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Constructivism in Mathematics Education

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Background

Constructivism is an epistemological stance regarding the nature of human knowledge, having roots in the writings of Epicurus, Lucretious, Vico, Berkeley, Hume, and Kant. Modern constructivism also contains traces of pragmatism (Peirce, Baldwin, and Dewey). In mathematics education the greatest influences are due to Piaget, Vygotsky, and von Glasersfeld. See Confrey and Kazak (2006) and Steffe and Kieren (1994) for related historical accounts of constructivism in mathematics education.

There are two principle schools of thought within constructivism: radical constructivism (some people say individual or psychological), and social constructivism. Within each there is also a range of positions. While radical and social constructivism will be discussed in a later section, it should be noted that both schools are grounded in a strong Skeptical stance regarding reality and truth: *Knowledge cannot be thought of as a copy of an external reality, and claims of truth cannot be grounded in claims about reality*.

The justification of this stance toward knowledge, truth, and reality, first voiced by the Skeptics of ancient Greece, is that to verify that one's knowledge is correct, or that what one knows is true, one would need access to reality by means other than one's knowledge of it. The importance of this skeptical stance for mathematics educators is to remind them that students have their own mathematical realities that teachers and researchers can understand only via models of them (Steffe, Cobb, & Glasersfeld, 1988; Steffe, Glasersfeld, Richards, & Cobb, 1983).

Constructivism did not begin within mathematics education. Its allure to mathematics educators is rooted in their long evolving rejection of Thorndike's associationism (Thorndike, 1922; Thorndike, Cobb, Orleans, Symonds, Wald, & Woodyard, 1923) and Skinner's behaviorism (Skinner, 1972). Thorndike's stance was that learning happens by forming associations between stimuli and appropriate responses. To design instruction from Thorndike's perspective meant to arrange proper stimuli in a proper order and have students respond appropriately to those stimuli repeatedly. The behaviorist stance that mathematics educators found most objectionable evolved from Skinner's claim that all human behavior is due to environmental forces. From a behaviorist perspective, to say that children participate in their own learning, aside from being the recipient of instructional actions, is nonsense. Skinner stated his position clearly:

> Science ... has simply discovered and used subtle forces which, acting upon a mechanism, give it the direction and apparent spontaneity which make it seem alive. (Skinner, 1972, p. 3)

Behaviorism's influence on psychology, and thereby its indirect influence on mathematics education, was also reflected in two stances that were counter to mathematics educators' growing awareness of learning in classrooms. The first stance was that children's learning could be studied in laboratory settings that have no resemblance to environments in which learning actually happens. The second stance was that researchers could adopt the perspective of a universal knower. This second stance was evident in Simon and Newell's highly influential information processing psychology, in which they separated a problem's "task environment" from the problem solver's "problem space".

> We must distinguish, therefore, between the task environment—the omniscient observer's way of describing the actual problem "out there"— and the problem space—the way a particular subject represents the task in order to work on it. (H. A. Simon & Newell, 1971, p. 151)

Objections to this distinction were twofold: Psychologists considered themselves to be Simon and Newell's omniscient observers (having access to problems "out there"), and students' understandings of the problem were reduced to a subset of an observer's understanding. This stance among psychologists had the effect, in the eyes of mathematics educators, of blinding them to students' ways of thinking that did not conform to psychologists' preconceptions (Cobb, 1987; Thompson, 1982). Erlwanger (1973) revealed vividly the negative consequences of behaviorist approaches to mathematics education in his case study of a successful student in a behaviorist individualized program who succeeded by inventing mathematically invalid rules to overcome inconsistencies between his answers and an answer key.

The gradual release of mathematics education from the clutches of behaviorism, and infusions of insights from Polya's writings on problem solving (Polya, 1945, 1954, 1962), opened mathematics education to new ways of thinking about student learning and the importance of student thinking. Confrey and Kazak (2006) described the influence of research on problem solving, misconceptions, and conceptual development of mathematical ideas as precursors to the emergence of constructivism in mathematics education.

Piaget's writings had a growing influence in mathematics education once English translations became available. In England, Skemp (1961, 1962) championed Piaget's notions of schema, assimilation, accommodation, equilibration, and reflection as ways to conceptualize students' mathematical thinking as having an internal coherence. Piaget's method of clinical interviews also was attractive to researchers of students' learning. However, until 1974 mathematics educators were interested in Piaget's writings largely because they thought of his work as "developmental psychology" or "child psychology", with implications for children's learning. It was in 1974, at a conference at the University of Georgia, that Piaget's work was recognized as a new field in mathematics education, one that leveraged children's cognitive development to study the growth of knowledge.

Smock (1974) wrote of *constructivism's* implications for instruction, not *psychology's* implications for instruction. Glasersfeld (1974) wrote of Piaget's *genetic epistemology* as a theory of knowledge, not as a theory of cognitive development. The 1974 Georgia conference is the first occasion this writer could find where "constructivism" was used to describe the epistemological stance toward mathematical knowing that characterizes constructivism in mathematics education today.

Acceptance of constructivism in mathematics education was not without controversy. Disputes sometimes emerged from competing visions of desired student learning, such as students' performance on accepted measures of competency (Gagné, 1977, 1983) versus attendance to the quality of students' mathematics (Steffe & Blake, 1983) and others emerged from different conceptions of teaching effectiveness (Brophy, 1986; Confrey, 1986). Additional objections to constructivism were in reaction to its fundamental aversion to the idea of truth as a correspondence between knowledge and reality (Kilpatrick, 1987).

Radical and Social Constructivism in Mathematics Education

Radical constructivism is based on two tenets: "(1) Knowledge is not passively received but actively built up by the cognizing subject; (2) the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality" (Glasersfeld, 1989, p. 114). Glasersfeld's use of "radical" is in the sense of fundamental —that cognition is "a *constitutive* activity which, alone, is responsible for every type or kind of structure an organism comes to know" (Glasersfeld, 1974, p. 10).

Social constructivism is the stance that history and culture precede and pre-form individual knowledge. As Vygotsky famously stated, "Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first *between* people …, then *inside* the child. (Vygotsky, 1978, p. 57)

The difference between radical and social constructivism can be seen through contrasting interpretations of the following event. Vygotsky (1978) illustrated his meaning of *internalization*—"the internal reconstruction of an external operation"— by describing the development of pointing.

> The child attempts to grasp an object placed beyond his reach; his hands, stretched toward that object, remain poised in the air. His fingers make grasping movements. At this initial state pointing is represented by the child's movement, which seems to be pointing to an object—that and nothing more. When the mother comes to the child's aid and realizes his movement indicates something, the situation changes fundamentally. Pointing becomes a gesture for others. The child's unsuccessful attempt engenders a reaction not from the object he seeks but *from another person* [sic]. Consequently, *the primary meaning*

> *of that unsuccessful grasping movement is established by others* [italics added]. (Vygotsky, 1978, p. 56)

Vygotsky clearly meant that meanings originate in society and are transmitted via social interaction to children. Glasersfeld and Piaget would have listened agreeably to Vygotsky's tale—until the last sentence. They instead would have described the child as making a connection between his attempted grasping action and someone fetching what he wanted. Had it been the pet dog bringing the desired item it would have made little difference to the child in regard to the practical consequences of his action. Rather, the child realized, in a sense, "Look at what I can make others do with this action." This interpretation would fit nicely with the finding that adults mimic infants' speech abundantly (Fernald, 1992; Schachner & Hannon, 2011). Glasersfeld and Piaget might have thought that adults' imitative speech acts, once children recognize them as imitations, provide occasions for children to have a sense that they can influence actions of others through verbal behavior. This interpretation also would fit well with Bauersfeld's (1980, 1988, 1995) understanding of communication as a reflexive interchange among mutually oriented individuals: "*The [conversation] is constituted at every moment through the interaction of reflective subjects*" (Bauersfeld, 1980, p. 30 italics in original).

Paul Ernest (1991, 1994, 1998) introduced the term social constructivism to mathematics education, distinguishing between two forms of it. One form begins with a radical constructivist perspective and then accounts for human interaction in terms of mutual interpretation and adaptation (Bauersfeld, 1980, 1988, 1992). Glasersfeld (1995) considered this as just radical constructivism. The other, building from Vygotsky's notion of cultural regeneration, introduced the idea of mathematical objectivity as a social construct.

> Social constructivism links subjective and objective knowledge in a cycle in which each contributes to the renewal of the other. In this cycle, the path followed by new mathematical knowledge is from subjective knowledge (the personal creation of an individual), via publication to objective knowledge (by intersubjective scrutiny, reformulation, and acceptance). Objective knowledge is internalized and reconstructed by individuals, during the learning of mathematics, to become the individuals' subjective knowledge. Using this knowledge, individuals create and publish new mathematical knowledge, thereby completing the cycle. (Ernest, 1991, p. 43)

Ernest focused on objectivity of adult mathematics. He did not address the matter of how children's mathematics comes into being or how it might grow into something like an adult's mathematics.

Radical and social constructivists differ somewhat in the theoretical work they ask of constructivism. Radical constructivists concentrate on understanding learners' mathematical realities and the internal mechanisms by which they change. They conceive, to varying degrees, of learners in social settings, concentrating on the sense that learners make of them. They try to put themselves in the learner's place when analyzing an interaction. Social constructivists focus on social and cultural mathematical and pedagogical practices and attend to individuals' internalization of them. They conceive of learners in social settings, concentrating, to various degrees, on learners' participation in them. They take the stances, however, of an observer of social interactions and that social practices predate individuals' participation.

Conflicts between radical and social constructivism tend to come from two sources: (1) differences in meanings of truth and objectivity and their sources, and (2) misunderstandings and miscommunications between people holding contrasting positions. The matter of (1) will be addressed below. Regarding (2), Lerman (1996) claimed that radical constructivism was internally incoherent: How could radical constructivism explain agreement when persons evidently agreeing create their own realities? Steffe and Thompson (2000a) replied that interaction was at the core of Piaget's genetic epistemology and thus the idea of intersubjectivity was entirely coherent with radical constructivism. The core of the misunderstanding was that Lerman on the one hand and Steffe and Thompson on the other had different meanings for "intersubjectivity". Lerman meant "agreement of meanings"—same or similar meanings. Steffe and Thompson meant "non-conflicting mutual interpretations", which might actually entail non-agreement of meanings of which the interacting individuals are unaware. Thus, Lerman's objection was valid relative to the meaning of intersubjectivity he presumed. Lerman on one side and Steffe and Thompson on the other were in a state of intersubjectivity (in the radical constructivist sense) even though they publicly disagreed. They each presumed they understood what the other meant when in fact each understanding of the other's position was faulty.

Other tensions arose because of interlocutors' different objectives. Some mathematics educators focused on understanding individual's mathematical realities. Others focused on the social context of learning. Cobb, Yackel, and Wood (1992) diffused these tensions by refocusing discussions on the work that theories in mathematics education must do—they must contribute to our ability to improve the learning and teaching of mathematics. Cobb *et al.* first reminded the field that, from any perspective, what happens in mathematics classrooms is important for students' mathematical learning. Thus, a theoretical perspective that can capture more, and more salient, aspects for mathematics learning (including participating in practices) is the more powerful theory. With a focus on the need to understand, explain, and design events within classrooms, they recognized that there are indeed social dimensions to mathematics learning and there are psychological aspects to participating in practices, and that researchers must be able to view classrooms from either perspective while holding the other as an active background: "[W]e have proposed the metaphor of mathematics as an evolving social practice that is constituted by, *and does not exist apart from*, the constructive activities of individuals" (Cobb, et al., 1992, p. 28, italics added).

Cobb *et al.*'s perspective is entirely consistent with theories of emergence in complex systems (Davis & Simmt, 2003; Eppstein & Axtell, 1996; Resnick, 1997; Schelling, 1978) when taken with Maturana's statement that "anything said is said by an observer" (Maturana, 1987). Practices, as stable patterns of social interaction, exist in the eyes of an observer who sees them. The theoretician who understands the behavior of a complex system as entailing simultaneously both microprocesses and macrobehavior is better positioned to affect macrobehavior (by influencing microprocesses) than one who sees just one or the other. It is important to note that this notion of emergence is not the same as Ernest's notion of objectivity as described above.

Truth and Objectivity

Radical constructivists take the strong position that children have mathematical realities that do not overlap an adult's mathematics (Steffe, et al., 1983; Steffe & Thompson, 2000a). Social constructivists (of Ernest's second type) take this as pedagogical solipsism.

> The implications of [radical constructivism] are that individual knowers can construct truth that needs no corroboration from outside of the knower, making possible any number of "truths." Consider the pedagogical puzzles this creates. What is the teacher trying to teach students if they are all busy constructing their own private worlds? What are the grounds for getting the world right? Why even care whether these worlds agree? (Howe & Berv, 2000, pp. 32-33)

Howe and Berv made explicit the social constructivist stance that there is a "right" world to be got—the world of socially constructed meanings. They also revealed their unawareness that, from its very beginning, radical constructivism addressed what "negotiation" could mean in its framework and how stable patterns of meaning could emerge socially (Glasersfeld, 1972, 1975, 1977). Howe and Berv were also unaware of the notion of *epistemic subject* in radical constructivism—the mental construction of a non-specific person who has particular ways of thinking (Beth & Piaget, 1966; Glasersfeld, 1995). A teacher needn't attend to 30 mathematical realities with regard to teaching the meaning of fractions in a class of 30 children. Rather, she need only attend to perhaps 5 or 6 epistemic children and listen for which fits the ways particular children express themselves (Thompson, 2000).

A Short List: Impact of Constructivism in Mathematics Education

- Mathematics education has a new stance toward learners at all ages. One must attend to learner's mathematical realities, not just their performance.

- Current research on students' and teachers' thinking and learning is largely consistent with constructivism—to the point that articles rarely declare their basis in constructivism. Constructivism is now taken for granted.

- Teaching experiments (Cobb, 2000; Cobb & Steffe, 1983; Steffe & Thompson, 2000b) and design experiments (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) are vital and vibrant methodologies in mathematics education theory development.

- Conceptual analysis of mathematical thinking and mathematical ideas is a prominent and widely used analytic tool (Behr, Khoury, Harel, Post, & Lesh, 1997; Glasersfeld, 1995; Lobato, Hohensee, Rhodehamel, & Diamond, 2012; Smith, diSessa, & Roschelle, 1993; Thompson, 2000).

- What used to be thought of as *practice* is now conceived as *repeated experience*. Practice focuses on repeated behavior. Repeated experience focuses on repeated reasoning, which can vary in principled ways from setting to setting (Cooper, 1991; Harel, 2008a, 2008b).

- Constructivism has clear and operationalized implications for the design of instruction (Confrey, 1990; Forman, 1996; M. A. Simon, 1995; Steffe & D'Ambrosio, 1995; Thompson, 2002) and assessment (Carlson, Oehrtman, & Engelke, 2010; Kersting, Givvin, Thompson, Santagata, & Stigler, 2012).

Cross-references
-Constructivist Teaching Experiment
-Socio-mathematical Norms in Mathematics Education
-Misconceptions and Alternative Conceptions in Mathematics Education

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