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On Professional Judgment and the National Mathematics Advisory Panel

Report: Curricular Content

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Abstract

The National Mathematics Panel's recommendations on curricular content are based on professional judgment of the panel members, not on research on learning and teaching algebra. As such, the panel's recommendations must be viewed in light of the political and ideological perspectives from which they were made.

The recommendations are also examined from the perspective that inattention to meaning is a root problem of mathematics teaching and learning, yet the report does not consider issues of mathematical meaning or their implications for reform.

President Bush directed the National Mathematics Advisory Panel “to use the best available scientific evidence” to devise a report addressing, among other things, “instructional practices, programs, and materials that are effective for improving mathematics learning” and “ideas for strengthening capabilities to teach children and youth basic mathematics, geometry, algebra, and calculus and other mathematical disciplines” (National Mathematics Advisory Panel, 2008a, p. 7)¹. As I argue below, the Panel’s response falls short of the President’s charge in two ways. First, the report fails to produce a trustworthy argument for the curricular recommendations it gives. Second, the recommendations themselves address only surface aspects of the problems in American mathematics education.

Trustworthiness

The Panel defined “best available scientific evidence” to mean results gotten from experimental studies that test hypotheses, use random selection and randomized assignment to treatments, and that have been replicated (MPFR, p. 81). Though the Panel gave two standards of quality – one for evidence and one for effects of interventions, they are both made from an experimental design perspective. It is instructive that the Panel’s quality of evidence scheme has a “suggestive evidence” category, but to be suggestive a study must still use statistical controls. Moreover, to be above “low quality” as an investigation of effect a study must have at least a “moderately large” probability sample (MPFR, pp. 83-85). So, by the Panel’s definitions, most qualitative studies, especially teaching experiments and design experiments, fall under the Panel’s low-quality category and cannot

¹ I will use “MPFR” to refer to the math panel final report.

provide even suggestive evidence for future research. As a result, the report is not informed by a large portion of basic mathematics education research that investigates fundamental processes of classroom learning and teaching. This last point is expanded in a later section.

The MPFR conforms for the most part to its standards when it does cite a source of evidence when making an evidence-based claim or recommendation. Unfortunately, the panel does not refrain from taking stances for which it has no empirical evidence that meets its standard. In particular, the report of the Task Group on Conceptual Knowledge and Skills (National Mathematics Advisory Panel, 2008b)², which developed Chapter 4 (*Curricular Content*), stated, “It should be noted that there is no direct empirical evidence to support the effectiveness of any lists discussed in this section for success in algebra course work” (TGCKS, p. 31). The Task Group nevertheless went on to construct recommendations that were “guided by professional judgment” (TGCKS, pp. ix, x, 2, 5, 15, 31, 40). However the Math Panel’s final report was less forthcoming in regard to the TGCKS’s reliance on professional judgment, stating

A small number of questions have been deemed to have such currency as to require comment from the Panel, even if the scientific evidence was not sufficient to justify research-based findings. In those instances, the Panel has spoken on the basis of collective professional judgment, but it has also endeavored

² I will refer to the task group’s report as TGCKS

to minimize both the number and the scope of such comments.

(MPFR, p. 12)

Despite the MPFR's claim of having minimized both the number and scope of comments based on "collective professional judgment", the curricular content group's recommendations, all of which were arrived at by professional judgment, are in Chapter 4 of the final report, and there are no other recommendations in that chapter. However, the phrase "professional judgment" appears just once in Chapter 4.

It is important to understand the significance of the Panel's deep reliance on the TGCKS's professional judgment in formulating its curricular recommendations. First, doing so belied the Panel's own standards of quality and evidence and its claim that it would make evidence-based recommendations. Second, while the Panel chose to rely on some of its members' professional judgment in formulating its recommendations, it consciously chose to ignore the professional judgment of more reform-minded experts who had perspectives that differed from theirs.

Moreover, the TGCKS heavy reliance on professional judgment to make its recommendations calls into question its members' qualifications for exercising such judgment. For example, what biases did they have? How familiar were they with, and how well did they understand, the basic research literature on mathematics teaching and learning? The Panel's deep reliance on professional judgment in formulating its curricular content recommendations, without a transparent effort to represent competing viewpoints, leaves the Panel open to the

charge that it might be composed of the most politically connected members of the community, but not the most knowledgeable regarding mathematics learning, teaching, and curricula.

Chapter 4 is written to convey a solid scholarly grounding when often there is none. For example, the claim, that “fractions, when properly taught, introduces students to the use of symbolic notation and the concept of generality” (TGCKS, p. 41) seems innocuous. However, in the TGCKS report, the warrant for this claim comes from one reference to one expository article by one of the panel’s members (Wu, 2001). The reference is omitted from the MPFR, however, giving the statement the appearance that it is a straightforward generalization from literature that the Panel found to support it. Now, one might say that literature does exist that suggests that fractions, properly taught, is a site where students can learn to reason generally, but the Panel did not find it. Rather, it expressed its opinion and hid the fact that it was expressing ungrounded opinion. That is, the Panel’s violation of its own evidence standards in formulating its recommendations creates the impression that they picked their evidence to suit their biases.

The filter that the TGCKS applied to the research literature on mathematics learning, teaching, and curricula produces an ironic result. By focusing only on investigations of efficacy or effect that passed through their self-defined evidence and quality standards, they do not cite a single result from basic research in mathematics education. At the same time, they found that studies of effect and efficacy of instructional or curricular approaches that meet their

standards are rare. The net effect, then, is that Chapter 4 of the final report is informed neither by basic research in mathematics education nor is it grounded in high quality studies of efficacy and effect. Put another way, the Panel's content recommendations convey neither a reason for believing that they will work (because they are uninformed by any mechanisms from basic research by which they might be expected to work), nor do they inspire confidence that they will be effective (because of the paucity of experimental comparisons).

Had the Panel adopted a more inclusive attitude toward literature it could examine, it would have found a plethora of important issues it should address—in students' preparation for algebra and in the very nature of algebra for which they should be prepared. Regarding students' preparation for algebra it is well known that many students do not make a transition from arithmetic to algebra. The literature on epistemological and cognitive obstacles to mathematical learning makes clear that the way a student understands a particular idea can enable or obstruct their learning of ideas that depend upon it (Booth, 1981; Carlson, 1998; Harel, 1998; Herscovics, 1989; Herscovics & Linchevski, 1994; Schneider, 1991; Sierpiska, 1992). For example, students in U.S. elementary grades rarely write arithmetic sentences to represent a situation, whereas Russian, Chinese, and Japanese students do (Fuson, Stigler, & Bartsch, 1989). The rarity of this practice in U.S. elementary grades has the effect that when students move from arithmetic to algebra they have developed a predisposition to understand expressions as something they should calculate rather than to understand them as representations to interpret (Herscovics & Linchevski, 1994; Kieran, 1992; Linchevski &

Herscovics, 1996). Regarding the nature of algebra, it is well known that a strong focus on algebra as symbol manipulation has the effect that students rarely develop the idea that variables represent quantities whose values vary, which has long-term negative effects for students success in calculus and analytic geometry (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Oehrtman, Carlson, & Thompson, 2008; Schneider, 1991, 1992; Thompson, 1994a; Trigueros & Jacobs, 2008).

Finally, it is important to note that the Final Report has a decidedly martial tone. By that I mean that it focuses on a need for students to exhibit specific correct mathematical behavior. The tension within the Panel over this issue is reflected in its discussion of whether to refer to algorithms, standard algorithms, or “the” standard algorithms (November 28, 2007 meeting transcript, pp. 115-120, 168-170). The “the” faction won—the Final Report always takes it as a primary goal that knowing an algorithm that produces correct results is not satisfactory. Rather, students must learn specific (“the standard”) algorithms in arithmetic.

In this same regard, there is a stark contrast between the tone of the 2008 Japanese Course of Study (“COS“; Japan Ministry of Education, 2008) and the MPFR’s very martial tone. The COS is quite concerned with children’s thinking; the MPFR and the TGCKS report are more concerned with children’s behavior. A comparison of the MPFR and TGCKS report with the Japanese COS also highlights the Panel’s inattention to the role of meaning in learning mathematics. By “meaning” I do *not* mean “definition”. Rather, I use “meaning” in the sense of that which comes to mind to make a word, phrase, observation, etc. sensible and comprehensible. Anyone familiar with the Japanese K-9 curriculum knows the

importance it gives to students' construction of meaning as the basis for skill. The comparison also highlights an internal contradiction in the Panel's report—it calls repeatedly for greater coherence in the mathematics curriculum while seemingly being unaware that coherence is a property of a body of meanings, not a property of a list of topics. Thus, the TGCKS provides no guidance to the nation about how to increase the likelihood that mathematics instruction and curricula will end up with students creating coherent mathematical meanings during their schooling.

Recommendations for Curricular Content

With the above said about the weak warrant given for the recommendations in Chapter 4 and about the strong skills-without-meaning perspective taken by the Panel, there is little to say about the substance of Chapter 4's content recommendations. They revolve around a list of 27 “major topics of school algebra” and a list of 11 “benchmarks” for what the Panel considers “critical foundations” for school algebra. Unfortunately, the list of major topics is as informative about what the Panel envisions being taught as is a textbook's table of contents. Likewise, the benchmarks are little more than statements that students will be proficient with the procedures that are taught at a particular grade level. Anyone looking for guidance on actually implementing the Panel's vision of preparing students for algebra will not find it in this report.

One would hope that the TGCKS report could have provided insights not included in the Panel Report. It does not. The one area that showed promise but remained unfulfilled was the TGCKS emphasis on the importance that students

develop strong facility with fractions. The MPFR stated the following recommendation.

The curriculum should allow for sufficient time to ensure acquisition of conceptual and procedural knowledge of fractions (including decimals and percents) and of proportional reasoning. The curriculum should include representational supports that have been shown to be effective, such as number line representations, and should encompass instruction in tasks that tap the full gamut of conceptual and procedural knowledge, including ordering fractions on a number line, judging equivalence and relative magnitudes of fractions with unlike numerators and denominators, and solving problems involving ratios and proportion. The curriculum also should make explicit connections between intuitive understanding and formal problem solving involving fractions. (p. 29)

Few people in mathematics education would disagree with this recommendation. But preceding it is a sequence of paragraphs talking about the necessity that children be able to “quickly and easily retrieve basic number facts” (MPFR, p. 28), which sets an unfortunate tone for interpreting the recommendation, and about how little is understood about the relationship between “informal knowledge and the learning of formal mathematical fractional concepts and procedures” (MPFR, p. 28). This latter statement is not true. The complexities of this relationship are well understood and have been explicated thoroughly (Ball, 1993; Cramer, Post, & delMas, 2002; Empson, Junk,

Dominguez, & Turner, 2006; Heller, Post, Behr, & Lesh, 1990; Mack, 2001; Norton, 2008; Sowder, Bezuk, & Sowder, 1993; Steffe, 2001; Thompson & Saldanha, 2003; Tirosh, 2000). But this understanding did not show up in the literature that the Panel surveyed. The Panel would have served the nation well had it explained why this understanding has not affected the treatment of fractions in school texts nor the general psychological literature on fractions.

Competing visions of school algebra and preparation for them

It should be clear that there is little chance that the MPFR will lead to improvements in students' learning of arithmetic or algebra. Its emphasis on proficiency with standard procedures in arithmetic and its lip service to "conceptual understanding" will do little to address the fundamental problem of mathematics education in the United States—namely, the systematic inattention to students' development of meanings that will support an interest in mathematics that results in taking more, and higher level, coursework.

A variety of sources suggest that incoherence and meaninglessness is a prominent feature of school mathematics in the United States. The Third International Mathematics and Science Study report of 8th-grade mathematics instruction in the United States, Germany, and Japan states this clearly.

Finally, as part of the video study, an independent group of U.S. college mathematics teachers evaluated the quality of mathematical content in a sample of the video lessons. They based their judgments on a detailed written description of the content that was altered for each lesson to disguise the country

of origin (deleting, for example, references to currency). They completed a number of in-depth analyses, the simplest of which involved making global judgments of the quality of each lesson's content on a three-point scale (Low, Medium, High). Quality was judged according to several criteria, including the coherence of the mathematical concepts across different parts of the lesson, and the degree to which deductive reasoning was included. Whereas 39 percent of the Japanese lessons and 28 percent of the German ones received the highest rating, none of the U.S. lessons received the highest rating. Eighty-nine percent of U.S. lessons received the lowest rating, compared with 11 percent of Japanese lessons [and 34% of the German lessons]. (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999, p. iv)

The TIMSS video sampling technique drew representative samples from each of its participating countries (Stigler et al., 1999). That no U.S. lesson's content received the highest quality rating from these mathematicians and that 89% of the U.S. lessons' content received the lowest quality rating suggests a general inattention in U.S. instruction to meaning in general, let alone meanings that students develop. Instead, U.S. lessons tended to focus on having students do things and remember what they had done. There was little emphasis on having students develop robust and generalizable meanings. However, lack of meaning in instruction is a problem that cannot be addressed by a revised list of arithmetic or

algebra topics. It will require a systemic re-orientation toward students' long-term development of usable and powerful meanings and to an increase in their proclivity to reason with them. To attain a focus on students' development of powerful mathematical meanings and using them in their reasoning, which includes coming to a consensus on what accomplishing it looks like, will require sustained political and intellectual leadership. Unfortunately, the MPFR provides neither one. Whether all students learn the same computational algorithms is a distraction.

The MPFR conveys a stance that meaning is not relevant. How is this so? By, for example, speaking of solving equations as a necessary skill at which students must become proficient, but not raising the importance that students develop meanings for equations —“Where do equations come from? Why are they important? What, precisely, do they represent?”—and develop meanings for equivalent expressions and equivalent equations (Ernest, 1987; Kirshner, 1989). It stresses the importance of linear functions, but not the importance of establishing coherent understandings of direct variation and constant rate of change as a foundation for them (Clagett, 1968; Confrey, 1994; Harel, Behr, Lesh, & Post, 1994; Hart, 1978; Karplus, Pulos, & Stage, 1983; Thompson, 1994b). It stresses the importance of proficiency with standard arithmetic algorithms, but does not admit the importance of, or subtleties of, having students develop orientations to “making sense” of problems and situations in which conventional procedures might be used (Cobb et al., 1991). It does not recognize that a single-minded focus on proficiency is likely one of the major sources of the mathematics anxiety

that is rampant in our population (Bessant, 1995; DeBellis & Goldin, 2006; Skemp, 1979; Wilensky, 1997). It fails to recognize the importance of what Harel (1998) has called “the necessity principle,” the principle that students learn mathematics more meaningfully when ideas are introduced to satisfy an intellectual need for them, and that such needs are engineered by thoughtfully crafted curricula and instruction. For example, were the question, "Could we rewrite this expression so that we know it represents the same number, but does more useful work?" then properties such as distributivity can be made *necessary* in a context. The MAA report *Algebra as a Gateway to a Technological Future* (McCallum et al., 2007) does a nice job of making this argument.

Finally, the vision of algebra reflected in the MPFR content recommendations is a skills-based foundation for advanced symbolic manipulation and abstract algebra (especially the algebra of polynomial forms). It completely ignores algebra as a preparation for calculus, which would entail strong emphases on variable as varying magnitude (Trigueros & Jacobs, 2008), covariation and function (Carlson *et al.*, 2002; Oehrtman *et al.*, 2008), rate of change and accumulation (Confrey & Smith, 1995; Thompson & Silverman, 2008), and modeling (Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Lehrer, Schauble, Carpenter, & Penner, 2000; Smith, Haarer, & Confrey, 1997; Verschaffel, De Corte, & Vierstraete, 1999). Even successful university calculus students have difficulty solving problems that depend upon understanding ideas variable as varying magnitude and function as a relationship between variables (Carlson, 1998; Selden, Mason, & Selden, 1994; Selden, Selden, Hauk, & Mason,

2000; Selden, Mason, & Selden, 1989). All these examples point to the importance that the specific meanings that students create for important ideas involving foundational ideas have significant long-term repercussions in their mathematical development.

Most of the studies cited here are qualitative studies or have small samples that were not selected at random. They nevertheless are, as a whole, suggestive that the TGCKS focus on skills is misplaced. For the TGCKS to use the literature I cited, however, would have required that its members know and value it. With a few exceptions there is no evidence that they did. Instead, they relied on a Defense Department group to sift their literature for them. It is thus disappointing that the MPFR missed a once-in-a-generation opportunity to address foundational problems in the nation's mathematics education. Instead, it gives untrustworthy arguments for its curricular recommendations and the recommendations themselves address only surface aspects of the problems in American mathematics education.

Finally, I should point out that most of the studies I've cited in criticism of the MPFR and the recommendations created by the TGCKS are, according to the Panel's standards of evidence and quality, weak (or no) evidence and low quality. There is no doubt that research in mathematics education needs more experimental studies that test hypotheses and add to theory confirmation. However, the funding has never been adequate to support both research into basic mechanisms of learning and teaching and to support large scale, randomized experimental studies.

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