Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education.* WISDOMe Mongraphs (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming.

QUANTITATIVE REASONING AND MATHEMATICAL MODELING[1,2]

Patrick W. Thompson
School of Mathematical and Statistical Sciences
Arizona State University

I first provide a bit of historical background on a theory of students' development of algebraic reasoning through quantitative reasoning. The quantitative reasoning part of the theory gained some popularity, but its most important features (at least in my thinking), the parts explicitly related to algebraic reasoning, received little notice. I then point to important work that extended the quantitative reasoning part of this theory in important ways (e.g., Lobato, Reed, Ellis, Norton, Castillo-Garsow, and Moore) and discuss how the "little noticed" aspects could inform that work in useful ways. I also discuss how a focus in school mathematics on quantitative reasoning and its extension into algebra could be leveraged with regard to students' engagement in authentic mathematical modeling. Finally, I extend the theory of quantitative reasoning, with a focus on quantitative covariation, to include reasoning with magnitudes and discuss examples of how rich, coherent understandings of magnitudes can be foundational for advanced reasoning in analytic geometry, calculus, differential equations, and analysis.

In 1990 I presented a theory of quantitative reasoning (Thompson, 1990) that proposed meanings for quantity, quantitative structure, and quantitative reasoning that could provide a foundation for theories of how to promote and support algebraic reasoning. I never attempted to publish that paper, as its purpose was to concentrate my thinking rather than to generate a publication. To a large extent, that strategy worked. A number of articles and book chapters followed from it, each examining a small part of the theory, and that theory continues to be at the core of my current work with theories of instruction and learning in the context of instruction. But none of this work revisited the theory in its entirety. Since then I've gained, I hope, perspective on what are the central aspects of quantitative reasoning, and I hope to take this opportunity to revisit them.

Central Tenets of a Theory of Quantitative Reasoning

A quantity is in a mind, it is not in the world

Johnson (2008) gives a fascinating account of Lavoisier's experiments leading up to the discovery of oxygen in the late 1700s. In the mid-centuries (circa 1600-1800, especially 1690-1760), when scientists witnessed something burning they *saw* a release of phlogiston—the stuff that objects contain that revealed itself in flames. Phlogiston was a real substance to them. The original object, before burning, contained phlogiston and calx—calx being the stuff that stays behind like ash after phlogiston leaves. Their mental model of combustion was of something being released, not something being combined. Attempts to quantify phlogiston—to determine how much of it a substance contained—built, over many years, to Lavoisier's conclusion that the model in which phlogiston played a key role was not viable, and that there is no such thing as phlogiston. In other words, phlogiston was a quantity until it was a mistake.

[1] Preparation of this paper was supported by National Science Foundation Grant No. EHR-0353470. Any conclusions or recommendations stated here are those of the authors and do not necessarily reflect official positions of NSF.
[2] Many ideas expressed in this paper developed through intense conversations with the late Jim Kaput over the period 1985-1995. It is impossible to untangle my and Jim's thinking without his participation. I apologize to Jim's memory for my inability to do this.

The point that quantities are mental constructions, and that their creation is often effortful, is central to mathematics education. Too often quantities, such as area and volume, are taken as obvious, and hence there is no attention given to student's construction of quantity through the dialectic object-attribute-quantification. Instead, textbook writers and teachers just use them to teach.

Jason Silverman and I (Silverman & Thompson, 2008) pointed this out as revealed in Marty Simon's work with teachers on helping them understand multiplication as a quantifying operation (Simon & Blume, 1994). We noted that many of Simon's pre-service teachers took a region as a unit of area, and that a measure of area was simply copying that region repeatedly to cover another region and counting the number of copies. Thus, while measuring a larger rectangle with a unit rectangle those students objected when Simon turned the unit rectangle 90 degrees after covering the first row. They saw Simon's method producing "overlapping areas". They did not see Simon using the smaller rectangle as a ruler, whereby he used its edge to measure the lengths of the larger rectangle's edges.[3] They did not see either measured area or a unit of area as being composed of two lengths combined multiplicatively.

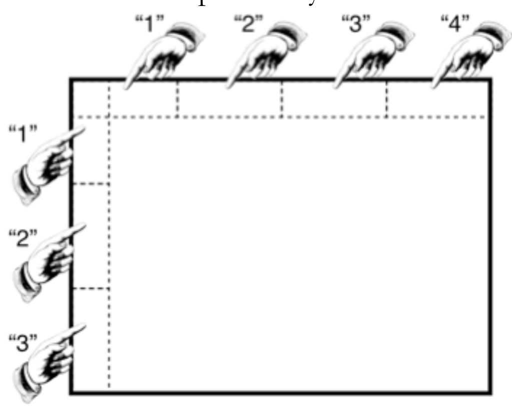

Figure 1. An "erred" measurement process (from Silverman & Thompson, 2008)

In another article I described Simon's students' way of thinking—of units of area or units of volume as objects to be counted—as one-dimensional area and one-dimensional volume (Thompson, 2000) and discussed the obstacles this way of thinking poses for understanding Euclidean dimension in a way that supports understanding fractal dimension. In that article I also described children reconceptualizing multiplication as a quantitative operation. I asked a group of 5th-graders the following question:

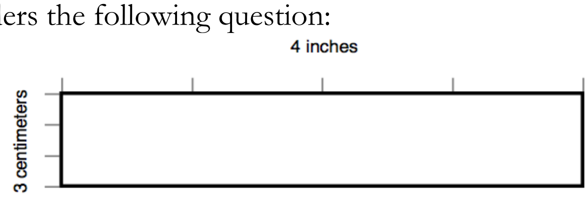

Figure 2. We have a rectangle that is 4 inches wide and 3 centimeters high. Suppose we did the silly thing and multiplied 3 by 4 to get its area. We would get 12. But 12 what?

The ensuing discussion went on for 35 minutes before one child asked timidly, "Would it be 12 rectangles that are 1 cm by 1 inch?" In the next 10 minutes children worked to understand how it was that (1) it made sense, in multiplying length by width, they were somehow generating rectangles,

[3] I hasten to make clear that Simon *intended* to have his students experience the dilemma of whether "turning the unit rectangle" was a legitimate way to measure the area of a larger rectangle.

and (2) that the only thing they needed to know about a "covering collection" was the unit-length of each side of the basic area unit. To understand multiplication as a quantification of area they needed to reconceptualize one unit of area ("amount of region") as a multiplicatively-structured object.

To pre-service teachers for whom area was a thing to be covered by chunks of area, measuring area as if by a ruler made no sense. Nor did their conception of area have the property of bi-proportionality—that if you make one dimension *n* times as large and another dimension *m* times as large, the resulting area is *nm* times as large. Only covering and counting made sense. For area to entail proportionality, they needed to develop a different quantity that they would call area—one that is created multiplicatively. To return to the example of phlogiston, those for whom the theory of phlogiston explained combustion, phlogiston existed. It was a substance that was key to their understanding of combustion. They saw it manifested in everyday experience. But they had not quantified it. In the process of attempting to quantify phlogiston, and thereby gain greater insight into the nature of combustion, new realizations emerged.

The stance that quantities are in minds, not in the world, is for educators, not scientists. Of course scientists are well aware that the best they can do is build models of the way the world "really" works. But they could not do science if they did not take their models as depicting reality "as it is". Educators, however, must be constantly aware that what is in students' minds is what matters. The stance that quantities are in minds, not in the world, actively supports being attentive to how students conceive situations and serves as an active reminder that their conceptions of even the most "obvious" aspects of a situation are likely to differ from ours in important ways.

Quantification is a *really big deal*

In (Thompson, 1994d) I said this about quantification.

> Quantification is a process by which one assigns numerical values to qualities. That is, quantification is a process of direct or indirect measurement. One does not need to actually carry out a quantification process for a quantity in order to conceive it. Rather, the only prerequisite for a conception of a quantity is to have a process in mind. Of course, a person's grasp of a process may change as he or she re-conceives the quality and the process in relation to one another.

I wish I had not said this, for it is commonly understood that I meant precisely what I said—that quantification is merely a process of assigning numbers to an attribute of an object. I didn't mean this at all, and the remainder of that article, which analyzed one child's construction of speed as a quantity, documented the dialectic between her progressive conceptualization of motion as entailing the simultaneous accumulation of two quantities (distance and time) and conceptualizing a quantification of that simultaneous accumulation. Here conceptualization of each—speed as a quantity and proportional correspondence as her quantification of it—went hand in hand. They were inseparable.

Luis Saldanha and I (Thompson & Saldanha, 2003) developed another example of the power of quantification as a root of mathematical thinking. We considered the question, "What is torque and how might one quantify it?" While I would never claim that our conceptual analysis was definitive with respect to learning the idea of torque, I believe it does reflect the dialectic between conceptualizing a quantity and conceptualizing its quantification. The central questions were,

- What is the object one conceives when conceiving torque?
- What is the attribute of that object that one quantifies?

- What is one unit of torque?

The answer to (1) can only be that it is a system that you are quantifying—a system that involves something turning around a fixed point that somehow behaves differently the farther from that fixed point you are. The answer to (2) of course involves (1). The attribute of that system is something like "amount of twist", the awareness that it takes more shoulder strength to hold a pail of water farther from your body than closer. The answer to (3) is more complicated. It is the realization that (a) any quantification of torque must take into account, simultaneously, the distance that a force is applied from the fixed end and the amount of force being applied (and thus a unit of twistiness is say, Newton-meter or foot-pound), and (b) that *amount* of twistiness is proportional to each of those components—if we apply the same force at thrice the distance we get thrice the amount of twistiness. If we apply thrice the force at the same distance, we get thrice the twistiness. Thus, whatever amount of twistiness is produced by 1 force-unit at 1 distance-unit, a force of n force-units at 1 distance unit will be n times as great, and a force of 1 force-unit at a distance of m distance-units will be m times as great. Put more formally, if $T(1,1)$ is the amount of twistiness produced by 1 force-unit applied at 1-distance unit from a fixed point, then $T(n,1) = nT(1,1)$ and $T(1,m) = mT(1,1)$. Therefore $T(n,m) = mnT(1,1)$. Thus, we "multiply force times distance" to quantify torque.

As an extension of this example regarding torque, and building off the ways of thinking described in (Thompson & Silverman, 2008), if students combine an understanding of torque with an understanding of accumulation, then it is not hard for them to imagine a method for conceptualizing the total torque generated by an object of varying density that has one end fixed. Imagine slicing the object into super-thin pieces that are vertical to the axis of twist, imagining each slice to be of uniform density, and summing the torque generated by each as if each is generates torque independently. The sum is your total torque. However, this method is *unavailable* to students who have not conceptualized torque and accumulation appropriately: torque as constituted multiplicatively by a force acting at a distance from a fixed point and a total amount of a quantity as generated by accumulating little bits of it.

The dialectic between students conceptualizing a quantity and conceptualizing a quantification of it is so important that I'll share a third example. A class of 8th-grade students in central Illinois was curious about why grain silos explode (the news reported one having exploded the prior day). I decided to leverage that curiosity into an ad-hoc unit on quantification—on creating the quantity "explosiveness" and how one might measure the explosiveness of a grain silo. Students were stymied until someone remembered, from science, that an explosion is just fast burning, and so something must burn to produce a silo explosion. Then one student remembered throwing a pinch of flour into a campfire and seeing it burst into flames. Oh, grain dust is what burns! But if grain dust is all on the ground, you could hit it with a blowtorch and it wouldn't burn. What's going on? Another student recalled that burning is just oxidation, and that oxidation happens only at a material's surface, so maybe when grain dust is all on the floor there just isn't enough exposed surface for it to burn. Where does that take us? Oh! When dust is on the floor, only the top, thin layer is exposed to air. When dust is "in the air", each individual dust particle is surrounded by air! Where does that take us? Well, maybe if we think about the amount of grain dust (by volume) in relation to the amount of surface area exposed that will give us a measure of explosiveness? But wait, doesn't the size of the silo matter? Oh, you need to think of the amount of surface area in relation to the volume of dust, per cubic foot of air in the silo. They ended, after several days of intense debate, with a unit of measure being cm^2 of "dust surface area" per cm^3 of "dust volume" per ft^3 of "silo volume" as their unit in which they would measure a silo's explosiveness[4]. They figured that there would be a low range of measurements that indicate "not explosive", a mid-range of measurements that indicate "it depends on other things" (e.g., the exact composition of the dust), and a high range of measurements that indicate "highly explosive". They wondered about how to figure the total particulate surface area of some amount of dust, but they were confident that this was possible if you assumed all dust particles had a given diameter.

Whether students ended with a good measure of explosiveness or a poor measure actually is not relevant to the significance of their activity. Rather, the significance is that in coming up with a measure that satisfied them, they needed to build from models they already had (combustion happens at an object's surface; explosions happen by rapid oxidation[5]) and they needed to engage in a dialectic between conceptualizing the attribute being measured (explosiveness and how things exploded) and the way in which they would measure it (by comparing relative amounts of related quantities).

Another aspect of this example is important for school mathematics. Once students had conceptualized explosiveness (how it works) and measures of it, we could start doing mathematics. "Suppose there are two silos having the same volume and having the same volume of grain dust in the air, but the diameter of individual particles' in one silo is twice the diameter of particles in the other. How explosive is one silo relative to the other?" On the other side of this example is the point that, without students having a working model of explosions and without them having conceptualized a measure of explosiveness, that question would have been meaningless to them.

To come full circle regarding quantification, I wish that instead of saying that quantification is the process of assigning numerical values to a quality I had said this:

> *Quantification is the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit.*

Sometimes the process of quantification might take minutes, other times years, and other times generations.

This expanded definition of quantification, to me, provides an explicit link between science education and mathematics education. We must take quantification seriously if we want to develop instructional and curricular strategies for having students' mathematics be of use to them in dealing with their worlds.[6] By "their worlds" I include more than students' physical worlds. Economics abounds with quantities that students conceptualize poorly (e.g., principal, interest, compounded interest, annual interest rate, monthly interest rate, profit, loss, revenue, expense, marginal cost, etc.), and which, properly conceptualized, would enable people to make wise decisions and avoid costly mistakes. Issues of scale have their conceptual roots in quantification. Paulos' (1988) classic book on

[4] I still find it fascinating that in this sophisticated conversation they felt that they needed "small units" like cm^2 and cm^3 to measure attributes of dust particles, but cm^3 was too small a unit to measure the volume of a silo.

[5] They did not consider that chemical explosions do not involve oxidation, but, again, this hardly matters here.

[6] Bob Mayes noted that many science educators find my meaning of quantitative reasoning "too mathy". My reply, to reiterate a point that I made in (Thompson, 1994b), was that while math education must take scientific reasoning more seriously as a source of mathematical reasoning, science education must take mathematical reasoning more seriously as a source of scientific reasoning. I still maintain that science education could profit from attending with greater clarity to quantity and quantitative reasoning as I describe them here.

innumeracy is an account of the consequences of our populace's failures in quantification—of failing to conceptualize aspects of our world so that they are measurable in some unit, which completely obstructs their ability to reason quantitatively.

Quantitative operations are not the same as numerical operations (but they are related)

Steffe (1991a, 1991b) provided what I consider to be the foundational analysis of the mental operations that generate quantity. In these papers he describes the operations of segmenting and unitizing as foundational for generating concepts of measured quantity. So, in the strict sense that Steffe develops, operations that generate quantity and operations that generate number are tightly related.

Without contradicting Steffe, I'd like us to enter the world of children and teachers. Children often employ numerical operations that have no quantitative significance. They might have quantitative roots, but at the moment of employing them they often do not reflect operations on quantities. They might even be highly structured with regard to numerical underpinnings, entailing composite units and multiplicative relationships. But again, they can do this without imagining something being quantified.

In 1993 I wrote about six 5^{th}-grade children's additive reasoning (Thompson, 1993). In that article I defined *difference* as a quantity resulting from comparing two quantities additively—"the amount by which one quantity exceeds or falls short of another". Thus, a difference, as a result of a comparison, is a quantity. But it is a quantity that exists within a structure: (Quantity A, Quantity B, Result of Comparing). I also gave examples of how, in a complex situation, one could potentially use any arithmetic operation, not just subtraction, to evaluate that quantity. I also gave an example of the children using subtraction but not knowing what quantity they had just evaluated. This example deserves repeating for the present conversation. It involves a rather lengthy excerpt, so I apologize in advance. I will focus upon lines 57-92, but those lines make sense only in the context of the discussion that precedes them.

Problem 2:
Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B.
The two fellows argued about which one stood taller over his sister.
It turned out that Brother A won by 17 centimeters.
Brother A was 186 cm tall.
Sister A was 87 cm tall.
Brother B was _____ cm tall.
Sister B was _____ cm tall.
Put numbers in the blanks so that everything works out.[7]

1. PT: Okay. Do this one as a team. Put numbers in the blanks so that everything works out.
   Pause as children read the text.
2. Jill: … (inaudible) … Well, actually, Brother B has to be at least 17 cm shorter than Brother A.
3. Molly: But it says that Brother A won by 17 cm.
4. Jill: I know.
5. Molly: So that means …

[7] The children had solved a problem using the same context, but with numbers instead of blanks, so they had already sorted out the assumption that Brother A compared himself to Sister A and Brother B compared himself to Sister B.

6. Jill: So that means he (Brother B) has to be at least 17 cm shorter …
7. Molly: Yeah, at least.
8. Jill: No wait … no it doesn't, because … not it doesn't, because …
9. Molly: It says she could be shorter than him if the difference is small. *Pause.* But if Brother A won by 17 cm, that means that Sister B must be at least 17 cm taller than Sister A.
10. Jill: Uh-uh (no). If Brother B is shorter, then she can't be.

    Pause.
11. Jill: Well we could put …
12. Molly: Like the last one we had.
13. Jill: Well, you could put that Brother B was 150 cm tall and Sister B was 149 cm tall.
14. Peter: How do you know that would work?
15. Molly: But Brother A still has to win by 17 cm.
16. Don: Oh, so Brother B … yeah would be 17 … oh no, he wouldn't … But he could be 17 cm shorter. He doesn't have to be, but he could be.

    Pause.
17. Don: I don't think it matters what we put in there.
18. Jill: Ummm … Brother A had to win by 17 cm.
19. Don: (Quietly.) I don't think it matters.
20. Molly: Peter: } Yes it does. Brother A has to win by 17 cm.
21. Molly: As long as he wins by 17 cm, you could put anything in.
22. Peter: But what do you mean that he wins by 17 cm?
23. Molly: He has to be 17 cm taller than Sister A … than …
24. Jill: Brother B is than Sister B.
25. Peter: Oh, I know.

    Pause.
26. PT: You have any numbers in there yet?
27. All: No.
28. PT: Good discussion! … Are you trying to figure out what the first number is?

    Pause.
29. Jill: Wait. Lets find out the difference between Brother A and Sister A.
30. Don: It's easy.
31. Molly: It's 99.
32. Don: It's 98 … 99.
33. Molly: So, Sister B must be 17 cm … must be 99 minus 17.
34. Jill: She could be.
35. Molly: She could be, because he has to win by 17 cm.
36. Peter: 99 minus 17 … 83.
37. Don: It's 82.
38. Peter: She could be 82. But what would Brother B be?
39. Molly: So, she could be 82 cm tall. But she doesn't *have* to be 82 cm tall.
40. Jill: She could be 82 cm or lower.

41. Molly: He'd [she'd?] have to win by 17 cm …
42. Peter: Brother B?
43. Molly: Brother A.
44. Jill: But then Brother B can't be … we could make Brother B be 90 … we could make Brother B be the same thing.
45. Peter: You could make Brother B be 100.
46. Molly: No, you couldn't, because, then he'd be … I don't know.
47. PT: What does the 82 mean?
48. Jill: That's the difference between Sister A and Brother A.
49. PT: The 82 is?
50. Jill: Don: No, it isn't. 99 is.
51. Molly: But he had to win by 17 cm, so we took away 17 from 99 … So Brother B would have to be the same as Brother A. [To Jill] We could do it that way.
52. Jill: Uh-huh (yes) …
53. Peter: 186 …
54. Jill: Then Brother B would win.
55. Peter: No he wouldn't
56. Molly: Yes he would.
57. PT: Could I ask you to think about something … Try this. Think about what the 82 means.
Long pause.
58. Jill: *inaudible* … okay, so the difference between Brother A and Sister A and subtracted … uh … how much … he won … by (to Molly) …
Pause.
59. Molly: Huh?
60. Jill: … We took the difference between Brother A and Sister A. Does everybody understand that?
61. All: Yeah.
62. Jill: And then we took away how much Brother A won by.
63. All: Yeah.
64. Molly: Yes … yes … yes!
65. Peter: (Giggles at Molly.) So 186 …
66. PT: (To Jill) And you got 82, right? Now, what does the 82 stand for?
Pause.
67. Jill: Uhhhhh …
68. All: (Giggles.)
69. PT: Well, what does that 99 stand for?
70. Molly: How much the difference is … Oh, so the 82 could be the difference between Brother B and Sister B.
71. PT: Is that what it is?
72. Don: It could.
73. PT: Or is that how tall Sister B is?
74. Molly: It couldn't be how tall she is because then Brother B would win.

75. Peter: No it wouldn't. He could be 82 or more.
76. Jill: She has to be shorter than 90 cm … or taller …
77. Don: It doesn't matter.
78. Peter: This would work … 185 and 82.
79. PT: Would Brother A win by 17 cm, Peter?
80. Peter: … no.
81. Molly: He has to win by 17 cm, so you could have Sister B be 127 cm shorter than Sister A.
82. Peter: It could be 99. That would be 17 difference.

Pause.

83. PT: Let's think again. The 99 stands for what?
84. Jill: Don: How much difference there is between Sister A and Brother A.
85. Molly: I know. I know. You can subtract 82 minus 186 and then you'd get 104. And 104 is 17 more than 87.

Pause.

86. Don: Huh?
87. PT: Explain it again, Molly. But instead of talking about the arithmetic, talk about what it is that you are thinking.
88. Molly: The first thing we did was the difference between Brother A and Sister A, and it was 99. And minus 17 is 82.
89. Jill: But what was the 82 for?
90. Molly: Huh?
91. Jill: What does the 82 stand for?
92. Molly: I don't know.

The structure of this problem is that it entails a quantitative difference that compares two quantitative differences (see Figure 3). In lines 1-56 the children determined that Brother A stood 99 cm taller than Sister A. They conceptualized the quantitative difference between their heights and subtracted to evaluate it. In lines 57-92 they intuited that Brother A winning the argument by 17 cm meant that the difference they found possessed an excess of 17 cm over something. In line 70 Molly wondered (correctly) whether the 82 (99-17) could be the difference between Brother B and Sister B, but her reply to my question in lines 71 and 73 suggested that this was not a necessary conclusion for her.[8] The remainder of the excerpt suggested to me that though they knew that there was a comparison between Brother B's and Sister B's heights, that knowledge did not bring to mind a quantitative structure in which the result of that comparison existed as a quantity in its own right. Thus, my interpretation was that these students could compare two quantities additively to form a quantitative difference, but their operation of comparison was not such that knowing that a quantity is a difference brings with it the awareness that two other quantities must be compared to produce it. This explains their inclination to try to put "82" somehow into the height of Brother B or Sister B. Not thinking of a difference as a quantitative structure is similar to students being given a speed and not thinking automatically that distance and time must be involved, too.

[8] Earlier parts of the article show clearly that the children felt free to push back when I questioned their conclusions and they were confident in them.

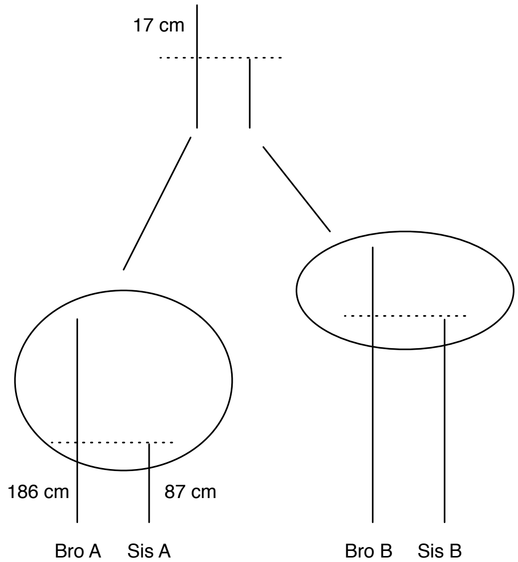

Figure 3. Structure of the problem as a comparison of differences.

So, to be clear, in my usage, quantitative operations are those operations of thought by which one constitutes situations quantitatively. Numerical operations are the operations by which one establishes numerical relationships among their measures. Quantitative and numerical operations are certainly related developmentally, but in any particular moment they are not the same even though in very simple situations children (and teachers[9]) can confound them unproblematically.

Dispositions that allow students to create algebra from quantitative reasoning
Ellis (2007) gives a compelling account of the important role that quantitative reasoning plays in students' operations of generalizing, and of the role that generalizing plays in students' development of algebraic reasoning. She outlines a number of generalizing actions students employ that, when grounded in quantitative reasoning, produce more and stronger generalizations than when students attempt to generalize from numerical or calculational patterns. I have little to offer that extends Ellis' work, but I believe I can say something about other processes at play both within and in addition to the generalizing actions she identified.

In (Thompson, 1990) I identified a number of dispositions and a process of abstraction that can aid students' construction of algebraic reasoning from quantitative reasoning. Briefly, the are:

A disposition to represent calculations
It is common in Asian and Russian primary mathematics curricula to have students write the arithmetic they would perform to resolve a situation without calculating the result. This practice has a salutary effect: Students can see the values of the quantities that are described in the situation and see where these values fit into a solution. Thus, unevaluated expressions are reminders to the student writing them, and hints to students reading them, to the underlying quantitative structure giving rise to the calculation. This provides opportunities for pseudo-empirical abstraction (Piaget, 2001), which is to reason about the products of one's reasoning as if reasoning about objects in the world, which is propaedeutic for students' eventual reflective abstraction of their reasoning.

Students' disposition to write numerical formulas for their calculations produces many opportunities for thinking about what one does with quantities' values in specific situations and with

[9] And researchers.

patterns of known information. Numbers then become almost symbolic. Thus, when a student has exercised repeatedly the disposition to write numerical formulas, they feel little resistance to using a letter to represent a value that they do not know. Thus, a disposition to write numerical formulas for evaluating a quantity leads to a disposition to write formulas that involve representations of to-be-known values of quantities.

A disposition to propagate information

Quantitative reasoning provides the means for imagining a situation quantitatively. But quantitative reasoning that entails only quantitative operations cannot answer questions about a quantity's numerical value. Learning how to evaluate a quantity is part of quantifying it. For example, the process of evaluating a ratio entails proportional correspondence: If 7 dollars corresponds to 3 loaves of bread, then 1/3 of 7 dollars corresponds to 1/3 of 3 loaves (whence division to evaluate a ratio, assuming constancy of price).

But often in complex quantitative settings we have partial information, whence the need to propagate information (as in the example of Brothers A and B). Thus, to resolve the situation students must add information they derive to their understanding of the setting and then attempt to resolve the situation further. Propagation is made possible by two conditions: being aware of quantitative structure and being aware of numerical operations to perform to evaluate a quantity in that structure.

A disposition to propagate information is essential for students to deal with complex quantitative situations. Moreover, students need to be able to evaluate quantities both canonically and non-canonically. Here is a detailed list of canonical and non-canonical operations that is essential for students to propagate inferred information throughout a quantitative structure.[10]

1) *Canonical Arithmetic for Quantitative Relationships: Arithmetic operation to evaluate a quantity that results from a quantitative operation.*

Structure	*Arithmetic Operation to Evaluate the Resultant Quantity*
- A quantity is the result of an additive combination of two quantities[11]	Addition
- A quantity is the result of an additive comparison of two quantities	Subtraction
- A quantity is the result of a multiplicative combination of two quantities	Multiplication
- A quantity is the result of a multiplicative comparison of two quantities	Division
- A quantity is the result of an instantiation of a rate	Multiplication
- A quantity is the result of a composition of ratios	Multiplication

[10] This list is sufficient for middle school mathematics and most of high school. It can easily be extended.

[11] "Additive combination" includes disjunctive combinations, such as "Put 3 apples with 4 oranges." The unit for this particular combination is APPLE|ORANGE—any item in this combination is an apple or an orange. Paying attention to the unit avoids the asymmetry of Schwartz's (1988) scheme of calling the resulting unit a "fruit." That is, if the unit of the combination is fruit, then from "Sally has 7 [fruit], four of which are [orange]" we cannot conclude that she has 3 apples. However, from "Sally has 7 [apple|orange], four of which are [orange]" we can, indeed, conclude that she has 3 apples.

- A quantity is the result of a composition of rates Multiplication

2) *Non-canonical Arithmetic for Subordinate Quantities in a Quantitative Relationship: Arithmetic operation to evaluate a quantity that is an operand of a quantitative operation.*

In the case where the value of a resultant quantity is known but the value of one of the operand quantities is unknown, students must decide upon an inferred numerical operation that is based on knowledge of the canonical operation.

- If $a = b * c$, then $c = a \div b$ and $b = a \div c$.
- If $a = b \div c$, then $c = b \div a$ and $b = a * c$.
- If $a = b + c$, then $c = a - b$ and $b = a - c$.
- If $a = b - c$, then $c = b - a$ and $b = a + c$.

I imagine that students could internalize these relationships in any of several ways depending on the sophistication of their quantitative reasoning. The ideal case would be that they construct these non-canonical operations as generalizations from a combination of quantitative and numerical operations.

3) *Any time a quantity is evaluated or has a formula constructed for it, every structure of which that quantity is part is examined to see if anything new can be inferred about it. If new information can be inferred, then an inference is made and propagation continues.*

If a student has both a disposition to write formulas for quantities (either arithmetical or algebraic) and a disposition to propagate information, then they are positioned to write complex algebraic formulas to represent values of quantities in a situation.

Students who are disposed to write formulas for evaluating a quantity are still not positioned to write equations. They must also be willing to write formulas to evaluate a quantity for which they already know a value or for which they have already written a formula. When they have a formula for evaluating a quantity together with a value that the formula must produce, they have an equation. When they have two formulas for evaluating a single quantity's value, they have an equation.

A disposition to think with abstract units

Mathematicians typically say that a scalar is a "dimensionless" quantity—a ratio between two quantities in the same unit. This, however, is more an artifact of an arithmetic of units than of conceptual analysis. There are many situations where it is not only sensible, but *desirable* to retain units in a rate involving quantities having the same unit. Here is one example.

> Jane is a golfer with a keen eye but a weak swing. She can accurately estimate distances, but consistently hits the ball $\frac{4}{5}$ ths as far as she needs to hit it. If Jane is 150 yards from where she wants the ball to go, what estimate should she use in order to hit the ball far enough?

Jane has a constant "hit rate". The value of that rate is $\frac{4}{5}$. The unit of that rate is "yd/yd."

Together these mean that there are $\frac{4}{5}$ th yards of "hit" per each yard of distance to the hole. To consider $\frac{4}{5}$ as a scalar in this situation would be unnatural. A unit of "yd/yd" follows naturally from conceiving the situation quantitatively (and with the knowledge that golfing distances are measured in yards).

The interpretation of Jane's hit rate as having a unit of *yd/yd* is not necessary, and perhaps seems unnatural to a sophisticated reader. The statement "… hits the ball $\frac{4}{5}$ ths as far as she needs to hit it" does not, itself, imply a unit for Jane's hit rate. Rather, it is the unit *yd* for Wants to Hit and *yd* for Estimated Distance that suggests the unit *yd/yd* for Jane's hit rate.

The crux of the issue of scalars resides in the question: Can one quantitatively interpret the statement "… hits the ball $\frac{4}{5}$ ths as far as she needs to hit it" independently of specific context (i.e., units of other quantities)? Yes, by conceiving of a situation in terms of *parametric* units. The idea of parametric units is the same as that of parametric values. You know that a unit will be associated with a quantity, but it need not be given specifically to know something about the quantity. A unit can be attached to the quantity when one's understanding of the situation becomes specific enough to warrant it.

The notion of parametric units can be made explicit by putting a question mark as the first character in a unit-name. The question mark indicates that the unit is a parameter; the name itself is used as a matching variable so that, once bound by the matching process in checking the consistency of a quantitative structure, the same name implies the same unit wherever it is used within the structure of a quantity's unit.[12] Thus, we could say that the unit of Jane's hit rate is 4/5 *?L1/?L1*, meaning that this works as long as she measures her estimate and her actual hit length in the same unit. The actual unit would be determined to make the quantitative structure coherent (e.g., yards in Jane's case).

A disposition to reason with magnitudes

I follow Wildi (1991) in the way I speak about quantities' magnitudes. The idea he tries to capture is, for example, that someone's height is the same regardless of the unit in which you measure it. A person has the same "amount of height" whether measured in inches, centimeters, meters, miles, kilometers, or light years. He captures this formally with the equation $\|Q\| = m(Q_u)\|u\|$, which he intended as "the magnitude of quantity Q is the measure of Q, in units of u, times as large as the magnitude of u." To put it another way, and to continue this example, a measure of a person's height is the magnitude of that person's height in relation to the magnitude of the unit in which you measure it. Of course the definition is circular, but it is circular only to the extent that we must define the magnitude of a unit – the amount of a quantity that the unit comprises—outside the formula. But this is common: The magnitude of one foot is the length of *this* (as we hold up a titanium bar borrowed from the U.S. Bureau of Standards).

The idea of quantitative magnitude is nontrivial for elementary grades students. It must be raised in context repeatedly, with specific quantities, over their schooling. One way to develop the idea of quantitative magnitude is to take seriously the idea of unit conversions, but not by rote. Rather, the idea of unit conversions should be built upon the idea that if the measure of a quantity is M_u in units of u, then its measure is $12M_u$ in units of magnitude (1/12) $\|u\|$ and its measure is

[12] Whether the same parameter-name needs to imply the same binding throughout a structure is unclear. If a non-parametric unit name were to be thought of as binding to itself, then the answer would have to be that the same parameter-name in different quantity's units would have to represent the same binding. However, it seems entirely reasonable that a person could be thinking of parametric units for different quantities, use the same "mental symbol" in constructing the quantity's units, but intend them to be different. In essence, the issue is whether bindings are local to a quantity-as-conceived or global to the person-as-conceiver. I imagine that either could be the case for any given individual.

$(1/12)M_u$ in units of magnitude $12\|u\|$. If a unit of length named *Fred* is 3 times as long as the unit *Gertrude*, then my height in *Freds* will be 1/3 my height in *Gertrudes*.

Thinking of a quantity in terms of its magnitude involves the same conceptual operations as thinking about inverse variation. Indeed, students can build the conceptual operations of inverse variation reflectively from reasoning about quantitative magnitudes. Also, when a student understands magnitude by a scheme of operations that entails inverse variation, then she can take a unit-less length to represent a quantity's magnitude with the full knowledge that this length actually represents a class of unit-measure pairs. She can also reason about a quantity whose value is labeled, say, "15.3" without units by understanding that the unit's magnitude is 1/15.3 of the quantities' magnitude—even when neither is stated or known. I suspect that students who can resolve complex situations using only magnitudes are also adept at reasoning with parametric units.

Quantitative Reasoning, Covariation, and Generalization[13]

To segue into issues of quantitative reasoning and modeling I shall first examine ideas of covariation and generalization. The importance of covariational reasoning for modeling is that the operations that compose covariational reasoning are the very operations that enable one to see invariant relationships among quantities in dynamic situations. The importance of generalization for mathematical modeling is that generalizing is what one does when representing relationships among quantities in static situations, and imagining those relationships to remain the same under changed circumstances is tightly related to covariational reasoning.

Covariation

Quantitative reasoning is often portrayed as a foundation for arithmetic and algebra, which it is. What is less well understood is the central role that quantitative reasoning can play in students' learning and using concepts that are foundational to calculus and differential equations. There is the obvious role that quantitative reasoning plays in making sense of situations in which calculus is embedded and about quantities in them. The less well understood aspect of quantitative reasoning is the foundation it provides for holding in mind invariant relationships among quantities' values as they vary in dynamic situations. An elementary school student might envision a situation in which one tree is three times as tall as another. What she will find difficult is to imagine this relationship holding over time as the trees grow—imagining that the taller tree is *always* three times as tall as its neighbor as each of them grow. The two instances reveal a stark contrast between two ways of thinking. In the first instance, "three times as tall" is a description of a static relationship. In the second instance, "three times as tall" is an invariant relationship between two quantities whose values vary. The second instance involves a way of thinking that is foundational for the concepts of variable and function in calculus.

The mathematics of variation involves imagining a quantity whose value varies. It is easy to say what variation is, but it is nontrivial for students to develop the conceptual operations that allow them to see variation in quantities and quantitative situations. The concept of variation of which I speak is, for me, best captured like this: Suppose x represents a quantity's value. That the quantity's value varies is to say that one *anticipates* its measure having different values at different moments in time. So a varying quantity's value might be represented as $x = x(t)$, where t represents (conceptual) time. I prefer, though, to characterize variation mathematically in a way that captures the idea that

---

13 The discussion in this section builds considerably from earlier papers on rate of change (Thompson & Thompson, 1996; Thompson, 1994a, 1994d; Thompson & Thompson, 1994) and on concepts of function (Saldanha & Thompson, 1998; Thompson, 1994a, 1994c). Carlos Castillo-Garsow has helped me understand why these ideas were not evident in those papers from a reader's perspective.

variation always happens over an interval. So, let D = domain(t), the values of conceptual time over which t ranges. Let D_ε be the domain of t, but in intervals of size ε, which means that $D = \bigcup_t (t, t+\varepsilon)$,[14] where t can take any value of conceptual time. In other words, D is not partitioned into intervals of size ε. Instead, one *anticipates* that it will be *covered* with intervals of size ε. I emphasized "anticipate" to convey that I do not envision people actually imagining a cover of an interval. The purpose of speaking of a covering instead of a partition is that one can imagine a quantity's value varying within an interval in chunks, but with the immediate and persistent realization that any chunk of completed variation can be re-conceived as entailing continuous variation.

With the above characterization in place, variation in one variable, x, can be expressed as $x_\varepsilon = x(t_\varepsilon)$, where t_ε represents the interval $[t, t+\varepsilon)$ and t varies through conceptual time.[15] I recognize immediately the difficulties with this formulation, but what I hope to capture is the idea that in thinking of a varying magnitude, even at momentary times, we are imagining a small interval of conceptual time through which the variation happens.[16]

This formulation of variation tries to capture the conceptual operations behind an image of a growing magnitude, like a line segment getting longer. A particular, static, magnitude has a value of $x(t)$. But in thinking of the magnitude varying, we imagine it varying in microscopic bits, each bit itself entailing variation upon close inspection. The importance that students build this image early is implicated by studies showing that calculus students do *not* see variables varying (Carlson, 1997, 1998; Jacobs, 2002; Trigueros & Jacobs, 2008).

I can now represent a conception of two quantities' values covarying as $(x_\varepsilon, y_\varepsilon) = (x(t_\varepsilon), y(t_\varepsilon))$. I intend the pair $(x_\varepsilon, y_\varepsilon)$ to represent conceiving of a multiplicative object—an object that is produced by uniting in mind two or more quantities simultaneously. Examples are

- A rectangle's length, width, and area;
- A point's location in two directions simultaneously from a reference point;
- The value of one quantity and the value of another quantity that is related to it in some natural or deterministic way;
- The values of two quantities as they each vary in time, such as an object's location and its velocity at that location.

This formulation of covariation resembles, in many ways, traditional definitions of a function defined parametrically and of definitions of a relation between two sets. But classical definitions of parametric functions, set relations, etc. do not make an explicit representation of how people imagine variables varying or how they unite two functions to make an invariant relationship between them. Marilyn Carlson and Mike Oehrtman have written extensively about the importance of covariational reasoning as a foundation for students' understandings of function (Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson & Oehrtman, 2005; Carlson, Persson, & Smith,

---

14 I am not concerned at this moment with whether intervals should be open, closed, or half open/half closed. I *am* concerned with capturing the dynamic between covering and varying.

15 That is, think of a value that varies by infinitesimal amounts, and varies within those amounts as well. In other words, variation, as a way of thinking, is recursive. We define variation within a ε-sized interval in the same way we define variation within intervals it contains and within any interval that contains it. I am indebted to Carlos Castillo-Garsow for discussions that led to this characterization of variation.

16 The phrase "through which the variation happens" is of paramount importance. Even when thinking of variation happening in bits, we want students to imagine that variation happens *within* bits as well.

2003; Oehrtman, 2002; Oehrtman, Carlson, & Thompson, 2008). What I've offered here, I believe, is a new way of thinking about covariation and its relationship to quantitative reasoning.[17]

In summary, there are two considerations in examining students' construction of quantitative covariation. The first is conceiving the quantities themselves and images of them that entail their values varying. The second is to conceptualize the multiplicative object made by uniting those quantities in thought and maintaining that unit while also maintaining a dynamic image of the situation in which it is embedded. This act, of uniting two quantities conceptually within an image of a situation that changes while staying the same, is nontrivial. Yet it is at the heart of using mathematics to model dynamic situations.

Two recent studies examined aspects of students' abilities to model dynamic situations through a focus on ways of understanding covariation. Carlos Castillo-Garsow (2010) focused students' understandings covariation and rate of change and their implication for understanding exponential growth. Kevin Moore (2010) examined students' construction of trigonometric functions through a focus on ways of understanding angle measure and the covariation of angle measure and trigonometric ratio. These two studies are instructive for ways in which a focus on quantitative reasoning opens new perspectives on students' understanding of functions and their abilities to engage in mathematical modeling.

Castillo-Garsow (2010), in a teaching experiment involving two high school junior Algebra II students, used the context of interest-bearing bank accounts as the general theme by which he broached issues of covariation, rate of change, and exponential growth. He scaffolded tasks that portrayed the value of an account growing according to different bank policies regarding how they reported the value of an account. Some banks reported account values at the moment of inquiry as the amount at the beginning of the interest period, and increased the reported value only at the end of the interest period. Other banks reported accounts' values as having grown linearly from the moment of initial deposit to the moment of inquiry. The two policies would produce the same amount of earned interest at the end of an interest period, but resulted in different reported values during an interest period. The first policy (increase only at end of interest period) produces a step function for the value of an account in relation to the number of years it is on deposit. The second policy produces a linear function of account value in relation to the number of years it is on deposit.

Castillo-Garsow then crossed the two reporting policies with a policy of compounding earned interest —some banks reported accounts' values as changing only at the end of a compounding period and other banks reported accounts' values as having grown linearly within a compounding period. The first policy produces a step function that grows geometrically, and its rate of change of account value with respect to the number of years on deposit is zero almost everywhere. Where it is not zero, the rate of change does not exist. The second policy produces a

[17] A complete analysis of variation and covariation would certainly involve looking at the development of concepts of simultaneity and time. The idea of simultaneity has its roots in what Inhelder and Piaget originally called "multiplicative operations" (e.g., Inhelder & Piaget, 1969, pp. 42-43) and in students' construction of time as a quantity (Feagans, 1980; Glasersfeld, 1996; Kamii & Russell, 2010; Lovell & Slater, 1960; Natsopoulos & Abadzi, 1986; Piaget, 1970). By "multiplicative operations" Inhelder and Piaget meant the construction of logical "and"—the operation of having in mind two attributes of an object simultaneously. An object could be a triangle, and it could be blue. Inhelder and Piaget claimed that it is while constructing concrete mental operations that a child comes to see an object as being simultaneously a triangle and blue. They characterized multiplicative operations as being the source of multiple classification (sort objects on two characteristics simultaneously) and multiple seriation (order objects on two measures simultaneously). The idea of conceptual time is that of a measured duration, where operations of measure vary with the sophistication and development of the person imagining it. A full discussion of both simultaneity and time is clearly not possible in this paper.

continuous function whose rate of change of account value with respect to number of years on deposit is proportional to the account's value at the beginning of the compounding period. This rate of change exists at all moments in time except at the end of a compounding period.

In an extension of the continuous model, Castillo-Garsow also introduced a banking policy that he called the "per capita" policy—each dollar earns interest at a simple rate. The model is extended by how the bank treats the interest accrued by each dollar—either by putting it in a separate, non-interest account or by adding it to the account that earned it.

Though Castillo-Garsow's results are too intricate to convey in this brief account, there are two aspects to his experiment that are important for the present discussion. The first point is the potentially powerful distinctions that a focus on quantitative reasoning can introduce into studies of learning and teaching. It is by maintaining a tight focus on quantitative reasoning that Castillo-Garsow developed the intricate interaction among ways of thinking about covariation, rate of change, and exponential growth that he embodied in tasks. The second point is that his analysis of covariation and rate of change reinforce the first point made in this paper—that it is useful to see quantities as being in minds, not as being in the world. In particular, Castillo-Garsow found that how one imagines that quantities' values vary has dramatic implications for how one conceives situations that embody variation. The two students in Castillo-Garsow's teaching experiment, whom he called Tiffany and Derek, saw variables varying in two very different ways. Tiffany saw variables varying in completed chunks. Derek had a mixed view of variation—variables could vary in chunks, but they also varied continuously within completed chunks. Tiffany had developed an understanding of rate of change that amounted to seeing how much something changed in total, and then distributed that change to any sub-interval of time according to the fraction that the sub-interval's length was of the overall interval. For example, if an account earned interest at the rate of 8% per year, then in 3 months the account would earn 1/4 of what it would have earned in one year. But the one-year-change was the only change that occurred in Tiffany's thinking. In Tiffany's thinking, the account's earned interest did not pass through the 1/4-year amount on its way to the one-year amount. Rather, the 1/4-year amount was an amount to be computed ad hoc. Derek, who thought of time passing continuously and thought of an account having a value at each moment in time, thought more in terms of the account changing at the rate of a number of dollars per year for some number of years, and the number of years needn't be a whole number. Earned interest accrued continuously. Tiffany could not see that the accounts rate of change with respect to time was proportional to the account's value. Derek saw this relationship intuitively.

Moore's (2010) teaching experiment is the only study of which I am aware that looked closely at students' understanding of angle measure quantitatively. By this I mean that if a degree or a radian is a measure of something, it must be some attribute of an angle. Moreover, the quantification process must result in the attribute and the unit in which it is measured being related naturally, and it must mean that a degree and a radian are essentially the same thing, except one is a scaled version of the other. Moore began with a suggestion I had made in (Thompson, Carlson, & Silverman, 2007) and in (Thompson, 2008), which was that to see angle measure as a result of a quantification process, then the attribute of an angle had to be something like "openness" and its measure must be a subtended arc length measured in a unit of arc that is proportional to a circle's circumference.[18] Moore built an instructional unit upon the *problematique* of what one measures to "measure an angle", and that problematized the question of what such a measure means. His targeted meaning of angle measure was that one measures an angle's *openness* by measuring the arc

[18] After publishing these articles, I learned from David Bressoud that what I proposed was, in fact, the conceptualization of angle measure that had predominated until the early 1900's, when triangle trigonometry gained ascendance (Bressoud, 2010).

that it subtends in a circle centered at its vertex. The matter of *unit* is important, because a unit of absolute length (e.g., cm) does not work. Circles of different radii will give different measures of openness for the same angle, and we do not want that. We want the size of the circle to be immaterial. Moore's solution was to take arc length measured in any unit that is proportional to a circle's circumference as the unit of arc length. Arc lengths that are 1/360th a circle's circumference are traditionally called *one degree*. Europeans sometimes use an arc length that is 1/400th a circle's circumference (called a *grad*). Arc lengths that are the length of a circle's radius also work.[19] In effect, Moore's proposal for conceptualizing angle measure as measured arc, in effect, turns every circle into a unit circle.

The idea of variation in an angle's measure turns out to be nontrivial for many students and teachers. I asked a group of teachers with whom I worked, and who thought of trigonometry in terms of SOH-CAH-TOA[20], to show an angle of 30 degrees. I then asked them to show how to think about that angle increasing in measure. Figure 4 shows the diagrams they drew. The triangle on the left is one with a 30° angle. The figure on the right is their depiction of that angle varying. Notice that to vary the angle's measure they varied the length of the side opposite the angle. This group of teachers also could not say what was being measured when an angle in their triangle has a measure of 30°. Clearly, this conception of an angle's variation will cause their students immense problems when attempting to envision the behavior of, say, sin(x) as the value of x varies. Indeed, they will have a hard time building a coherent meaning of what x represents.

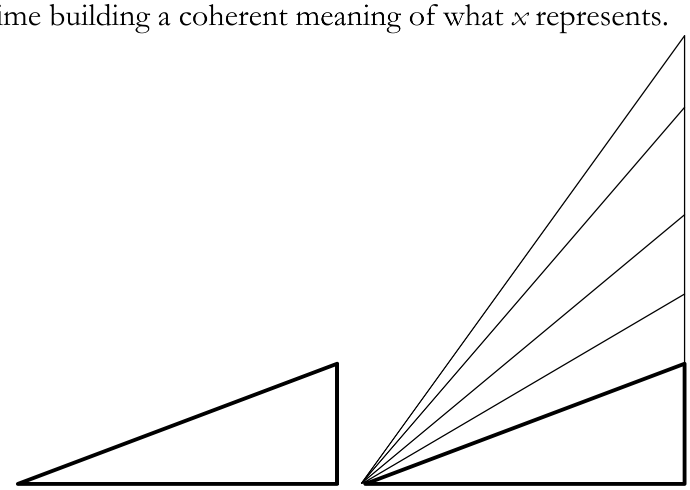

Figure 4. Teachers' version of an angle of 30° and how to envision increasing it.

Moore (2010) conducted his teaching experiment with three precalculus students who had no discernable concept of angle measure or covariation of quantities. His intense focus on angle measure had the effect that two students developed strong understandings of sine and cosine as functions of an angle measure. Moore argued that it was their explicit attention to quantities and covariation of quantities that enabled them to build a coherent body of meanings by which they envisioned sine and cosine as a relationship between angle measure and vertical and horizontal displacement from a circle's center. His focus on quantitative reasoning also revealed that the one student who did not thrive failed to do so because of a persistent inattention to quantities in settings.

19 It is incorrect to call this unit a *radian*. A radian is an equivalence class of arc lengths each subtended by an angle that subtends an arc of length one radius on any one of them.

20 SOH-CAH-TOA: *S*ine is *O*pposite over *H*ypotenuse, *C*osine is *A*djacent over *H*ypotenuse, *T*angent is *O*pposite over *A*djacent.

I am puzzled that mathematicians seem to have so little interest in the idea of covariation. I suspect that one reason is their abiding interest in structure, and covariation, on its face, can seem unstructural. I suspect also that their personal development of covariational reasoning happened without their awareness and simply became a legacy way of thinking. Finally, mathematical reforms of from 1800 to 1930 banished ideas of motion, time, and infinitesimals from the concept of function (Kleiner, 1989; Kleiner, 2001). The reason to banish motion and time was to accommodate the creation of functions that pushed the boundaries of then current definitions, such as functions that are continuous everywhere but differentiable nowhere and functions that are continuous on the irrational numbers but discontinuous on the rational numbers. These functions are defined using limits, and such limit function cannot be imagined in terms of covarying quantities.

Generalization

I already mentioned the work of Amy Ellis. She is investigating the role that quantitative reasoning plays in students' abilities to make mathematical generalizations from their work in applied settings (e.g., Ellis, 2007). Her work has the potential of explaining the historically mixed results that others have found when investigating the impact of students' use of concrete manipulatives on their mathematical learning (Caglayan & Olive, 2010; Cobb, 1995; Izsak, Tillema, & Tunc-Pekkan, 2008; Kratzer & Willoughby, 1973; Meira, 1991; Sowell, 1989; Thompson, 1992; Wearne & Hiebert, 1988). In essence, it could be that the effectiveness of using concrete materials depends upon the extent to which students are reasoning quantitatively while using them, the nature of their quantification of the materials (i.e., the kinds of quantities they construct), and the extent to which they form generalizations from their quantitative reasoning. More broadly, Ellis' work on generalization in algebra has implications for thinking about mathematical modeling. A mathematical model is a generalization of one's understanding of a situation's inner mechanics—of "how it works".

The idea of transfer of learning is integral to the idea of generalization. Joanne Lobato has examined the age-old issue of transfer-of-learning, but from a actor-centered perspective (Lobato, 2006; Lobato & Siebert, 2002) instead of from a problem-centered perspective. Lobato's work is tightly connected to the idea of quantitative reasoning as described here, most specifically with regard to students' understanding of ratio-as-measure (Lobato & Thanheiser, 2002). Her approach to transfer is more than simply an application of a theory of quantitative reasoning. Rather, she takes the stance that it is when a knower's ways of thinking remain invariant across settings that they have "transferred" their learning from one setting to another. The traditional stance on transfer is that one transfers successfully by applying a solution method used to solve one problem to a problem that is isomorphic to it (isomorphic modulo solution method). Lobato found that way of thinking problematic with regard to transfer as a learning mechanism and as a construct to use in fine-grained analyses of students' thinking. Instead, she defines transfer as taking place when the student sees two situations as being essentially the same. To see the difference between Lobato's and the traditional way of thinking about transfer I'll refer to a phenomenon appearing in two different studies of quantitative reasoning (Thompson, 1994d; Thompson & Thompson, 1994). In both studies children were presented with situations that asked them to determine an amount of time to cover a given distance at a given speed. One could say that students divided distance by speed to get time. But we discovered that this characterization is inaccurate. What the children did was to imagine measuring the given distance in units of a speed-length. The number of speed-lengths contained in the given distance told them the number of seconds that an object traveled (one second for each speed-length). Thus, they divided to find the number of speed-lengths in the given distance. When next given the task of finding a speed that would have a moving object cover a given number of feet in a given number of seconds, children resorted to guessing a speed and testing to see how well it worked.

From a traditional perspective, this story has nothing to do with transfer. The problems were not isomorphic and the students did not use the same solution method. But from a quantitative perspective this story has everything to do with the idea of transfer. In both cases children assimilated the situation of relating distance, time, and speed to a scheme of operations that entailed measuring a distance with another distance to get a time. In the latter case, their goal was to find a speed-length that would make an object travel a given distance in a given time. In the service of that goal, the children imagined measuring the given distance in units of a guessed speed-length, the guess being guided by an estimate of how many speed lengths it would take to create the given number of seconds. In a sense, they were searching for a ruler of the right length by which to measure the to-be-traveled distance. The larger point, however, is that the children saw the two situations as essentially identical. They transferred their way of thinking about distance, speed, and time from the first setting to the second by using that same way of thinking to understand them both. Lobato's reconceived view of transfer has tremendous implications for the design of instruction and curricula that aim at both respecting and fostering students' abilities to see mathematical similarity across settings.

Quantitative Reasoning and Modeling

Mathematical modeling is simply mathematics in the context of quantitative reasoning. It is the use of mathematical notation and methods to express and to reason about relationships among quantities. Indeed, quantitative reasoning together with propagation of formulas covers most of what can be called modeling in K-12 mathematics. But both school and university texts in the United States give little attention to quantitative reasoning as I've described it here. Let me give a few examples of ways the two can be used reflexively.

- In algebra, simple equations are taught as forms to operate upon. $x/5 = 17$, for example is offered as an opportunity to multiply both sides by 5. Approached quantitatively, it would be an opportunity to reason about relative size: "17 is one-fifth as large as *x*. This means that *x*, being 5 one-fifths of *x*, is five 17's, or $x = 5*17$."
- In my earlier discussion of quantitative reasoning and algebra, complex expressions become models of the quantitative relationships that give rise to expressions and equations through the propagation of information within a quantitative structure. A complete expression or equation, to a student mindful of how she got it, then represents the quantitative relationships she created in comprehending the situation.

It is often unrecognized that quantitative reasoning plays an important role in using calculus in applied settings. For example, Moore, Weber, and Carlson (in preparation) describe students solving "the box problem" (Figure 5), a standard calculus optimization problem. One result was that many students were stymied from the outset because they did not understand how one could make a box by folding the sheet of paper. More commonly, students did not know how to write an expression that represented the length and width of the box's base. Some students could write the expression $V = x(11 - 2x)(8.5 - 2x)$ as the box's volume given a cutout of size *x*, but they could not say what in the box varies and what stays the same if you vary the value of *x*. Finally, and I found this most surprising, some students who set up the problem appropriately, and determined the maximum volume, nevertheless thought that cutting a square from each corner was simply a condition of the problem, that you needn't cut squares in order to make a box. They did not realize that cutting anything other than a square, or cutting different sized squares, would produce a "box" whose sides would not fit together. Moore et al. argue that all these difficulties are clearly grounded in problems of quantitative reasoning.

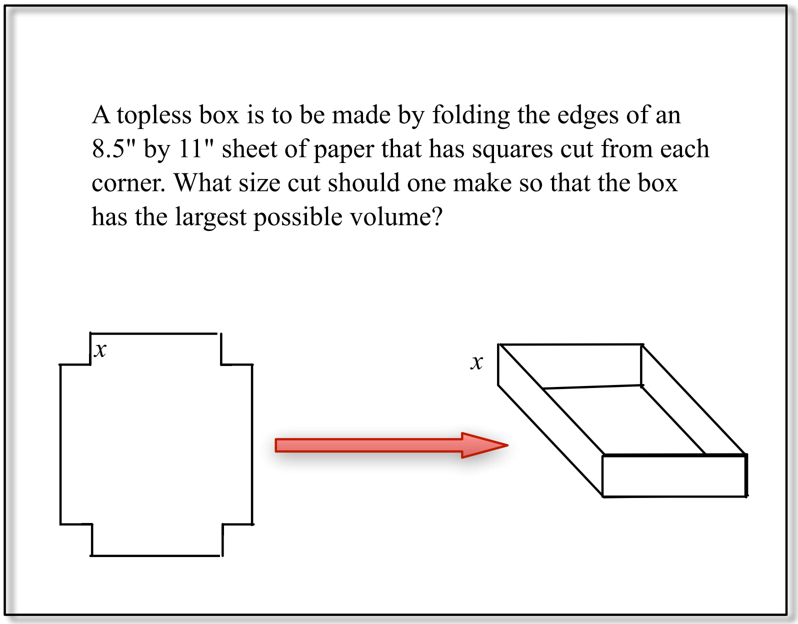

Figure 5. The Box Problem

Other studies in our group are finding interesting connections between students' difficulties in calculus and their impoverished quantitative reasoning. Carla Stroud (in preparation) found that students' understanding of instantaneous rate of change has no connection whatsoever to constant or average rate of change. In their thinking, velocity is a property of an object, just as its weight is a property. This revealed itself in their responses to this question.

> When the Space Shuttle is launched it speeds up continuously until its main thruster stops burning. During this time it is never going at a constant speed. What, then, would it mean to say that the Space Shuttle was going precisely 182.3784 miles/hour at precisely 2.1827 seconds after liftoff?

All 11 of her subjects, who had completed Calculus I or Calculus II, said something to effect that it would mean that at 2.1827 seconds its speedometer read 182.3784 miles/hour. In their thinking, instantaneous speed is simply the number at which the object's speedometer's needle points were you to freeze time. No one ever wondered about how such a speedometer might work or how you could ascertain its correctness. Nor did they relate it to average speed over an interval of time and a limiting process. Their study of calculus produced the idea that calculus has formulas that magically produce these numbers that speedometers point at.

My group's current project is to develop a calculus that is firmly grounded in two principles: That calculus is about quantities and their covariation, and in Guershon Harel's necessity principle (Harel, 1998, 2008). The necessity principle, briefly, is that an idea should arise as a result of students feeling an intellectual need—that the idea provides a resolution of some problem or dilemma. While I cannot describe this effort here, I must say that a focus on quantitative reasoning fits perfectly with Harel's dictum of mathematics arising out of intellectual necessity.

References

Bressoud, D. M. (2010). Historical reflections on teaching trigonometry. *Mathematics Teacher, 104*(2), 107-112 (with 113 pages of supplement).

Caglayan, G., & Olive, J. (2010). Eighth grade students' representations of linear equations based on a cups and tiles model. *Educational Studies in Mathematics, 74*(2), 143-162.

Carlson, M. P. (1997). Obstacles for college algebra students in understanding functions: What do high-performing students really know? *AMATYC Review, 19*(1), 48-59.

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In *Research in Collegiate Mathematics Education, 3*, CBMS Issues in Mathematics Education (Vol. 7, pp. 114-162). Washington DC: Mathematical Association of America.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Carlson, M. P., & Oehrtman, M. C. (2005). Key aspects of knowing and learning the concept of function. *Research Sampler*. Washington, DC: Mathematical Association of America. Retrieved February 5, 2006, from http://www.maa.org/t_and_l/sampler/rs_9.html

Carlson, M. P., Persson, J., & Smith, N. (2003, July). Developing and connecting calculus students' notions of rate-of-change and accumulation: The fundamental theorem of calculus. In N. Patemen (Ed.), *Proceedings of the 2003 Meeting of the International Group for the Psychology of Mathematics Education - North America*, (Vol 2, pp. 165-172). Honolulu, HI: University of Hawaii.

Castillo-Garsow, C. C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth.* Unpublished Ph.D. Dissertation, Arizona State University, Tempe, AZ.

Cobb, P. (1995, July). Cultural tools and mathematics learning: A case study. *Journal for Research in Mathematics Education, 26*(4), 362-385.

Ellis, A. B. (2007). The influence of reasoning with emergent quantitites on students' generalizations. *Cognition & Instruction, 25*(4), 439-478.

Feagans, L. (1980). Children's understanding of some temporal terms denoting order, duration, and simultaneity. *Journal of Psycholinguistic Research, 9*(1), 41-57.

Glasersfeld, E. v. (1996, September). *The conceptual construction of time.* Paper presented at the Conference on Mind and Time. Neuchatel, Switzerland. Available at http://www.oikos.org/Vonglasoct1.htm.

Harel, G. (1998, June-July). Two dual assertions: The first on learning and the second on teaching (or vice versa). *Mathematical Monthly, 105*(6), 497-507.

Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, Part I: focus on proving. *ZDM Mathematics Education, 40*, 487-500.

Inhelder, B., & Piaget, J. (1969). *The early growth of logic in the child: Classification and seriation*. New York: W. W. Norton.

Izsak, A., Tillema, E. S., & Tunc-Pekkan, Z. (2008). Teaching and learning fraction addition on number lines. *Journal for Research in Mathematics Education, 39*(1), 33-62.

Jacobs, S. (2002). *Advanced placement BC calculus students' ways of thinking about variable.* Unpublished doctoral dissertation, Arizona State University, Tempe, AZ.

Johnson, G. (2008). *The ten most beautiful experiments*. New York: Knopf.

Kamii, C., & Russell, K. A. (2010). The older of two trees: Young children's development of operational time. *Journal for Research in Mathematics Education, 41*(1), 6-13.

Kleiner, I. (1989). Evolution of the function concept: A brief survey. *College Mathematics Journal, 20*(4), 282–300.

Kleiner, I. (2001). History of the infinitely small and the infinitely large in calculus. *Educational Studies in Mathematics, 48*(2-3), 137-174.

Kratzer, R. O., & Willoughby, S. S. (1973). A comparison of initially teaching division employing the distributive and Greenwood algorithms with the aid of a manipulative material. *Journal for Research in Mathematics Education, 4*, 197–204.
Lobato, J. (2006). Alternative perspectives on the transfer of learning: History, issues, and challenges for future research. *Journal of the Learning Sciences, 15*(4), 431-449.
Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *Journal of Mathematical Behavior, 21*(1), 87-116.
Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio as measure as a foundation for slope. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions: 2002 Yearbook of the NCTM* (pp. 162-175). Washington, DC: National Council of Teachers of Mathematics.
Lovell, K., & Slater, A. (1960). The growth of the concept of time: A comparative study. *Journal of Child Psychology and Psychiatry, 1*(3), 179-190. Available at http://dx.doi.org/10.1111/j.1469-7610.1960.tb01992.x.
Meira, L. L. (1991, April). *On the transparency of material displays*. Paper presented at the Annual Meeting of the American Educational Research Association. Chicago, IL.
Moore, K. C. (2010). *The role of quantitative reasoning in precalculus students learning central concepts of trigonometry.* Unpublished Ph. D. dissertation, Arizona State University, Tempe, AZ.
Natsopoulos, D., & Abadzi, H. (1986). Understanding linguistic time sequence and simutaneity: A literature review and some new data. *Journal of Psycholinguistic Research, 15*(3), 247-273.
Oehrtman, M. C. (2002). *Collapsing dimensions, physical limitation, and other student metaphors for limit concepts: An instrumentalist investigation into calculus students' spontaneous reasoning.* Unpublished Dissertation, University of Texas at Austin, Austin, TX.
Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, DC: Mathematical Association of America. Available at http://pat-thompson.net/PDFversions/2008MAAFunctions.pdf.
Paulos, J. A. (1988). *Innumeracy: Mathematical illiteracy and its consequences*. New York: Hill and Wang.
Piaget, J. (1970). *The child's conception of time*. New York,: Basic Books.
Piaget, J. (2001). *Studies in reflecting abstraction* (R. L. Campbell, Trans.). New York: Psychology Press.
Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America*, (Vol 1, pp. 298-304). Raleigh, NC: North Carolina State University. Available at http://pat-thompson.net/PDFversions/1998SimulConVar.pdf.
Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*, 499-511. Available at http://pat-thompson.net/PDFversions/2008SilvermanThompsonMKT.pdf.
Simon, M. A., & Blume, G. W. (1994). Building and understanding multiplicative relationships: A study of preservice teachers. *Journal for Research in Mathematics Education, 25*(5), 472–494.
Sowell, E. J. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics Education, 20*, 498–505.
Steffe, L. P. (1991a, April). *Composite units and their schemes*. Paper presented at the Annual Meeting of the American Educational Research Association. Chicago, IL.
Steffe, L. P. (1991b). Operations that generate quantity. *Journal of Learning and Individual Differences, 3*(1), 61–82.
Thompson, A. G., & Thompson, P. W. (1996, January). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*(1), 2-24.

Available at http://pat-thompson.net/PDFversions/1996Talking%20about%20Rates%202.pdf.

Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebraic*. Unpublished manuscript, Center for Research in Mathematics & Science Education. Available at http://pat-thompson.net/PDFversions/ATheoryofQuant.pdf.

Thompson, P. W. (1992). Notations, conventions, and constraints: Contributions to effective uses of concrete materials in elementary mathematics. *Journal for Research in Mathematics Education, 23*(2), 123–147. Available at http://pat-thompson.net/PDFversions/1992Notation&Constraints.pdf.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165–208. Available at http://pat-thompson.net/PDFversions/1993Complexity&AddStruct.pdf.

Thompson, P. W. (1994a). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*(2-3), 229–274. Available at http://pat-thompson.net/PDFversions/1994Rate&FTC.pdf.

Thompson, P. W. (1994b, April). *Bridges between mathematics and science education*. Paper presented at the Research blueprint for science education conference. New Orleans, LA.

Thompson, P. W. (1994c). Students, functions, and the undergraduate mathematics curriculum. In E. Dubinsky, A. H. Schoenfeld & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1*, Issues in Mathematics Education (Vol. 4, pp. 21–44). Providence, RI: American Mathematical Society.

Thompson, P. W. (1994d). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 179–234). Albany, NY: SUNY Press.

Thompson, P. W. (2000). What is required to understand fractal dimension? *The Mathematics Educator, 10*(2), 33-35.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education*, (Vol 1, pp. 45-64). Morélia, Mexico: PME. Available at http://pat-thompson.net/PDFversions/2008ConceptualAnalysis.pdf.

Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*, 415-432. Available at http://pat-thompson.net/PDFversions/2007JMTETasks.pdf.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds.), *Research companion to the Principles and Standards for School Mathematics* (pp. 95-114). Reston, VA: National Council of Teachers of Mathematics.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Washington, DC: Mathematical Association of America. Available at http://pat-thompson.net/PDFversions/2008Accum.pdf.

Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education, 25*(3), 279–303. Available at http://pat-thompson.net/PDFversions/1994Talking%20about%20Rates%201.pdf.

Trigueros, M., & Jacobs, S. (2008). On developing a rich conception of variable. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education*, MAA Notes (pp. 3-13). Washington, DC: Mathematical Association of America.

Wearne, D., & Hiebert, J. (1988). A cognitive approach to meaningful mathematics instruction: Testing a local theory using decimal numbers. *Journal for Research in Mathematics Education, 19*, 371–384.

Wildi, T. (1991). *Units and conversions: A handbook for engineers and scientists*. New York: IEEE Press.