that you would win this bet were you to take it?

    Part 1 - Discuss how you are thinking about this situation in order to formulate an answer to the question.

    Part 2 - We will watch video excerpts of students attempting to make sense of this situation. Discuss what they seem to struggle with.

2. How might we help students reason about the situation in a way that is coherent and consistent with ideas of repeated sampling and probabilistic situations we have been discussing?

Activity 4 is non-conventional in the sense that, unlike a typical math problem, it does not have a correct answer. We designed this situation to get at the idea that a particular event could be seen as an outcome from different stochastic processes, and thus the probability of this event differs depending on how one conceives of the stochastic process. Our purpose in engaging the teachers in this activity was to understand the ways in which the teachers responded to this type of situations. Discussion around this activity lasted for 138 minutes in total. Teachers' initial interpretations could be divided into four categories that led to three different answers: "1/2," "1/3," and "either 1 or 0." (See Table 12)

[Insert Table 12]

Betty, Sarah, Lucy, Henry, and Linda believed that the probability was 1/2.

Excerpt 20

20. Betty Okay, I took it from the two cards that were on the table, I mean, from the two cards that were red, that had red on them, not from the white-white because that seemed to be out of the picture. So I took those two possibilities, so it is either going to be on the other side, red or white, so that's, to me, one out of two.

…

33. Terry What is the probabilistic situation? What is the process that is being repeated?

34. Betty Well it is— to me, as it is, maybe I messed up=

35. Terry Lucy?

36. Lucy Flipping the card that is right in front of you, flipping it up.

37. Terry Okay, so that's the repeatable process…

38. Betty Every time you drew a red card.

39. Henry That card only=

40. Terry =Just the one card that's=

41. Henry ='Cause if it is the ah, white… well, I don't (inaudible)

42. Sarah: From either of the cards that would give you …

43. Henry I wasn't looking at it though from any of the three cards, I was just looking at it from the, the two that would be red or white.

44. Sarah From just that one card.

45. Betty To me that is the only thing that is being repeated – rather than drawing from the basket to start with or whatever…or the sack whatever.

46. Terry All right, Linda you were going to say something.

47. Linda It would be like drawing from a bag of cards that are red/red or red/white.

48. Terry So what are you saying there, are you agreeing that the probabilistic situation is flipping that card over?

49. Linda Yeah, it is the same thing, or, or having a bag of red and—red stuff and white stuff, because the issue is…that the issue is not the side facing you, you already know what that is.

50.	Sarah	Did you=
51.	Linda	=So you're down to picking between, picking things from a sample space of red vs. white.

…

631.	Sarah	Well, but I know, I mean, Betty was looking at this one card and we're looking at—and I'm thinking like Betty is – we've got two things going on here, you asked Betty what she *thought*, when I thought about it I thought about the way she did: you've got a card laying there that's red, it is been drawn out and to be red up it either has to be red or white on the bottom, there's only two outcomes, you flip it over, it is a one out of two possibilities on that particular part, but if you want it stated probabilistically…

There were two important assumptions made by this group of teachers. First, they believed that since a card was already drawn and a red side was up, the white/white card should be eliminated from their consideration, because had it been drawn, it could not have rendered the outcome of a red side facing up. The second assumption, more tacit than the first one, was that it did not matter which side of the card was visible. Only the color of the visible side mattered. As Linda said, "the issue is not the side facing you, you already know what that is." Table 13 illustrates how the teachers conceived of the outcomes of the process.

[Insert Table 13]

There were two ways the teachers conceived of the situation: One group of teachers, Betty, Lucy, and Linda, conceived of a stochastic process; Sarah conceived of the situation non-stochastically. The stochastic process that Betty and Lucy conceived was "flipping the card whenever there is a red card comes up" (lines 36 and 38)[2]. Linda equated this process to that of repeatedly "drawing from a bag of [two] cards that are red/red or red/white." In other words, were Linda to draw the R/R card, the other side would be red; were she to draw the R/W card, the other side would be white. Sarah saw the situation as: There was one card with red on top, the bottom was unknown but there were two possibilities, red and white (line 63). Therefore the probability was the relative proportion of these two possible outcomes. There was no image of a stochastic process in Sarah's conception of the situation. To summarize, here we saw the teachers giving the same interpretation to probability (as relative proportion of observed outcomes out of all possible outcomes) with Lucy, Betty, and Linda having a stochastic conception (path 1-2-3-9), and Sarah having a non-stochastic conception (path 1-9).

Nicole gave a different answer: The probability of winning the bet was 1/3. Nicole had conceived of a two-stage sampling process. This process was "Pull a card from the bag, place one side on the table, and look at the other side of the card." Table 14 illustrates how she thought about the outcomes of this process.

[Insert Table 14]

Nicole evidently believed that picking a card randomly from the bag and placing it on the table produced 6 possible outcomes of the color of the upper side. There were two ways of placing the red/red and white/white cards: side A up or side B up. Since it was known that a red side faced up, three outcomes that had white on the upper side would be eliminated, hence the crossing out of WW, WW, and WR. Next, out of the three possible outcomes that had red on the upper side, only one had a white on the other side. Therefore, the probability was 1/3. Note that Nicole's conception of the situation was stochastic, and probability was the relative proportion of observed outcomes out of all possible outcomes (path 1-2-3-9). It is also important to note that Nicole focused on the outcomes being pairs of sides, one side up and one side down, and not simply on the color of the upper side.

Nicole's explanation is in the following excerpt:

Excerpt 21

145 Nicole Why can't the clown, also say – it doesn't say this here – but why can't the clown just pull out a card and slap it down and let's assume it is white this time, and now the clown says, um… what's the probability the other side is white?

suggested that the situation she conceived of was "The clown plays the game repeatedly, drawing a different card, and making the bet with reference to whatever color is up." This conceptualization of the situation allowed her to accommodate to hypothetical scenarios where white cards faced up.

John offered the third interpretation, that is, the situation was not probabilistic.

Excerpt 22

132. John I just wanted to say that, I think that we're wrong, I believe that the probability is one or zero, because we're talking about *that* card. You want to repeat the process, why don't you flip it over one time you know the answer, so there's no repeating it=

133. Sarah That's a good call.

134. John This is not a probabilistic situation, would be my answer. This process cannot be repeated.

135. Terry So if you go, if you go to that booth at the fair, the clown is always going to have the same cards sitting there.

136. John No…no, no. See what this says, this says, we do=

137. Terry But you, in order for this to make sense you have to think about it, probabilistically.

138. John Yup, well, that's what I'm— the way I'm looking at it is that card's been sat down on the table, or, the card is there, it is that card, it is not asking what's the probability it pulls it out of the hat, it is after he's pulled it out of the hat and put it down, here's the card sitting on the table. The process being repeated is not pulling it—keep pulling it out of the hat=

John argued that both 1/2 and 1/3 were wrong and that the probability was either 1 or 0 because he believed that the situation was just a one-time event. The situation was about the clown and the one person who was betting the color of the down side of that one card, *once*. There were two possibilities: if the other side is red, the probability is 0, and you lose the bet; if the other side is white, then the probability is 1 and you win the bet (path 1-5).

After the teachers laid out their initial interpretations and answers to the question, they debated on which interpretation should be the correct one. It is not within the scope of this paper to delve into the details of that debate. Briefly, each teacher was committed to his/her own interpretation, and spoke from his/her own perspective. Thus, as a group, they were not able to reach an agreement on which interpretation was correct. Pat commented that a situation could have multiple interpretations and that an answer would be valid as long as it is consistent with the way one conceives the situation and the interpretation (that led to the answer) is consistent with the text. However, the teachers kept defending their own point of view and dismissed the others as being wrong or not viable. Nicole proposed to use computer simulation to resolve the conflict. The discussion quickly fell back to the same debate "What is the situation?" as each teacher argued for, and tried to set up, the simulation in a way that fit with how he/she conceived of the situation. Pat tried to convey the idea that one's simulation was essentially determined by his/her conceptualization of the situation, and therefore a simulation could not, in principle, resolve the question of which conceptualization was correct. However, only one teacher, Nicole, came to understand this idea. Despite Pat's repeated explanation, Henry and Terry[3] kept proposing simulation as a means of proving the correctness of their own interpretations, believing that there should only be one correct simulation to run, which would then lead to the correct answer.

The debate remained at the level of each person trying to convince others that they were wrong by reiterating his or her own view. At this juncture, Pat attempted to engage the teachers in reflective conversations about their debate by raising the question, "How do you resolve the conflict without restating what you think is true?" His intention was to push the teachers to reflect on the assumptions behind their thinking. Pat understood the fundamental distinction between the two camps—those who believed the probability to be 1/2, and those who argued for 1/3—was that the first camp approached the problem with a fixation on the question that was asked while the second camp focused on the underlying situation behind the question. Having an orientation to the underlying situation allows one to envision the space of all possibilities and to solve all the questions relevant to a situation, whereas an orientation to the question often leads one to choose only relevant information and thus prevents one from making sense of the entire situation as a whole (recall the Urn & Marble example).

Pat's intention, by raising the question of "How to resolve the conflict," was to invite the teachers to think about instructional design, that is, in designing probability instruction the teachers should avoid this type of conflicts by always orienting the students to think about the underlying situation as opposed to the question that is being asked. To participate in this reflective conversation required that the teachers take their debate as an object of reflection and consider the potential instructional action that would resolve the conflict had it occurred in their classrooms. However, the teachers were not able to take the collection of, and the conflict of, different interpretations as an object of thought. Rather, they understood Pat as saying "How do we decide which interpretation is correct?" and thus kept arguing for what they believed was the correct interpretation and proposing to use simulation to resolve the differences. When Pat reiterated that simulation could not validate interpretation, John stated in frustration that to resolve the differences they should simply tell the other party they were wrong, and that he would rather be told he was wrong than engage in a discussion like the one they were having. Our follow-up interviews revealed that several other teachers also shared this frustration.

The teachers' belief that the discussion was about which interpretation was correct implied that they did not think of the discussion as an experience upon which they could reflect on both conceptual understanding of probability and the pedagogy of teaching probability. Their frustration suggested that they did not think of the discussion as being productive and that they did not benefit from struggling with this conflict. They did not separate their identity of teacher from their identity of learner/problem solver and try to ask the question: "What can I learn from this in terms of teaching?" Rather, they seemed to have anticipated the seminar discussion to be a model that they would imitate in their classrooms, and their frustration was, in part, a result of the breakdown of that anticipation.

The pedagogical conversation culminated with Pat explaining his intention. He wanted the teachers to experience the dilemma of conflicting interpretations of a probability situation and to witness how irreconcilable it could be once it occurs in the classroom. He hoped that they came to see this dilemma as something to be avoided by cultivating in students an orientation to make sense of the problem situation before answering specific questions.

In the post-interviews, we presented a scenario similar to the Clown and Card scenario.

> Students in a probability & statistics class were presented the following problem situation:
>
> > Three prisoners — John, Michael, and Quincy — share a cell block. The Warden announced that, in celebration of the upcoming new year, one of the three would be released, but he will not say in advance who it will be.
> >
> > Michael knew his chance of being released is 1/3. To improve his chances, he said to the Warden "I know you cannot tell me who will be released. But, will you tell me the name of one prisoner who will *not* be release?"
> >
> > The Warden told him a name of someone other than Michael. Michael left, confident that his probability of being released had risen to 1/2.
> >
> > Is Michael right? Explain.
>
> Two students responded like this:
>
> <u>Student 1</u>
>
> "Michael is right.

> Once the Warden tells him the name of who will not be released, then the Warden only has two prisoners (including Michael) from whom to choose. If the Warden will make his choice randomly, that means there's a 50%, or 1/2, chance that Michael will be the one released".

Student 2

> "Michael is wrong.
> His chance of being released is 1/3 because if the Warden chooses at random from the three prisoners, then each one is equally likely to be chosen! The Warden speaks to Michael after making his selection, so what he tells Michael has no effect on his chance for release".

- Please comment on these two students' responses.
- Suppose that you are the teacher in this class. How might you handle the situation of your students being split evenly between these two responses?

This scenario, too, could be subject to multiple interpretations, as presented by the utterances of Students 1 and 2. Both interpretations took probability as relative proportions of outcomes, and the differences between the interpretations (and thus, conceptualizations of outcomes) could be accounted for by one's assumption about when the warden makes his decision. If the warden makes his decision after telling Michael the name, the probability of Michael's release would be 1/2. If the warden had made his decision prior to telling Michael the name, the probability of Michael's release would be 1/3. Both interpretations are viable because the statement of the situation does not exclude either one, and thus both probabilities should be accepted as correct in light of the students' explanations. Moreover, one could interpret the Warden scenario non-stochastically, as a one-time decision process never to be done again. In that case, questions of probability are immaterial.

Teachers' comments on the students' responses showed that almost all the teachers saw the similarity between this scenario and the Clown and Cards scenario, in that they are both subject to multiple interpretations. All teachers except Alice and Lucy thought that both interpretations were viable, depending upon when the warden made his decision, with Henry leaning towards Student 2's interpretation. Lucy and Alice stood firmly behind Student 2's interpretation, and believed that the warden had made his decision before talking to Michael.

The next question asked the teachers how they might handle a hypothetical situation of their students being split evenly between these two responses. All the teachers acknowledged the legitimacy of both interpretations. Most of the teachers were comfortable accepting students' different interpretations as long as they provide sufficient justifications. However, many teachers commented that they preferred not to have such discussion in their class, and that a well-designed problem should not be open to multiple interpretations. These comments betrayed their commitment as teachers to achieve consensus in the classroom. Linda and Alice provided solutions to achieve this consensus—by either adopting a convention (Linda), or laying certain groundwork so as to avoid multiple interpretations and potential dispute (Alice). In summary, teachers' responses to this interview question suggested that most of them had adjusted their ways of thinking with respect to situations involving multiple interpretations that are not excluded by the problem text, in that all teachers were open to the idea that a probability situation could be interpreted differently.

Summative analysis.

Finally we look at each teacher's interpretations of probability across all the problem scenarios.

[Insert Table 15 to Table 22]

These summative tables revealed that five out of all eight teachers (John, Linda, Sarah, Betty, Lucy) had a situational conception of probability. That is, their interpretations of probability vary according to the particularity of the situation or context in which the probability is embedded. Nicole, Henry, and Alice had a stochastic conception of probability. Henry also had a non-situational conception of probability. He was able to distinguish a situation from conceptions of the situation, and offer multiple interpretations of one situation.

Discussion

This paper explored the teachers' conceptualizations and interpretations of probability situations as they engaged in designed activities that purported to reveal their understandings of probability. We structured the activities and interview questions around two important ideas: (a) A stochastic conception of probability is one that supports thinking about statistical inference. Thus, we want to know the extent to which the teachers reasoned stochastically, and what difficulties the teachers might have experienced in reasoning stochastically; (b) We wanted the teachers to understand that the situation in which you act is the situation as you conceive it to be. That is, a probability situation can be interpreted either non-stochastically or stochastically, depending on how one conceives of the underlying process behind the stated situation. Moreover, when a situation is conceived non-stochastically, there are multiple ways to think about what "probability" means in that situation, and similarly, with a stochastic interpretation. The multiplicity of these ways of thinking and interpretations were presented in the theoretical framework for probabilistic reasoning. We believe that as teachers of probability, they must be aware of, and be able to control, different interpretations of probability situations so that they will be equipped to understand students' various conceptions of probability.

With respect to the first question regarding the extent to which teachers reasoned stochastically, we found that in the beginning of the seminar, most of the teachers had a non-stochastic conception of probability. In the beginning of the second week, when we began to focus on probability, we saw a turning point during the teachers' discussion on the PowerPoint slide 4: *Rain & Temperature*. This slide presented two probability situations. While one group of teacher consisting of Henry, Nicole, and John argued that the two situations could be interpreted both non-stochastically and stochastically, another group consisting of Terry and Linda insisted that one had to be non-stochastic and the other stochastic. At the end of this debate, there appeared to have been a shared understanding among the teachers that a situation is not stochastic (probabilistic) in and of itself, and that it is how one conceives of it that makes it stochastic. However, post-seminar interview revealed that only two teachers were controlling the way they interpreted probability situations. They were aware that a situation could be conceived of differently. A majority of the teachers were inclined to interpret probability situations either non-stochastically or stochastically.

In the discussion around the Clown & Cards scenario, teachers were confronted with the idea that a probability situation could be interpreted differently depending on the underlying process one conceives. Teachers struggled with this idea. The amount of confusion and frustration they experienced was beyond their comfort level. Although due to limited space we did not go into details and instead focused this paper on teachers' content understanding, we have learned a valuable lesson from our attempt at engaging teachers in pedagogical conversations about the Clown & Cards scenario. That is, conversations designed to elicit and enhance pedagogical content knowledge are difficult when teachers are still developing and refining their content knowledge. The follow up interview revealed that, at the end of the seminar, the teachers had become receptive to the idea of multiple interpretations of probability situations, and that they would accept the legitimacy of various interpretations provided that they are justified. However, their preference in avoiding discussions of this idea in classroom revealed that they might have seen it as a source of confusion rather than as a matter of fact.

Summative analysis of teachers' interpretations across all situations revealed that three teachers, Nicole, Alice, and Henry, had predominantly stochastic conceptions. Sarah had a non-stochastic conception. The remaining teachers' conceptions of probability were situational: Their interpretations of particular probability situations were contingent upon how these situations were stated.

Contributions and Implications

The most salient finding of this study was the theoretical framework that emerged from the analyses of teachers' understandings of probability. This theoretical framework, in comparison to prior relevant research studies, opens up the "black box" of probabilistic reasoning: It explicates: what constitutes a coherent and powerful understanding of probability, the non-conventional ways of understandings that people might have, and how these various ways of understandings relate to, or develop into, a coherent understanding. The model can be used to design assessments of probabilistic thinking. One could design questions to elicit the reasoning behind students' answers to a probability question so that each of three core questions can be answered with respect to the decisions the student made. In this way, the framework provides a tool for understanding and supporting students' learning of probability by allowing us to model their understandings of probability and generate insights about possible instructional interventions to support their learning.

This study also provides a rich description of the kinds of difficulties the teachers experienced in developing coherent and powerful understandings of probability. It also yields conjectures about what it is that might have hindered the teachers' attempts in doing so. The set of descriptions and conjectures provide an insight into the complexity in understanding probability, and what we should reasonably expect of the understandings of the content knowledge of high school teachers who teach, or are going to teach, probability. In the general population, we should expect a complicated mix of understandings of probability that are often incoherent and highly compartmentalized, which do not support teachers' attempts to develop coherent pedagogical strategies regarding probability.

These theoretical frameworks and our knowledge of the teachers' understandings, together, provide insights into how instruction of probability and statistical inference should be designed in future professional development that aims to support teachers' learning of probability and statistical inference. For example, we learned that teachers' understandings of probability and statistical inference were highly compartmentalized. Their conceptions of probability were not grounded in the conception of distribution, and thus did not support thinking about statistical inference. The implication of this finding is that instruction regarding probability and statistical inference must be designed with the principal purpose of helping teachers to develop understanding of probability and statistical inference that cut across their existing compartments. A powerful conception of probability that supports reasoning in statistical inference builds heavily on the conception of distribution of outcomes. To develop a stochastic conception, one has to develop a series of ways of thinking that include: (a) conceiving of an underlying repeatable process, (b) understanding the conditions and implementations of this process in such a way that it produces a collection of variable outcomes, and (c) imagining a distribution of outcomes that are developed from repeating this process. This series suggests a strategy for instructional design for professional development for probability: Start by engaging teachers in activities that support their building an image of *distribution of outcomes* from a random experiment, and ask probability questions about these distributions. The purpose of these activities is to broach the concept of *probability* and *distribution of outcomes*, and to help teachers develop a stochastic conception of probability—the meaning that probability of an event is a long-run expectation and that probability "regions" are regions of a distribution. Then, teachers should be engaged in an actual process or simulation of repeated sampling, and in discussions of features of the resulting distributions of sample statistics. In doing so, we can help teachers develop a stochastic conception of probability in the context of repeated sampling, and build connections among concepts of *probability*, *distribution of sample statistics*, and *p-value*; the ideas essential to statistical inference. Finally, one could move on to topics in statistical inference. Through systematic scaffolding in helping teachers to develop an increasingly rich conception of *distribution of sample statistics*, this strategy would allow teachers to develop a stochastic conception of probability in regard to taking a sample, and hence lay a foundation for understanding statistical inference.

Limitations

There are limitations of this study, some of which are unavoidable due to the very nature of the study. Due to the challenges of any systemic attempt to identify teachers' probabilistic understanding in the field of statistics education, this study was highly exploratory. First, we worked with a small group of teachers to allow the opportunity for each teacher to reveal their understandings in the seminar. This small sample size does not support making claims about the prevalence of this study's central findings to a broader population of teachers. Second, when we designed and conducted the seminar, we have but an emergent understanding of what it means to understand probability coherently and how such understandings develop. As a result, some data seemed weak or inadequate in hindsight. For example, the first post-interview question provided an opportunity for us to explore the teachers' stochastic conception of probability. At the time when the interview was conducted, the only distinguishing element that the interviewers believed that distinguished a stochastic conception from a non-stochastic conception was whether a teacher conceived of a repeatable process underlying a probability situation. Therefore, the questioning stopped once that information was collected. However, these data turned out to be insufficient in probing how well the teachers understood the repeatable process and whether they had constructed an image of a distribution of outcomes generated from that process, which we learned at the end of the analysis are important benchmarks for stochastic conception.

Next steps

This study is an early step of a larger research program, which aims to understand ways of supporting teachers' learning and their transformations of teaching practices into ones that are propitious for students' learning in the context of probability and statistics. As a precursor, this study has developed initial frameworks for understanding teachers' (in general, learners') understandings of probability. In the mean time, it also suggested a number of different directions for future research. The most immediate follow-up work would be to refine these frameworks, that is, to test their viability by working with a broader audience and revising them accordingly. Another contribution from this research work would be to generate insights about the prevalence of people's particular conceptions and understandings. These results could then inform instructional design of probability.

Although we documented the changes in teachers' thinking across time, in most cases we do not know *how* these changes occurred. This was partially due to the fact that this study was not designed as a traditional design experiment that takes teacher change as its primary focus. Naturally, one of the follow-up studies would be to design a teaching experiment that explicitly focuses on ways of supporting teachers' development of coherent probabilistic understanding, given what we learned in this study about what it means for them to have such understanding and what difficulties they might encounter in developing this understanding.

In another equally important, perhaps more challenging, line of work, we would focus on teachers' pedagogical content knowledge (Shulman, 1986) in probability. The goal of supporting teachers' learning is to support teachers' teaching, which we hope will ultimately lead to enhanced learning opportunities and experiences for their students. Although our research highlighted the importance and necessity of addressing teachers' content knowledge, we are cognizant of the complexity and the challenges involved in articulating and developing teachers' pedagogical content knowledge with regard to the teaching of probability. A design research methodology (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) seems to be the most desirable approach to pursue this endeavor in the near future.

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Footnotes

[1] We use the phrase "probability situation" to distinguish from "probabilistic situation." We use the former to refer to a situation that someone might conceive as involving probability. We use the latter to imply that someone has conceived of a probability situation stochastically.

[2] We agree with a reviewer's comment that the process described by Betty and Lucy was not adequately stochastic in the sense that the conditions of the process were not well specified—they did not say how a card got onto the table.

[3] Note that Terry, the workshop host, had turned herself into a participant in this discussion.

Table 1

Demographic information on seminar participants

Teacher	**Years Teaching**	**Degree**	**Stat Background**	**Taught**
John	3	MS Applied Math	2 courses math stat	AP Calc, AP Stat
Nicole	24	MAT Math	Regression anal (self study)	AP Calc, Units in stat
Sarah	28	BA Math Ed	Ed research, test & measure	Pre-calc, Units in stat
Betty	9	BA Math Ed	Ed research, FAMS training	Alg 2, Prob & Stat
Lucy	2	BA Math, BA Ed	Intro stat, AP stat training	Alg 2, Units in stat
Linda	9	MS Math	2 courses math stat	Calc, Units in stat
Henry	7	BS Math Ed, M.Ed.	1 course stat, AP stat training	AP Calc, AP Stat
Alice	21	BA Math	1 sem math stat, bus stat	Calc hon, Units in stat

Table 2

Overview of seminar topics

Week	Monday	Tuesday	Wednesday	Thursday
June 11 -- June15	Data, samples, and polls "Is this result unusual?": Concrete foundations for inference and hypothesis testing	Statistical unusualness Statistical accuracy Distributions of sample statistics	Margin of error Putting it all together	Students' understandings of distributions of sample statistics Analysis of textbook treatments of sampling distributions
June 18 -- June 22	Textbook analysis of probability intro Probabilistic vs. non-probabilistic situations	Conditional probability Contingency tables and conditional probability	More conditional probability Uses of notation	Analysis of textbook definitions of probability Data analysis: Measures of association

Table 3

Theoretical constructs in probability framework

Ways of thinking	1	Q1	Is there an image of a repeatable process?
	2	Q2	Are the conditions of the process specified?
	3	Q3	Is there an image of a distribution of outcomes?
Meanings for probability	4	OA	Outcome approach
	5	ANA	Outcome is A or non A, prob.=1 or 0
	6	PH	Proportionality heuristic
	7	ANA	Outcome is A or non A, prob.=50%
	8	APV	Probability as relative proportion: all possible values of random variables
	9	APO	Probability as relative proportion: all possible outcomes
	10	RF	Probability as relative frequency: distribution of all outcomes

Table 4

Examples of conceptions & interpretations of probability

Path	Interpretation of Probability	**Example I:** *What is the probability that I will get a sum of 11 when I throw two dice?*	**Example II:** *What is the probability that 18 out of 30 people favor Pepsi?*
1-4	Outcome approach	11 is my favorite number, so I believe the probability is going to be 80%.	I'm a coke drinker. I think it is unlikely that 18 out of 30 people favor Pepsi.
1-5	Outcome is A or non A, Prob.=1 or 0.	I will either get an 11 or not. The probability is 1 if I do get an 11, 0 if I don't.	The probability is 1 if 18 out of 30 favor Pepsi, is 0 if not.
1-2-6	Proportionality heuristic	N/A	The likelihood of 18/30 is high if a larger sample reflects similar or the same proportion.
1-6	Proportionality heuristic	N/A	A person either likes or doesn't like Pepsi (implying that population parameter is 50%), thus the likelihood of 18/30 is high.
1-7 or 1-2-7	Outcome is A or non A, Prob.=50%	I will either get an 11 or not. The probability of getting 11 is 50%.	The outcome is either 18 or not 18, so the probability is 50%
1-8 or 1-2-8	8 All possible values of random variables	There are 11 possible outcomes: 2, 3, … 11, and 12. The probability of getting 11 is 1/11.	N/A
1-9 or 1-2-3-9	9 All possible outcomes	There are 36 possible outcomes: (1, 1), (1, 2)…, and (6, 6). Two of these outcomes (5, 6) and (6, 5) will give the sum of 11. The probability of getting 11 is 2/36.	There are 31 possible outcomes. The probability is 1/31.
1-2-3-10	Probability as relative frequency	If I repeatedly throw two dice, what fraction of the time will I get a sum of 11?	If we repeatedly take samples of 30 people, what fraction of time will get results of 18 people favoring Pepsi?

Table 5

Number of marbles in urns

Urns	# of red marbles	# of white marbles
A	2	5
B	5	4
C	3	9

Table 6

Overview of the activities and interviews

Themes of activities and interviews & duration			
	Activity (A) and Interview (I)	**Day**	**Duration**
Stochastic and non-stochastic conception	A-1 Chance and likelihood	1	29 m.
	A-2 Movie Theatre scenario	3	106 m.
	A-3 PowerPoint presentation	5	67 m.
	I-1 Five probability situations		
	I-2 Gambling		
Multiple interpretations of probabilistic situation	A-4 Clown and Cards	6	138 m.
	I-3 Three Prisoners		

Table 7

Teachers' conceptions of probability situations in Activity 1

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1Q1	John	N						Y			
	Nicole	Y	Y	Y							Y
	Betty	N									
	Linda	N			Y						
	Henry	Y	Y	Y							Y
D1A1Q2	John	N					Y	Y			
	Nicole	N			Y						
	Sarah	Y	N								
	Lucy	Y	N				Y				
	Betty	N					Y				
	Linda	N					Y	Y			

Instruction for reading Table 7 (and similar ones in the rest of the paper)
The first column "locator" tells where an interpretation is located. For example, "D1A1Q2" means "Day 1, Activity 1, Question 2". Columns numbered 1-10 are theoretical constructs of the probability framework. Detailed description is provided in Table 3. "Y" and "N" in the coding window means "Yes" and "No". Examples: a. the "Y" in column 1 and row 4 means "For Activity 1 Question 1, Nicole conceived of a repeatable process." b. the "Y" in column 7 and row 3 means "John's interpretation of probability, in Activity 1 Question 1, is "probability of an event A is 50% because there can be only two outcomes, A or non A." Codes across a row provide the information about a path. For example, row 3 denotes John's non-stochastic conception, path 1-5. Codes across a column tells the number of instances in which teachers exhibits a particular way of thinking. For example, column 1 tells the number of instances the teachers did or did not conceive of a repeatable process.

Table 8

Teachers' conceptions of probability situation in Activity 2

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D3A2	John	N					Y				
	Nicole										
	Sarah	N			Y						
	Lucy										
	Betty	N			Y						
	Linda	N			Y						
	Henry	Y	N								Y
	Alice	Y	Y	Y							Y

Table 9

Teachers' conceptions of probability situations in Activity 3

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D5A3S1	Linda	Y	N							Y	
D5A3S2	Linda	N								Y	
D5A3S3	Linda	N								Y	
D5A3S4Q1	John	Y	Y	Y							Y
	Nicole	Y	Y	Y							Y
	Linda	N				Y					
	Henry	Y	Y	Y							Y
	Terry	N				Y					
D5A3S4Q2	John	Y	Y	Y							Y
	Nicole	Y	Y	Y							Y
	Linda	Y	Y	Y							Y
	Henry	Y	Y	Y							Y
	Terry	Y	Y	Y							Y

Table 10.

Teachers' conceptions of probability situations in Interview 3-1 (3^{rd} interview, question 1)

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
I3-1Q1	John	N				Y					
	Nicole	Y	Y	Y							Y
	Sarah	Y									
	Lucy	N									
	Betty	Y	Y								
	Linda	Y	Y	Y							Y
	Henry	Y	Y	Y							
	Henry	N				Y					
	Alice	Y	Y	Y							
I3-1Q2	John	N				Y					
	Nicole	Y									
	Sarah	Y				Y					
	Lucy	N				Y					
	Betty	N				Y					
	Linda	N				Y					
	Henry	N				Y					
	Henry	Y	Y								
	Alice	Y	Y	Y							Y
I3-1Q3	John	Y	Y	Y							Y
	Nicole	N			Y						
	Sarah	N									
	Lucy	N									
	Lucy	Y									
	Betty	Y	Y	Y							Y
	Linda	Y	Y	Y							Y
	Henry	Y									
	Alice	Y	Y	Y							Y
I3-1Q4	John	Y									
	Nicole	Y	Y								
	Sarah	Y	Y	Y							Y
	Lucy	Y									
	Betty	Y	Y	Y							Y
	Linda	Y	Y	Y							Y
	Henry	Y	Y	Y							Y
	Henry	N				Y					
	Alice	Y	Y	Y							Y
I3-1Q5	John	Y									
	Nicole	Y	Y	N							
	Lucy	N									
	Lucy	Y									
	Betty	N				Y					
	Alice	Y	Y	Y							Y

Table 11

The choices teachers made in Interview 3-2 (3rd interview, question 2)

Name	1	2
John	*a*	*b*
Nicole	*a*	*b*
Sarah	*a*	*a*
Lucy	*a*	*a*
Betty	*a*	*a*
Linda	*a*	*b*
Henry	*b*	*b*
Alice	*a*	*a*

Table 12

Teachers' conceptions and interpretations of probability situation in Clown & Cards scenario

Locator	Name	P.	1	2	3	4	5	6	7	8	9	10
			Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D6A4	John	1,0	N				Y					
	Nicole	1/3	Y	Y	Y						Y	
	Sarah	1/2	N								Y	
	Lucy	1/2	Y	Y	Y						Y	
	Betty	1/2	Y	Y	Y						Y	
	Linda	1/2	Y	Y	Y						Y	
	Henry	1/2										
	Alice											

Table 13

Outcomes conceived of by Betty, Lucy, Sarah, and Linda

	Up	Down
1st	R	R
2nd	R	W
	~~W~~	~~R~~
3rd	~~W~~	~~W~~

Table 14

Outcomes conceived of by Nicole

Card	Side A	Side B
R/R	Up (R)	Down (R)
	Down (R)	Up (R)
~~W/W~~	~~Up (W)~~	~~Down (W)~~
	~~Down (W)~~	~~Up (W)~~
R/W	Up (R)	Down (W)
	~~Down (W)~~	~~Up (R)~~

Table 15

John's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1Q1	John	N						Y			
D1A1Q2	John	N					Y	Y			
D3A2	John	N					Y				
I2-3	John	Y	Y	Y							Y
D5A3S4Q1	John	Y	Y	Y							Y
D5A3S4Q2	John	Y	Y	Y							Y
I3-1Q1	John	N				Y					
I3-1Q2	John	N				Y					
I3-1Q3	John	Y	Y	Y							Y
I3-1Q4	John	Y									
I3-1Q5	John	Y									
D6A4	John	N				Y					

Table 16

Nicole's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1Q1	Nicole	Y	Y	Y							Y
D1A1Q2	Nicole	N			Y						
I2-3	Nicole	Y									
D5A3S4Q1	Nicole	Y	Y	Y							Y
D5A3S4Q2	Nicole	Y	Y	Y							Y
I3-1Q1	Nicole	Y	Y	Y							Y
I3-1Q2	Nicole	Y									
I3-1Q3	Nicole	N			Y						
I3-1Q4	Nicole	Y	Y								
I3-1Q5	Nicole	Y	Y	N							
D6A4	Nicole	Y	Y	Y							Y

Table 17

Sarah's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1Q2	Sarah	Y	N								
D3A2	Sarah	N			Y						
I2-3	Sarah	Y	Y	N			Y				
I3-1Q1	Sarah	Y									
I3-1Q2	Sarah	Y				Y					
I3-1Q3	Sarah	N									
I3-1Q4	Sarah	Y	Y	Y							Y
D6A4	Sarah	N									

Table 18

Lucy's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1Q2	Lucy	Y	N				Y				
I2-3	Lucy	Y	Y	Y							Y
I3-1Q1	Lucy	N									
I3-1Q2	Lucy	N				Y					
I3-1Q3 (1)	Lucy	N									
I3-1Q3 (2)	Lucy	Y									
I3-1Q4	Lucy	Y									
I3-1Q5 (1)	Lucy	N									
I3-1Q5 (2)	Lucy	Y									
D6A4	Lucy	Y	Y	Y							Y

Table 19

Betty's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1Q1	Betty	N									
D1A1Q2	Betty	N					Y				
D3A2	Betty	N			Y						
I2-3	Betty	Y	Y	N			Y				
I3-1Q1	Betty	Y	Y								
I3-1Q2	Betty	N				Y					
I3-1Q3	Betty	Y	Y	Y							Y
I3-1Q4	Betty	Y	Y	Y							Y
I3-1Q5	Betty	N				Y					
D6A4	Betty	Y	Y	Y							Y

Table 20

Linda's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1Q1	Linda	N			Y						
D1A1Q2	Linda	N					Y	Y			
D3A2	Linda	N			Y						
I2-3	Linda	Y	Y	Y							Y
D5A3S1	Linda	Y	N							Y	
D5A3S2	Linda	N								Y	
D5A3S3	Linda	N								Y	
D5A3S4Q1	Linda	N				Y					
D5A3S4Q2	Linda	Y	Y	Y							Y
I3-1Q1	Linda	Y	Y	Y							Y
I3-1Q2	Linda	N				Y					
I3-1Q3	Linda	Y	Y	Y							Y
I3-1Q4	Linda	Y	Y	Y							Y
D6A4	Linda	Y	Y	Y							Y

Table 21

Henry's conceptions of probability situations

Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D1A1Q1	Henry	Y	Y	Y							Y
D3A2	Henry	Y	N								
I2-3	Henry	Y	Y	Y							Y
D5A3S4Q1	Henry	Y	Y	Y							Y
D5A3S4Q2	Henry	Y	Y	Y							Y
I3-1Q1 (1)	Henry	Y	Y	Y							
I3-1Q1 (2)	Henry	N				Y					
I3-1Q2 (1)	Henry	N				Y					
I3-1Q2 (2)	Henry	Y	Y								
I3-1Q3	Henry	Y									
I3-1Q4 (1)	Henry	Y	Y	Y							Y
I3-1Q4 (2)	Henry	N				Y					

Table 22

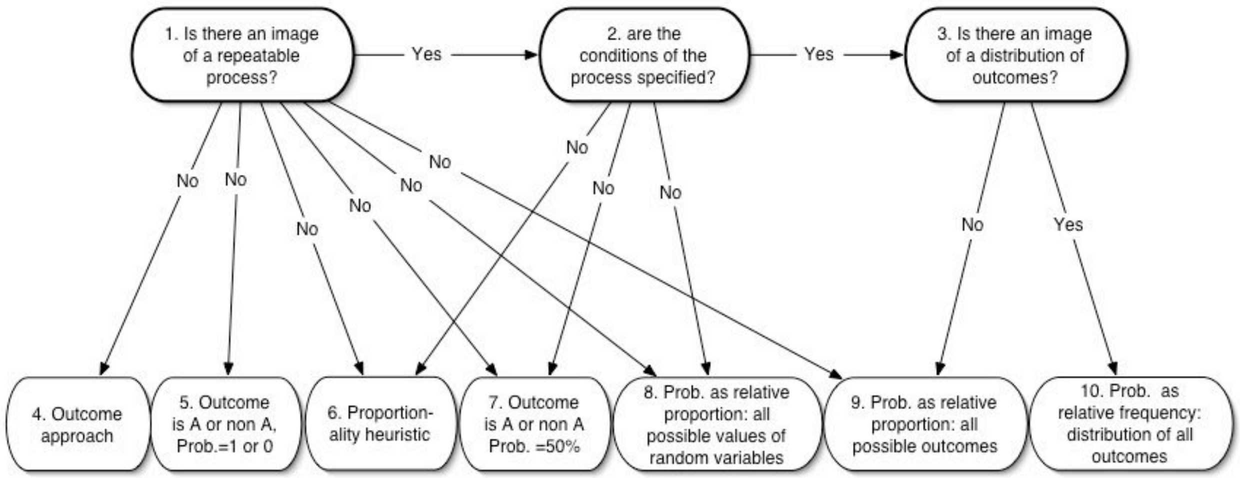
Alice's conceptions of probability situations

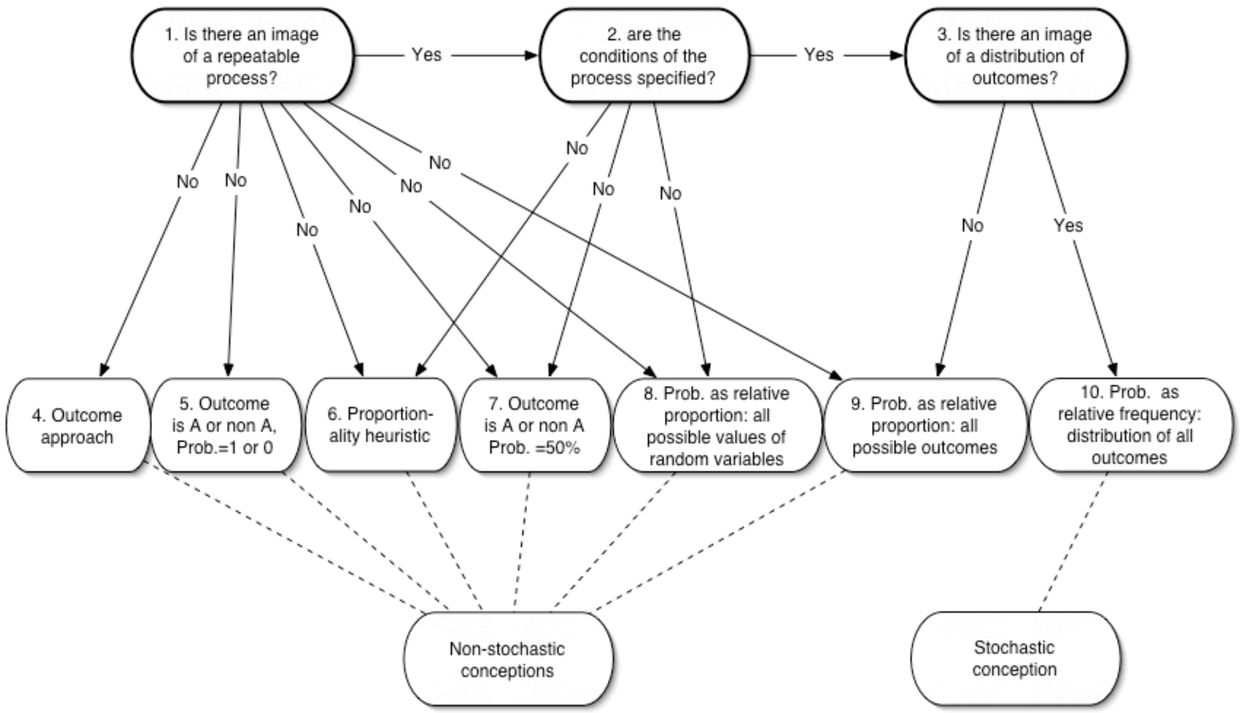
Locator	Name	1	2	3	4	5	6	7	8	9	10
		Q1	Q2	Q3	OA	ANA	PH	ANA	APV	APO	RF
D3A2	Alice	Y	Y	Y							Y
I2-3	Alice	Y	Y	Y							Y
I3-1Q1	Alice	Y	Y	Y							
I3-1Q2	Alice	Y	Y	Y							Y
I3-1Q3	Alice	Y	Y	Y							Y
I3-1Q4	Alice	Y	Y	Y							Y
I3-1Q5	Alice	Y	Y	Y							Y

Figure Captions

Figure 1. Ways of thinking and interpretations of probability

Figure 2. Theoretical framework for probabilistic understanding


1. Is there an image of a repeatable process?
Yes
2. are the conditions of the process specified?
Yes
3. Is there an image of a distribution of outcomes?
No
No
No
No
No
No
No
No
No
No
Yes
4. Outcome approach
5. Outcome is A or non A, Prob.=1 or 0
6. Proportion-ality heuristic
7. Outcome is A or non A Prob. =50%
8. Prob. as relative proportion: all possible values of random variables
9. Prob. as relative proportion: all possible outcomes
10. Prob. as relative frequency: distribution of all outcomes


1. Is there an image of a repeatable process?
Yes
2. are the conditions of the process specified?
Yes
3. Is there an image of a distribution of outcomes?
No
No
No
No
No
No
No
No
No
No
Yes
4. Outcome approach
5. Outcome is A or non A, Prob.=1 or 0
6. Proportion-ality heuristic
7. Outcome is A or non A Prob. =50%
8. Prob. as relative proportion: all possible values of random variables
9. Prob. as relative proportion: all possible outcomes
10. Prob. as relative frequency: distribution of all outcomes
Non-stochastic conceptions
Stochastic conception