"Why use $f(x)$ when all we really mean is y?"

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Every high school mathematics teacher who has taught function notation has heard the question: *Why use f(x) when all we really mean is y?* This short article responds to that question by addressing the idea of function notation from historical and pedagogical perspectives.

History

Notations are a means of representing something conventionally. As such, we cannot understand the invention of function notation without first understanding the idea its inventors intended to capture with it. Clearly, function notation was invented to represent functions. But the idea of function itself evolved and changed over time. Kleiner described the idea of function as “3700 years of anticipation …. [and] 300 years in intimate connection with problems in calculus and analysis” (Kleiner, 1989, p. 282). For 3700 years mathematicians and scientists envisioned two quantities changing together but they had no means to represent “changing together”. They could represent values of the two quantities, but they did not have a way to represent a relationship that existed between them.

Kleiner described the next 300 years, after the advent of algebraic notation, as mathematicians’ attempt to express relationships between variables in a way that they could speak of *relationships yet to be defined explicitly.* For example, you know that you had a specific height at each moment in time since your birth, but you have no way of knowing exact heights at exact moments in time. How might you represent your height at each moment of elapsed time since birth without knowing the precise relationship between your height and elapsed time? In other words, mathematicians wanted to be able to say something like “my height had a value at each moment of elapsed time” symbolically.

Cajori’s history of mathematical notations (Cajori, 1928, 1929)[1] gives several examples of different attempts to represent relationships between two variables’ values. It is worth noting that the difficulty they had was in representing the relationship, not in representing the values. Early attempts were unsatisfying because the notation named the relationship, but not what the relationship related (e.g., $x\rceil 1\rfloor$ to represent a function of x and $x\rceil 2\rfloor$ to represent a second function of x). In a 1697 letter from Johann Bernoulli to Gottfried Leibniz, Bernoulli wrote about a new notation he had invented that represented both the variable used to define a function, a name for the function, and a value of the function for a value of x. “[By Xx] I understand that it has been given by X of x, and the constants of the quantities” and “Xx is the same quantity of x and constants combined.” (FC, 642). By Bernoulli’s convention, “X2” would represent the value of the function X when x has a value of 2.

In 1734, Euler created our modern notation, writing “If $f\left(\frac{x}{a}+c\right)$ denotes any function of $\frac{x}{a}+c$ …”. But Euler’s notation was not taken up generally for several decades. Even Euler adjusted his notation so that $f{:}(\mathrm{x})$ denoted the value of a function of one variable and $f{:}(x,y)$ denoted the value of a function of two variables (FC, 643). Another notation in use during Euler’s life was that all

[1] Many of my statements rely on Florian Cajori’s excellent two-volume history of mathematical notations. Cajori numbered his paragraphs across the two volumes, so I will state citations to Cajori’s history as (FC, paragraph number).

functions were named by a Greek letter, such as ϕx to name a function (ϕ) defined in the variable x. It was not until the early 1800's that textbooks began using Euler's original notation consistently (FC, 645). However, ideas like a function's domain, codomain, image, and range had yet to be thought of because the problems mathematicians worked on did not demand them.

Today there are a number of conventions for representing relationships between the values of two variables. Each emphasizes a different aspect of the idea of function. For example, if one wants to emphasize that a function named h is a relationship between two sets A (the set of inputs) and B (the set of outputs), one could write $h: A \to B$ or $A \xrightarrow{h} B$. To represent the natural log function from the positive reals to the reals we could write $\ln: \Re^+ \to \Re, x \mapsto \log_e(x)$ to mean that "ln" is the name of the function, it takes positive real numbers as input, and outputs real numbers according to the rule that when given a value of x, ln outputs $\log_e(x)$. The second notation (function over the arrow) would be $\Re^+ \xrightarrow{\ln} \Re, \ln(x) = \log_e(x)$, which conveys the same information as the first convention.

Pedagogy

The above historical discussion was about adult mathematicians trying to devise a notational system for representing functions, where their personal concepts of function were quite mature. High school students usually will not have a mature understanding of function, even by the standards of 1700. The typical approach taken by textbooks is to just begin using "*f*(*x*)" where they had been using "*y*", leading students to ask the natural question, "Why not just use *y*?"

The issue we must address, as Harel (2013) puts it, is how we might *necessitate* function notation—how we might make function notation a solution to what students recognize is a problem. Two ways that I find particularly useful are (1) to create situations where students need to re-use a formula repeatedly, and (2) create situations where students need to name a process before, or without, having an opportunity to define it.

Re-use formulas

To employ the *re-use formulas* strategy students need a computing device that will allow them to define functions. Automatic evaluation of function values is crucial to the success of this approach. My examples use a computer program called *Graphing Calculator* ("GC" for short; Avitzur, 2011).

Students like formulas. Well, at least they like formulas in comparison to functions. Formulas give explicit rules as to what to do, and many (too many) students prefer rules to thinking any day of the week. The idea behind the *re-use formulas* strategy is to make an explicit reliance on a rule as inconvenient as possible.

Figure 1 presents the well-known Box Problem. What I present here regarding function notation should follow a thorough discussion of the situation, especially how one can make a box by cutting square corners from a rectangular sheet. The discussion should also bring out the facts that the box's height will be the length of the square's side, the box's width will be the sheet's width minus what is cut out, and the box's length will be the sheet's length minus what is cut out. They will also need to recall, or be reminded, how to compute the volume of a rectangular cylinder (area of base times height).

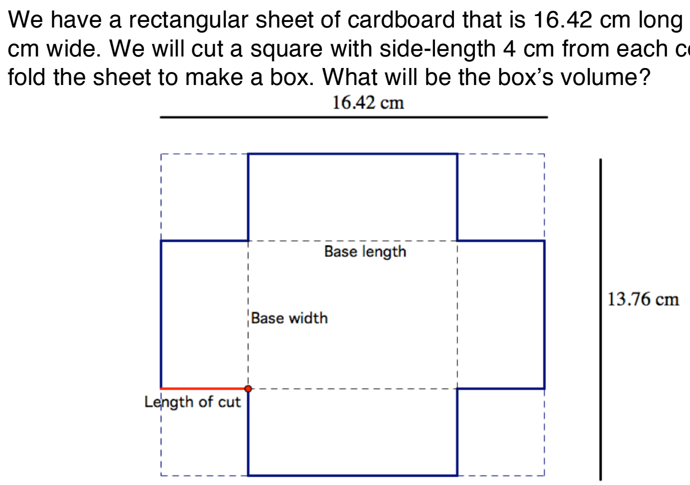

Figure 1. The box problem.

Here is an outline of the *re-use formulas* approach to necessitating function notation. Plain type is something you would say. *Italic type* is a comment.

- *Reason together to conclude that volume is base area times height. Base area is width times length. So base area is* $(13.76-2*4)(16.42-2*4)$*, and volume is therefore* $4(13.76-2*4)(16.42-2*4)$*. Now calculate.*
- What would the volume be if the square's side length is 2.3 cm? What would the volume be if the square's side length is 3.8 cm? What would the volume be if the square's side length is 5 cm? What side length for the square makes the box have the largest possible volume?
- Anyone tired of calculating?
- What formula are you using to calculate volumes? $V=c(13.76-2c)(16.42-2c)$, *where c is the side-length of the square cutout.*
- Does knowing this formula help us calculate values faster than we've been calculating them? *Not really.*
- Let me show you something. There's a way to name this formula *and* have the formula calculate itself automatically.
- *Define* $V(c)=c(13.76-2c)(16.42-2c)$ *in your computing device while projecting your device's display on a projector screen. Explain that c is the cutout's width in cm and that V(c) produces the box's volume in cubic cm given a value of c.*
- *Type V(2.3). The display shows 249.02376. Have students interpret the display. Type V(3.8). The display shows 206.45856. Have students interpret the display. Type V(5). The display shows 120.696. Have students interpret the display. See Figure 2. I've yet to see a student who doesn't think this is cool.*

- How much does the volume change were we to increase the cutout length from 4.1 cm to 4.3 cm? *Calculate this by typing V(4.3) – V(4.1). The display shows -13.87296. Have students interpret V(4.3), V(4.1), V(4.3) – V(4.1), and -13.87296.*

$$V(c) = c\,(13.76 - 2c)\;(16.42 - 2c)$$
$$V(2.3) \quad = 249.02376$$
$$V(3.8) \quad = 206.45856$$
$$V(5) \quad = 120.696$$
$$V(4.3) - V(4.1) \quad = -13.87296$$

Figure 2. Function *V* defined, and values of *V* that are computed by GC when given values of *c*.

- How might we think about what (the computing device) is doing when we type *V*(number)? *It is important that students develop a mental model of what the device is doing to produce numbers. It won't take long for them to propose something like a function machine.*
- How might we use our definition of *V* to graph the relationship between volume and cutout length?
- Suppose the cardboard sheet was 28 cm by 40 cm. Must we start over from scratch? *No. Just change the function definition.*

I do not mean to convey the impression that I predict the above outline will match what happens in your classroom. But it should give an idea about how to orchestrate a lesson that necessitates the idea of function notation as a convenient way to re-use formulas.

It is necessary to address the conventions that lie behind function notation after an informal introduction to function notation like *re-use formulas*. The convention for using function notation is that you write the name of the rule, the variable that represents the value on which the rule acts, and then the rule that defines the function. Figure 3 unpacks the terms and conventions having to do with function notation.

$$\overbrace{\overbrace{\overbrace{V}^{\text{Name}}\;(\;\overbrace{u}^{\text{Input}}\;)}^{\text{Output}} = \overbrace{u(13.76-2u)(16.42-2u)}^{\text{Rule for how to produce an output}}}^{\text{Function definition}}$$

Figure 3. Parts of a function definition.

We often use the phrases "name of rule" and "name of function" interchangeably. The values that we put into the rule are called *input values*. The number that results from applying the rule to a specific input value is called an *output value*, represented above by *V*(*u*). The symbol *V*(*u*) is an example of *representing the function's output values* for varying values of the input *u*. The act of using function notation to represent a relationship between two quantities' values is called *defining a function.*

It is important to notice that "=" in the definition of *V* in Figure 3 does not mean that two numbers are equal – that the number represented by $V(u)$ is the same as the number represented by $u(13.76-2u)(16.42-2u)$. Rather, this use of "=" means "is defined to be". That is why we cannot say that the right-hand-side represents the output of *V*. Instead, it is like the part of a dictionary definition that sits aside the word that is being defined.

Name a process before defining it

There is a theorem in number theory (call it Theorem A) that says:

> Suppose that a and b are integers, $a > b$, $a, b \neq 0$. Suppose that r is the remainder of a divided by b. If d is the greatest common divisor of a and b, then d is the greatest common divisor of b and r.

A corollary of Theorem A is that if $r = 0$, then b is the greatest common divisor of a and b.

Theorem A is the basis for Euclid's algorithm for computing greatest common divisors (an important computation in cryptology). The algorithm goes as follows. I will use "rem" to name a function that produces the remainder of one integer divided by another and "gcd" to name a function that produces the greatest common divisor of two integers.

You are given values of a and b, $b \neq 0$. To compute gcd(a,b):

- If rem(a,b)=0, then gcd(a,b) = b.
- If rem(a,b) $\neq 0$, then compute gcd(b,r).

We have, essentially, defined gcd, namely:

$$\gcd(x,y)=\begin{cases} y \text{ if } \operatorname{rem}(x,y)=0 \\ \gcd(y,\operatorname{rem}(x,y)) \text{ if } \operatorname{rem}(x,y)\neq 0 \end{cases}$$

But we have not defined rem even though we have said how we will use its output. Thus, we *need* a definition for the function that produces the remainder of one number divided by another.

To define rem, we must think about how we would compute a remainder: calculate *a/b* as a decimal number, subtract the integer part, and then multiply the remaining decimal part by *b*. The function that gives the integer part of a number is called the "floor" function, denoted $\lfloor x \rfloor$. So,

$$\begin{aligned}\operatorname{rem}(a,b) &= b\left(\frac{a}{b}-\left\lfloor\frac{a}{b}\right\rfloor\right)\\ &= a-b\left\lfloor\frac{a}{b}\right\rfloor\end{aligned}$$

Figure 4 shows rem and gcd in action within GC. It also shows that gcd is defined for two negative integers. But the example of gcd(-58,-4) shows that the number that gcd produces when x and y are negative is not their greatest common divisor. GC says it is -2 when it should be 2. I invite you to adjust the definition of gcd so that it works for negative integers. You might also adjust the definition of gcd so that it produces an acceptable output even when its inputs are not integers.

$$\mathrm{rem}(82, 19) \quad = 6$$
$$\gcd(82, 19) \quad = 1$$
$$\gcd(14208, 5661) \quad = 111$$
$$\gcd(-58, -4) \quad = -2$$

Figure 4. The functions rem and gcd in action.

A Misuse of Function Notation

It is common for teachers and textbooks to write statements like $C = f(x) = 3x + 2$. This is actually a misuse of function notation and it confounds meanings of "=". But just like any common practice, it has its pros and cons.

On the positive side:

1) Writing $C = f(x) = 3x + 2$ is convenient. It does two things in one line. It says that $C = f(x)$ and it says that $f(x) = 3x + 2$.

2) Many textbooks write it this way.

3) Many people have the habit of writing statements like this.

On the negative side:

4) Writing $C = f(x) = 3x + 2$ reinforces the attitude that $f(x)$ is superfluous. By transitivity of equality, $C = 3x + 2$, so, as many students ask, why not just write that?

5) Writing $C = f(x) = 3x + 2$ confounds two meanings of "=". The first use of "=" in $C = f(x) = 3x + 2$ represents equality between two numbers. The second use of "=", as mentioned earlier, means "is defined as".

6) Many people say that you must write $C = f(x) = 3x + 2$ so that f has an output variable. But functions have outputs. They do not have output *variables*. $f(x)$ represents the output of f when it is evaluated at a value of x. The phrase *output variable* confuses the ideas of function and variable.

7) Writing $C = f(x) = 3x + 2$ confounds ideas of formula and function. We write $A = L \times W$ to say that the area of a rectangle is the product of its width and length. But suppose L and W vary with time? We wouldn't write "$A = L \times W$ after an elapsed time of t minutes". Nor

would we write "$A = L(t) \times W(t)$ after an elapsed time of t minutes". Rather we would write "$A(t) = L(t) \times W(t)$ after an elapse time of t minutes".

8) Finally, the statement $y = f(x)$ derives from the mathematical concept of a function's graph. Technically, the graph of a function f:A → B is defined as $G_f = \{(x, y) \in A \times B \mid y = f(x)\}$. The statement $y = f(x)$ is simply a shorthand for saying that, given a value of x, the y-coordinate of the corresponding point on f's graph is $f(x)$.

My recommendation

Try to keep two ideas separate. One is the use of a variable to represent a quantity's value. The other is to say that a quantity's value is a function of some other quantity's value. Don't confound these ideas. Instead of writing "$C = f(x) = 3x + 2$" to communicate two ideas in one statement, write something like "$C = f(x)$, where $f(x) = 3x + 2$". And write this after developing the idea that f represents a relationship between two quantities' values. One quantity has the value x and the other has the value $f(x)$.

Why is function notation difficult for students?

A primary source of students' difficulty with function notation is that they only see it where "y" could be used just as well. The textbook says they must use $f(x)$ when there really is no need for it. They rarely see function notation used in settings where using it actually enables them to do things that they otherwise could not.

A second source of students' difficulty with function notation is that they have not internalized the system of conventions in Figure 3 that gives function notation its meaning. A common way of thinking among students is that the entire left hand side of Figure 3 is the function's name—that the name of the function in Figure 3 has four characters—and that the right hand side of Figure 3 is the only part that matters. Thus, they see nothing wrong with function definitions like $f(n) = \frac{x(x+1)}{2}$ because the right hand side tells them how to compute a number and the left hand side doesn't matter.

The moral of this short article is that students must see a need for function notation, must internalize its conventions in order to use function notation productively, and must find it useful to do interesting mathematics.

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