

FUNCTION NOTATION AS AN IDIOM¹

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Functions play a large role in mathematics education beginning in middle school. The aim of this paper is to investigate the meaning teachers hold for function notation; namely, we suggest that many teachers view function notation as a four-character idiom consisting of function name, parenthesis, variable and parenthesis. Many of the teachers who engaged in tasks aimed at exploring teachers' meanings for function notation responded in a manner suggestive of viewing function notation idiomatically.

INTRODUCTION

The function concept permeates mathematics education beginning in middle school. Studies show that learning the function concept is challenging, even for high performing undergraduates (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998). While research continues to extend our understanding of function, there is still a need for exploring *meanings* associated with function. Wilson (1994) draws attention to the understanding of function for one particular pre-service teacher; this case study of the pre-service teacher's thinking about functions as "computational activities" sets the tone for the goal in this paper of delving into the meanings teachers hold. Our focus shifts from the function concept in general, however, to function notation in particular.

THEORETICAL FRAMEWORK

The use of function notation is ubiquitous in mathematics beyond middle school. It is also commonly a teacher's experience that, at some moment during its introduction, some student will ask, "Why use $f(x)$ when all we really mean is y ?" (Thompson, 2013b). Teachers' abilities to answer this question will be based in their meanings for function notation, its conventions, and its uses. As such, our goal in this paper is to explore teachers' meanings for function notation.

We consider meanings to be constructed by an individual to organize his or her experiences. Creating meaning entails constructing a scheme through repeated reasoning and reconstruction to organize experiences in a way that is internally consistent (Piaget & Garcia, 1991; Thompson, 2013a; Thompson, Carlson, Byerley, & Hatfield, in press).

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The function concept is a significant and complex element of mathematical fluency that permeates school mathematics beginning in grade 8. We choose to focus on the notational aspect of functions for the sake of brevity. In particular, we recall that a function's definition consists of the components shown in Figure 1.

$$\begin{array}{c}
 \text{Function definition} \\
 \hline
 \begin{array}{c}
 \text{Output} \\
 \hline
 \begin{array}{c}
 \text{Name} \quad \text{Input} \\
 \hline
 \underbrace{f}_{\text{Name}} \left(\underbrace{x}_{\text{Input}} \right) = \underbrace{x(3x+1)(x^2-14x)}_{\text{Rule for how to produce output}}
 \end{array}
 \end{array}
 \end{array}$$

Figure 1. A scheme for defining a function using function notation (from Thompson, 2013b).

For the working mathematician, the notation $f(x)$ itself represents the value of the function f when given a value of x , with or without the associated defining rule. One may refer to the function f , specify the input variable x which is to be used to describe the rule, or call the entire $f(x)$ to stand for the output of the function. The latter notation may be used to introduce a rule or to hold the place of an unknown or complex rule in the definition of another function. We suspect that many teachers do not have this meaning for function notation, and instead employ the four-character idiom—function name, parenthesis, variable, parenthesis (e.g. $f(x)$), in its entirety, as the name of a function.

METHODOLOGY

Thompson (2013a) argued the need for understanding teachers' mathematical meanings because those meanings are passed on to the students. Investigating teachers' in-the-moment meanings serves as a starting point for professional development to help teachers develop meanings that are more productive for teaching for coherence. In light of this, we designed tasks to explore teachers' meanings for function, specifically focusing on whether function notation holds the same meaning for teachers as what we as researchers think of when using function notation. The tasks were administered to 100 high school teachers as part of a larger assessment. A team developed scoring rubrics to characterize meanings revealed in teacher responses (Thompson & Draney, under review).

Tasks

The first task we will discuss consists of two parts. Part A asks the teacher to complete a function definition by filling in blanks and Part B presents sample student work for the teacher to explore (Figure 2).

Part A addresses function notation as an idiom directly. We suspect that teachers who view function notation as a four-character idiom will read " c of v " as the name of the new function, and fill in the blanks with r 's and u 's because the functions referenced are read " w of u " and " q of r ". It is unlikely for teachers who see function notation idiomatically to tease out the function name " w " from the notation $w(u)$ because they view the " u " as part of the name.

Here are two function definitions.

$$w(u) = \sin(u-1) \text{ if } u \geq 1$$

$$q(r) = \sqrt{r^2 - r^3} \text{ if } 0 \leq r < 1$$

Part A. Here is a third function c , defined in two parts, whose definition refers to w and q . Place the correct letter in each blank so that the function c is properly defined.

$$c(v) = \begin{cases} q(_) \text{ if } 0 \leq _ < 1 \\ w(_) \text{ if } _ \geq 1 \end{cases}$$

(on next page)

Part B

James, a student in an Algebra 2 class, defined a function f to model a situation involving the number of possible unique handshakes in a group of n people. He defined f as:

$$f(x) = \frac{n(n+1)}{2}$$

According to James' definition, what is $f(9)$?

Figure 2. Item 1 Variable mismatch. © 2013 Arizona Board of Regents. Used with permission.

In the case where a teacher fills in the blanks of Part A with “ v ”, we included Part B to explore how the teacher addresses variable mismatch in student work. The sample work provided gives an ill-defined function $f(x) = \frac{n(n+1)}{2}$ and requests the value of $f(9)$. In particular, Part B reveals the degree to which teachers’ take notice of and can explain the problem of variable mismatch in a the definition of a function. A teacher who reads function notation idiomatically will be unbothered by the mismatched variables x and n in the function definition, and will compute $f(9) = 45$. Teachers who do not read function notation idiomatically will, at least, observe that James’ function always produces the output $\frac{n(n+1)}{2}$ regardless of the input x . Ideally, a teacher would be able to further explain to James that his function is ill-defined because the variable x has not been defined.

The second task we include in this paper was designed to evaluate teachers’ tendency to use function notation in the rule of another function’s definition. In particular, we wanted to see if function notation served the purpose of representing a varying quantity for the teachers. So we designed a situation that necessitated the use of function notation to model a quantitative situation that was familiar to teachers (Figure 3).

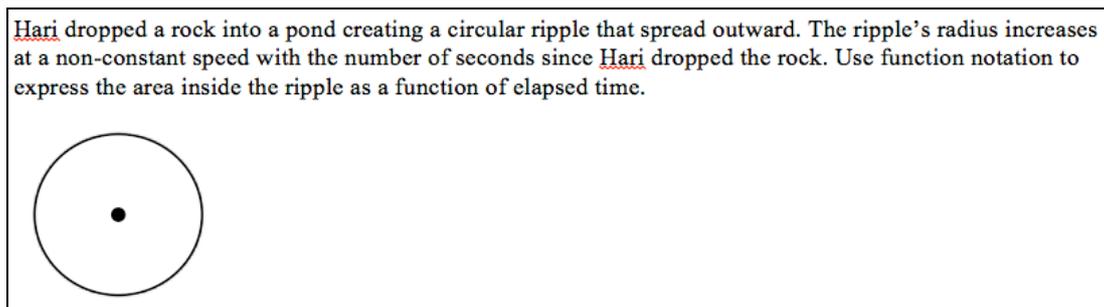


Figure 3. Task 2 Rock into pond. © 2013 Arizona Board of Regents. Used with permission.

We anticipated that teachers might use function notation on the left-hand-side of the function's definition but not use function notation to represent the circle's radius as a function of time. We included the phrase "at a non-constant rate" to describe the growth of the circle's radius so that teachers would not assume unthinkingly that the radius increases at a constant rate and hence model the scenario with $r = kt$.

RESULTS

Ninety-seven of the 100 teachers who were given Task 1 responded. Table 1 shows the distribution of responses. The table has two main points of interest. First, 48 of 97 teachers filled in the blanks of Part A with u 's and r 's or otherwise did not use the variable v (e.g. some teachers wrote q or w in the blanks). This supports our suspicion that many teachers read function notation idiomatically, as explained in the task design. They saw what was written to the left of the equal sign as the name of what was written to the right of the equal sign. They did not parse the definition according to the scheme in Figure 1. For those teachers who filled in the blanks with something other than u , r or v , we suspect that the " c of v " on the left hand side of the function definition is read idiomatically by the teachers. These teachers are unlikely to identify " c " as the function name and " v " as the input variable, instead reading " c of v " as the entire function name.

Variable Mismatch	$f(9)=45$	f is constant but $f(9)=45$	$\frac{n(n+1)}{2}$	f is ill-defined	Totals
Fill in Blanks					
Other	12	0	2	1	15
u 's and r 's	26	3	2	2	33
v in 1-3 blanks	2	1	1	3	7
v in all blanks	17	1	12	12	42
Totals	57	5	17	18	97

Table 1. Teacher responses to Task 1.

The next number that stands out in Table 1 is that 57 of 97 teachers responded to Part B by substituting 9 for n in James' definition to obtain a value of 45 handshakes. This

number shows that most teachers were untroubled by the variable mismatch in James' definition and used the function definition as if it were written $f(n) = \frac{n(n+1)}{2}$.

Of the 33 teachers whose Part A response suggested idiomatic thinking of function notation, only 7 wrote responses that suggested an awareness of something awry with James' definition. Only 2 of those 33 were explicit about the variable mismatch being problematic to the function definition.

Moreover, almost half of the teachers who filled in the blanks of Part A with v 's substituted 9 in the right side of James' definition in Part B. We suspect that these teachers are aware of the practice of using a variable consistently in function definitions, but this practice did not keep them from overlooking the variable mismatch in James' definition.

While the goal of Task 2 was to reveal teachers' usage of function notation in modelling scenarios, it became evident in scoring that this item could be used to gain insight into whether teachers view function notation idiomatically. In Table 2, we compare teacher responses in Task 1 and Task 2. For Task 1, we categorized the pairing of responses in Part A and Part B as Low, Medium or High based on the degree to which responses revealed a tendency to use variables consistently and identified the problematic nature of James' function definition. Likewise, teacher responses to Task 2 are ranked as Low, Medium or High based on the use of function notation within the model described by the teacher.

Fifty of 87 teachers gave Low responses, meaning that (1) they did not use function notation or (2) they used function notation only on the left-hand-side of their model and used variables inconsistently (see Figure 4). Another 16 of 87 gave Medium responses, which include responses that used function notation on both sides of the model but used variables inconsistently.

		Task 2			<i>Totals</i>
		Low	Medium	High	
<i>Totals</i>	Low	39	9	9	57
	Medium	1	2	2	5
	High	10	5	10	25
	<i>Totals</i>	50	16	21	87

Table 2. Teacher responses to Task 1 and Task 2

We suspect that teachers who gave solutions like that in Figure 4 view function notation idiomatically. In particular, this teacher might read “ f of x ” as the entire name of the function describing the area of the circle, making the variable mismatch on the right-hand-side of the definition a non-issue for this teacher. In fact, if we look to the same teacher's response on Task 1, he filled in the blanks with r and u and

computed $f(9)=45$ as if the variable discrepancy was not present. In this manner, this teacher seems to consistently read function notation idiomatically.

A handwritten equation $f(x) = \pi r^2 \cdot t$ is enclosed in a rectangular box. The handwriting is in black ink on a white background.

Figure 4. Sample teacher response to Task 2

We look at another teacher’s responses to both items to try and describe a viable model for his thinking with regard to function notation (Figure 5). Notice this teacher gave the highest-level response to Task 1 Part A, filling in all the blanks with v . However, the teacher computed the value of $f(9)$ in Task 1 Part B by substituting 9 into the right hand side of the James’ function definition. It is possible that this teacher is loosely aware of the need for consistency in variable usage in defining a function, but this consistency is not required to evaluate a function value. We suspect that this teacher sees the left hand side of a function definition as a label for the function name and the right hand side of a function definition as “where the math happens”. Looking on to his response to Task 2, we see the teacher uses “ A_{circle} ” to introduce his model for area. This unconventional notation to reference the circle’s area reinforces the idea that what appears on the left side of the equal sign is a label.

Task 1 Part A	Task 1 Part B
$c(v) = \begin{cases} q(v) & \text{if } 0 \leq v < 1 \\ w(v) & \text{if } v \geq 1 \end{cases}$	$\frac{9(10)}{2} = 45 \text{ unique handshakes if 9 people are in grp.}$
<p style="text-align: center;">Task 2</p> <p style="text-align: center;">as a function of elapsed time.</p> <div style="text-align: center;"> $A_{circle} = \pi r^2$ <p style="margin-left: 100px;">↑</p> $r(t) = \text{Some expr. to find the radius in terms of time since Hari dropped rock}$ </div> <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> $A_{circle} = \pi (r(t))^2$ </div>	

Figure 5. Sample teacher responses to Tasks 1 and 2

<p>Task 1 Part A</p> <p>I DK?</p>	<p>Task 1 Part B</p> $f(x) = \frac{n(n+1)}{2}$ $f(9) = \frac{9(9+1)}{2}$ $\frac{90}{2} = 45 \text{ units hand/Mark}$
<p>Task 2</p> $a = \pi r^2$ $f(a) = \pi (r \cdot t)$ $f_a = \pi (r \cdot t)^2$	

Figure 6. Sample responses from teacher on both tasks

Figure 6 gives one final example of one teacher's responses to both tasks. This teacher did not give a response to Task 1 Part A, used James' definition by evaluating the right hand side by substituting $n=9$ in Task 1 Part B and struggled to introduce function notation in his response on Task 2. In fact, the teacher appears to have attempted to convert the area formula for a circle into a model using function notation by introducing the phrasing "f-parenthesis-variable-parenthesis" on the left hand side. Collectively, this teacher's responses suggest a lack of importance, from the teacher's perspective, to consistent usage of variables (as seen in Task 1 Part B and Task 2—in which the variables a , r and t are all utilized in the model) and the possibility of holding a meaning for function notation as nothing more than a conventional label of the left side of an equal sign.

DISCUSSION

Our goal for this research was to explore teachers' mathematical meanings about function notation. We suggest that our results, though specific to meanings for function notation, support Thompson's claim that attending to meanings must be central to our work as mathematics educators (Thompson, 2013a). Responses to our tasks reveal that many teachers read function notation idiomatically. Consequently, we suspect these teachers view only the content to the right of the equal sign as the mathematically relevant portion of a function definition as described in Figure 1. This type of reasoning leads to a need for describing a rule to model scenarios rather than employing function notation to represent a varying quantity. We suggest further research be conducted to explore the extent to which teachers who view function

notation idiomatically use function notation to represent varying quantities as opposed to developing rules to model scenarios.

Another area of interest is to look at what meanings teachers hold for the notational devices used for operations on functions. For instance, what meaning do the following equations have for a teacher who views function notation idiomatically?

$$(f \pm g)(x) = f(x) \pm g(x)$$

$$(f \circ g)(x) = f(g(x))$$

Since the content on the left hand side of the equal sign is no longer of the simple form—letter, parenthesis, variable, parenthesis—does the left hand side still serve as a label? Does it add confusion, is it ignored, does the teacher simply focus on the right hand side? Under these conditions, what meanings do these teachers convey in their classrooms while teaching function notation and operations on functions? Further research ought to investigate such questions, as classroom discussions regarding function and operations on functions are likely impoverished when the teachers leading such discussion hold idiomatic meanings for function notation.

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