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A Conceptual Approach to Calculus Made Possible by Technology

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Calculus reform and using technology to teach calculus are two longtime endeavors that appear to have failed to make the differences in student understanding predicted by proponents. We argue that one reason for the lack of effect is that the fundamental structure of the underlying curriculum remains unchanged. It does not seriously consider students' development of connected meanings for rate-of-change functions and accumulation functions. We report an approach to introductory calculus that takes coherence of meanings as the central criterion by which it is developed, and we demonstrate that this radical reconstruction of the ideas of calculus is made possible by its uses of computing technology.

KEYWORDS mathematics education, fundamental theorem of calculus, technology, derivative, integral, instrumentation

David Tall, in reflecting on 40 years of calculus reform, said, "After reform projects have attempted a range of different approaches using technology, what has occurred is largely a retention of traditional calculus ideas now supported by dynamic graphics for illustration and symbolic manipulation for computation" (2010, p. 2). Tallman and Carlson (under review) found that Tall's observation actually reflects a broader calculus culture. In a national sample of 150 Calculus I final exams with 3,735 items, Tallman and Carlson found that (a) less than 15% of the items required students to do anything more than recall a procedure; (b) over 90% of the exams had more than

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70% of items classified as "remember" or "recall and apply procedure"; and (c) only 2.7% of the exams had 40% or more items requiring students to demonstrate or apply understanding. Tallman and Carlson's analysis of calculus final exams stands in stark contrast with beliefs of instructors who submitted them. Over 68% of the instructors claimed that they asked students to explain their thinking often or very often on final exams, yet only 3% of the instructors' items actually requested an explanation. It seems that, while the reform of calculus has had an impact on calculus rhetoric, it has not had an impact on what is expected that students learn. In this article we share our attempt to dramatically transform the calculus that students are expected to learn.

Calculus can be thought to address two fundamental situations: (a) you know how fast a quantity is changing and you want to know how much of it there is, and (b) you know how much of a quantity there is and you want to know how fast it is changing. It turns out that these situations are two sides of a coin—and realizing this was the breakthrough behind Newton's calculus (Baron, 1969; Boyer, 1959; Bressoud, 2011; Thompson, 1994a). Typical approaches develop these two parts of calculus independently and then tie them together with the Fundamental Theorem of Calculus (FTC). A common outcome, however, is that students understand derivatives as what you get by applying differentiation rules and that integration is about finding areas bounded by curves. The FTC is superfluous to their understanding of either. Derivatives are not about rate of change and integrals are not about accumulation. As one student said about integration, "I don't understand how a distance can be an area."

In this article, we describe a course that approaches introductory calculus with the aim that students build a reflexive relationship between concepts of accumulation and rate of change, symbolize that relationship, and then extend it to have broader reach. We also describe how it is only with technology that this approach is possible.

The building of this reflexive relationship unfolds in two phases. In Phase 1 students address Situation 1—they develop accumulation functions from rate of change functions. It is via Phase 1 that, as Bressoud (2009) urged, students put the integral into the Fundamental Theorem of (Integral) Calculus. In Phase 2 students address Situation 2—they develop rate of change functions from accumulation functions. It is important to note that accumulation and rate of change are never treated separately. Ideas of rate of change are central to building accumulation functions, and ideas of accumulation are central to building rate of change functions. In a very real sense, the FTC is present within every day of instruction.

In broad outline:

Phase 1: Build Accumulation from Rate of Change

Step 1. Students conceptualize continuous quantitative variation and covariation[1]; conceptualize functions as invariant relationships between the values of covarying quantities.
Step 2. Students conceptualize constant rate of change as two quantities covarying so that changes in one are proportional to changes in the other.
Step 3. Students build accumulation functions from constant rate of change functions.
Step 4. Students build accumulation functions from constant rate of change step functions; accumulation occurs over each interval at a constant rate.
Step 5. Students build approximate accumulation functions from exact rate of change functions by using step functions to approximate rate of change functions.
Step 6. Students define exact accumulation functions and represent them in open form as $F(x) = \int_a^x f(t)dt$, where $f(t)$ is a rate of change function and dt is the size of "an infinitesimal change." Definite integrals are then just specific values of exact accumulation functions: $\int_a^b f(x)dx$ is simply the value of F at $x = b$.

Phase 2: Build Rate of Change from Accumulation

Step 7. Students reverse direction: They rethink "amount functions" as accumulation functions. Thus, any function that gives an amount of something (like area of a square as a function of its side length) can be thought of as that amount having accumulated with respect to variation in its argument. That is, if f is a function whose values give amounts of some quantity with respect to values of another quantity, then $f(x) = \int_a^x r(t)dt$ for some rate of change function r and for some reference point a. The central problem is to determine r.
Step 8. Students build approximate rate of change functions (as average rate of change over intervals) to generate rate of change step functions from exact accumulation functions.
Step 9. Students derive exact rate of change functions in closed form from exact accumulation functions in closed form.[2]
Step 10. Students expand the closed-form quest to develop closed-form representations of functions that are sums, quotients, products, and composites. We extend further to the case of trigonometric, exponential, and logarithmic functions.

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1 *Covariation* is the mental process of coordinating the values of two quantities as they vary simultaneously (Saldanha & Thompson, 1998; Thompson, 2011). As Carlson and colleagues (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Larsen, & Lesh, 2003) make clear, covariational reasoning is essential for reasoning about functions as models of dynamic events, and it is nontrivial for students to develop.

2 A *function* is expressed in closed form when it is defined succinctly in terms of familiar functions, or algebraic operations on familiar functions. The function f defined as $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ is in open form, whereas f defined as $f(x) = e^x$ is in closed form.

Step 11. Students formalize the relationship between accumulation and rate of change—that has been employed throughout—by stating it as the Fundamental Theorem of Integral Calculus.

Step 12. Students understand the fact that every rate of change function has an accumulation function. Some accumulation functions can be expressed in closed form; most cannot.

This development has four central features. The first feature is that all steps are permeated with the ideas of variation, covariation, and function as an invariant relationship between covarying quantities. In other words, it is a demand of the course that students think that variables vary—always.

The second feature is that students must capture processes of variation, change, and accumulation symbolically—and their symbolizations must work. By "work" we mean that when students create a function as a solution to a problem and define that function within Graphing Calculator (GC; Avitzur, 2011), the program we use in the course, the function must be both syntactically coherent (or otherwise it will not return a value) and they must argue that their function actually answers the question asked. If they use a summation to define an approximate accumulation function A, then GC must be able to compute values $A(x)$ for any value of x in A's domain. Students must be able to graph $y = A(x)$ and make sense of the graph in relation to the problem situation.

The third feature is that students' construction of meaning is central. Powerful meanings suggest courses of possible action in problematic situations to a person having them. The course is designed to aid students' development of understandings and ways of thinking that enable them to make sense of calculus. One student, who struggled with the idea that $\Delta y/\Delta x$ gives an average rate of change, exclaimed one day, "Oh! You can't *see* the average rate of change *in* the graph! You have to *compare* the change in y to the change in x in your mind!" Course assessments, both summative and formative, provide information on the actual meanings that students have developed.

The fourth feature of the course is that students accept open-form representations of a function as actually representing the function. One way in which GC supports students' acceptance of open-form definitions of functions is that students can use them in GC in exactly the same way as functions defined in closed form. The open-form expression $\int_0^x \cos(t)dt$ represents the exact accumulation of a quantity that changes at a rate of $\cos(t)$ within the interval $0 \le t \le x$ just as exactly as does the close-form expression $\sin(x)$. It is an intellectual achievement for students to understand each of $\int_0^x \cos(t)dt$ and $\sin(x)$ in its own terms and then to determine that the two expressions actually define the same function. Their realization that functions expressed in open form have equal intellectual status as functions expressed in closed form is at the heart of Steps 5, 8, and 12.

THE ROLE AND PLACE OF LIMITS IN CALCULUS

Davis and Vinner (1986) claimed that *getting closer and closer to* is an unavoidable way of thinking about limits. Others since then (e.g., Cornu, 1991; Ferrini-Mundy & Graham, 1994; Oehrtman, 2009; Roh, 2008; Williams, 1991) support Davis and Vinner's claim and make clear that, despite our best efforts, a sophisticated understanding of limit is not within the grasp of most calculus students.

A formal definition of limit is not part of this calculus course. We avoid limits by talking about the ideas of *essentially equal to* and *close enough for practical purposes*. We grant that these phrases are inexact and are not rigorous. However, we also argue that calculus is not the proper venue to teach ideas of analysis. Students' understandings of analysis—rigorous treatments of limit, continuity, etc.—are best taught as subtleties in concepts that students already understand well. As the lead author says to his colleagues, "Our course on advanced calculus, to students, is advanced nothing. They do not understand calculus well enough to understand the issues being addressed, to understand how ideas of advanced calculus are an advance."

WHAT IT LOOKS LIKE

It requires a semester's worth of instructional and curricular materials to support students through Steps 1–12, so it is not possible in this article to illustrate each step. Instead we will illustrate Steps 1, 5, and 8 and then discuss the FTC and its centrality to the course.

We must clarify that both our and students' use of GC is embedded in all that we do. We use it every day to create animations and to share examples for class discussion. It, along with a Bamboo tablet, is our blackboard. GC displays mathematical expressions in standard mathematical notation. So, in terms of mathematical writing, our projected computer screen looks the same as if we had written on a whiteboard.

Students use GC as a notebook. They define functions in it, type explanations of their functions, and state their solutions to problems. They also write explanations of their solutions on printouts of their function definitions and graphs. None of these uses is unique to this course. What makes the course unique is the conceptual development that is designed into it and that GC makes possible.

Step 1: Conceptualizing Variation, Covariation, and Function

Jacobs and Trigueros (Jacobs, 2002; Trigueros & Jacobs, 2008) showed convincingly that even the best high school calculus students have very weak understandings of variable. Variables did not vary in these students' thinking. Variables were, in effect, letters that stood for constants—but replaceable constants. The common input-output image of a function that so many texts promote is compatible with the image that variables are replaceable constants. But calculus is the mathematics of change, so the variable-as-constant meaning is largely incompatible with developing an understanding of calculus. Variables, in students' thinking, must vary if students are to understand calculus coherently.

There is a potential conundrum in trying to think coherently about a variable varying. If a variable is a letter that stands for a number, how can we think about those numbers changing without thinking that we replace one number with another? The solution is to take humans out of the "number changing" process! But to do this requires that we first make the idea of quantity clear.

A quantity, in this usage, is a mental construct. It is an object conceptualized in such a way that it has one or more attributes that are measurable (Saldanha & Thompson, 1998; Thompson, 1994b, 1996, 2011). A variable, in relation to quantities, is a letter that stands for the value of a quantity whose magnitude varies. Students do not need to think about themselves replacing one value with another. The fact that, in specific contexts, a quantity's value varies takes care of the variation. We can imagine your height since your birth having a value (a measure) at each moment in time. No one replaced the value of your height from moment to moment. It changed on its own without anyone's help. In this sense, the use of h to represent your height carries with it the understanding that the value of h varies. This meaning of variable should be developed in grades K-8. In the United States it is rarely developed, even in calculus (Carlson, 1998; Jacobs, 2002; Trigueros & Jacobs, 2008).

We employ a number of computer simulations to convey to students what we mean by "think covariationally." Figure 1 shows one that builds upon the famous "bottle problem" first developed in the 1980s at the Nottingham University Shell Centre (UK) and later studied by Marilyn Carlson (Carlson, 1997, 1998; Carlson, Oehrtman, & Engelke, 2010). The version we describe here shows the outline of a bottle that is being filled with water and asks students to imagine a graph of water's volume in relation to the water's height in the bottle.

The display in Figure 1 shows several controls. Two controls allow the student to adjust the bottle's shape. The slider "BottomShape" controls how bulged the bottom is; the slider "TopShape" controls how slanted the top is (see Figure 2). Finally, the slider h controls the water's height in the bottle. It is a feature of GC that you can animate the display by setting one slider to vary its value automatically. The slider h varies automatically between 0 and 4.25 upon clicking its "Play" button (see Figure 1). As h varies, the shaded region varies accordingly and the value of "Volume" varies likewise.

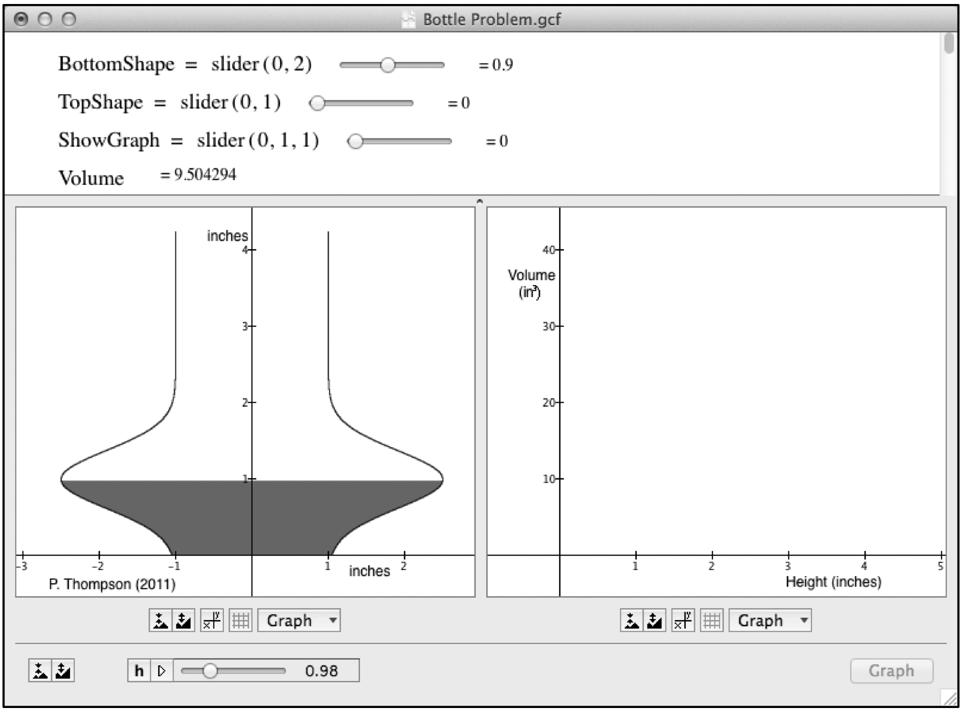

FIGURE 1 The bottle problem display (color figure available online).

Students are presented the task of envisioning the water's volume in relation to the water's height for different bottle shapes, and to represent that relationship in a hand-drawn graph. They are asked to attend to the fact that every water height has a volume, and to explain why the volume seems to increase more rapidly for some changes in height than others. They can check their hand-drawn graphs by moving the "ShowGraph" slider to 1. The displayed graph is also dynamic, "revealing" itself as the value of *h* varies (Figure 3).

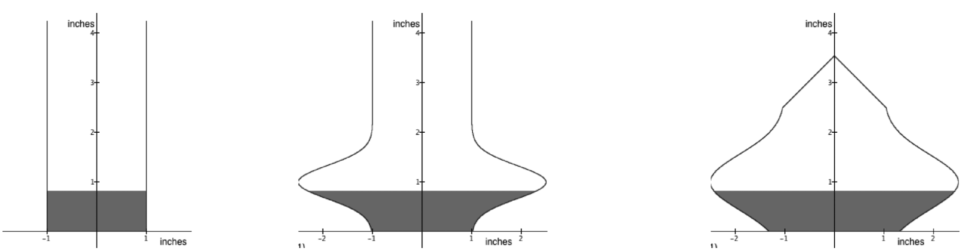

FIGURE 2 The bottle can have different shapes depending on the values of "TopShape" and "BottomShape. " In any shape, the water level rises as the value of *h* increases (color figure available online).

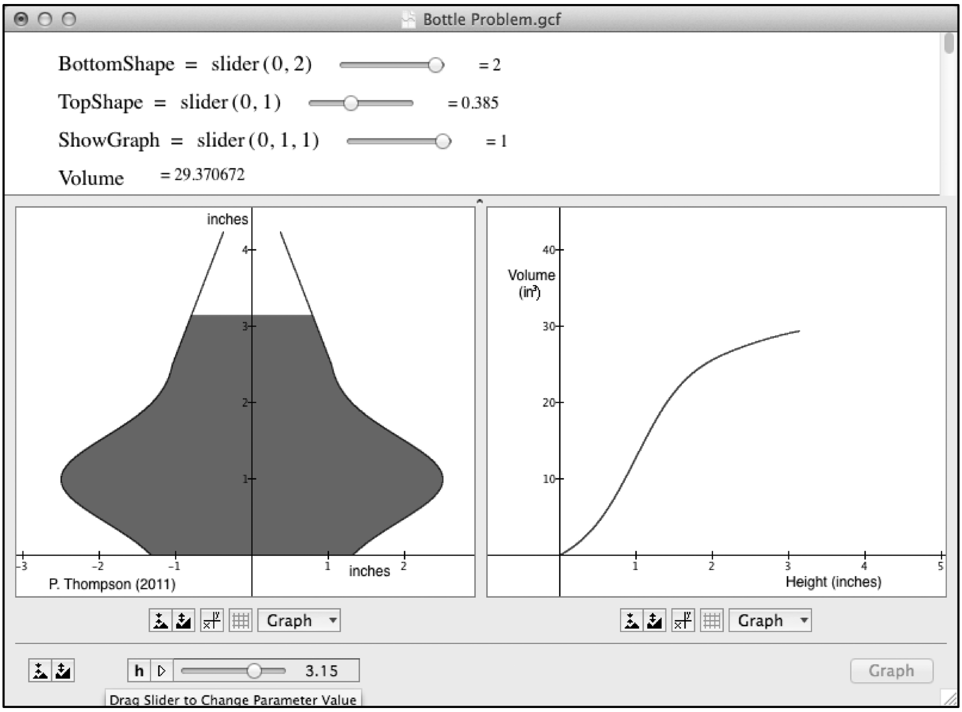

FIGURE 3 GC shows a graph of the water's volume with respect to its height as the water's height varies (color figure available online).

Subsequent questions ask students to predict, before showing a graph, at what heights the change in volume will happen least rapidly, most rapidly; at what heights the rate increases, decreases, or is constant (again, relative to various bottle shapes and features of them).

We should point out that GC is a graphing program with some Computer Algebra System (CAS) capabilities. It is not a programming environment. All displays, such as the animations in Figures 1–3, are the graphs of mathematical functions or relationships. Appendix A explains the mathematics behind Figure 3.

Students' concepts of function and their understanding of function notation are also addressed in Step 1. Problems and activities from Step 1, and from Step 8, are given in Appendix B.

Step 5: Build Approximate Accumulation Functions from Exact Rate of Change Functions

The idea of an exact rate of change function is problematic in traditional calculus and cannot be used until the derivative has been defined. We have found, however, that even after traditional instruction on derivative, students think of instantaneous rate of change not as the value of a derivative function but instead as a reading from an object's personal speedometer (Stroud, 2010; Weber, Tallman, Byerley, & Thompson, 2012). Their meaning of instantaneous rate of change does not entail ratios, limits, or the idea of rate of change as we normally mean it. While this is a problem for traditional approaches to calculus (students' meanings of instantaneous rate of change are incoherent even after instruction on it), this meaning is adequate for the purpose of defining accumulation functions from rate functions. Several weeks afterward we uncover the problematic nature of *instantaneous speed* when we define rate of change functions from exact accumulation functions.

The strategy for building an approximate accumulation function from an exact rate of change function is to first convert the exact rate of change function into a step function (Figure 4). The reason is that to generate an amount of change, we need to pretend that the quantity changes at a constant rate with respect to a small change in its argument. The problem posed to students is this:

> How can we change, say, $f(x) = \cos(x)e^{\sin(x)} + 2$ into a step function r that is constant over intervals of size Δx and whose values approximate the values of f?

The task to approximate a function with a step function is a challenge for them. How do you define a function that takes any given function and approximates it with a step function? We can imagine readers asking why this is a legitimate problem for calculus students. It is a legitimate problem because without such a method, students cannot compute functions whose values are approximately equal to a quantity's accumulation.

The strategy for defining accumulation functions from exact rate of change functions that we settle upon as a class is essentially this:

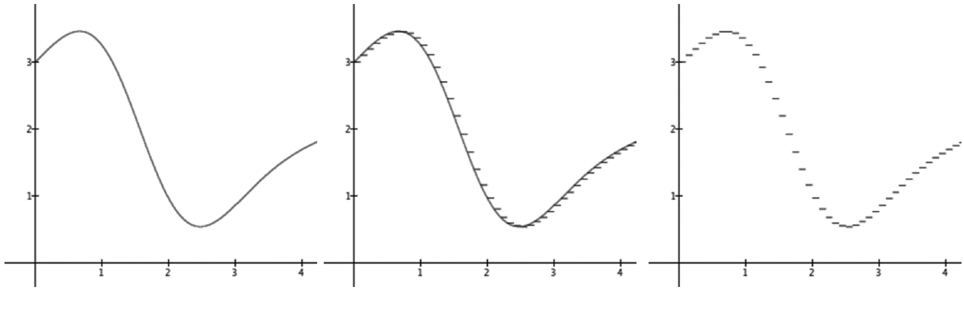

FIGURE 4 Define a step function that approximates a given rate of change function over intervals of size Δx (color figure available online).

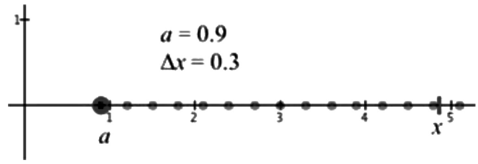

FIGURE 5 Given values of a and Δx, what is the value of the left end (or middle, or right end) of the Δx-interval containing the value of x? (color figure available online).

1. A step function is constant throughout each Δx-interval. Define a function that assigns the value of f at the left (or middle, or right) of each Δx-interval to every value of x within the Δx-interval.
2. In order to assign the value of f at the left endpoint of each Δx-interval, students need to identify the left-end of the Δx-interval that the value of x is within (see Figure 5).
3. Students define the functions left, mid, and right as in Figure 6. Two things are key to the definition of the function *left*—the meaning of division and the use of the floor function. The expression $\frac{x-a}{\Delta x}$ gives the number of Δx-intervals between the values of a and x, including a fraction of an interval if x is not an endpoint. The floor function, symbolized as $\lfloor u \rfloor$, gives the greatest integer less than or equal to u. Thus, $\lfloor \frac{x-a}{\Delta x} \rfloor$ gives the number of complete Δx-intervals between the values of a and x. The expression $a + \Delta x \lfloor \frac{x-a}{\Delta x} \rfloor$ gives the left end of the Δx interval that contains the value of x. Therefore, left(x, a, Δx) turns every value of x within a given Δx-interval into the value of the interval's left end.
4. Define r as $r(x, a, \Delta x) = f(\text{left}(x, a, \Delta x))$; r is a step function whose value within any Δx-interval is the value of f at the left end of the interval. r is the step function we sought. The step function in Figure 4 was created by graphing $y = r(x, 0, 0.1)$, r defined as stated here, and $f(x) = \cos(x)e^{\sin(x)} + 2$.

Students typically define the approximate accumulation function A so that $A(x, a, \Delta x)$ gives the approximate accumulation from a to x, over complete intervals of size Δx, of the quantity that accumulates at a rate of $f(x)$ at each value of x, as in

$$A(x, a, \Delta x) = \left(\sum_{k=1}^{\lfloor \frac{x-a}{\Delta x} \rfloor} r\left(a + (k-1)\Delta x, a, \Delta x\right)\Delta x \right)$$

$$\text{left}(x, a, \Delta x) = a + \Delta x \left\lfloor \frac{x-a}{\Delta x} \right\rfloor$$

$$\text{mid}(x, a, \Delta x) = \text{left}(x, a, \Delta x) + \Delta x / 2$$

$$\text{right}(x, a, \Delta x) = \text{left}(x, a, \Delta x) + \Delta x$$

FIGURE 6 Definitions of functions to give the left endpoint, midpoint, and right endpoint of the Δx-interval that x is in.

They are surprised when A, defined previously, gives a step function when graphed. We then devote a lesson to imagining the accumulation function varying as x varies *within* a Δx-interval. We end with a definition of A that accounts for accumulation both over completed Δx-intervals and accumulation within the Δx-interval that contains the value of x.

$$A(x, a, \Delta x) = \left(\sum_{k=1}^{\lfloor \frac{x-a}{\Delta x} \rfloor} r\left(a + (k-1)\Delta x, a, \Delta x\right)\Delta x \right) + r(x, a, \Delta x)\left(x - \mathrm{left}(x, a, \Delta x)\right),$$

or more simply,

$$A(x, a, \Delta x) = \left(\sum_{k=1}^{\lfloor \frac{x-a}{\Delta x} \rfloor} f\left(a + (k-1)\Delta x\right)\Delta x \right) + r(x, a, \Delta x)\left(x - \mathrm{left}(x, a, \Delta x)\right)$$

The definition of A, though visually daunting, has a very simple structure when interpreted according to the quantities that it evaluates. The expression

$$\sum_{k=1}^{\lfloor \frac{x-a}{\Delta x} \rfloor} f\left(a + (k-1)\Delta x\right)\Delta x$$

represents the completed accumulation—accumulation over complete Δx-intervals—between a and x. The expression $r(x, a, \Delta x)\left(x - \mathrm{left}(x)\right)$ represents the accumulation that has occurred within the incomplete interval containing the value of x—the interval within which x is currently varying. The accumulation function has the structure of a linear function within any Δx-interval, and thus has the structure of a piecewise-linear function over the interval $[a, x]$.

$$A(x, a, \Delta x) = \overbrace{\left(\sum_{k=1}^{\lfloor \frac{x-a}{\Delta x} \rfloor} f\left(a + (k-1)\Delta x\right)\Delta x \right)}^{\text{completed accumulation}} + \overbrace{r(x, a, \Delta x)\left(x - \mathrm{left}(x, a, \Delta x)\right)}^{\text{varying accumulation}}$$

It is instructive to note that, since A is piecewise-linear, it always changes at a constant rate within any Δx-interval and that rate of change is the value of f at the left (or middle, or right) of the interval. Noticing this is the first step in an explicit conceptualization of the FTC.

> Departments of transportation are very concerned with how steep a highway is, because to qualify for federal matching funds no stretch of highway can have a rate of change of elevation with respect to distance greater than 8/100 ft/ft (i.e., a rate no greater than 8 ft of change in elevation per 100 feet of highway).
>
> The Colorado Department of Transportation puts a graph on highway maps that shows the highway's rate of change of elevation at spots on the highway that are not far apart.
>
> The graph below shows one highway's rate of change of elevation map over a stretch of road. It shows the highway's rate of change of elevation (in feet per hundred feet) at each number of miles over the first 5.5 miles from Colorado-Utah border. This rate of change of elevation map is modeled by the function $r(x) = \cos(x)e^{\sin(x)} + 2$. (Remember, this is *an example.*) Define a function that gives the road's *elevation* at each distance from the border, given that the road's elevation at the border is 5,326 feet.

FIGURE 7 A context for developing accumulation functions from rate of change functions.

The development of the generalized accumulation function is done within the context of concrete situations. We do not have space to show how this is done, but Figure 7 gives one context that does a lot of work for the course.

It is important to note that, in this course, no context used in the construction of accumulation functions involves computing an area under a curve. We consider it a travesty that calculus curricula take *area under the curve* as the essential meaning of integration. We point out in one lecture, weeks after students have begun creating and using accumulation functions, that the product $f(a+(k-1)\Delta x)\Delta x$ just happens to be the product you would use to compute the area of a rectangle having side lengths $f(a+(k-1)\Delta x)$ and Δx if we are graphing functions within a rectangular coordinate system. So *area under a curve* is a very small idea in the realm of using accumulation functions to model phenomena.

In concluding this discussion of Step 5, we would like to make a brief comment about Step 6—defining exact accumulation functions. As noted earlier, we do not discuss limits, but we do discuss the ideas of *indistinguishable from* and *essentially equal* to. In playing with the effects of making values of Δx very small, students discover that at some point one or more of three things happen: (a) making Δx smaller makes no appreciable difference in the values their function produces; (b) the smaller the value of Δx, the slower the computer produces results; and (c) you can always make Δx so small that the computer's estimates go crazy and become unreliable. We then introduce the notation $\int_a^x f(t)dt$. We introduce it simply as a matter of notation to designate an accumulation function for which Δx is so incredibly small that making it smaller makes no appreciable difference in the values it produces.[3] This is actually a rather sly move, because GC

[3] We also discuss the need to use a letter other than x in the integrand when we use x in the integral's upper limit. (See Thompson & Silverman, 2008, pp. 44–45, for this explanation.)

allows this open-form integral in function definitions (see Line 9 of Figure 8 in Appendix A). In other words, students start using standard integral notation to define functions that have the same meaning as the approximate accumulation function (defined as a summation) that they worked so hard to create. Students then see standard integral notation as a convenience. It does not introduce anything new.

Step 8: Build Approximate Rate of Change Functions from Exact Accumulation Functions

As stated earlier, any function that gives an amount of one quantity in relation to the value of another can be thought of as an accumulation function. The function $v(x) = \frac{4}{3}\pi x^3$ gives the volume of a sphere of radius-length x. If we think of x varying from 0, then $v(15)$ gives the accumulated volume of a sphere that grew in size from radius-length 0 to radius-length 15. Put another way, $v(x) = \int_0^x r(t)dt$ for some rate of change function r. The problem is to determine r.

Our strategy for constructing an exact rate of change function r for an accumulation function F so that $F(x) = \int_a^x r(t)dt$ for some number a is this:

1. Students construct an approximate accumulation function from the exact accumulation function. The approximate accumulation function will be piecewise linear over Δx-intervals, and the rate of change of the piecewise function over each Δx interval will be the function's average rate of change over that interval.
2. The constant-rate step function is the function that has, for each value of x in a Δx-interval, the value of the accumulation function's average rate of change over that interval.
3. The exact rate of change function is the function (if there is one) that is indistinguishable from the rate function generated by using infinitesimally small values of Δx.

While this strategy outlines the general approach, we make one adjustment that simplifies matters significantly. Instead of looking at the average rate of change of F over fixed Δx-intervals, we think of a sliding interval of length Δx (Thompson, 1994a; Weber et al., 2012). We can then define the general open-form rate of change function for any exact accumulation function F as $r(x) = \frac{F(x+\Delta x)-F(x)}{\Delta x}$.

In contrast to traditional approaches to developing the derivative, we follow Tall (1986) in letting the value of x in the definition of r vary while keeping the value of Δx fixed. Thus, instead of the difference quotient being a number for every value of Δx, it defines a function r for every value of Δx.

If r converges as the value of Δx decreases, then it converges to a function, and we call that function F's *exact rate of change* function. We do not dwell on a deep meaning of *converges*, although we do look at examples where it is clear that convergence does not happen (especially in the context of discussing the Mean Value Theorem). It is not uncommon for students to see that $r(x)$ converges differently for different functions F. What they notice, put in standard language, is that $r(x)$ converges pointwise in some instances and uniformly in others. Later, we glance at "super wrinkly" functions for which $r(x)$ does not converge at all, anywhere.

There is one important difference between the development of exact accumulation functions from exact rate functions and the development of exact rate functions from exact accumulation functions. In constructing exact rate of change functions we can often manipulate the approximate rate of change function's definition symbolically to see a function we already know. In other words, we can usually derive closed-form representations of exact rate of change functions; whereas, we are rarely able to derive closed-form representations of exact accumulation functions. However, as we develop a repertoire of exact rate of change functions in closed form that are derived from exact accumulation functions in closed form, we expand our capacity to work backward from a closed-form rate of change function to a closed-form accumulation function.

THE FUNDAMENTAL THEOREM OF CALCULUS

The FTC is often stated like this:

> Suppose f is continuous on a closed interval $[a,b]$.
> Part I. If the function F is defined by $F(x) = \int_a^x f(t)dt$ for every x in $[a,b]$, then F is an antiderivative of f on $[a,b]$.
> Part II. If G is any antiderivative of f on $[a,b]$, then $\int_a^b f(x)dx = G(b) - G(a)$

We have argued elsewhere (e.g., Thompson, 1994a) that we should think of Part I as the FTC's essence, and think of Part 2 as a relatively straightforward corollary. It is Part 1 that students have difficulty conceptualizing. Indeed, Phase 1 of the course can be characterized as having the goal that students conceptualize $F(x) = \int_a^x f(t)dt$. The intricate relationship between accumulation and rate of change is the essence of the FTC. It is by building accumulation from rate of change and rate of change from accumulation, that the FTC is present each day of instruction. We do not wait until the end to magically link derivatives and integrals.

To understand that $\int_a^x f(t)dt$ is an antiderivative of f means, first, to understand that $\int_a^x f(t)dt$ is a function of x. So Phase I is structured to have students develop the idea that accumulation can be conceived as a function. The meaning we intend that students develop is that $\int_a^x f(t)dt$ is the function that gives the accumulation of a quantity over the interval $[a, x]$ whose accumulation changes at a rate of $f(t)$ for each value of t in $[a, x]$. That is, this intended understanding of accumulation *entails* the FTC—sort of. After Phase I students know that $\int_a^x f(t)dt$ is a function of x and they know that $\int_a^x f(t)dt$ has $f(x)$ as its rate of change. But, at the end of Phase 1, they do not know that $\int_a^x f(t)dt$ is an antiderivative of f. This is because the concept of antiderivative is based on the concept of derivative, which is developed in Phase 2.

The importance of $\int_a^x f(t)dt$ being an antiderivative of f is that all antiderivatives of f differ by at most a constant. Computing individual values of

$$A(x, a, \Delta x) = \left(\sum_{k=1}^{\lfloor \frac{x-a}{\Delta x} \rfloor} f\left(a + (k-1)\Delta x\right)\Delta x \right) + r(x, a, \Delta x)\left(x - \text{left}(x, a, \Delta x)\right)$$

is painfully slow for small values of Δx even on the fastest computers. But it is (for infinitesimal values of Δx) essentially an antiderivative of f—a function that has f as its exact rate of change function. So, if we know a closed-form function, one that is also an antiderivative of f, and one that we know how to calculate efficiently, then we can use it in place of the conceptually clear but hard to compute summation. The inefficiency of this open-form approximation of an exact accumulation function is a primary motive for deriving closed-form rate of change functions from closed-form accumulation functions in Phase 2. In deriving closed-form rate of change functions and, thinking backward with the chain rule, product rule, etc., we build up a repertoire of easy-to-compute functions that we can use in place of hard-to-compute open-form approximate accumulation functions. The fact that we can use exact rate of change functions in many useful applications is a bonus.

That $F(x) = \int_a^x f(t)dt$ is a function, and that it is also an antiderivative of f, is driven home through problems like these:

Problem 1. The function q defined as $q(x) = xe^{\cos x}$ does not have an elementary antiderivative. But it has an antiderivative. Name one.

Problem 2. Define G as $G(x) = \int_a^x t\cos(t)dt$. Define a slider a that takes on values between -5 and 5. Graph $y = G(x)$. Vary the value of a. Why does the graph change the way it does?

Problem 3. A ball is hanging by a 2-foot rubber cord from a board. The ball is then given a sharp push downward and left free to bounce up and down. Its vertical location relative to its resting point is given by the function $f(x) = -e^{-x/12}\sin\left(\frac{\pi}{2}x\right)$, where x is the number of seconds since the ball was pushed. Define a function that will compute the ball's total distance traveled x seconds after being pushed.

Problem 4. Let k be a differentiable function of x. Define the function K as $K(x) = \int_a^x \frac{d}{dt}k(t)dt$. For what values of a will $K(x) = k(x)$? Why?

An answer to Problem 1 is $Q(x) = \int_0^x te^{\cos t}dt$. By Part I of the FTC, Q is an antiderivative of $xe^{\cos x}$. In exploring Problem 2, students see that the graph of G moves vertically as they vary the value of a. The answer to Problem 2, then, is that since G is an antiderivative of $xe^{\cos x}$ for all values of a, and since all antiderivatives of $xe^{\cos x}$ differ by a constant, by varying the value of a we are varying the value of the constant that is added to $\int_0^x te^{\cos t}dt$. The essential insight to Problem 3 is that the ball's total distance traveled is accumulated by moving in bits of distance, where each bit is made by the ball moving at a positive rate for an infinitesimal amount of time. Upon this insight, it is clear that the ball's total distance traveled after x seconds is $T(x) = \int_0^x \left|\frac{d}{dt}f(t)\right| dt$. The essential insight to Problem 4 is that both $K(x)$ and $k(x)$ are antiderivatives of $\frac{d}{dx}k(x)$. Thus, by Part 2 of the *FTC*, $K(x) = k(x) - k(a)$, and therefore $K(x) = k(x)$ for values of a such that $k(a) = 0$. Students can answer the first part of Problem 4 by defining a as a slider, graphing $K(x)$ and $k(x)$ for various functions, and then noticing that the graphs coincide when a is such that $k(a) = 0$. However, it is students' insight into the FTC that supports their answer to the "why" part of Problem 4.

MATHEMATICS FIRST, TECHNOLOGY SECOND

This approach to introductory calculus is driven by research on learning and understanding rate of change and research on students' difficulties in calculus. The aim was to teach schemes of meaning that would lend coherence to students' thinking about accumulation and rate of change. Technology entered the picture as a means to that end. As we have illustrated, and will explain more fully, technology made the conceptual development possible. But it was the intended conceptual development that determined the technology that was needed.

WHY THIS APPROACH IS IMPOSSIBLE WITHOUT TECHNOLOGY

The development of accumulation functions as first-class functions depends on students being able to represent them and have their representations behave like functions they accept: They give it an input and it produces an output; they graph it; they transform it (shifting its graph left, right, up, or down as they would any function); and so on. They develop a familiarity with open-form accumulation functions that makes them, in the students' experience, real. None of this is possible without employing computing technology. The fact that GC uses standard mathematical notation enhances students' experiences even more. The notation "becomes alive," and errors in students' notation lead to displays and results that students do not anticipate. In other words, students' work with GC allows them to judge the validity of their mathematics by how it *works*. Students know that something is amiss if their choice of a closed form representation of $\int_a^x \frac{1}{1+t^2} dt$ does not produce a graph that matches the graph of $y = \int_a^x \frac{1}{1+t^2} dt$. Similarly, they know something is amiss if their choice of a closed-form rate of change function for $f(x) = x^{\cos x}$ appears not to coincide with the graph of $y = \frac{f(x+0.0001)-f(x)}{0.0001}$. Technology allows students to form the mindset that open-form definitions of mathematical functions are not just legitimate mathematical definitions; they are often trusted definitions if one is unsure about a closed-form result.

We will close by observing that students' acceptance of GC is a slow process. At first they insist upon using their handheld calculator when it would be easier to use GC. At the end of the course, however, students prefer to use GC, for the very reasons we noted in the previous paragraph—the mathematics is live. We believe that what happens is best captured by what Verillon and others have called *instrumentation* (Artigue, 2002; Thomas & Holton, 2003; Vérillon & Rabardel, 1995; Zehavi, 2004). By *instrumentation*, Verillon and Rabardel mean that people get to know an object or artifact through using it to the point that its affordances and constraints become part of their thinking and reasoning. It becomes *ready at hand* in Heideggar's terms (Winograd & Flores, 1986). What is unique in the present course is that GC's transformation into an instrument for students' mathematics happens hand in hand with students' construction of the mathematics for which GC becomes instrumental.

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APPENDIX A

The Mathematics Behind the Bottle Problem Animation

Figure 8 gives the entire workings of the bottle problem animation.

Here is how the mathematics in Figure 8 align with the computer display and animation:

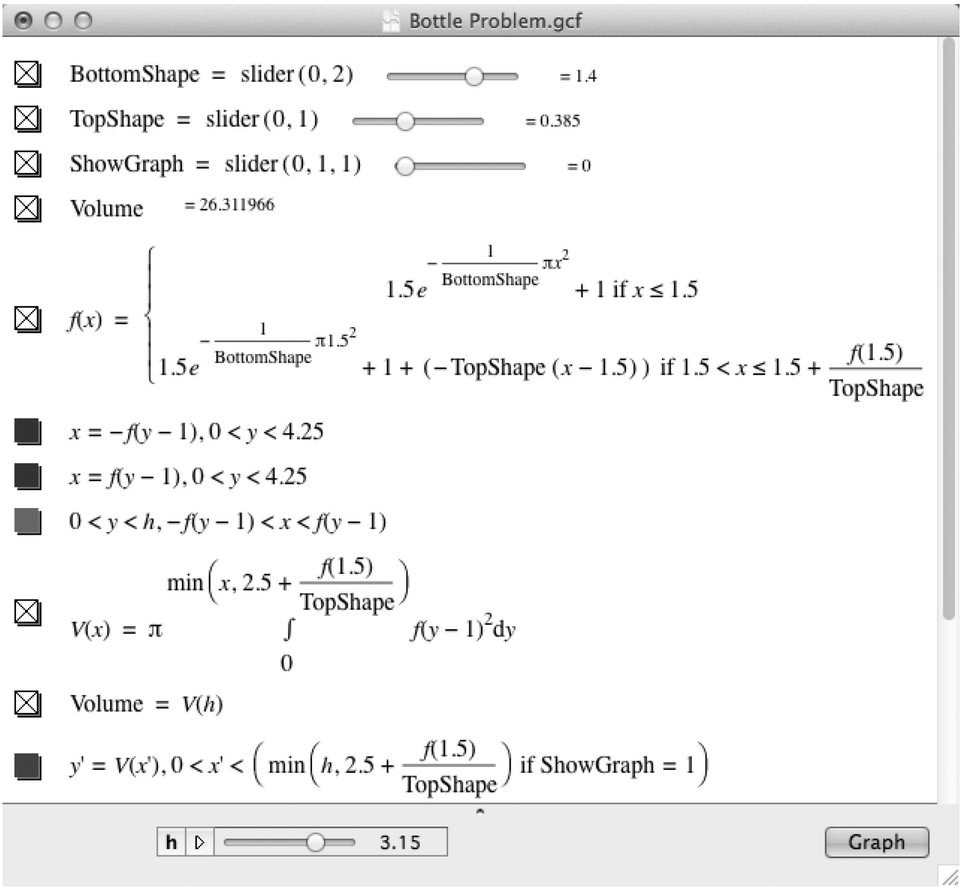

FIGURE 8 The mathematical functions and relationships behind the Bottle Problem animation. In some operating systems, this file will work only if "$f(1.5)$" in part 2 of Line 5 is expanded to "$1.5e^{\frac{-1}{\text{BottomShape}}\pi 1.5^2} + 1$" (color figure available online).

- The bottle's outline is the graph of a function f that is defined piecewise. The definition of f appears in Line 5 of Figure 8. The first part of f's definition is a scaled version of the probability density function of a normal distribution up to $x = 1.5$, with the value of "BottomShape" used as the distribution's standard deviation.
- The second part of f's definition is the point-slope formula applied to the point $(y, x) = (1.5, f(1.5))$ and a rate of change of "TopShape."
- Lines 6 and 7 generate the bottle's outline by graphing $x = f(y - 1)$ and $x = -f(y - 1)$, $0 < y < 4.25$.
- Line 8 shades the region consisting of points (x, y) such that $0 < y < h$ and $-f(y - 1) < x < f(y - 1)$.
- Line 9 defines the function V, which is the volume of revolution calculated using the "slab" method. The radius of each slab is $f(y - 1)$. The limits of

integration are 0 and the smaller of the value of x and the bottle's largest possible height.

- Line 11 graphs $y' = V(x')$ for $0 \le x' \le$ the minimum of h and the bottle's height. The use of y' and x' is a signal to GC to draw the graph in the right pane.
- As the value of h varies, the mathematics expresses itself as an animation that shows a bottle being filled and an unfolding graph of its volume versus its height.

APPENDIX B

Examples of Problems, Exercises, and Activities for Steps 1 and 8

Step 1 – Concept of Function and Function Notation

Green Globs (Adapted from Dugdale, 1982, Figure 9). This exercise is an adaptation of Sharon Dugdale's classic game Green Globs. The difference between this and Dugdale's original game is that Dugdale's game did not require students to create a composite function. In solving this problem, students first find a function whose graph passes through the points (in this case, $y = \frac{(x-3)^3}{8}$), but to satisfy the problem's request students must graph $y = g(\frac{x-2}{2})$.

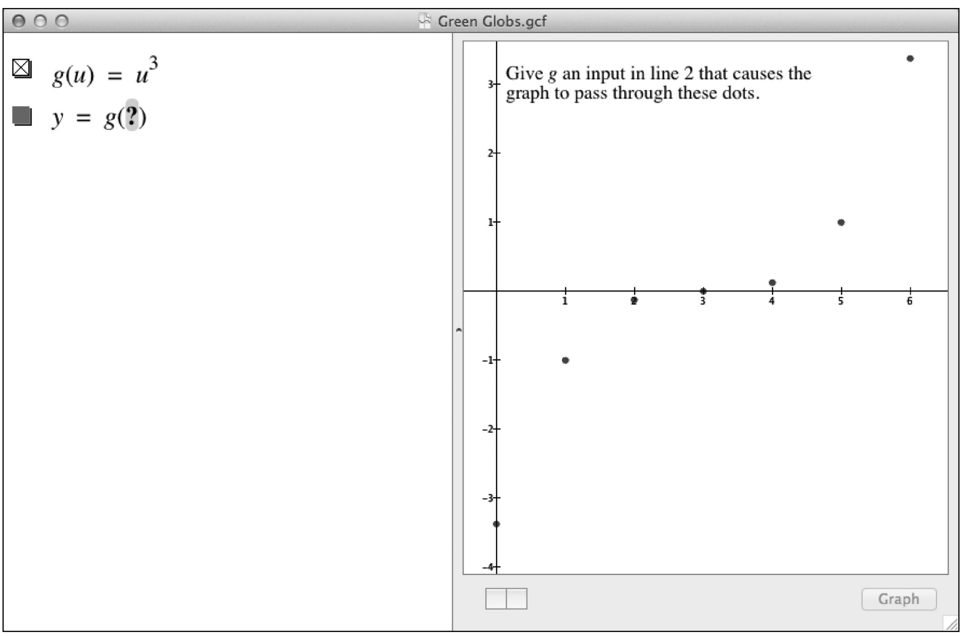

FIGURE 9 Screen from activities inspired by Green Globs (color figure available online).

A model for the blood-alcohol-concentration (BAC) of a 73-kg (160-lb) person t hours after drinking V milliliters of alcohol is $c(t)=\frac{V}{40}\left(e^{-0.85t}-e^{-t}\right), 0 \le t$. A value of c is a number of milliliters of alcohol in the bloodstream per liter of blood (again, for a 73-kg person). A 12-ounce can of beer, a 1.5-ounce shot of whiskey, and a 5-oz. glass of wine each contain approximately 18 ml of alcohol.

1) Explain why it is reasonable that the BAC function is the difference of two exponential functions.

2) Kevin, a 73 kg male, chugs three cans of beer, one every 0.5 hours (he's at a party). Graph Kevin's BAC with respect to the number of hours since taking his first gulp. (You will need to change the definition of c so that it is defined for real numbers less than 0.)

3) Generalize the definition of c so that it takes five inputs:

   a) The number of hours since a person took his/her first gulp,
   b) His/her body mass,
   c) The amount of alcohol consumed per gulp,
   d) The number of gulps consumed in total, and
   e) The number of hours between successive gulps

   and outputs his/her BAC at that number of hours since the first gulp. Explain any assumptions you make in order to define this new function.

4) Blood Alcohol Content below 0.05 is considered safe for driving. The National Football League has a policy that a stadium cannot serve beer past the end of the 3rd quarter. Each quarter of football lasts, on average, 0.75 hours, with a 0.25-hour break after the 2nd quarter.

   Katie, who has a body mass of 50 kg, drank a 20-oz glass of beer during the first quarter of the game and another during the third quarter. Is it safe for Katie to drive home after the game? If not, how long must she wait? *Assume that Katie consumes each glass of beer in 4 equal amounts spread equally across a quarter of play, and that she reaches her car 30 minutes after the game ends.*

5) Here is a graph of BAC with respect to time of a person who drank 5 beers taking 1/4-beer sips at the beginning of every half hour. Why does this person's BAC seem to be relatively stable during the period from 5 hours to 10 hours after starting?

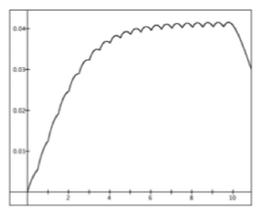

FIGURE 10 A sequence of problems that focuses on uses of function notation in modeling (color figure available online).

Blood Alcohol Concentration (Adapted from Stacey, 2010). We use the Blood Alcohol Concentration (BAC) problem (Figure 10) primarily to place students in a position where they must modify a function definition in order to use it more broadly than was intended in its initial design. For example, Question 2 can be answered with the graph of $y = c(x) + c(x - 0.5) + c(x - 1), x \ge 0$. The graph shows Kevin's BAC over time since his first can of beer. But to make this expression work, students must redefine c so that $c(x) = 0$ for $x < 0$. This is because $x - 0.5$ is less than 0 for the first half hour and therefore $c(x - 0.5)$, with c's original definition, is undefined for the first half hour. Redefining c so that $c(x) = 0$ for $x < 0$ says, in effect, that Kevin's BAC is unaffected by beers he is yet to consume.

MAT 270 Fall 2011 Rate of Change of Accumulation
P. Thompson Due: Oct 9, 2011

For simplicity, I will use h instead of Δx. But I use them with the same meaning – they are the width of the intervals over which we presume the accumulation function has a constant rate of change.

Do this for each of 1-5. Answer each set of questions on a separate sheet of paper:

- Describe how to think of the given function as an accumulation function.
- State the meaning of $r(x) = \frac{f(x+h) - f(x)}{h}$ in relation to the accumulation function it is based on. (Do this in complete sentences, and do it in every problem. You must practice saying what $r(x)$ means.)
- Expand and simplify the definition of $r(x)$.[1]
- Express the simplified quotient as a function that it is essentially equal to r for small values of h (see lecture slides from 10/7/11 and the end of the video from 10/7/11 for reminders of what this means).

  We will call the function that is essentially equal to $r(x)$ the accumulation function's *exact rate of change* function.

- Graph the accumulation function's exact rate of change function. Click a point on the graph, Record the point's coordinates and say what they mean.
- Graph $r(x)$ for several values of h (positive and negative). How small must h be for the graph of r to be indistinguishable from the graph of f's exact rate of change function over the domain shown in *GC*? Does your answer to, "How small must h be ...?" change when you zoom in on the graph of f? Explain.
- Redefine $f(x)$ as an integral. For example, we determined that $r(x) = 2x$ is the exact rate of change function for $f(x) = x^2$. We can therefore write $x^2 = \int_a^x 2t\,dt$ for some number a. (We will soon learn how to determine the appropriate value of a.)

1. $f(x) = 3x^2$
2. $f(x) = (2x-3)^3$
3. $f(x) = x^4$
4. $f(x) = x^7$
5. $f(x) = x^{10}$

FIGURE 11 An assignment from Phase 2, Step 8.

Step 8. Students Build (and Investigate) Rate of Change Functions

Figure 11 shows a problem used to reinforce the idea that any function taken as an accumulation function can be thought of as an integral built from the function's rate of change function.

Figure 12 shows an assignment that focuses on relationships between the tangent to a function's graph at a point $(a, f(a))$, the function's rate of change with respect to x at $x = a$, and information that a function's rate of change at a value in its domain tells us about the function's behavior in neighborhoods around that value.

MAT 270 Fall 2011 Rate of Change and Tangents to Graphs
P. Thompson Due: Monday, Oct 17

In GC, define a = slider(-10, 10). Then do the following for each of the accumulation functions in 1-3:

a) Graph f in GC
b) Define the function's rate of change function r_f in closed form, if you can. Otherwise use the open form definition of r_f.
c) Set the slider a to some number.
d) State the meaning of $r_f(a)$ for that value of a.
e) Define a function t whose graph is tangent to the graph of f at $x = a$. You will have done it correctly if the graph of $y = t(x)$ is tangent to the graph of f for every value of a.

1. $f(x) = x^3 - 2x^2 - x$
2. $f(x) = x^4 - 3.75x^3 + 5x^2 - 2.81x + 0.562$
3. $f(x) = \cos\big(\sin\big(4\cos(x)\big)\big)$

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4. Examine the graphs of f in relation to the graphs of r_f in each of 1-3. Graph $y = f(x)$ and $y' = r_f(x')$. *The prime notation in GC does not mean derivative. Rather, it tells GC to graph the function in a second window.*

a) Look at values of x where the value of $r_f(x)$ is negative. What does $r_f(c)$ being negative for some value of c tell you about how f is behaving over small regions around $x = c$? Why is that true?
b) Look at values of x where the value of $r_f(x)$ is positive. What does $r_f(c)$ being positive for some value of c tell you about how f is behaving over small regions around $x = c$? Why is that true?
c) Look at values of x where the value of $r_f(x)$ is 0. What does $r_f(c)$ being zero for some value of c tell you about how f is behaving over small regions around $x = c$? Why is that true?

FIGURE 12 An assignment to help students conceptualize relationships between a function's rate of change at a point, the tangent to the function's graph at that point, and what a value of the function's rate of change tells us about the function's behavior.