ONE APPROACH TO A COHERENT ~~ALGEBRA~~ K-12 Mathematics : OR, IT TAKES 12 YEARS TO LEARN CALCULUS[1]

Patrick W. Thompson
Arizona State University, USA

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In this presentation I develop an argument that we should think more broadly about the school mathematics curriculum, that we should cease worrying about what algebra is and instead drop the mantra "Algebra for all" in favor of the mantra, "Calculus for all." Of course, all mantras suffer the weakness that, by definition, repeating them soothes the mind without having to think deeply about what they mean. Nevertheless, "Algebra for all" has a particularly insidious effect on mathematics education because (1) it takes for granted that "to do algebra" is to apply rules to strings of letters and numerals, and (2) it implies we need not think hard about why we want students to learn whatever we claim they should. Of course, the mantra "Calculus for all" faces similar objections. The difference is that when we say that all students should learn calculus, people take for granted that we do *not* mean what is typically taught in a current Calculus I course, and therefore we have a greater opportunity to clarify what we do mean. So "calculus for all" is not yet a mantra because it is not taken uncritically or unthinkingly.

A central tenet of my argument is that it takes 12 years to learn calculus, and learning it begins with learning much of what is normally thought of, in the United States anyway, as the province of arithmetic and algebra, but with subtle twists. In arithmetic, we need to have a target that students understand quantities and their magnitudes, and to have students experience a steady diet of thinking about situations involving variable quantities (i.e., quantities whose magnitudes vary in some natural way). These are rarely done in the United States. It might be done widely outside the United States, but the only curricular approach that emphasizes reasoning about magnitudes of which I'm aware is the arithmetic curriculum of Davydov and El'Konin. However, many curricula outside the U.S. do, indeed, ask students to deal regularly in their arithmetic curricula with situations that involve quantities whose values vary.

I follow Wildi (1991) in the way I speak about quantities' magnitudes. The idea he tries to capture is, for example, that someone's height is the same regardless of the unit in which you measure it. A person has the same "amount of height" whether measured in inches, centimeters, meters, miles, kilometers, or light years. He captures this formally with the equation $\|Q\| = m(Q_u)\|u\|$, which he intended as "the magnitude of quantity Q is the measure of Q, in units of u, times as large as the magnitude of u. To put it another way, and to continue this example, a measure of a person's height is the magnitude of that person's height in relation to the magnitude of the unit in which you measure it. Of course the definition is circular, but it is circular only to the extent that we must define the magnitude of a unit – the amount of a quantity that the unit comprises—as being defined outside the formula: A foot is the length of *this* (as we hold up a titanium bar borrowed from the U.S. Bureau of Standards).

The idea of quantitative magnitude is nontrivial for elementary grades students. It must be raised in context repeatedly, with specific quantities, over their schooling. One way to develop the idea of quantitative magnitude is to take seriously the idea of unit conversions, but not by rote. Rather, the idea of unit conversions should be built upon the idea that if the measure of a quantity is M_u in units of u, then its measure is $12M_u$ in units of magnitude (1/12) $\|u\|$ and its measure is $(1/12)M_u$ in units of magnitude $12\|u\|$. If a unit of length named *Fred* is 3 times as long as the unit *Gertrude*, then my height in *Freds* will be 1/3 my height in *Gertrudes*.

Thinking of a quantity in terms of its magnitude involves the same conceptual operations as thinking about inverse variation. Indeed, students can build the conceptual operations of inverse variation reflectively from reasoning about quantitative magnitudes. Also, when a student understands magnitude by a scheme of operations that entails inverse variation, then she can take a unitless length to represent a quantity's magnitude with the full knowledge that this length actually represents a class of unit-measure pairs. She can also reason about a quantity whose value is labeled, say, "15.3" without units by understanding that the unit's magnitude is 1/15.3 of the quantities' magnitude—even when neither is stated or known.

I should also say that I am using "quantity" in a rather technical way. I *do not* mean something that exists in a world independent of a person conceiving it. Rather, I mean that a quantity is in the eye of the beholder. A quantity is an attribute of an object that a person has conceived as somehow being measurable. That is, a quantity is a scheme of mental operations by which a person conceives the object, an attribute of it, and the possibility of measuring that attribute. Different people looking at what we (as observers) take to be the same object can (and quite often do) see different quantities. This point is salient with regard to teachers and students talking about quantities. Teacher must not assume that their students are thinking of the same quantities as they. It is safer to assume, at the outset of a discussion, that teachers and students see different quantities.

With the preliminaries said, I'll make some comments about the slides in the online presentation at http://tpc2.net/Pathways/.

STRANDS IN SCHOOL (MATH)

I find it useful to think of the school mathematics curriculum as organized in three "strands" that lead from arithmetic through algebra and to calculus: (1) the mathematics of quantity; (2) the mathematics of variation; and (3) the mathematics of representational equivalence (Figure 1). These three strands can be woven into a fabric that covers much of what we hope students learn from arithmetic through algebra to calculus. Also, though they are highly related, any one can be given prominence for a period of time while the others move to the background—not to be forgotten, but to provide support.

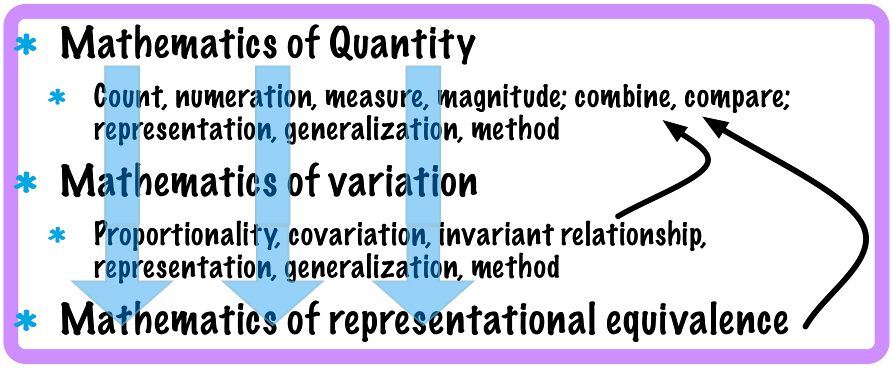

Figure 1. Three strands in school mathematics.

Mathematics of Quantity

The mathematics of quantity falls under what I have described elsewhere as quantitative reasoning (Thompson, 1990b, 1994b, 1995). Quantitative reasoning is when a person conceives a situation as composed of quantities (as described above) and relations among them. Arithmetic reasoning occurs when someone reasons about quantities values in terms of relationships among quantities. Algebraic reasoning happens when one or more of the quantities has an indeterminate value and one represents the arithmetic implied by those relations. Many of these ideas are developed in (Thompson, 1990a, 1993, 1995). I include geometric reasoning under quantitative reasoning to the extent that the person reasoning about geometric objects has conceived them so that that something about them is measurable. Geometric objects themselves are not quantitative, although it is hard to go very far in geometry before thinking of such things as length and angle measure.

Mathematics of Variation

The mathematics of variation involves imagining a quantity whose magnitude varies. To say what variation is is easy, but it is nontrivial for students to develop the conceptual operations that allow them to see variation in quantities and quantitative situations. The concept of variation of which I speak is, for me, best captured like this: Suppose x represents a quantity's magnitude. That the quantity's magnitude varies is to say that one imagines its magnitude having different values at different moments in time. So a varying quantity's magnitude might be represented as $x = x(t)$, where t represents (conceptual) time. I prefer, though, to characterize variation mathematically in a way that captures the idea that variation always happens over an interval. So, let $D = \text{domain}(t)$, the values of conceptual time over which t ranges. Let D_ε be the domain of t, but in intervals of size ε. Then, variation in one variable, x, can be expressed as $x_\varepsilon = x(t_\varepsilon)$, where t_ε represents the interval $[t, t+\varepsilon)$ and t varies through conceptual time.[2] I recognize immediately the difficulties with this formulation,

> Note to self: Point forward to students' study of science and engineering. Ideas of variation are not just key, they are *foundational* to understanding major ideas in each.

[2] That is, think of a value that varies by infinitesimal amounts, and varies within those amounts as well. In other words, variation, as a way of thinking, is recursive. We define

but what I hope to capture is the idea that in thinking of a varying magnitude, even at momentary times, we are imagining a small interval of conceptual time through which the variation happens.[3]

This formulation of variation tries to capture the conceptual operations behind an image of a growing magnitude, like a line segment getting longer. A particular, static, magnitude has a value of $x(t)$. But in thinking of the magnitude varying, we imagine it varying in microscopic bits, each bit itself entailing variation upon close inspection. The importance of building this image early is implicated by studies showing that calculus students do *not* see variables varying (Jacobs, 2002; Trigueros & Jacobs, 2008).

Given the importance of students' conceiving of variables as varying magnitudes, the natural question is how to focus explicitly on helping them develop this way of thinking. In my mind, it is by including variation persistently in discussions of quantity. For example, in first or second grade it is common to present children with situations like "Three birds were on a fence. Four more joined them. How many birds are on the wire now?" It would not be a distraction to ask children to imagine an undetermined number of new birds joining the group (let them decide the rate) and developing a table of the number of birds in relation to the time since the birds began arriving. In later grades students are asked to calculate areas of rectangles. Instead of calculating areas for multiple, individual rectangles, teachers could portray settings in which the length of one rectangle is stretched smoothly, and having students make a table of the rectangle's areas at particular moments of the stretching process.[4] Activities like this would also provide students with occasions to imagine dynamic situations that entail invariant relationships between quantities' values (like area as a function of length given a constant width).

The examples given above actually highlight the fact that any conception of variation actually involves two quantities, and that to imagine variation of one quantity we must, albeit tacitly, introduce another whose value varies in tandem with the one of interest. That is, any conception of variation also entails co-variation. Sometimes the covariation is tacit in students' thinking. In a one-student teaching experiment (Thompson, 1994b), the fifth-grader in this study had to extract time from her image of an increasing distance in order to conceive speed as a quantity. There is a growing literature on the foundational importance of covariation in students' conceptions of dynamic situations (see Oehrtman, Carlson, & Thompson, 2008 for a review of this literature).

With the above characterization of variation, covariation, which I have defined elsewhere

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variation within a ε-sized interval in the same way we define variation within the interval that contains it. I am indebted to Carlos Castillo-Garsow for discussions that led to this characterization of variation.

[3] The phrase "through which the variation happens" is of paramount importance. Even when thinking of variation happening in bits, we want students to imagine that variation happens *within* the bits as well.

[4] It is worth mentioning that the use of tables is powerful in regard to highlighting variation and covariation only to the extent that we persistently raise the question of how to imagine what happens *between* entries.

as keeping in mind two quantities' values simultaneously as their values vary (Saldanha & Thompson, 1998), can now be expressed as $(x_\varepsilon, y_\varepsilon) = (x(t_\varepsilon), y(t_\varepsilon))$.

Concepts of variation and covariation can be highlighted in almost any quantitative setting. They arise most vividly in situations conceived to involve proportionality or rate of change. This fact also points to the strong connection among algebraic reasoning, covariational reasoning, and quantitative reasoning.

Mathematics of Representational Equivalence

Arithmetic is representational at its start. At least it can be if seen that way. When "3+2" is seen as a command to compute, it is not representational. It is an imperative. But when "3+2" is seen as representing an act of combining 3 things with 2 things, it is representational. It represents something like, "the number of things you get when taking 3 things and 2 things as one number". Figure 2 reflects someone seeing the dots as two numbers and then re-seeing them as one number gotten from combining the two numbers of dots. This is not the same as computing. Computing is a method to find out what the combined amount actually is. Luis Saldanha and I made this same point when we talked about seeing, say, (328×421) as a number that is 328 times as large as 421 and 421 times as large as 328. We do not know, at this moment, what that number is, but we trust that there is such a number, that it is represented by (328×421), and that it has these relationships with 328 and 421. Moreover, 328, as a fraction of this number, is 1/421 as large. Similarly, 421 is 1/328 as large as this number. This is what I mean by using arithmetic representationally. It is hardly done in the U.S. Rather, students are taught to see arithmetic expressions as commands to compute. This is not to say that computation is not an important part of arithmetic. Rather, I mean that computing would be far more meaningful if it were seen as answering a question like, " (328×421) represents a number that is 421 times as large as 328. I wonder what that number is?"

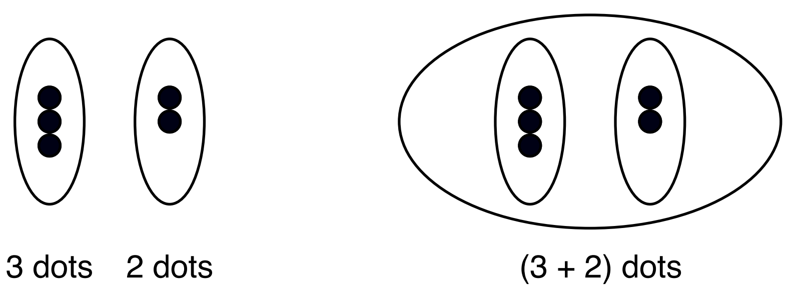

Figure 2. Representation as seeing and re-seeing.

Were the representational aspect of arithmetic to receive greater attention, then the representational aspect of algebra would be far less traumatic for students. We could draw on children's established practice of using expressions to represent re-seeing a situation, which is consistent with a practice of emphasizing reflective abstraction and reflective discourse in instruction.

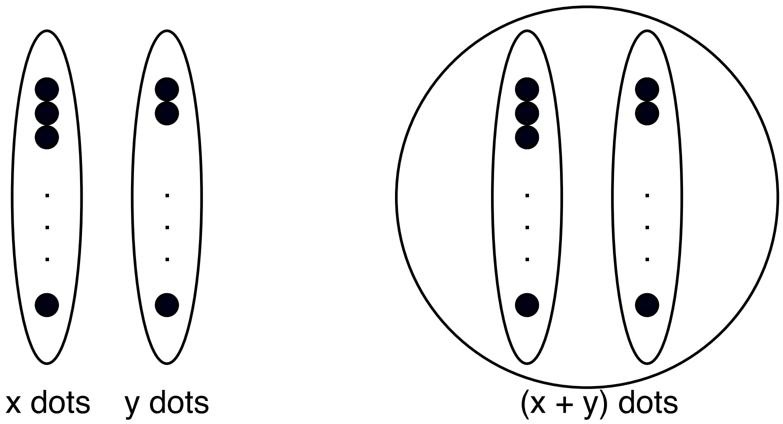

Figure 3. Leveraging representational practices in arithmetic.

Using arithmetic representationally can also provide segues into structural properties of numbers and quantitative relationship. "How can a number be 26 times as large as 7 and 7 times as large as 26 simultaneously?" It is a matter of *seeing the generality in your actions of seeing* (Figure 4). It is a matter of *seeing the generality in your actions of seeing* that A being 7/6 as large as B means that B is 6/7 as large of A, or that $y = (7/6)x$ means that $(6/7)y = x$ (Figure 5). It is a matter of *seeing generality in your actions of seeing* that you see that Figure 6 demonstrates that 1/2 is indeed larger than 1/4 despite appearances to the contrary.

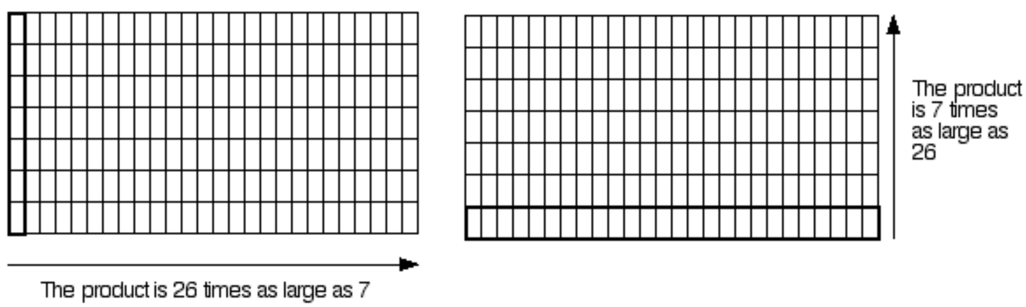

Figure 4. One number is, simultaneously, 26 times as large as 7 and 7 times as large as 26. Therefore, "(26x7)" and "(7x26)" represent the same number.

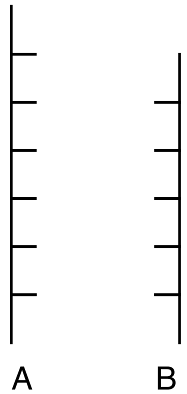

Figure 5. Seeing that A being 7/6 as large as B means B is 6/7 as large as A. Generalization via reflective abstraction: A being m/n as large as B means that B is n/m as large as A.

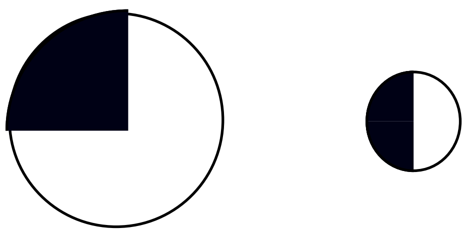

Figure 6. Seeing that 1/2 is larger than 1/4 because the pie pan with half a pie has a larger relative portion filled with pie than the pie pan with 1/4 pie. It is "more full".

It is worth mentioning that not all forms of abstraction are equally powerful. Piaget knew this when he distinguished among empirical, pseudo-empirical, and reflective abstraction (Piaget, 2001). Figure 7 illustrates what Piaget called pseudo-empirical abstraction. It is abstracting from the results of having acted instead of abstracting from the actions of producing the results. Reflective abstraction producing the same symbolic expression as in Figure 6 might come from looking at (4+9) and (9+4), etc. and seeing that you will get the same number either way because you could count from right to left as easily as you can count from left to right. That realization allows you to see that if you interchange the addends but count from right to left, you are counting the same objects in the same order, and therefore will get the same result. (My presentation develops the distinction between pseudo-empirical and reflective abstraction in the context of deriving the point-slope formula for linear functions.)

2 + 3 = 5 and 3 + 2 = 5. So 2 + 3 = 3 + 2
4 + 9 = 13 and 9 + 4 = 13. So 4 + 9 = 9 + 4
…
$x + y = y + x$
Figure 7. Example of pseudo-empirical abstraction.

The Three Together: Mathematical Schemes

The three strands in interaction, each receiving appropriate emphasis, and always with the other two in the background, builds a foundation for algebraic reasoning that simultaneously builds a foundation for schemes of meanings that are crucial for understanding the calculus.

Two aspects of algebraic reasoning that I have not addressed, but which are important both for algebra and for calculus, are ideas of accumulation and rate of change. (I do not have time to develop this here, but I do develop these ideas in (Thompson & Thompson, 1996; Thompson, 1994a, 1994b, 1996; Thompson & Silverman, 2008; Thompson & Thompson, 1992, 1994).

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