Schemes For Thinking With Magnitudes: A Hypothesis About Foundational Reasoning Abilities In Algebra[1,2,3]

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Abstract

The word "magnitude" often is used loosely. People sometimes use it to refer to a general sense of quantitative size. At other times they use it to speak about large differences in size, as in "an order of magnitude larger". In this article we provide a conceptual analysis of the idea of magnitude to show that it has vastly different levels of meaning, and that these levels are characterized by particular schemes and ways of thinking about quantities and their measures. We also demonstrate that the construct "thinking with magnitudes" has a high likelihood of being an important research site for clarifying students' successes and difficulties in algebra and higher mathematics. We contend that much of past research on numerical and quantitative reasoning is clarified by attending to levels of students' ability to think with magnitudes.

Introduction

Smith and Thompson (2007) made the case that students who become accustomed to reasoning quantitatively in elementary school would be positioned to understand algebra as a generalization of their reasoning. At the end, they briefly recapped Thompson and Saldanha's (2003) explication of the notion of fraction as a scheme that entails a capacity to reason about reciprocal relationships of relative size (e.g., A is 3/7 as large as B if and only if B is 7/3 as large as A) and argued that this capacity is an important foundation for algebraic reasoning. In this article we clarify that a conception of reciprocal relationships of relative size between quantities is just the starting point for constructing meanings of number and quantity that are foundational to learning algebra and calculus. We describe mental operations that are involved in different levels of the development of a magnitude scheme, offer a new way to think about schemes and their development, discuss data involving high school teachers' meanings for magnitude and relative magnitude, and explain why reasoning with magnitudes and relative magnitudes is important for learning higher mathematics.

Concepts of Magnitude

The idea of magnitude, at all levels, is grounded in the idea of a quantity's size. A quantity, however, is not something in the world. It is a person's conception of an object and an attribute of it, and a means by which to measure that attribute. Anyone's understanding of a quantity's size will be colored by his or her conception of the quantity being considered and by his or her understanding of how it might be measured (Thompson, 1993, 1994b, 1995, 2011). For example, 110 high school geometry students were asked what they were measuring when they measured an angle; 42% of them said that they were measuring the angle's area (Thompson, 2013, p. 73). Were these students to have *really* meant what they said, they would be forced to say that the two angles in *Figure 1* have different measures because they "enclose" different areas. To these students, any reference to an angle's magnitude would have little meaning because even when thinking of size, an angle would not have a unique size. In other words, the discussion here presumes, simply for convenience, that someone thinking of a quantity has made the significant intellectual achievement of conceiving it coherently.

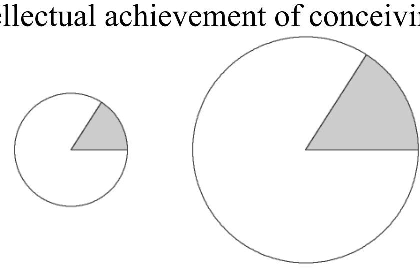

Figure 1. Two angles each with measure 1 radian but having different "sizes".

There are five levels of meaning for a quantity's magnitude that are largely undifferentiated in the research literature. They are:

- A person has a sense of the extent of a quantity. She is capable of making perceptual judgments of whether Quantity A is larger or smaller than Quantity B. We will call this sense of magnitude *Awareness of size*.

- A person equates a quantity's size and its measure, where "measure" means "number of units". We call this sense of magnitude *Measure magnitude*.

- A person conceives of the size of Quantity A relative to a unit B and they are both measured in a common unit. This meaning of magnitude is developed through the coordination of measurement and number schemes. We will call this sense of magnitude *Steffe magnitude*. It is characterized by the ability to think of size relative to a composite unit (a unit of units).

- A person conceives of Quantity A measured in unit *a* as having a Steffe magnitude while at the same time conceiving of its unit *a* as having a Steffe magnitude. In addition, the person sees the pair (Steffe magnitude of A, Steffe magnitude of *a*) as exemplifying a relationship that is invariant with changes in *a*. We will call this sense of magnitude *Wildi magnitude*.

- A person conceives of the Wildi magnitude of Quantity A in relation to the Wildi magnitude of Quantity B, and sees the *relationship* as invariant with respect to changes in in either or both the unit of B and the unit of A. We will call this sense of magnitude *Relative magnitude*. It is worth noting that high-level scientific reasoning that involves physical quantities typically involves conceiving of relative magnitudes.

Measure Magnitude

Thinking of size as a Measure magnitude entails an additive understanding of measure, and therefore of magnitude. A person conceiving of magnitude as measure does not distinguish the two. Something has a length of 12 feet, and the person conceiving of 12 feet conceives its length as containing 12 parts, each "one foot" in length. The parts have the same length, but only because if one part had a different length it would not be called "a foot". A person thinking with Measure magnitude will likely think that a length stated as 13 feet is smaller than a length stated as 396.24 centimeters because 396.24 is larger than 13. The question of equivalence does not arise. In Steffe's system, Measure magnitude involves two levels of units—a quantity taken as a measured whole and a conception of the quantity segmented into parts, each part taken nominally to be the same size and collectively taken to constitute the whole.

Steffe Magnitude

Thinking of size as a Steffe magnitude entails thinking of measurement multiplicatively and reciprocally. For Quantity B to be 7/3 as large as Quantity A means that B is 7 times as large as 1/3 of A (*Figure 1*, left). Thinking of size as a Steffe magnitude also involves reciprocity of measured size. B being 7/3 times as large as A means that A is 3/7 as times as large as B (*Figure 1*, right). B being 7/3 times as large as A gives a measure of B in units of A. A being 3/7 times as large as B gives a measure of A in units of B. In Steffe's system, Steffe magnitude involves three levels of units—a unit in which both B and A are measured and B in units of A (or A in units of B).

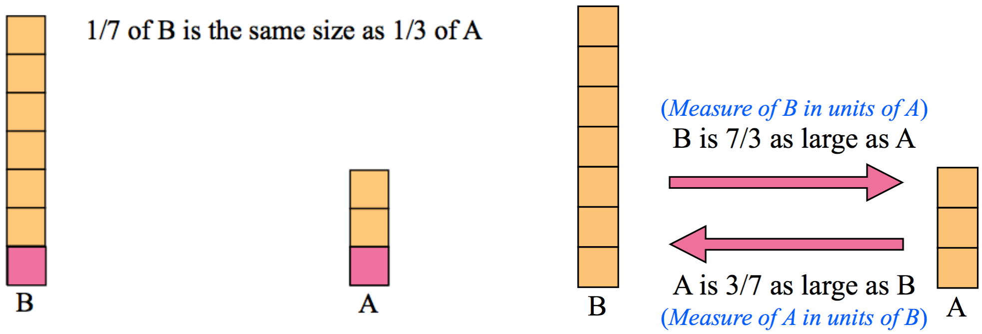

Figure 2. Steffe magnitude entails reciprocal relationship of relative size. B measured in units of A is 7/3, which implies that A measured in units of B is 3/7 (and vice versa).

Wildi Magnitude

Thinking with Steffe magnitudes supports conceiving numerically of a quantity's size relative to the size of its unit. But the size of the quantity's unit, when thinking with Steffe magnitudes, is a Measure magnitude. Two advances can be made in thinking with Steffe magnitudes. The first is to conceive a quantity's unit as having a Steffe magnitude, as opposed to a Measure magnitude, relative to a unit *u*. The second is to conceive the implications for the measure of B when its unit is re-expressed in another unit. When a person thinks of a quantity's magnitude in this way, we will call it thinking with *Wildi* magnitudes.

We will use the notation $m(B)_b$ to represent the measure of quantity *B* in units of *b* and we will use $\|B\|$ to represent the Wildi magnitude of *B*. If we conceive *b* as having a Steffe magnitude, and we think of a new unit *b'* such that $m(b')_b = 3/4$, then we can conclude immediately (because of being able to think with Steffe magnitudes) that $m(b)_{b'} = 4/3$, or that $\|b\| = \frac{4}{3}\|b'\|$. Suppose that $m(B)_b = k$. Inside every one of *k* *b*-units (or part of a *b*-unit) is (4/3) *b'*-units (or a proportional part of 4/3 of a *b'*-unit). This means that if $m(B)_b = k$, and if $m(b')_b = 3/4$, then $m(B)_{b'} = \frac{4}{3}k$. This line of reasoning is depicted in *Figure 3*.

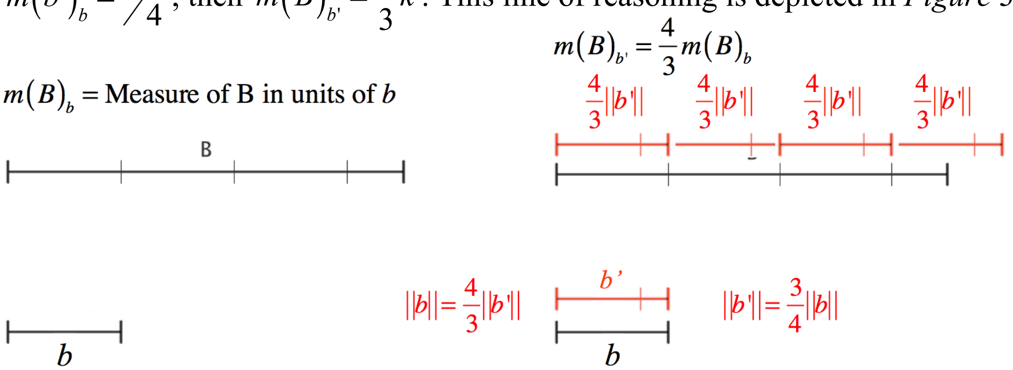

Figure 3. Relationship between measures under change of unit when thinking with Wildi magnitudes.

In general, if $m(B)_b = k$ and $\|b'\| = r\|b\|$, then $m(B)_{b'} = \frac{1}{r}m(B)_b$. In words, this says that if a quantity *B* is measured in units of *b*, and the magnitude of *b*' is *r* times as large as the magnitude of b, then the measure of B in units of b' is ($1/r$) times as large as the measure of B in units of b.

Wildi (1991) defined the magnitude of a quantity Q as the measure of Q, with respect to a unit u, times the magnitude of u. We express this symbolically as $\|Q\| = m(Q)_u \|u\|$. It is important to understand that the magnitude of u is not a measure. It is the size of an object having the attribute that is being measured and that is taken to have one unit of that attribute. The American unit "foot", in the not-too-distant past, was the length of a specific platinum bar held in Washington DC at the National Bureau of Standards.

The power of Wildi's definition is that it makes explicit the fact that the magnitude of a quantity is invariant with respect to a change of unit. If a quantity Q has a length of 22 inches, then $\|Q\| = 22\|inch\|$, meaning that the magnitude of Q is 22 times as large as the magnitude of an inch, while at the same time $\|Q\| = \frac{22}{12}\|foot\|$, because a foot is 12 times as long as an inch.

A person thinking that Q has a length of 22 inches with an image of having laid 22 inches alongside Q is thinking of the Measure magnitude of Q. A person thinking that Q has a length of 22 inches with an image of the length of Q as being 22 times as long as the length of an inch is thinking of the Steffe magnitude of Q. A person who says that Q has a magnitude of 22 inches while understanding that the magnitude of Q will remain the same with a change of unit, and who anticipates *how* the measure of Q will change by knowing the relationship between units, is thinking of the Wildi magnitude of Q. When a person anticipates that any measurement of Q with respect to an appropriate unit can be expressed in any other appropriate unit by some conversion without changing Q's magnitude, she possesses Wildi's meaning of magnitude.

The phrase "any other appropriate unit" hides a sophisticated understanding of the quantity in question and units in which it can be measured. It expresses a confidence that comes with having a scheme that anticipates such conversions for that quantity. For example, John's height in feet is 6.17. A person who thinks with Wildi magnitudes, and who knows the appropriate relationships, can anticipate that John's height can be measured in light years—and it will remain the same height. A light year is the distance light travels in a year. The speed of light is (approximately) 186,282 miles/sec, so a light year is 186,282×(the number of seconds in a year) *miles*. This number of miles can be converted into a number of feet, the reciprocal of which gives the number of light years in one foot. Hence John's height in light years is 6.17 times as large as the magnitude of one foot expressed in light years. The confidence shown in expressing John's height as a number of light years also entails a facility with quantitative, numerical, and arithmetical operations that are deeply rooted in (reflectively abstracted from) having reasoned many times about quantities in settings that require a change of units.

We stress that when someone conceives of a quantity's size as a Wildi magnitude that this is more than being able to convert from one unit to another. Thinking of a quantity's size as a Wildi magnitude entails the anticipation that the quantity's magnitude is invariant with *any* change in unit. Put another way, to think with Wildi magnitudes requires an operational scheme of meanings for the quantity in question, its size, how one measures it, and the formation of its unit. This scheme involves arithmetical operations (multiplication and division), quantitative operations (proportional correspondence), measure (as a multiplicative relationship between two quantities' magnitudes), quotient (as a measure of relative size of measures), and fraction (hierarchical relationships of size). We will say more about schemes in the next section. We should also point out that we do not claim that a person who has a scheme for Wildi magnitudes has the general ability to convert from one unit to another. One must understand the quantity under consideration and the meaning of its units. To think about the number of watts that is equivalent to 465 horsepower, one must understand the quantity *power* and its quantification.

As an aside, we suspect that the US Congress' 1976 effort to have the U.S. convert to the metric system (Wikepedia, 2013) failed largely because of the popular inability to think in terms of Wildi magnitudes (and perhaps even Steffe magnitudes). People were not proficient in converting among units, nor did not have imagistic benchmarks for metric units that allowed them to think intuitively about how large was some feature of an object when its measure was stated metrically. This is similar to students' and teachers' lack of imagistic benchmarks for directions associated with 1 radian from horizontal, 2 radians from horizontal and so on, despite having imagistic benchmarks for directions associated with angle measures given in terms of "π" (e.g., π/3, π/4, and π/2).

Relative Magnitude

We measure a quantity in relation to a unit. "Clearly, the unit must be of the same nature as the quantity being measured" (Wildi, 1991, p. 58). We measure an object's weight by comparing it with a standard weight; we measure an amount of twist by comparing it with a standard amount of twist; we measure a light's luminance by comparing it with a standard luminance. But we also measure quantities that are made by relating two quantities of different natures. We measure speed by relating a distance traveled and the amount of time required to travel that distance. We measure force by relating an object's mass with its acceleration. As explained by Johnson (this volume) and Steffe (this volume), a quantity created by relating two quantities is called an *intensive* quantity.

Upon traveling 62 miles in 2.7 hours, we do not divide 62 miles by 2.7 hours to see how many durations of length 2.7 hours are contained in 62 miles. Rather, we use the logic of proportional correspondence (Thompson, 1994b; Thompson & Thompson, 1994) to reason that, assuming a constant speed, 1 hour of driving (1/2.7 of 2.7 hours) will correspond to 22.96 miles (1/2.7 of 62 miles). In other words, division of measures determines the relative size of two measures. When driving at a constant speed such that in 2.7 hours one drives 62 miles, the number of miles driven will always be 22.96 times as large as the number of hours spent driving. We point out here that thinking of relative magnitude relies on being able to use distributed partitioning (Steffe & Olive, 2010). One must employ the reasoning that $1/n$ of m units is the same as m/n of one unit.

The most rudimentary measure of an intensive quantity is to determine the relative size of its constituent quantity's measures, as in the example of driving 62 miles in 2.7 hours. But what could we mean by the *magnitude* of an intensive quantity? We propose the term *relative magnitude* to convey the idea that the person conceiving an intensive quantity conceives of the constituent quantities as having Wildi magnitude. A person who conceives an intensive quantity as constituted by quantities having Wildi magnitude is positioned to see the *relationship* between quantities as being invariant with change of unit in either or both of its constituent quantities.

A person who conceives distance and time as having Wildi magnitudes will understand that a car's speed of 60 miles per hour is the same speed as when duration is measured in one second. The car will travel 1/3600 as far in one second as it will in one hour, because $\|s\| = \frac{1}{3600}\|hr\|$. Thus a speed of 60 miles per hour can be expressed as 60/3600 miles per second. Similarly, a speed of 60 miles per hour can be expressed as $5280 \cdot 60/3600$ feet per second because $\|ft\| = \frac{1}{5280}\|mi\|$. The reasoning behind these conversions entails using reciprocal relationships of relative size (in relating the magnitude of one unit to the magnitude of another) and two applications of proportional correspondence (expressing relative size with respect to the first unit, then re-expressing relative size in terms that relate the measure relative to the second unit in terms of measure relative to the first).

We will use the notation $\text{RelMag}(R,S)_{r,s}$ to represent the relative magnitude of quantities *R* and *S* in the context of measuring *R* in units of *r* and measuring *S* in units of *s*. The general form of reasoning that supports re-expressing relative size so that it keeps relative magnitude invariant is summarized below.

- $m(B)_{b'} = \frac{1}{k} m(B)_b$ when $\|b'\| = k\|b\|$
- $\text{RelMag}(R,S)_{r,s} = \dfrac{m(R)_r}{m(S)_s}$
- $\|r'\| = k\|r\|,\ \|s'\| = j\|s\|$
- $\text{RelMag}(R,S)_{r',s'} = \dfrac{m(R)_{r'}}{m(S)_{s'}} = \dfrac{\frac{1}{k}m(R)_r}{\frac{1}{j}m(S)_s} = \left(\dfrac{j}{k}\right)\left(\dfrac{m(R)_r}{m(S)_s}\right) = \left(\dfrac{j}{k}\right)\text{RelMag}(R,S)_{r,s}$

Thinking of an object *B*'s density as a relative magnitude of its mass and its volume involves seeing *B*'s mass and *B*'s volume as having Wildi magnitudes. Suppose *B* has a mass of 5 pounds (avoirdupois) and a volume of 27 cubic inches. Understood as Wildi magnitudes, we anticipate that 5 pounds and 27 cubic inches can each be re-expressed in any other appropriate unit without changing the relative magnitude of the *B's* mass and volume. Because we understand that relative magnitude is measured by a quotient of measures, we know that $\text{RelMag}(B_{mass}, B_{vol})_{lb,in^3} = \dfrac{m(B_{mass})_{lb}}{m(B_{vol})_{in^3}} = \dfrac{5}{27}$. *B*'s density is 5/27 pounds per cubic inch. Suppose we are told that $\|jib\| = \frac{1}{738}\|pound\|$ and that $\|jab\| = \frac{1}{15}\|in\|$. First, we immediately conclude that $\|jab^3\| = \left(\frac{1}{15}\right)^3 \|in^3\|$. We then go on to reason,

$$\begin{aligned}
\text{RelMag}\left(B_{mass}, B_{vol}\right)_{jib,jab^3} &= \frac{m\left(B_{mass}\right)_{jib}}{m\left(B_{vol}\right)_{jab^3}} \\
&= \frac{\frac{1}{\left(1/738\right)} m\left(B_{mass}\right)_{lb}}{\frac{1}{\left(1/15^3\right)} m\left(B_{vol}\right)_{in^3}} \\
&= \left(\frac{738}{15^3}\right) \frac{m\left(B_{mass}\right)_{lb}}{m\left(B_{vol}\right)_{in^3}} \\
&= \left(\frac{738}{15^3}\right) \text{RelMag}\left(B_{mass}, B_{vol}\right)_{lb,in^3} \\
&= \left(\frac{738}{15^3}\right) \frac{5}{27}
\end{aligned}$$

It is important to notice that a relative magnitude of 5/27 is a number that tells you that the magnitude of the relationship between B_{mass} and B_{vol} is 5/27 times as large as the magnitude of the relative magnitude of one pound and one cubic inch—the number of pounds is 5/27 times as large as the number of cubic inches. Similarly, the relative magnitude of B_{mass} and B_{vol}, when B_{mass} is measured in jibs and B_{vol} is measured in cubic jabs, is 0.04 times as large as the relative magnitude of one jib and one cubic jab—the number of jibs is 0.04 times as large as the number of cubic jabs. Moreover, a person reasoning with relative magnitudes, during the whole process, will not lose sight of the fact that *B*'s density is the same in either representation. *B*'s density is 5/27 lb/in^3 and it is 0.04 jibs/jab^3. Both express the relative magnitude of *B*'s mass and *B*'s volume.

It is clear to us that a requirement for anyone to think with relative magnitudes is that her scheme for Wildi magnitudes is operational. We say this because the scheme for Wildi magnitude, as exemplified in the discussion of *B*'s density, must be used on both B_{mass} and on B_{vol}—simultaneously. We suspect that it is through attempts to reason about relative magnitudes that the scheme for Wildi magnitudes actually becomes operational.

As for practical examples of thinking with relative magnitude, we offer the situation below.

> In the United States, a car's fuel efficiency is measured in miles traveled per gallon of gasoline consumed. In Europe, a car's fuel efficiency is measured in liters of gasoline consumed per 100 kilometers traveled. Rachel's Ford Focus gets 36 mi/gal on the highway. She took her car to France while studying at the Sorbonne and now wants to sell it (in France). How should she state her car's fuel efficiency in a newspaper ad?

As a research item we see it offering opportunities to probe these issues:

- How do individuals think about the quantities involved? At what level of magnitude do they think of them?

- Do people think of fuel efficiency as a quantity? If so, do they see fuel efficiency as being invariant with change of units?
- In what ways are individuals' abilities to think with magnitudes related to their familiarity with unit conversions? Are advanced forms of thinking with magnitudes dependent on conversions only as matters of information?

Comments on Magnitude and Relative Magnitude

Our discussion of magnitude and relative magnitude started with the notion of magnitude as awareness of size, then as thinking of measure as magnitude, then thinking of size relative to a composite unit, then thinking of magnitude as an amount that is invariant with change in unit, then thinking of the magnitude of intensive quantities wherein the *relationship* between magnitudes remains invariant with changes in units. We suspect that there is a developmental trajectory lurking within these levels of meaning of "magnitude", but not one that is spontaneous. In the United States, given its current state of mathematics education, the trajectory cannot even be called purposive. We see no attention given to the idea of magnitude and its development—not within curriculum, within teacher education, and rarely within mathematics education research.

We have a strong suspicion that thinking with magnitudes is highly related to students' abilities to reason numerically, especially with fractions, and with numbers generically. We also suspect that children's development of algebraic reasoning and calculus reasoning is strongly dependent upon their abilities to think with magnitudes. Higher levels of thinking with magnitudes involves reasoning with general relationships and general numbers, so students whose thinking is at the higher levels of magnitude will be able to ground their reasoning about algebra concretely within their understandings of magnitudes. We do not know this as a fact, because research on mathematics learning has not attended to issues of magnitude. It seems, to us, that this is a rich area for future research. Our intention here is to provide a framework for thinking about magnitudes that might support this research.

We have used the phrase "thinking with magnitudes" repeatedly. It should now be clear that this phrase is ambiguous. A student whose meaning of magnitude is at any of these levels will think with the meanings of magnitude that he or she holds. The phrases "thinking with magnitudes" and "a quantity's magnitude" can only have meaning when we are specific about the meanings of magnitude that we are attributing to particular persons in particular settings or when we are specific about the meanings we are attributing to an epistemic student who we take as having those meanings. Thus, "thinking with magnitudes", by itself, can mean no more than that students employ *their* meanings of magnitudes in their thinking.

Magnitude Schemes

Having spoken of schemes repeatedly we feel obliged to say what we mean by a scheme and to speak to the matter of how schemes develop.

Cobb and Glasersfeld (1983) and Glasersfeld (1995, 1998) proposed that, to Piaget, a scheme was a three-part mental structure: a condition that would trigger a scheme, an action or system of actions, and an anticipation of what the action should produce. We believe that Cobb and Glasersfeld said less than they meant, as their wording suggests that a scheme is like a condition-response pair. This interpretation fits better with Piaget called a *schema* of action

(Piaget, 1968, p. 11; Piaget & Inhelder, 1969, p. 4). Piaget spoke of a child's sucking schema, for example.[4] Having worked with Glasersfeld over many years, and having used the concept of scheme with Glasersfeld to describe organizations of complex thought processes, we offer an elaboration of Glasersfeld's definition that we believe makes its explanatory power more evident.

Piaget's use of "scheme" often was quite utilitarian. It allowed him to speak of mental organizations that supported flexible thinking and reasoning without having to specify the contents of those organizations, and he spoke of schemes at different levels of sophistication without elaborating their contents. Montangero and Maurice-Naville (1997, p. 155) supplied a compendium of Piaget's various uses of "scheme". We quote it in its entirety. Years in brackets are the year of the original French publication.

1) "We shall apply the term 'action schemata' to whatever, in an action, can thus be transposed, generalized, or differentiated from one situate to another: in other words, whatever is in common between various repetitions or superpositions of the same actions" (*Biology and Knowledge*, [1967], p. 7).

2) "A scheme is the structure or the organization of actions which is transferred or generalized when this action is repeated in similar or analogous circumstances" (*The Psychology of the Child*, [1966] 1969, p. 11, footnote not translated in the English version)

3) "These patterns [schemes] being none other than the whole gamut of actions capable of [being actively repeated]" (*The Psychology of Intelligence*, [1948] 1950, p. 8).

4) "The system, composed of determined and completed movements and perceptions, reveals the dual character of being structured (hence of itself structuring the perception or comprehension) and of constituting itself from the outset inasmuch as it is a totality" (*The Origins of Intelligence in Children*, [1936] 1977, p. 417.

5) [Schemes are] organized totalities whose internal elements are mutually implied (*The Origins of Intelligence in Children*, [1936] 1977, p. 445).

6) "The scheme of an action is neither perceptible (one perceives a particular action, but not its scheme) nor directly introspectible, and we do not become conscious of its implications except by repeating the action and comparing its successive results" (*Mathematical Epistemology and Psychology*, [1961] 1966, p. 235).

We must remind ourselves that by "action" Piaget meant "all movement, all thought, or all emotions that responds to a need" (Piaget, 1968, p. 6). As such, we see immediately that the organizations of which Piaget spoke in these six definitions of scheme are mental organizations. Schemes are organizations of mental activity that express themselves in behavior, from which we, as observers, discern meanings and ways of thinking. Scheme is a theoretical construct that we impute to individuals to explain their behavior.

[4] We agree with Montangero and Maurice-Naville (1997, p. 154) when they say that Piaget made a strong distinction between the ideas of schema and scheme. A schema is much simpler than a scheme, having an organization that supports stimulus-response behaviors.

To capture the complexity of thinking with magnitudes (or the complexity of thinking of constant rate of change, derivative, integral, function, and so on) we are forced to expand Cobb and Glasersfeld's definition of *scheme*. We define a scheme as *an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity.*

Thompson (1994c, 1996) explained the vital role that imagery plays in Piaget's meaning of scheme and its development. Thompson characterized an image as

> By "image" I mean much more than a mental picture. Rather, I mean "image" as the kind of knowledge that enables one to walk into a room full of old friends and expect to know how events will unfold. An image is constituted by coordinated fragments of experience from kinesthesia, proprioception, smell, touch, taste, vision, or hearing. It seems essential also to include the possibility that images entail fragments of past affective experiences, such as fearing, enjoying, or puzzling, and fragments of past cognitive experiences, such as judging, deciding, inferring, or imagining[5]. Images are less well delineated than are schemes of actions or operations (Cobb & von Glasersfeld, 1983). They are more akin to figural knowledge (Johnson, 1987; Thompson, 1985) and metaphor (Goldenberg, 1988). A person's images can be drawn from many sources, and hence they tend to be highly idiosyncratic. (Thompson, 1994c, pp. 229-230)

Thompson (1994c) went on to explain the ways in which the notion of image is intertwined with Piaget's concept of scheme and of mental operation. He pointed out three levels of imagery in Piaget's work. The first level of imagery is when a child engages in deferred imitation. Deferred imitation is when a child acts the imitated behavior to have an image of it. The second level of imagery is an image of a desired state and actions that can attain it, but the actions and image are intermingled, so it is easy for the child to "lose his image" when his present situation obstructs acting in a way he intended. The third level of imagery is operative.

> [This is an image] that is dynamic and mobile in character … entirely concerned with the transformations of the object. … [The image] is no longer a necessary aid to thought, for the actions which it represents are henceforth independent of their physical realization and consist only of transformations grouped in free, transitive and reversible combination … In short, the image is now no more than a symbol of an operation, an imitative symbol like its precursors, but one which is constantly outpaced by the dynamics of the transformations. Its sole function is now to express certain momentary states occurring in the course of such transformations by way of references or symbolic allusions." (Piaget, 1967, p. 296)

Our reason for dwelling on Piaget's notion of image is because, in developing a scheme, a student must develop images of having reasoned in particular ways, meaning she must develop recollections of "momentary states" in having reasoned. Students "construct stable

[5] Tom Kieren and Susan Pirie (Kieren, 1988; Kieren & Pirie, 1990; Kieren & Pirie, 1991; Pirie & Kieren, 1991) make it evident that the act of imagining can itself inform our images.

understandings by repeatedly constructing them anew" (Thompson, 2013, p. 61). To construct a scheme, students must repetitively engage in the reasoning that will become solidified in that scheme in order to have occasions to develop the imagery that supports it (Cooper, 1991; Harel, 2008a, 2008b, 2013). It seems trite to say so, but we are compelled to point out that for students to develop schemes for reasoning with magnitudes, they must attempt to reason with magnitudes and about magnitudes.

Piaget was aware that there are types of organizations of mental activity that exemplify schemes at their highest level. These are schemes that support flexible, innovative, creative thinking by making connections among meanings and ways of thinking that are typical of high forms of thought. He called these organizations *groupings*. Groupings of mental actions are characterized by the properties of

- Composition (two actions within the grouping can be carried out in thought as one action)
- Associativity (chains of actions within the grouping can be restructured mentally)
- Reversibility (initial states of actions can be viewed in retrospect in relation to their consequences)

What we have characterized as "thinking with relative magnitudes" might be better characterized as a grouping. It constitutes an organization of meanings and ways of thinking that, within their domain, form a complex, closed, and operational system in which students can compose actions flexibly and keep in mind the different parts of their reasoning process. We suspect that Piaget might have disagreed with our use of "grouping". His intent was to describe huge cognitive organizations, like a concept of space. We nevertheless believe that it is useful to distinguish between the vastly different levels of sophistication of, for example, a counting scheme and a relative magnitude scheme and we propose to use the term *grouping* to distinguish the highest levels. We are not wed to this usage of *grouping*. But we think "scheme" is used too commonly without differentiating among vastly different levels of cognitive behavior.

Understandings, Meanings, and Ways of Thinking

In our prior discussion of thinking with magnitudes we used the words "understanding", "meaning", and "way of thinking" advisedly. With our definition of scheme and description of the role of imagery in schemes, we can be more precise about our use of these affiliated terms.

In the past, Thompson used "to understand" in a Piagetian tradition—as to assimilate to a scheme (Skemp, 1961, 1979; Thompson, 2013; Thompson & Saldanha, 2003). Thus, an understanding was always what a person understood. It was non-judgmental regarding correctness or appropriateness of the person's understanding. Likewise, Harel (2008a, 2008b, 2008c) introduced the phrase "way of understanding" to distinguish between "understand" as conveying a judgment that a person has a normatively correct understanding of a mathematical idea and "way of understanding" as the system of meanings that person used to understand a situation, an inscription, or a mathematical utterance. While the two usages had the same intent—to emphasize that mathematics is personal, and even "standard" mathematics rests upon personal meanings that are negotiated within a community—they were not completely aligned. As a result of recent collaborations, Harel and Thompson have developed a system for addressing issues of understanding, meaning, and ways of thinking that they hope lends coherence to the use of these terms (Table 1).

Table 1

Definitions of understanding, meaning, and ways of thinking. (Thompson & Harel, in preparation; Thompson, Harel, & Thomas, in contract)

Construct	Definition
Understanding (in the moment)	Cognitive state resulting from an assimilation
Meaning (in the moment)	The space of implications existing at the moment of understanding
Understanding (stable)	Cognitive state resulting from an assimilation to a scheme
Meaning (stable)	The space of implications that results from having assimilated to a scheme. The scheme is the meaning. What Harel previously called Way of Understanding
Way of Thinking	Habitual anticipation of specific meanings or ways of thinking in reasoning

This system for the use of "understanding", "meaning" and "way of thinking" continues Harel's and Thompson's quest to decouple "understand" and "correct understanding". They do this by resting their system on the idea of assimilation. They rely on Piaget's characterization of assimilation as, in effect, giving meaning.

> Assimilating an object to a scheme involves giving one or several meanings to this object, and it is this conferring of meanings that implies a more or less complete system of inferences, even when it is simply a question of verifying a fact. In short, we could say that an assimilation is an association accompanied by inference. (Johnckheere, Mandelbrot, & Piaget, 1958, p. 59) as translated by (Montangero & Maurice-Naville, 1997, p. 72)

Table 1's first entry (*Understanding in the moment*) describes a person who has an understanding of something said, written, or done in the moment of understanding it. Technically, all understandings are understandings-in-the-moment. Some understandings, however, might be a state that the person has struggled to attain at that moment through functional accommodations to existing schemes (Steffe, 1991), and is easily lost once the person's attention moves on. This type of understanding is typical when a person is making sense of an idea for the first time. The *meaning of an understanding* is the space of implications that the current understanding mobilizes—actions or schemes that the current understanding implies, that the current understanding brings to mind with little effort.[6] An understanding is *stable* if it is the result of an assimilation to a scheme. A scheme, being stable, then constitutes the space of implications resulting from the person's assimilation of anything to it. The scheme is the meaning of the understanding that the person constructs in the moment. As an aside, schemes provide the "way"

[6] Recall that one of Piaget's definitions of scheme was (p. 10 of this paper), "[Schemes are] organized totalities whose internal elements are mutually implied."

in Harel's "way of understanding". Finally, Harel and Thompson characterize "way of thinking" as when a person has developed a pattern for utilizing specific meanings or ways of thinking in reasoning about particular ideas.

Thus, a scheme for magnitudes is a meaning for magnitudes. The five meanings of magnitude we listed earlier are schemes. A person who understands a situation as involving magnitudes has assimilated the situation to a scheme for magnitudes. The use of a relative magnitude scheme routinely in dealing with situations that a person sees as involving relative magnitudes shows that the person has a way of thinking with relative magnitudes. The ability to explain what a relative magnitude is shows that the person has a way of thinking *about* relative magnitudes.

Comments on the Development of Schemes

We hope that our characterization of levels of meaning of "magnitude" conveys the large number of moving parts in even mid-level ways of thinking about magnitude. The question now is, "How might students develop these ways of thinking?" In the current age of learning progressions we can anticipate advice like this. Children must

- Learn about size and become skilled at comparing sizes
- Learn to measure
- Learn to change a unit of measure
- Learn to divide measures
- Learn to change units of measures, then divide

While the above is certainly a caricature of learning progressions, it also captures the common idea (which harkens from the days of Gagné, 1977) that students must attain a certain level of proficiency with prerequisite skills before they can master subsequent skills.[7] It also fits the goal that every item be unidimensional by those who use Rasch or IRT models to assess students' level in a progression—that it assesses behavior that is too simple to be the expression of a scheme. Instead of thinking in terms of learning progressions and unidimensional items to assess students' placement in them, we think of scheme development in terms of the formation of learning clouds. We will use proportionality as an example.

Our approach is to give an evolutionary view on the progression of proportional reasoning in learners over ages 6 to 18. We hesitate to use the word "progression" in this context because of the image it often conveys of one thing happening after another—a progression in steps. Instead, we hope to convey an image of parallel developments of ways of thinking that are always in interaction and yet which constitute the span of those curricular concepts that compose proportional reasoning. We speak of any one way of thinking that we take as foundational for a particular curricular concept as also participating in the activity of other curricular concepts and ways of thinking. The result is an ensemble of meanings and ways of thinking such that, at every moment in the child's development of proportional reasoning, any two aspects of proportional reasoning entails some common ways of thinking while at the same time involving ways of thinking that are unique to themselves—whence the idea of a *developmental cloud*.

We describe, for example, ways of thinking about multiplicative comparisons that underlie the curricular concepts *constant rate of change* and *measurement* (thus explicating how,

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[7] We hasten to point out that the work on learning progressions by Rich Lehrer and Leona Schauble (Lehrer, 2013) departs dramatically from this description.

as curricular concepts, they are related) and how they each involve ways of thinking that are unique (thus explicating how, as curricular concepts, they differ). We, as observers of students' thinking, anticipate that the distinction between ways of thinking that are highly similar yet distinct often is inherited from earlier ways of thinking that evolved tightly with concrete situations that children made meaningful. Covariation, for example, has partial roots in double counting—the coordination of two counting sequences (e.g., computing 6+5 by counting 7 (is 1), 8 (is 2), and so on).

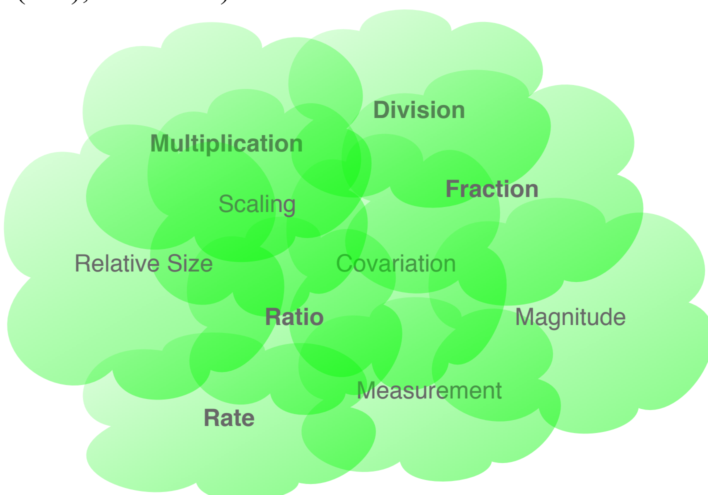


Figure 4. The Proportionality "Cloud": Aspects of Proportional Reasoning

Proportionality has many aspects. Each entails ways of thinking about quantities, relationships among quantities, and relationships among ways of thinking. As a result, proportional reasoning appears in various guises in different contexts and different levels of sophistication. It is also worth noting that these ways of thinking are involved in service to each other and that they do not align with conventional categorizations of curricular topics.

Another concern with the use of "progression" is that its common meaning often constrains one to speak of one idea at a time, in isolation of others. But we must consider, for example, multiplication as a conceptual site for exercising ways of thinking that underlie related curricular ideas like measurement, fraction, division, and constant rate of change. To do that, we focus on the evolution of ways of thinking that are at the foundation of a particular curricular concept while simultaneously pointing out how these ways of thinking participate in other curricular concepts. Our aim for this approach to explicating a curricular concept's progression is to provide a platform for talking about how a teacher can focus on one idea while simultaneously incorporating and building upon related ideas.

We accept the cloud metaphor as a way to think of scheme development for all major mathematical ideas. Thus, we see the development of schemes for magnitude, function, variable, rate of change, derivative, integral, differential equation, and so on as all having the character of learning clouds. Two challenges come with this metaphor: (1) making it precise through models of individual students' learning, and (2) developing a methodology for profiling the state of a student's cloud. The latter will be especially challenging. Students will have some ways of thinking that are more or less advanced than others, which will impact the development of other parts of the cloud that interact with it and with which it interacts. Also, the patterns of cloud formation will vary across students. Some students will be more advanced in one area and less advanced in others in comparison to other students, which will produce different patterns of functioning.

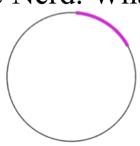
One final note: Our descriptions of ways of thinking that participate in other ways of thinking (e.g., ways of thinking about partitioning that participate in ways of thinking about measuring) are not about natural cognitive development. We take as axiomatic that students' experiences in instruction both facilitate and constrain their opportunities to think about particular ideas and how they are related.

Middle and High School Mathematics Teachers' Understandings of Magnitude

While writing this paper, and becoming clearer in our own thinking about the nature and significance of reasoning with magnitudes, we decided to collect data on individual's understandings of magnitude. We piloted early versions of items in *Figure 5* with 114 undergraduate mathematics students. Results from the pilot were that college mathematics students reason about magnitudes very poorly. After revising the items and rubrics for scoring them based on the pilot, we gave the items in *Figure 5* to 112 middle and high school mathematics teachers who participated in summer MSP workshops in the Midwest (60) and the Southwest (62). We surveyed middle and high school teachers' thinking about magnitude to get a sense of opportunities that students might have for thinking about magnitudes. Our presumption is that teachers provide a gateway to students' futures, so understanding teachers' capabilities can give a more complete understanding of students' opportunities to learn.

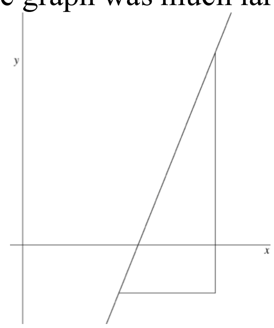
Our intention was to devise items that would reveal aspects of teachers' understandings of magnitude and their disposition to think with magnitudes. At the time we wrote the items we had not clarified the issue of relative magnitude, so our items focused on Wildi magnitudes. Questions 1 and 2 focused on converting a measure in one unit to a measure in another given a relationship between the units. We constructed Question 3 with enough information that teachers could estimate the relative magnitudes of changes in x and y by physically measuring Δy in units of Δx (exact measure was 2.5).

1) In Nerdland they measure lengths in Nerds. The highlighted arc measured in Nerds is 12 Nerds. In Rapland they measure lengths in Raps. One Rap is 3/4 the length of one Nerd. What is the measure of the highlighted arc in Raps?



2) A container has a volume of *m* liters. One gallon is 189/50 times as large as one liter. What is the container's volume in gallons? Explain.

3) *Part A*. The graph of a linear function $y = mx + b$ is given below. *x* and *y* are in the same scale. What is the numerical value of *m*? Explain your reasoning briefly. (The graph was much larger.)



Part B. What would be the numerical value of *m* if the *y*-axis were re-scaled so that the distance between 0 and 1 is 2 times as large as the original?

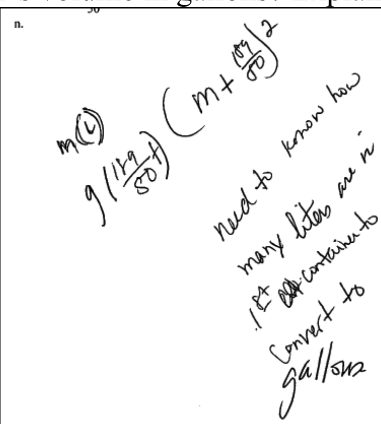
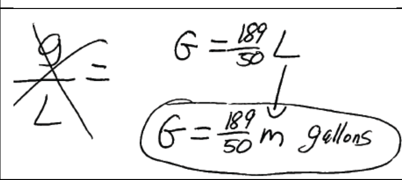
Figure 5. Items given to 112 middle and high school mathematics teachers.

We designed Question 1 so that it would be about length, but not about length straightforwardly. We gave information about the length's measure in one unit (Nerds) and a relationship between Nerds and a new unit having 3/4 the magnitude of a Nerd. We wanted to see if teachers would reason that if ||Rap|| = 3/4||Nerd||, then inside each Nerd is 4/3 Raps. Question 2 is identical to Question 1 with the exception that the new unit is larger than the original unit, the units (liters and gallons) and their sizes are commonly known but are not about length, and the original measure is represented with a letter instead of a numeral. The intended reasoning for Question 2 was twofold: (1) that ||gal||=(189/50)||liter|| implies that a gallon is larger than a liter, so the number of gallons must be smaller than the number of liters to have the same magnitude, and (2) that ||gal||=(189/50)||liter|| implies that ||liter||=(50/189)||gallon||, and therefore *m* liters would have the same magnitude as (50/189)*m* gallons. Questions 1 and 2 are very much like the well-known Student-Professor problem wherein subjects are told to write an equation to represent the relationship between a college's number of professors and number of students after they are told, "There are seven students for every professor" (Clement, Lochhead, & Monk, 1981; Clement, Lochhead, & Soloway, 1979; Rosnick & Clement, 1980; Wollman, 1983, 1986). A major difference between Questions 1 and 2 and Student-Professor problems is that our questions are about measures whereas Student-Professor questions are cardinalities of sets.

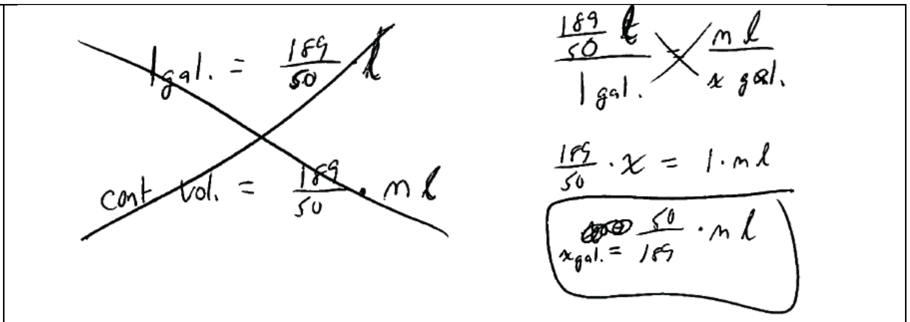
Question 3 is somewhat novel in comparison to Questions 1 and 2. We omitted markings on the axes while asking for a numerical value of *m* purposely to see whether teachers would actually compare the magnitudes of changes by comparing the lengths of the segments that represent them. We also anticipated that some teachers would think of the relative size of the two segments and not the relative magnitude of the quantities they represent. We therefore included a Part B that asked about the numerical value of *m* in the context of changing the scale of the *y*-axis. The value of *m* will not change, but if one is thinking about the triangle as invariant with change of scale, then one could think that the value of *m* is either doubled or halved. Our intent for Question 3 was to see if teachers understand slope as the relative magnitude of changes in two quantities and to see whether they took the segments representing changes as their objects of thought or took the segments as representations of magnitudes.

Each question had a rubric for coding responses to it. The rubrics put a response at one of four levels. Level 0 responses displayed a deep disconnect with the ideas involved. Level 1 responses were incorrect but showed evidence of struggling with the idea of units' magnitudes. Level 3 responses showed evidence of valid thinking, but after a false start. Correct responses without any attempt to convey reasoning were also put at Level 3. Level 4 responses displayed coherent understandings and ways of thinking about the underlying ideas. *Figure 6* shows examples of responses at Levels 0, 1, 3, and 4 for Question 2.

Question 2: A container has a volume of m liters. One gallon is 189/50 times as large as one liter. What is the container's volume in gallons? Explain.

Question 2 Level 0	
Question 2 Level 1	

Question 2 Level 3



Question 2 Level 4

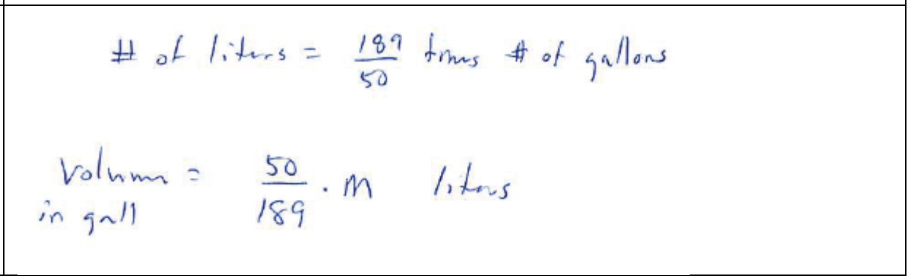


Figure 6. Examples of response levels for Question 2.

Table 2, Table 3, and Table 4 are also organized to show teachers' undergraduate major. *Math* means that a teacher received a B.S. or a B.A. in mathematics (25; only a few received a B.A. degree). *MathEd* means that a teacher received a B.S. or a B.A.E. in secondary mathematics education (36), which in terms of mathematics background is often similar to a B.A. in mathematics. *Other* means that a teacher received a degree in an area other than mathematics or math education but was teaching mathematics. Four teachers are listed as *ND*, which means that we do not have information on their degree.

We restricted our analysis of Question 3 to teachers who are actively teaching mathematics in a high school. Twelve (12) of 112 teachers taught in middle school exclusively. The rest taught high school.

Table 2
Rows are levels of Question 1 Level
Columns are levels of Major
Cell Entries: Count (Percent of Column Total)

	Math	MathEd	ND	Other	total
LV0	2	1	1	9	13
	(8)	(2.8)	(25)	(19.1)	(11.6)
LV1	12	15	3	14	44
	(48)	(41.7)	(75)	(29.8)	(39.3)
LV3	3	4	0	4	11
	(12)	(11.1)	(0)	(8.5)	(9.8)
LV4	8	16	0	20	44
	(32)	(44.4)	(0)	(42.6)	(39.3)
total	25	36	4	47	112
	(100)	(100)	(100)	(100)	(100)

Table 3

Table 3
Rows are levels of Question 2 Level
Columns are levels of Major
Cell Entries: Count (Percent of Column Total)

	Math	MathEd	ND	Other	total
LV0	5	3	3	10	21
	(20)	(8.3)	(75)	(21.3)	(18.8)
LV1	14	22	1	23	60
	(56)	(61.1)	(25)	(48.9)	(53.6)
LV3	5	4	0	12	21
	(20)	(11.1)	(0)	(25.5)	(18.8)
LV4	1	7	0	2	10
	(4)	(19.4)	(0)	(4.3)	(8.9)
total	25	36	4	47	112
	(100)	(100)	(100)	(100)	(100)

Table 2 shows levels of responses to Question 1 (Nerds and Raps). Approximately 50% of the responses were at Level 0 or Level 1 and 50% of the responses were at Level 3 or Level 4 (there was no Level 2). Responses at Level 0 and Level 1 showed no awareness of issues of magnitude. The most common response lower-level response was that 12 Nerds is the same as 9 Raps, which means they multiplied (3/4) and 12 and were not constrained by the thought that, since a Rap is smaller than a Nerd, it would require more Raps than Nerds to measure the same length. Table 1 also shows that teachers with a mathematics degree were less likely to give a high-level response than were teachers without a mathematics degree.

Table 3 shows that teachers in general were more than twice as likely to give a low-level response to Question 2 (72%) than a high-level response (28%). Table 3 also shows that this is true even for mathematics and mathematics education majors (though less so for math education).

The assignment of levels to responses for Question 3 is more complicated because of the question being in two parts. We coded Parts A and B independently, then combined the two into a leveled response. For Part A, we coded responses as Low-Level (LLA), Formulaic (FORM), Symbolic (SYM), Numerical Value between 2 and 3 but with no explanation (NV), and Numerical Value between 2 and 3 with a valid explanation (NVEX). FORM responses gave a formula for computing slope, like $\Delta y/\Delta x$ or $(y_2 - y_1)/(x_2 - x_1)$. SYM responses solved for m symbolically, as in $m = (y - b)/x$. NV responses gave a value of m between 2 and 3 but without an explanation. Numerical responses outside this range were coded LLA. No LLA numerical responses contained a valid explanation of where the number came from. Finally, NVEX responses gave a value for m between 2 and 3 and explained that the length of the vertical displacement was some number of times as large as the horizontal displacement (usually accompanied by tick marks on the segments).

For Part B (change of scale), we coded responses as Low-Level (LLB), Triangle Invariant (TI), Interpretation Issues (II), and Same Value (SV). LLB responses included statements like “I don’t know” or statements that we could not interpret with any coherence (e.g., “$x = 0$, $x = 2$, $y = 0$, $y = 2$”). A TI response indicated that the teacher imagined the triangle staying the same while the scale changed (thus an answer of “double” or “half”). An answer of “double” indicated that the teacher thought that the value represented by the interval [0,1] on the y-axis doubled, and hence the value of m doubled. An answer of “half” indicated that the teacher thought that the y-coordinates of the vertical displacement’s endpoints were halved, thus the value of m is halved. II responses indicated that the teacher was aware that an answer to Part B depended upon whether “y-axis rescaled” meant that *only* the y-axis is rescaled or both the y-axis and segment representing vertical displacement are rescaled.

Table 4 shows the assignment of levels to (Part A, Part B) response pairs. It shows that Part A responses were weighted more heavily than Part B responses. Indeed, an LLA response trumped all Part B responses. For example, “Same value” for Part B in the context of a Low-Level response to Part A meant only that the persons LLA response did not change.

Table 4
Columns are levels of Question 3 Part A. Rows are levels of Question 3 Part B. Cell entries are the level assigned that pair of responses.

Part B (Rows)	Part A (Columns)				
	Low-Level (LLA)	Formulaic (FORM)	Symbolic (SYM)	Numerical Value (NV)	NV and Explain (NVEX)
Low-Level (LLB)	0	1	1	1	2
Triangle Invariant (TI)	0	1	1	2	3
Interpretation Issues (II)	0	2	2	3	3
Same Value (SV)	0	2	2	3	4

Table 5 shows the results for Question 3. Of 100 high school mathematics teachers, 55% were at Level 2 or lower (61% if we count "no answer" among the lower levels), with minor differences among math, math ed, and other majors. We have received some criticism for not giving more credit to FORM and SYM responses. Our reply is that the item was intended to see whether teachers saw the triangle's vertical and horizontal sides as having magnitudes and that their formulaic or symbolic response suggests to us that the idea of magnitude is not something these teachers would emphasize in their teaching. Also, we have received criticism that our statement of re-scaling the *y*-axis is ambiguous. Our reply is that it is ambiguous only when one fails to think that the vertical and horizontal segments, as segments, connect points having coordinates that are dependent upon the axes. Also, the statement was not ambiguous to teachers who readily responded that the new value of *m* would be "half" or "double" its original value. It was not ambiguous to them because there evidently was no question in their mind as to the fact that the triangle was given, as opposed to being emergent from the conventions of graphing and the meaning of function.

Table 5
Rows are levels of Question 3 Level
Columns are levels of Major
Restricted to: High School Teacher
Cell Entries: Count (Percent of Column Total)

	Math	MathEd	ND	Other	total
LV0	5	6	0	9	20
	(20)	(16.7)	(0)	(25.7)	(20)
LV1	6	7	2	3	18
	(24)	(19.4)	(50)	(8.6)	(18)
LV2	5	3	1	8	17
	(20)	(8.3)	(25)	(22.9)	(17)
LV3	9	18	0	10	37
	(36)	(50)	(0)	(28.6)	(37)
LV4	0	1	0	1	2
	(0)	(2.8)	(0)	(2.9)	(2)
NoAns	0	1	1	4	6
	(0)	(2.8)	(25)	(11.4)	(6)
total	25	36	4	35	100
	(100)	(100)	(100)	(100)	(100)

Results to Questions 1-3 suggest to us that the majority of teachers in this sample are not, by and large, capable of bringing the idea of magnitude into their mathematical instruction even were they inclined to do so. We find this troubling, for two reasons. First, it suggests that the mathematics that teachers teach is not about generalized numbers. It is consistent with mathematics instruction that is about what to do with symbols. Second, when the idea of magnitude is suppressed or ignored in mathematics, then mathematics cannot be "a tool for science" as scientists would hope. Without an understanding of magnitude, mathematics is useless for a student's scientific understandings. At the same time we should say that we have criticized science education itself for ignoring the issues of quantity, unit, and magnitude (Thompson, 1994a, 2011, 2012).

Results in Table 5 are important also because of their implications for students' preparation for calculus. Slope understood as an index of "slantiness" or simply as a formula is inadequate to support understanding constant and average rate of change in a way that prepares students for calculus. Slope as a quotient of changes, where the quotient gives the relative magnitude of changes in two variables, is essential for understanding rate of change in calculus, especially when the changes are infinitesimal. When students' teachers are poorly positioned to think about quotient as a measure of relative magnitude, students, too, are poorly positioned to also learn to think of quotient as a measure of relative magnitude. Our experience teaching calculus at ASU is that students indeed enter calculus classes with very poor understandings of quotient and rate of change (Byerley, Hatfield, & Thompson, 2012).

Conclusion

We began with an explication of levels of meaning of "a quantity's magnitude" and the mental operations entailed by those meanings. We discussed five levels of meaning of magnitude that are largely unarticulated in the research literature.[8] They are: Awareness of size, Measure magnitude (magnitude and additive measure are the same thing), Steffe magnitude (size in relation to a composite unit together with reciprocity of size), Wildi magnitude (magnitude is invariant with change of unit), and Relative magnitude (relative size is invariant with change of unit in either or both quantities).

We characterized each meaning as a scheme of meanings and ways of thinking that builds on prior meanings of magnitude and that incorporates ever more sophisticated meanings of size as an invariant property of a quantity's magnitude. We also addressed the general notion of a scheme, and characterized the development of schemes as the formation of a learning cloud where many forms of thinking participate in each other's operation and in each other's development. This way of thinking about the development of meaning and ways of thinking conflicts strongly with current thinking about learning progressions and learning trajectories. Finally, we examined data from 112 mathematics teachers regarding their understandings of magnitude as it relates to changes of unit and to slope as a measure of relative magnitude, concluding that the teachers in this sample are poorly prepared to support students' learning about magnitudes and to support students' thinking with magnitudes.

[8] Steffe has encouraged us to use the word "stages" instead of "levels". The idea of a level is weaker than that of a stage. To say that a scheme is at a particular stage in its development means that there has a substantial reorganization among existing schemes and the construction of new schemes that supersede them. While we agree that our levels of meaning for magnitude probably constitute stages, we would like to see empirical investigations of this claim before making it.

Our hypothesis is that thinking at a high level with magnitudes and relative magnitudes produces an understanding of quantity that is very important for understanding physical quantities in science and that produces an understanding of generalized number that is important for understanding algebra and calculus. We suspect that a developed ability to reason with magnitudes and relative magnitudes "frees" students to focus on operational invariances in their reasoning. Put another way, we suspect that with a developed ability to reason with magnitudes students will be less likely to trip over numbers and their representations when dealing with sophisticated mathematical ideas.

We hope that readers see the framework we've offered here as profitable for future research. There are definite steps that need to be taken if it is indeed to be used profitably. The field must:

- Agree that, however we define it, the idea of magnitude is important for mathematics teaching and learning.
- Agree on what we mean that someone is thinking with magnitudes at some level.
- Agree on criteria for saying that a student's "magnitude thinking" is at a certain level.

While awaiting these agreements, or perhaps spurring them, the field must also attend to the theoretical foundation for understanding thinking with magnitudes and teaching and learning it. Specifically, the field must:

- Perform conceptual analyses of related mathematical ideas to see where thinking with magnitudes might play an important role. For example, conceiving of time as a magnitude might be important for conceiving of smooth continuous variation.
- Investigate what difference a student's or a teacher's level of thinking with magnitudes makes for learning or teaching other mathematical ways of thinking.
- Investigate ties between understanding science concepts and mathematical concepts that exist or could be profitably exploited for the benefit of each.

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