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TOWARD A FRAMEWORK FOR THE DEVELOPMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING

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Abstract

Shulman (1986, 1987) coined the term *pedagogical content knowledge* (PCK) to address what at that time had become increasingly evident – that content knowledge itself was not sufficient for teachers to be successful. Throughout the past two decades, researchers within the field of mathematics teacher education have been expanding the notion of PCK and developing more fine-grained conceptualizations of this knowledge for teaching mathematics. One such conceptualization that shows promise is *mathematical knowledge for teaching* – mathematical knowledge that is specifically useful in teaching mathematics. While mathematical knowledge for teaching has started to gain attention as an important concept in the mathematics teacher education research community, there is limited understanding of what it is, how one might recognize it, and how it might develop in the minds of teachers. In this article, we propose a framework for studying the development of mathematical knowledge for teaching that is grounded in research in both mathematics education and the learning sciences.

Key Words: Mathematical Knowledge for Teaching; Pedagogical Content Knowledge; Mathematics Teacher Education

Toward a Framework for the Development of Mathematical Knowledge for Teaching

It is widely accepted that teachers of mathematics need deep understanding of mathematics (Ball, 1993; Grossman, Wilson, & Shulman, 1989; Ma, 1999; Schifter, 1995). However it is axiomatic that teachers' knowledge of mathematics alone is insufficient to support their attempts to teach for understanding. Shulman (1986) coined the term *pedagogical content knowledge* (PCK), specific content knowledge as applied to teaching, in part, to address what at that time had become increasingly evident – that content knowledge itself is not sufficient for teachers to be successful. Pedagogical content knowledge is knowledge that lies at the confluence of content knowledge, knowledge of students' thinking (the understandings they bring to a particular class or lesson and how it can be capitalized upon), and knowledge of mathematics education and pedagogy (e.g., curriculum, particularly difficult concepts, and effective images and instructional aids).

Throughout the past two decades, researchers within the field of mathematics teacher education have been expanding the notion of PCK through the developing of more fine-grained conceptualizations of this knowledge for teaching mathematics. For example, Leinhardt and Smith (1985) highlight the importance of teachers' conceptual understandings of mathematics. Ball (1990), Ball and Bass (2000) and Thompson and Thompson (1996), note that teaching for understanding requires special *mathematical knowledge for teaching* (MKT).

Ball and her colleagues (Ball, Hill, & Bass, 2005; Ball, 1993; Ball, 2007; Ball & McDiarmid, 1990) have attempted to answer the question "what do teachers do in teaching mathematics, and in what ways does what they do demand mathematical reasoning, insight, understanding, and skill?" (p. 17). Their pioneering work has succeeded in identifying various examples of special ways in which one must know mathematical procedures and representations to interact productively with students in the context of teaching (Ball et al., 2005; Ball & Bass, 2003; Hill & Ball, 2004; Hill, Rowan, & Ball, 2005). They have also demonstrated that conceptual demands of teaching mathematics are, in fact, different than the mathematical understandings needed by other practitioners of mathematics (Ball, 1990; Ball, 1991). Others have shown that the mathematical knowledge needed for teaching mathematics is different from the mathematical knowledge taught in university mathematics classes (Kahan, Cooper, & Bethea, 2003). Further, Ball and her colleagues have succeeded in identifying a positive relationship between mathematical knowledge for teaching, as assessed by their instrument for measuring it, and student achievement (Hill et al., 2005).

Ball and her colleagues have focused on special ways teachers must know the mathematics that is visible during instruction, such as representations created during a computation and issues associated with the standard definitions of terms. Their focus is on the ways teachers treat this visible mathematics that are sensitive to students' understanding of it. We accept this focus as essential for identifying and sharing best practices of teaching. But we also ask the question, "What mathematical understandings allow a teacher to act in these ways spontaneously? How might these understandings develop?" For example, Thompson (2008) identifies special ways of understanding the idea of angle measure that are entailed in helping students see that degree measure and radian measure are intrinsically related in the same way that centimeters and inches are. Our reason for this focus is pragmatic:

> If a teacher's conceptual structures comprise disconnected facts and procedures, their instruction is likely to focus on disconnected facts and procedures. In contrast, if a teacher's conceptual structures comprise a web of mathematical ideas and compatible ways of thinking, it will at least be possible that she attempts to develop these same conceptual structures in her students. We

> believe that it is mathematical understandings of the latter type that serve as a necessary condition for teachers to teach for students' high-quality understanding (Thompson, Carlson, & Silverman, 2007, pp. 416-417).

Our perspective entails a fundamentally different foci than Ball's MKT: rather than focusing on identifying the mathematical reasoning, insight, understanding and skill needed in teaching mathematics, we focus on the mathematical understandings "that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas that does significant work in students' reasoning" (Thompson, 2008, p. 1). While we feel these research foci and conceptualizations of MKT are complementary, we have chosen our foci based on the observation that students (at least in the U.S.), and therefore teachers, seldom experience coherence or generativity in mathematics (Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999; Stigler & Hiebert, 1999) and that the work of teaching for understanding is predicated on coherent and generative understandings of the big mathematical ideas that make up the curriculum.

Research on *student* learning of mathematics is frequently theoretically grounded in some variant of constructivism (radical, social, etc.). Thompson (2002) notes that rather than explaining phenomena or prescribing actions, such a theoretical foundation or background theory serves "to constrain the types of explanations we give, to frame our conceptions of what needs explaining, and to filter what may be taken as a legitimate problem" (p. 192). We contend that there is no commonly accepted theoretical framework for research in mathematics *teacher* education and, as such, a research base for prospective or practicing mathematics teacher development is emerging relatively slowly and is particularly sensitive to the influence of political agendas and current trends in related fields. The purpose of this article is to propose a theoretical framework that extends a constructivist perspective to include the development of

MKT and in doing so, we call for others to extend or contest it. We firmly believe that systemic improvement in teacher quality is not possible without such an orienting framework for the field.

Extending the Construct of Pedagogical Content Knowledge

We begin with a discussion of the related construct of pedagogical content knowledge for mathematics teaching. In a review of the literature on PCK, Gess-Newsome (1999) proposes two models: integrative and transformative. In the integrative model of PCK, the relevant knowledge bases used in teaching are developed separately (or possibly in an incorporated manner) and the knowledge becomes integrated through the act of teaching. Under the integrative model, PCK does not exist as a domain of knowledge in itself. As Gess-Newsome (1999) notes, “The task of the teacher is to selectively draw upon the independent knowledge bases of subject matter, pedagogy, and context and integrate them as needed to create effective learning opportunities” (p. 11). An expert teacher, then, is one with organized knowledge bases that can be quickly and easily drawn upon while engaged in the act of teaching. This model of PCK is problematic for many reasons, including the fact that students tend to exit teacher preparation programs unable to effectively integrate their pedagogical knowledge bases with their procedural and often incoherent mathematical understandings (Carpenter, Fennema, Peterson, & Carey, 1988).

In this article, we adopt a model for the development of knowledge for teaching mathematics that is compatible with Gess-Newsome’s (1999) transformative model for PCK, but avoids the pitfalls of assuming independent knowledge bases from which teachers act. Gess-Newsome describes PCK as the result of a fundamental transformation of knowledge and the creation of new knowledge that, though possibly similar to existing mathematical or pedagogical understandings, possesses distinct characteristics that were not present in their original form (Gess-Newsome, 1999). The transformative model of PCK is, in some sense, a response to the shortcomings of traditional teacher education programs that employ the integrative model, which Mason (1999) described as "[...] often merely providing future and current teachers with an array of noncontextualized, unconnected activities, concepts, and demonstrations" (p. 277). In contrast, the transformative model necessitates purposefully integrated experiences that provide teachers with opportunities to extend and connect their mathematical and pedagogical understandings to create a "new" knowledge.

We believe that this transformative model is a solid starting point for thinking about the development of MKT. Despite the fact that since inception of PCK "transformation" has been at the core (Wilson, Shulman, & Richert, 1987), there are not many examples of these transformations in the literature. In the following section, we introduce our proposed framework for understanding this transformation en route to the development of mathematical knowledge for teaching.

Toward a Framework

In our framework, we see a person's MKT as being grounded in a personally powerful understanding of particular mathematical concepts and as being created through the transformation of those concepts from an understanding having pedagogical *potential* to an understanding that does have pedagogical power. As such, in this section, we describe what we mean by a personally powerful understanding, a pedagogically powerful understanding, and the proposed mechanism for the development of each.

Simon (2002, October) introduced the idea of a *key developmental understanding* (KDU) in mathematics as a way to think about understandings that are powerful springboards for learning, and hence are useful goals of mathematics instruction. He describes a KDU as a conceptual advance or a "change in the learner's ability to think about and/or perceive particular mathematical relationships” (p. 993). Individuals who possess a KDU tend to find different, yet conceptually related ideas and problems understandable, solvable and sometimes even trivial. Thompson and Thompson (1996) propose that evidence of such understanding includes one’s ability to solve a variety of both directly and indirectly related problems as *a consequence* of their understanding (as opposed to having been explicitly taught how to solve such problems). Simon’s Key Developmental Understandings are an example of mathematical understandings “that carry through an instructional sequence, that are foundational for learning other ideas, and that play into a network of ideas that does significant work in students’ reasoning” (Thompson, 2008, p. 1).

Teachers who develop KDUs of particular mathematical ideas can do impressive mathematics with regard to those ideas, but it is not necessarily true that their understandings are powerful pedagogically; it is possible for a teacher to have a KDU and be unaware of its utility as a theme around which productive classroom conversations can be organized. Developing MKT, then, involves transforming these personal key developmental understandings of a particular mathematical concept to an understanding of: (1) how this key developmental understanding could empower their students’ learning of related ideas; and (2) actions a teacher might take to support students' development of it and reasons why those actions might work. This transformation requires what Piaget called decentering: “the uniquely human ability to differentiate one’s own point of view from the point of view of another” (Wolvin & Coakley, 1993, p. 178) or attempting to see the world from the perspective of another. Viewed in this way, MKT is a second-order model: “[a model] observers may construct of the subject’s knowledge in order to explain their observations (i.e., their experience) of the subject’s states and activities” (Steffe et al., 1983, p. xvi).

A valid question to ask at this point is how might such understandings – both a personally powerful understanding (KDU) and MKT – develop in the minds of prospective and practicing teachers whose existing conceptions may or may not be powerful on a personal level. We believe that the answer to this question lies in the domain of cognitive psychology and note that Piaget proposed *reflecting abstraction* as the answer to similar questions (Piaget, 1977/2001; Steffe, 1991). In the sections that follow, we argue that the development of a KDU involves a first abstraction via reflecting abstraction and the development of MKT involves a second abstraction (where the development of a KDU is the first-order abstraction). Here we are applying Piaget's notion of reflective abstraction to the prospective or practicing teacher. An example is useful in making clear some of the subtleties of our proposed framework; in the next section, we present a discussion and analysis of one well-known example of teachers' mathematical development.

Teachers' Understanding of Area and Multiplicative Reasoning

Simon (1995) discusses one segment of a course designed to help prospective elementary mathematics teachers develop a deep understanding of the elementary mathematics curriculum. This segment centered on multiplicative relationships and employed the concept of area as a vehicle for developing an understanding of multiplicatively defined quantities. Simon specifically noted "My purpose was to focus on the multiplicative relationships involved, *not* to teach about area … [and to foster] a solid conceptual link between their understandings of multiplication and their understandings of measuring area" (p. 123). Simon divided the class into groups, gave each group a small cardboard rectangle and proposed the following problem:

> Determine how many rectangles, of the size and shape that you were given, could fit on the top surface of your table. Rectangles cannot be overlapped, cannot be cut, nor can they overlap the edges of the table. Be prepared to describe to the class how you solved the problem (Simon, 1995, p 123).

Simon's prospective teachers began the task by counting the number of cards that could be placed along the length of the table and the number of cards that could be placed along the width of the table and then multiplied the two quantities. During the whole class discussion that followed, Simon attempted to focus the conversation on the multiplicative relationships involved by asking his prospective teachers why they multiplied in this situation. Initial responses included "[…] 'cause that's the way we've been taught," or "[…] it's a mathematical law" (Simon, 1995, p. 124).

Simon then presented Molly's explanation for why she multiplied, namely that she was multiplying the number of groups by the number in each group and that when she was counting one way, she was counting one side, she was counting many rectangles in a group and when she was counting the other side, she was counting how many groups. It was clear to Simon that few of the prospective teachers had understood Molly's explanation. Simon then posed the question "What instructional situation might afford other prospective teachers the opportunity to construct understandings similar to Molly's" (p. 124) and hypothesized that he needed to pose tasks that focused on the relationships between the solution strategy (multiplying), counting the number of rectangles along the length and width, and the area of the table as covering the table with the unit of area and that problematized their view of area is simply the result of length times width. To do so, he devised two tasks: Double Counting and Turning Rectangles. The Double Counting problem focused on counting individual rectangles and counting groups of rectangles:

> Bill said, "If the table is 13 rectangles long and 9 rectangles wide, and if I count 1, 2, 3 […] 13 and then again 1, 2, 3 […] 9, and then I multiply, 13 × 9, then I have counted the corner rectangle twice." Respond to Bill's comment (p. 125).

In discussing his prospective teachers' response to the Double Counting problem, Simon noted that many prospective teachers began to "insist that we are really counting rows and columns …

[though] I suspect that some of the [prospective teachers] have latched onto the notion of rows and columns in an unexamined way" (p. 126).

The "Turned Rectangles" problem (see Figure 1), which had emerged in some of the prospective teachers' initial work, asked the question of whether the orientation of the rectangle mattered and was envisioned as having potential for supporting prospective teachers' understanding of multiplicatively related quantities such as area:

> Do you have to maintain the orientation of the rectangle for the second measurement or could (or should) you rotate the rectangle 90 degrees, using the same side of the rectangle to measure the both sides of the table.

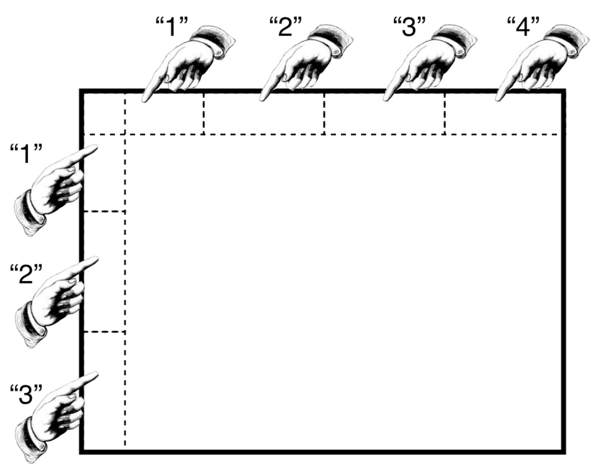

Figure 1: The "Turning Rectangles" problem.

The prospective teachers' responses to the Turned Rectangle problem were that the product of the two numbers would have no meaning (the product of 4×3 in Figure 1). Again, Simon posed an additional task to engage the prospective teachers in an activity from which he envisioned the multiplicative unit of area would emerge as a topic of discussion:

> Out in the hall I have two [rectangular] tables of different sizes. I used this method … where I measure across one way, turn the [rectangle], measure down the other way, and multiply… . When I multiplied using [this] method, on table A I got 32 as my answer and [when I measured]

> table B [using the same rectangle and the same method], I got 22. Now what I want to know is, [having used] the method of turning the rectangle, is table A bigger, is table B bigger, or don't you have enough information from my method to tell? (Simon & Blume, 1994, p. 480)

As the prospective teachers worked on the extension, some noticed that turning the unit was similar to using it as a ruler (measuring the rectangle's sides in units of the cardboard rectangle's side length) and that multiplying the two measures amounted to covering the rectangle with squares (see Figure 2).

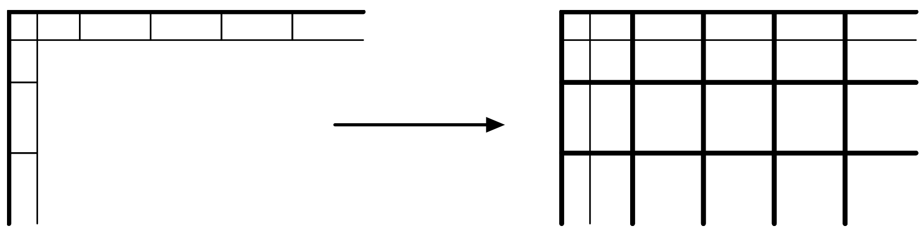

Figure 2. Seeing the unit rectangle as a ruler by which one measures the length of each side, creating unit squares in the process.

While Simon presented his analysis and discussion of these problems, in part, to articulate a model of teacher decision-making that he called the *Mathematics Teaching Cycle* and the constituent idea of *a hypothetical learning trajectory*, we presented his analysis for a different reason: The instructional setting presented was conceived by Simon as an environment propitious for prospective teachers' development. Simon's goal was, at least in part, to help his prospective teachers develop mathematical understandings of multiplication, and the vehicle for that development was a learning trajectory that included mathematical tasks designed to move the classroom conversation toward that goal. The critical question for our purposes is what mathematical understandings supported Simon's ability to conceive of the mathematical tasks that would support teachers' development along his hypothetical learning trajectory.

In thinking about the understandings that guided Simon's actions, we contrast his actions with typical instructional activities designed to support the prospective teachers' understanding of area as length times width: providing them with unit squares to cover the area and make the meaning of that calculation understandable. Piaget, Inhelder, and Szeminska (1948/1960) observed that while it is quite easy for children to cover an area and to multiply algorithmically, it is less straightforward for them to understand how two lines (length and width) produce an area when multiplied. They note that while squares are discrete quantities, length and width are lines that represent continuous quantities and that "it is impossible for children to understand how two lines (the length and width) can produce an area when multiplied" (Kamii & Kysh, 2006, p. 108). The learner must construct their understanding of area as a multiplicatively related quantity derived from two linear measurements. We contend that this understanding of multiplicatively related quantities, which also applies to volume (area times length), force (mass times acceleration), distance (rate times time), etc., is a key developmental understanding.

In the remaining sections, we unpack Simon's use of his KDU to illustrate our contention that while developing these personally powerful mathematical understandings are important, they are not sufficient for conceiving environments where teachers (prospective or practicing) might develop similar KDUs. We need to explain teachers' development of understandings that support their anticipation of a web of student understandings that comprise a KDU. It is in that respect that we present our theoretical framework.

Characterizing Mathematical Knowledge for Teaching

Unpacking the Learning Environments

Simon's instruction in the prior example aimed at having teachers develop a new way of thinking about area and their experiencing a fundamental shift in their understanding of multiplication – a KDU. An important question is how might someone develop an understanding of area as a multiplicative quantity when they already possess an understanding of area that is incompatible? As described previously, Piaget proposed *reflective abstraction* as the answer to how these new cognitive structures might develop (Simon, Tzur, Heinz, & Kinzel, 2004; Steffe, 1991; Thompson, 1985). Reflective abstraction is a process by which new, more advanced conceptions develop out of existing conceptions and involves abstracting properties of action coordinations in order to develop new cognitive structures. To illustrate this process, we present an example of such new cognitive structures and the abstraction process in the context of mathematics teacher education.

Simon noted that his class did develop a means of understanding the relationship between the larger rectangle's area and "multiplying the length times the width". With respect to this, one of the prospective teachers, Candy, commented,: "… it makes it confusing to try to look at the length times width. … You should really treat it as so many sets or so many groups, like nine groups …, thirteen groups of nine" (Simon, 1995, p. 126). This realization seemed to be the result of the prospective teachers visualizing a number of copies of their rectangle covering the surface to be measured. Teachers who reason in this way are likely looking for similarities and difference between the objects themselves, or "draw[ing] … information from objects and from the material or observable characteristics of actions," (Piaget, 1977/2001, p. 317) where, in this case, the action is understanding the characteristics of the collection. Piaget refers to this as empirical abstraction. Knowledge obtained through empirical abstraction, though it may be a transformation of previous knowledge, is not viewed as the development of new knowledge, because this is simply focusing on characteristics already in that object (Piaget, 1977/2001).

Individuals at this level might reason something like this: "If I cover the surface with copies of my rectangle and try to count all the rectangles, it would be easier if I counted the number in a row [or column] and then see how many rows [or columns] there are. The total area will be the same as the number of rows times how many rectangles there are in a row." In contrast, consider the prospective teachers who developed the ability to explain why measuring the lengths of sides to create square units was useful in describing areas. These teachers realized that they were not measuring the larger rectangle's area by covering it with smaller rectangles. They realized that they were using *the length of one side of the smaller rectangle to measure the length of the sides of the larger rectangle, and thus creating a unit of area out of units of length.* These latter prospective teachers developed new knowledge through reflective abstraction, not empirical abstraction. Instead of conceiving the smaller rectangle as something to use to "cover" the larger rectangle, they abstracted from their actions the notion that *measuring the sides of a rectangle with a common length induces a covering made of squares or parts of squares* and that *the salient aspect of putting down the cardboard rectangle is that they are using the length of one side as a common length.* It was this realization that allowed them to make sense of the area of the larger rectangles in units of "square sides." It is thus our claim that the prospective teachers who came to recognize the square as a *derived* unit of measurement developed new knowledge that was transformed, or abstracted, from applying their prior knowledge. Those prospective teachers who simply developed an understanding of why they multiply when calculating area did not fundamentally transform their knowledge, but simply augmented it via empirical abstraction.

Obviously, it is desirable for any teacher to develop deep personally powerful mathematical understandings. When teachers possess an understanding of area being measured by these “new units,” for example, it allows them to conceive of the complexity of teaching area in a different way. When encountering a student who is struggling with the notion of area, rather than relying on many different ways of saying, essentially, “multiply length times width,” a teacher could also focus on the development of the idea of area as “covering” and the relationship between linear measurements and area. This understanding of area as an n+1 dimensional, derived unit requiring the coordination of an n-dimensional quantity and a one-dimensional quantity, in turn could help students make sense of commonly problematic areas such as the relationship between area and volume.

In our recent work (Silverman, 2004; Silverman & Thompson, 2005), we note that such abstracted understandings are not sufficient for teachers to have the ability to present students with opportunities that position them to develop similar, consistent understandings. It is well documented that most teachers who do change their teaching practices do so in superficial ways (Stigler & Hiebert, 1999) and our work, which examined this result with a finer-grained analysis, indicated that changes in their teaching are a result of their pedagogical conceptualizations of the mathematics: both the sense they have made about the mathematics and their awareness of its conceptual development.

Refined Goals for Mathematics Teacher Education

Thompson, Philipp, Thompson, and Boyd (1994) and Cobb, Boufi, McClain, and Whitenack (1997) argue convincingly that students' participation in conversations about their mathematical activity (including reasoning, interpreting, and meaning-making) is essential for their developing rich, connected mathematical understandings. It therefore follows that it is in the context of instruction that supports reflective conversations that teachers’ development of KDUs is most probable. But original conversations that support collective reflection (and thus generating a reflective conversation) must be about something that transcends the conversations. Thompson (2002) used the phrase *didactic objects* to refer to things that exist at the boundary between original and reflective conversations. He explained that while reflective conversations can arise spontaneously, they can also be designed, and the design of objects about which the conversation centers can be done systematically.

Thompson (2002) describes this view of instructional design in mathematics as creating "a particular dynamical space, one that will be propitious for individual growth in some intended direction, but will also allow for a variety of understandings that will fit with where individual students are at that moment of time" (p. 194). With regards to this dynamical space, Thompson (1985) notes that when conceptualized in this way, instructional design requires that the objectives of instruction be stated in cognitive terms and that images of instruction be of a teacher choreographing conversations which have the possibility of stimulating reflective discourse around the desired mathematical idea. Instructional design, therefore, is not about teaching particular content; rather it is about formulating particularly powerful understandings and designing interactional spaces where others can come to understand the content in a similar and consistent way. It is also about designing instructional environments that take into account categories of "ways of thinking"[i] that students bring to instruction and which might be leveraged profitably to move the conversation forward.

Framework for Mathematical Knowledge of Teaching

This perspective on instructional design is just as applicable to mathematics teacher education as it is in the teaching of mathematics and it is in this vein that we propose the following framework for Mathematical Knowledge for Teaching. A teacher has developed knowledge that supports conceptual teaching of a particular mathematical topic when he or she (1) has developed a KDU within which that topic exists, (2) has constructed models of the variety of ways students *may* understand the content (decentering); (3) has an image of how someone else might come to think of the mathematical idea in a similar way; (4) has an image of the kinds of activities and conversations about those activities that might support another person's development of a similar understanding of the mathematical idea; and (5) has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas.

As a teacher thinks about the content to be taught, she envisions a student (other than the teacher) working through the material, easing through some problems and stumbling over others. The entire time, the teacher must ask herself “what must a student understand to create the understanding that I envision?” and “what kinds of conversations might position one to develop such understandings?” The prospective teacher must put herself in the place of *a student* and attempt to examine the operations that a student would need and the constraints the student would have to operate under in order to (logically) behave as the prospective teacher wishes a student to do. This is reflective abstraction.

A key developmental understanding might be viewed as a pedagogical action, where *action* is used in the Piagetian sense[ii]. Teachers are engaged in pedagogical actions when they wonder, “What might I do to help students think like what I have in mind?” Their question is posed in a domain specific manner, such as “How might I help my students think about logarithms as an accelerated condensing and recoding of the number line?” The development of MKT involves separating one’s own understanding from the hypothetical understanding of the learner (Steffe, 1994). When a person views a pedagogical action as if she is not an actor in the situation (even though she is), and when the person can separate herself from the action (and thereby reflect on it), the pedagogical action has been transformed into a pedagogical understanding. It is this understanding that is capable of being reflected upon, for the teacher now sees various alternatives that could have happened and has developed agency over the process. The teacher is also now able to see the "pedagogical power" of key developmental understanding.

When a teacher develops a key developmental understanding, his content knowledge becomes "related" to other content knowledge and extends his web of connections (Thompson & Saldanha, 2003). A key developmental understanding could then be viewed as knowledge that is assimilated to a scheme. This new understanding (and thus new knowledge) cannot be MKT because this transformed knowledge is not in and of itself pedagogical[iii]. At this point, this new knowledge is mathematical knowledge that has pedagogical potential. It is not until the teacher transforms this knowledge into knowledge that is pedagogically powerful that the teacher has developed MKT.

Concluding Comments

The mathematical knowledge required for teaching mathematics is one of the three strands of the research program identified by the RAND Mathematics Study Panel (2003) as integral for improving the quality of mathematics instruction in the United States. They note that one important line of work is to "extend current research on [the] mathematical knowledge needed for teaching other mathematical topics and to the realm of mathematical practices and their role in teaching" (p. 23). In this article, we extend this work by identifying a potential framework for recognizing MKT and for thinking about how it might develop in the minds of teachers. While identifying the specific knowledge needed for teaching particular mathematical content is important, it is a daunting task, considering the variety of mathematical content that comprises the K-12 mathematics curriculum. We present a framework that is not only informed by the work of mathematics teaching, but also a developmental trajectory for mathematics learning and the learning sciences. Our framework opens up the possibility for the goal of mathematics teacher education to shift from positioning prospective teachers to develop particular MKT to developing professional practices that would support teachers' ability to continually develop of MKT throughout their careers. These practices include the development of key developmental understandings, becoming reflectively aware of them, and placing them within a model of students' learning in the context of instruction.

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[i] We borrow this idea from Thompson and Saldanha (2000, October), who develop the notion of *epistemic subject* as a way to explain how a teacher can attend to the intellectual needs of an entire classroom without having to attend to the needs of every student in it. An epistemic subject is an idealized person who happens to think in a particular way. Our use of "categories of ways of thinking" is equivalent to Thompson and Saldanha's use of "epistemic subject."

[ii] Piaget defined the word broadly as any change to the perceptual input (Piaget, 1967).

[iii] Thus there is a transformation in the development of further refined and developed mathematical knowledge. It is, however, new mathematical knowledge, not pedagogical knowledge.

References

Ball, D., Hill, H., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator, 29*(3), 14-22, 43-46.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90*, 449-466.

Ball, D. L. (1991). Teaching mathematics for understanding: What do teachers need to know about subject matter? In M. Kennedy (Ed.), *Teaching academic subjects to diverse learners* (pp. 63083). New York: Teachers College Press.

Ball, D. L. (1993). Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. Carpenter, E. Fennema & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157-195). Hillsdale, NJ: Lawrence Erlbaum Associates.

Ball, D. L. (2007). *What kind of mathematical work is teaching and how does it shape a core challenge for teacher education*. Paper presented at the Judith E. Jacobs lecture given at the annual meeting of the Association of Mathematics Teacher Educators.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83-104). Westport, CT: Ablex Publishing.

Ball, D. L., & Bass, H. (2003). *Toward a practice-based theory of mathematical knowledge for teaching*. Paper presented at the 2002 Annual Meeting of the Canadian Mathematics Education Study Group.

Ball, D. L., & McDiarmid, G. W. (1990). The subject matter preparation of teachers. In W. R. Houston (Ed.), *Handbook of research on teacher education* (pp. 437-449). New York: Macmillan.

Carpenter, T., Fennema, E., Peterson, P. L., & Carey, D. (1988). Teachers' pedagogical content knowledge of students' problem solving in elementary arithmetic. *Journal for Research in Mathematics Education, 19*, 385-401.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflexive discourse and collective reflection. *Journal of Research in Mathematics Education, 28*(3), 258-277.

Gess-Newsome, J. (1999). Introduction and orientation to examining pedagogical content knowledge. In J. Gess-Newsome & N. G. Lederman (Eds.), *Examining pedagogical content knowledge* (pp. 3-20). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Grossman, P. L., Wilson, S. M., & Shulman, L. S. (1989). Teachers of substance: Subject matter knowledge for teaching. In M. C. Reynolds (Ed.), *Knowledge base for the beginning teacher* (pp. 23-36). Elmsford, NY: Pergamon Press, Inc.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's Mathematics Professional Development Institutes. *Journal for Research in Mathematics Education, 35*(5), 330-351.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(5), 371-406.

Kahan, J., Cooper, D., & Bethea, K. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of teachers. *Journal of Mathematics Teacher Education, 6*, 223.

Kamii, C., & Kysh, J. (2006). The difficulty of "length x width": Is a square the unit of measurement? *Journal of Mathematical Behavior, 25*, 105-115.

Leinhardt, G., & Smith, D. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 78*, 247-271.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mason, C. (1999). The TRIAD approach: A consensus for science teaching and learning. In J. Gess-Newsome & N. Lederman (Eds.), *Examining pedagogical content knowledge* (pp. 277-292). Boston: Kluwer Academic Publishers.

Piaget, J. (1967). *Six psychological studies*. New York, NY: Vintage Books.

Piaget, J. (1977/2001). *Studies in reflecting abstraction* (R. L. Campbell, Trans.). Sussex, UK: Psychology Press.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry*. London: Rutledge & Kegan Paul. (Original work published 1948).

RAND Mathematics Study Panel. (2003). *Mathematical Proficiency for all students: toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND.

Schifter, D. (1995). Teachers' changing conceptions of the nature of mathematics: Enactment in the classroom. In B. S. Nelson (Ed.), *Inquiry and the development of teaching: Issues in the transformation of mathematics teaching* (pp. 17-25). Newton, MA: Center for the Development of Teaching, Educational Development Center.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.

Silverman, J. (2004, April 12-17, 2004). *Comparing aspects of constructivist research methodologies in mathematics education: Modeling, Intersubjectivity, and Tool Use.* Paper presented at the annual meeting of the American Educational Research Association, San Diego, CA.

Silverman, J., & Thompson, P. W. (2005, October). *Investigating the relationship between mathematical understandings and teaching mathematics.* Paper presented at the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Roanoke, VA.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Simon, M. (2002, October). *Focusing on key developmental understandings in mathematics.* Paper presented at the Twenty-fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Athens, GA.

Simon, M., & Blume, G. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25*, 472-494.

Simon, M., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education, 35*(5), 305-329.

Steffe, L. P. (1991). The learning paradox. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 26-44). New York: Springer-Verlag.

Steffe, L. P. (1994). Children's constitution of meaning for arithmetical words: A curricular problem. In D. Tirosh (Ed.), *Implicit and explicit knowledge: An educational approach* (pp. 131-168). Norwood, NJ: Ablex.

Stigler, J. W., Gonzales, P., Kawanaka, T., Knoll, S., & Serrano, A. (1999). *The TIMSS Videotape Classroom Study: Methods and findings from an exploratory research project on eighth-grade mathematics instruction in Germany, Japan, and the United States*. (National Center for Education Statistics Report No. NCES 99-0974). Washington, D.C.: U.S. Government Printing Office.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Calculational and conceptual orientations in teaching mathematics. In A. Coxford (Ed.), *1994 Yearbook of the NCTM* (pp. 79–92). Reston, VA: NCTM.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*(1), 2-24.

Thompson, P. W. (1985). Experience, problem solving, and learning mathematics: Considerations in developing mathematics curricula. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 189-243). Hillsdale, NJ: Lawrence Erlbaum Associates.

Thompson, P. W. (2002). Didactic objects and didactic models in radical constructivism. In K. Gravemeijer, R. Leherer, B. VanOers & L. Verschaffel (Eds.), *Symbolizing, modeling, and tool use in mathematics education* (pp. 197-220). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Thompson, P. W. (2008). *Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education*. Plenary paper delivered at the 32nd Annual Meeting of the International Group for the Psychology of Mathematics Education. Morelia, Mexico. Volume 1, pp 1-18.

Thompson, P. W., Carlson, M., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal for Mathematics Teacher Education, 10*(4-6), 415-432.

Thompson, P. W., & Saldanha, L. (2000, October). *Epistemological analyses of mathematical ideas: A research methodology.* Paper presented at the Twenty-second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Tucson, Arizona.

Thompson, P. W., & Saldanha, L. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

Wilson, M. S., Shulman, L. S., & Richert, A. R. (1987). "150 different ways" of knowing: Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teacher thinking* (pp. 104-124). London: Cassell PLC.

Wolvin, A., & Coakley, C. (1993). *Perspectives on Listening*. Westport, CT: Greenwood Publishing Group.