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Calculus Students' Understandings of Division and Rate

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Abstract: We have conducted a preliminary investigation of university Calculus students' conceptions of division and rate of change because these ideas are used to define the derivative. We conducted exploratory interviews focused on building models of student understandings of division and rate. Retrospective analysis revealed the students interviewed had a variety of meanings for these concepts. Difficulty thinking about division as multiplicative comparisons of relative size was observed in multiple students. Additionally a student who explained rate as an amount added in equivalent x-intervals struggled to determine if a quantity was changing at a constant rate over unequally spaced x intervals. We hypothesize that difficulty conceptualizing division as quotient, and quotient as a measure of relative size[1] of two quantities, obstructs students' understandings of average and instantaneous rate of change. This research will further our goal of understanding student difficulties with derivatives.

Key words: *calculus, derivative, rate of change, division, student thinking, multiplicative thinking*

Introduction and Background

As Thompson and Saldanha urged, we take seriously the idea that "how students understand a concept has important implications for what they can do and learn subsequently" (Thompson & Saldanha, 2003, p. 1). Understanding is "what results from a person's interpreting signs, symbols, interchanges or conversations-assigning meanings according to a web of connections the person builds over time through interactions with his or her own interpretations of settings and through interactions with other people as they attempt to do the same" (Thompson & Saldanha, 2003, p. 12). We believe students build particular meanings for mathematical ideas by building on preexisting understandings (Steffe & Thompson, 2000a). Based on a conceptual analysis (Thompson, 2008) of the concepts of constant and average rate of change, we believe that conceptualizing division and rates as a multiplicative comparison of relative size is essential to understanding the derivative as a rate of change function. We interviewed university Calculus students to create models of their meanings for division and rate so that we can address the question "How do Calculus students understand division and rate?"

Our inquiry into Calculus students' meanings for division and rates of change emerged from observations of our own Calculus students and research on rates of change, division and derivatives. Asiala et al. (1997) summarizes a variety of studies that show that most Calculus students do not have a strong conceptual understanding of the derivative and struggle to solve non-routine problems. In Orton's (1983) study of student understanding of the derivative, he found that the rule where one divides the difference in y by the difference in x to obtain a rate was not elementary for a large number of students. Orton (1983) alluded to the possibility that

[1] It is more appropriate to say "relative magnitude" instead of "relative size" to account for comparisons of quantities of different physical dimensions (e.g., distance, time) but space is insufficient to explain this fully.

"one of the problems of learning about rate of change is that the ideas are basically concerned with ratio and proportion" (p. 243).

Carlson et al.'s (2002) study of 20 high-performing Calculus students revealed that most students struggled on tasks involving average and instantaneous rate of change. Although most students "were frequently able to coordinate images of the amount of change of the output variable while considering changes in the input variable", students were typically unable to coordinate changes in a function's average rate of change with uniform changes in the input (Carlson et al., 2002, p. 372). Most students did not understand situations where rates must be considered as multiplicative comparisons of changes in two variables. They were successful in describing rates of change as additive changes in the output.

Castillo-Garsow (2010) provided a model of one high performing secondary student's meaning for rate that could explain why students find understanding rates of change in Calculus challenging. For this student, an interest rate told her how much money to *add* to a bank account each year. Thinking of a rate as an amount added results in correct interpretations of situations as long as one always considers uniform changes in the independent variable. The student reworked problems with fractional amounts of one year into whole numbers of months so that the denominator of her division problem (change in money)/(change in time) was one unit. This allowed her to ignore division and consider additive changes in account balances. Simon and Blume (1994) cite studies indicating that many other students think additively when multiplicative thinking is more appropriate.

Coe (2007) conducted an in-depth study of three secondary math teachers' understandings of rates of change and revealed experienced teachers were not always able to articulate coherent connections between ideas of division, rate, and slope. For one teacher, Peggy, "the slope of a tangent gives a steepness that connects to speed in some contexts" (E. E. Coe, 2007, p. 176). Coe (2007) reported that in more than one instance Peggy "did not use her thinking of a ratio as a comparison of values" to understand slope (p. 195). Considering slope as an index of slantiness allowed this teacher to correctly answer many questions without thinking about division. Coe (2007) concluded that none of the teachers "could clearly explain the use of division to calculate slope" and "there was no evidence of quantitative understanding of the ratio" (p. 237).

The transcripts of students in Castillo-Garsow's (2010) and Carlson's et al.'s (2002) studies suggests that the students thought about rates of change additively. In problems that would prompt multiplicative thinking, the students invoked "workaround" strategies including only considering rates of change on increments of equal size (usually 1), and thought of speed and slope as indices instead of as ratios. Since understanding division as relative size is an essential mathematical component in many problems identified as obstacles for students, we investigated our students' meaning for division and rate to see if they had meanings for these topics that would allow them to understand derivatives.

Methodology

To build models of students' meanings for division we used Simon's (1993) descriptions of partitive and quotative meanings for division. These two meanings for division do not require multiplicative reasoning. A third model for division, relative size, requires students to reason multiplicatively; the relative size model for division calls upon a comparison between the size of one quantity with respect to another quantity (Thompson & Saldanha, 2003). Division as relative size allows students to be able to reason about non-integer divisors. If division is viewed partitively, it only makes sense to divide a number into n equal parts if n is an whole number.

In order to investigate the understandings/meanings that calculus students might have for division and rate of change, we conducted exploratory interviews with seven undergraduate calculus 1 students, guided by the theoretical perspectives of Steffe and Thompson (2000b). Our interview protocol contained tasks and questions that had been used in class or in other research on understandings of division (See Ball, 1990; Simon, 1993). For example, "Describe a situation where you would need to divide 6 by 3/4ths." or "How can you tell if your puppy is growing at a constant rate?" We conducted retrospective analysis to create models for students' understandings of division and rate. In our exploratory interviews, we attended to the idea that phrases students used such as "constant rate" do not necessarily mean the same thing to them as to us.

Preliminary Results

Preliminary results from our research confirm that individual students held various (and sometimes unproductive) meanings for division. Additionally, students with partitive meanings for division struggled to interpret answers to division problems involving decimals and struggled to provide a context where division by a fraction is needed to solve a problem.

Jack had strong quotative meanings for division but struggled to interpret the quotient as a measure of relative size. When asked to determine if a puppy was growing at a constant rate he explained that if it is measured on equally spaced intervals of time you can compare the changes in height using subtraction. He proposed if the changes in height are equal the puppy is growing at a constant rate. When asked what he would do if he had measurements corresponding to unequally spaced intervals of time, Jack could not use a multiplicative comparison to show the puppy was growing at a constant rate. Eventually he guessed that division might be an appropriate operation, but was unable to identify the expression "four units of height divided by two days" as a rate of growth. Jack's definition for proportionality referred to quantity A growing by *a* units every time quantity B grows by *b* units, which was consistent with his additive thinking about rate of change but distinct from thinking that changes in A are a/b times as large as changes in B.

Another student, Arlene, had been successful on high school Calculus assessments but had additive and procedural meanings for division. Arlene saw division as a command to perform a calculation. She also struggled to explain how 29.66 related to 0.236 when given the statement $7 \div 0.236 = 29.66$. Consistent with the findings of Ball (1990), Arlene's quotative meaning for division broke down when prompted to give a scenario where one would need to divide six by three-fourths. When asked to explain what $6 \div (3/4)$ meant, she invoked the rule of "skip-flip-and-multiply", explaining that this "is what we learned to do" and then gave a numerical answer instead of a meaning or a sensible scenario. Later on, Arlene could not explain why one divides in the slope formula, exclaiming, "I don't really see it as division…I see that there is division but when I think of it in terms of slope I don't, I don't see that." Like the teachers in Simon and Blume's (1994) study, Arelene was inexperienced in representing a physical situation with a mathematical relationship.

Don, who planned to teach high school math, revealed a dominant partitive scheme for division. Don stated that he would emphasize using the long division algorithm to his future students. As a real world example for 37 divided by 3, Don suggested to partition 37 pencils into 3 groups, and later modified his example to each pencil being a bag of 10 M&M's so that he can divide the M&M's into three equal groups. (Don didn't notice that multiplication by 10 doesn't make 37 divisible by three.)

Another mathematics education student, Cindy, possessed strong quotative meanings for division. She was able to correctly determine when division was an appropriate operation and construct situations where division by fractions was necessary. However, when explaining what an idea like proportional meant she used additive descriptions and struggled to explain why we divide when we find a slope. This strong student was able to correctly solve many problems but still offered primarily additive explanations.

Early Conclusions

Given our preliminary interviews we believe that it is possible that many Calculus students do not understand quotient as a measure of relative size and will be unable to make sense of average and instantaneous rate in the ways needed to understand derivatives. For example if one thinks of rate as an amount added, common explanations of the derivative which ask students to envision the numerator and denominator of a difference quotient becoming arbitrarily small do not make sense. If a student believes a rate is the amount added to the output instead of a multiplicative comparison, the rate is getting smaller and smaller in the limiting process because the change in y values is getting smaller and smaller. If they understand rates as an index of slantiness of a line, then the derivative is a way to measure a geometric property of a graph and they might not attend to the changing quantities being compared. We plan to conduct individual teaching experiments with pre-service secondary teachers to build models of how they understand division and associated concepts such as multiplication, rate, measure and fractions. We aim to understand why thinking of quotients as a measure of relative size appears to be so challenging.

Questions for the Audience

How can we promote understandings of division as relative size?
In the research that you do, are there any concepts related to division that students struggle with?
Can you think of any alternative explanations/models for our data?
Why do you suppose articulating meanings for seemingly elementary topics is so difficult?

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