Conceptualizing and reasoning with frames of reference

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Though commonly used in math and physics, the concept of frames of reference is not described cognitively in any literature. The lack of a careful description of the mental actions involved in thinking within a frame of reference inhibits our ability to account for issues related to frames of reference in students' reasoning. In this paper we offer a theoretical model of mental actions involved in conceptualizing a frame of reference. Additionally, we posit mental actions that are necessary for a student to reason with multiple frames of reference. This theoretical model provides an additional lens through which researchers can examine students' quantitative reasoning.

Keywords: Frames of Reference, Quantitative Reasoning, Theoretical Perspective

Consider the following problems that students encounter routinely in high school:

- Bobby is 3 years older than Lucy. When Bobby is x years old, how old will Lucy be?
- A particular engine can propel a boat at a maximum of 32 miles per hour. The boat travels 30 miles upstream from Port Adele to Port Chimney and then back, at maximum speed. The captain dropped a branch in the water before starting, estimating the downstream current as 6 mph. Considering just travel time, how long will the round-trip take?
- Yolanda and Sydney ran in the same marathon. Sydney ran 5/3 times as fast as Yolanda. If Sydney finished the 26.2-mile race in 4 hours, what was Yolanda's average speed?

Students often struggle to manage the dual perspectives required in each task (Bowden et al., 1992; Panse, Ramadas, & Kumar, 1994; Monaghan & Clement, 1999); for instance, the first scenario provides a comparison of Bobby and Lucy's age relative to Lucy's age, then switches to describing Bobby's age from Bobby's perspective, and finally asks for Lucy's age relative to Bobby's. A student must similarly tease apart the ways in which the framing of information about quantities in a scenario switches between two frames in the other two examples. In our own work investigating teachers' meanings on similar tasks, we identified a need to isolate the type of reasoning involved in answering the above tasks within quantitative reasoning.

Our search of the literature provided just a few references, all in physics education, that deal with tasks of this nature (Bowden et al., 1992; Panse et al., 1994; Monaghan & Clement, 1999). In line with the physics terminology, we choose to describe the extra layer of complexity in the above problems as issues of "frames of reference". In this report, we introduce what we mean by *a conceptualized frame of reference* and *reasoning with frames of reference*, and explain why this is an area that deserves attention by the math education community.

A definition of the noun phrase "frame of reference" would suggest that a frame of reference is an object external to the person reasoning with it. Such a perspective does not align with our goal of describing what it might mean for an individual to conceptualize a frame of reference.

Therefore, we articulate the mental activity involved in conceptualizing and reasoning with frames of reference. While the *products* of the mental activity we describe align with the classical definition for frame of reference as a coordinate system or a system of measures, our emphasis is on the mental actions a student must employ to conceptualize a frame of reference. In particular, we use the phrase "frame of reference" to refer to a set of mental actions through which an individual might organize processes and products of quantitative reasoning (Thompson, 2011). As such, conceptualizing frames of reference and quantitative reasoning are interrelated, with frames of reference providing an additional lens with which to look at quantitative reasoning.

Conceptualizing a Frame of Reference

An individual can think of a measure as merely reflecting the size of an object relative to a unit or he can think of a measure within a system of potential measures and comparisons of measures. An individual conceives of measures as existing within a *frame of reference* if the act of measuring entails: 1) committing to a unit so that all measures are multiplicative comparisons to it, 2) committing to a reference point that gives meaning to a zero measure and all non-zero measures, and 3) committing to a directionality of measure comparison additively, multiplicatively, or both.

Committing to a Unit

As an example, a student can think about the measure "4.5 feet" in different ways. If the student focuses only on the value "4.5" and sees the unit as of secondary (or perhaps no) importance, there is no meaningful connection between the unit and the value for this student. In contrast, if the student sees a multiplicative relationship between the unit and the value, this provides a meaning for the measure. In this second case, "4.5 feet" is a length that is 4.5 times as long as the length of an object that is taken as a standard foot. A student who sees this relationship and the importance of unit in establishing meaning for each measure has taken the first crucial step towards conceptualizing a frame of reference.

Committing to a Reference Point

As a demonstration, consider the phrases "distance Ben walked" and "distance Ben walked from his house today". Both phrases describe quantities. The first phrase is vague and leaves a reader wondering if the quantity described is Ben's distance walked today, Ben's distance walked in his room, or the distance Ben walked since his birth. As such, the ambiguity in the phrase "distance Ben walked" creates ambiguity in the meaning of a measure. Saying the measure of "distance Ben walked" is m units fails to provide usable information for an individual trying to reason about the situation. Moreover, the vagueness of "distance Ben walked" would make it possible for an individual to inadvertently change his meaning for "distance Ben walked" while reasoning within a complex situation. He might define formulas or expressions to model the situation without understanding that his inconsistent meanings for the quantity make his model incoherent. Another possibility is that two individuals can read a situation and internally ascribe different meanings to the quantity "distance Ben walked" (by assigning different reference points) without realizing that they have done so. They might then discuss a problem and never realize that they are talking past one another because they are operating and speaking within two different frames of reference.

The specificity of "the distance Ben walked from his house today" makes it a more useful description of a quantity. In particular, we can confidently say that if the measure of the quantity "distance Ben walked from his house today" is zero, then Ben hasn't left his house today. Similarly, if the measure of that quantity is b units, $b > 0$, then Ben walked b units outside of his house. The commitment to a reference point attributes a meaning to every measure of the quantity and avoids the problems associated with ambiguity described above.

Committing to a Directionality of Measure Comparison

Consider a student designing a study to investigate the relationship between people's weight and Vitamin C consumption. The student plans to weigh each participant at the start and at the end of a two-month period, during which the participants will consume various amounts of Vitamin C daily. The student plans to examine the changes in the participants' weights. This student could imagine these comparisons in two different ways. If the student is oriented to think always of positive changes, then the student would make the following kinds of statements: "Josh is 6 pounds heavier at the end of the study" and "Wanda is 6 pounds lighter at the end of the study". In this case, the student has not thought of the comparison of measures within a frame of reference. Rather, the student adjusted his description so that a comparison always results in a positive number. Such adjustments constantly alter the directionality of comparison in order to think of the larger measure relative to the smaller. Should the student be asked what a participant's change of 1.5 pounds means, he could not say definitively whether the participant gained or lost weight.

Alternatively, suppose that the student commits to a comparison of "pounds heavier at the end than at the beginning". The additive comparison that the student has in mind is the post-weight minus the pre-weight. Here, the student would make statements like: "Josh is 6 pounds heavier" and "Wanda is –6 pounds heavier." In these statements, the student made use of the same direction in comparing the measures. Unlike the other case, the student now definitely interprets a change of 1.5 pounds as the individual weighed 1.5 pounds more at the end of the experiment than at the beginning.

We note that this commitment to the directionality is crucial when making multiple comparisons. For instance, most students can mentally shift between "heavier than" and "lighter than" when comparing two people's weights. However, the activity of comparing three or more people's weights proves much more difficult without committing to a directionality within a frame of reference.

An analogous commitment to a directionality when comparing measures holds for multiplicative comparisons. A student thinking within a frame of reference will be able to say "x is 3 times as large as y" and "y is one-third as large as x." A student who avoids committing to a directionality of comparison will only be able to make the first statement, possibly because of a discomfort with non-integers.

As a final note, we emphasize that we are *not* suggesting people should commit to a single reference point or a single directionality of comparison for their entire engagement in a task. In fact, it is often the case that while solving problems, an individual must conceptualize more than one frame of reference. The commitments we refer to only occur *within* the act of conceptualizing one frame of reference; a student can choose to work with a different frame of reference for the same quantity within one context, but while working within one frame, he works consistently with the choices of reference point and directionality of comparison he made in order to conceptualize that frame of reference. The conceptualization of multiple frames of reference then requires further mental actions to bring information from multiple frames together, an activity we call *reasoning with multiple frames of reference*.

Reasoning with Multiple Frames of Reference

We identify two types of reasoning that a student might employ when engaging in a task that necessitates conceiving of multiple frames of reference. The first type is that a student *coordinates multiple frames of reference* when he finds the relationship between one or more quantities' measures in two frames, such that he can determine a measure given in one frame from a measure given in the other. A student who has coordinated two frames of reference could, given an event's representation in one frame, represent that event in another frame in order to compare similar quantities. The second type of reasoning is that of a student *combining multiple frames of reference* when he considers two different quantities simultaneously within their respective frames of reference. Below we discuss the mental actions that are associated with each type of reasoning.

Coordinating Multiple Frames of Reference

A student coordinates multiple frames of reference by carrying out three sets of mental actions. She must first recognize the need to transform the measures of quantities measured in different frames of reference into measures measured in the same frame of reference. Second, a student must coordinate known *measures* of quantities in different frames in order to answer her question. Third, she must use those known measures to coordinate the *frames*.

We illustrate these mental actions in the context of the task presented in Figure **1**.

Two children, Alice and Bob, walk together from school to home. Alice starts measuring the distance they have traveled by counting the sidewalk squares they have crossed since passing the tree. Bob starts counting the sidewalk squares they have crossed since passing the stop sign and noticed that there were 3 squares between the tree and the sign. Let u be the number of sidewalk squares Alice has counted. Write an expression that gives Bob's count of sidewalk squares.

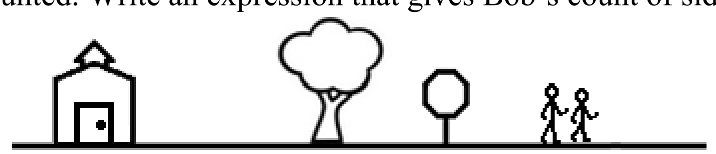


Figure 1. The Alice and Bob task.

Before beginning to coordinate multiple frames of reference, the student must first recognize that Alice and Bob each conceived of a comparable quantity within separate frames of reference. The student's recognition of this fact coincides with her envisioning what a distance of zero squares means to both Alice and Bob. The student must recognize that for Alice, "zero squares" means that the children are at the tree; likewise the student understands that "zero squares" to Bob means that the children are at the stop sign.

While the student could answer the prompt with a statement such as "Let v represent the number of squares that Bob has counted", she may feel the need to make use of the given definition for u. However, in attempting to use u, she imagines shifting from Alice's measurements (and frame of reference) to Bob's measurements (and frame of reference). The student anticipates that for the shift to work, she needs to find a commonality between the two frames of reference. The stem of the task in Figure **1** provides the student with a useful point of commonality between the frames. The student knows that Alice and Bob walk along the same path, counting the same sidewalk squares, with Alice starting to count at a tree and, three squares later, Bob starts counting at the stop sign. The stop sign serves as a point of commonality between the two frames of reference. The student knows that for Alice the stop sign is 3 squares from the tree. Likewise, she knows that Bob views the stop sign as 0 squares from itself. Thus, a measure of 3 squares for Alice, 3_{Alice}, is the same point along the path as 0 squares for Bob, 0_{Bob}. In establishing the link $3_{\text{Alice}} \equiv 0_{\text{Bob}}$, the student has coordinated known measures of comparable quantities from two different frames of reference. To fully coordinate the two frames of reference, the student must establish the relationship between the measure of a quantity in one frame of reference and the measure of the comparable quantity in other frame of reference. The student imagines that if Alice and Bob are at the stop sign and move forward one square, then both of Alice's and Bob's counts will increase by one; thus $4_{\text{Alice}} \equiv 1_{\text{Bob}}$. She anticipates that as they keep moving forward *any amount*, both Alice and Bob will increase their counts (e.g. they move forward another 0.5 squares, $4.5_{\text{Alice}} \equiv 1.5_{\text{Bob}}$). Likewise, she imagines that if Alice and Bob moved backward one square, their counts would increase by -1; thus $2_{\text{Alice}} \equiv -1_{\text{Bob}}$. In examining these connections based from the point of commonality, the student anticipates that Bob's count will always be 3 squares less than Alice's count. This supports the student in expressing Bob's count as $u - 3$ using Alice's frame of reference.

Coordinating multiple frames of reference is cognitively demanding. It requires that a student conceive each frame as a valid frame, be aware of the need to coordinate quantities' measures within them, and carry out the mental process of finding a relation between the frames while keeping all relative quantities and information in mind.

Combining Multiple Frames of Reference

A student *combines* frames of reference when she considers multiple quantities that exist within separate frames of reference simultaneously. Combining frames of reference is a separate act from coordinating frames of reference. When combining frames of reference, the student does not have a goal of expressing measures of one or more quantities in terms of different frames. Rather, the student's goal is simply to hold quantities from multiple frames of reference in mind concurrently. In the above section, the student would have combined Alice's frame of reference with Bob's frame of reference had she stated "Alice and Bob's home is both u squares from the tree *and* $u - 3$ squares from the stop sign". As a further example, coordinate systems allow us (mathematicians, teachers, and students) to represent the measures of different quantities simultaneously when those measures stem from potentially different frames of reference. Figure **2** shows two examples of this; a coordinate system combining Alice's and Bob's frames of reference as well as a coordinate system for air temperature in Fahrenheit and Celsius. Students' acts of joining two or more number lines that represent measures of (one or more) quantities in different frames of reference, and anticipating that ordered pairs (or n-tuples) give information about the measures in relation to each other, is the heart of combining multiple frames of reference.

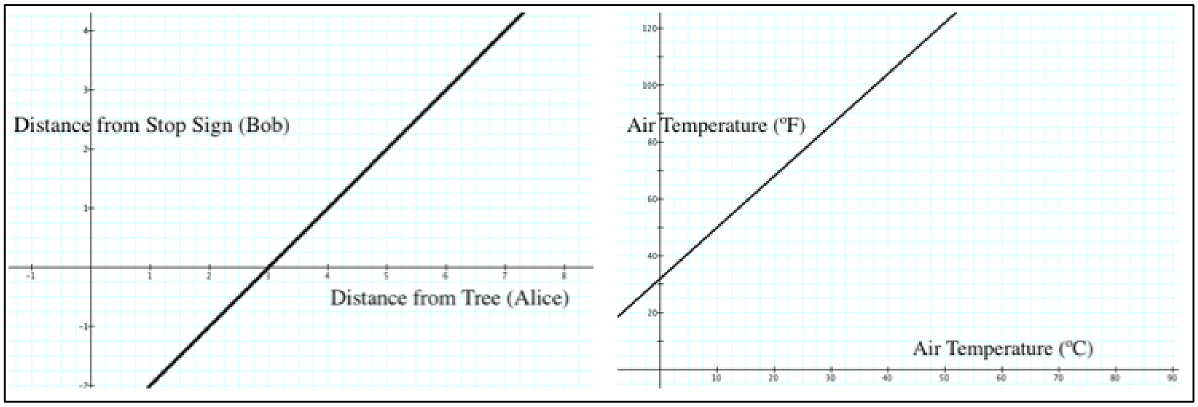


Figure 2. Examples of coordinate systems as combining multiple frames of reference.

Coordinating and Combining Multiple Frames of Reference

We note that when the student imagines a point (an ordered pair) along either line in Figure 2 as representing the measures of quantities in different frames of reference, she has combined the frames. If, however, she sees the line not just as representing a set of coordinated measures of quantities, but as a transformational relation between values of the quantities, she sees the graph as representing a functional relationship between the quantities.

Placing Our Theoretical Perspective amongst Others

Our interest in frames of reference and reasoning with frames of reference came about in an unexpected way. While analyzing teachers' responses to two items intended to target proportional thinking and rate of change, we found that teachers' responses to both items revealed struggles with coordinating quantities measured in what we came to realize were different frames of reference. Bowden et al. (1992) looked at the different approaches students used to analyze problems that involved an object moving inside another moving object (such as vector addition or proportional reasoning) and concluded that few students focused on "distinguishing frames of reference" (p.263-264). Bowden et al. noted that they attempted to characterize students' meanings based on their entire transcripts; however, Bowden et al. did not explain what they meant by "frames of reference". Rather, they used "frame of reference" as the possession of some object, e.g. "the frame of reference of the boat," Likewise they did not explain what they meant by "students' meanings." Monaghan and Clement (1999) wrote that computer simulations helped students develop mental imagery and ability to switch between frames of reference (e.g., as in a scenario involving a moving car and a plane flying overhead). However, they did not define or explain what they meant by frames of reference other than using pointers as Bowden et al. did. In further work they continued to use the construct of frames of reference without explicating what they meant by it (Monaghan & Clement 2000). Panse et al. (1994) investigated and identified "alternative [unproductive] conceptions" that students had about frames of reference, such as the idea that a frame of reference was a concrete object with boundaries or that a frame of reference is defined by the existence of a concrete object. While they did valuable work in describing alternative conceptions that hindered students' ability to reason about physical situations, they did not describe their normative conception of frames of reference. In all literature focusing on the idea of frames of reference or student thinking thereof, the authors presume that they and their readers share a common understanding of what "frames of reference" entails.

Expanding the Theory of Quantitative Reasoning

The few times an author (usually of a textbook) did explicitly describe what he or she meant by a frame of reference, the description focused on a frame of reference as an object or objects. Typical definitions range from "a coordinate system with a clock" (Young and Freedman 2011) to "a rigid system of 3 orthogonal rods welded together" (Carroll 2004) to "a set of observers at rest relative to each other" (de Hosson et al. 2010), with no further discussion about how students must conceptualize a frame of reference in order to reason with them. Such definitions support a student in focusing on the object of a frame of reference itself. In contrast, a key moment in developing our theory was when we began framing the question as "How does a student think about measures within a frame of reference?" As we said earlier in this manuscript, we defined a fully conceptualized frame of reference by stating that "An individual conceives of *measures as existing within a frame of reference* if the act of measuring entails [three commitments]." In other words, the mental actions, behaviors, and skills that we traditionally associate with someone "understanding frames of reference" (whatever that means) have nothing to do with how one thinks about frames of reference and everything to do with how one thinks about quantities.

In 1993 in his first article about quantitative reasoning, Thompson defined a quantity by saying that a "person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it" (Thompson, 1993). He also added in an unpublished 1990 paper that this includes implicitly or explicitly thinking of appropriate units (Thompson, 1990). We find this to be a useful definition that provides a place to start thinking and talking about quantities, especially with younger children. However, curricula that seek to emphasize quantitative reasoning have highlighted further aspects of quantities, such as measuring a quantity in relation to a reference point (Carlson et al., 2013).

Therefore, we define the idea of a *framed quantity*, which refers to when a person thinks of a quantity with commitments to unit, reference point, and directionality of comparison. As an example, consider a person who thinks about measuring how far Yolie has traveled as she walks her dog, understanding that appropriate units would be linear units such as feet, meters, and miles. This person is thinking about a quantity. In contrast, a person thinking about measuring Yolie's displacement to the east from her front door in meters is conceiving of a framed quantity. Not only does this person's mental construction have all the aspects of a conceptualized quantity, but it also shows a commitment to a unit (meters), reference point (front door) and directionality of comparison (displacement to the east yields positive measures). In other words, the quantity is so well defined that any measure value contains all the necessary information to understand its meaning. If x = Yolie's displacement to the east from her front door (meters), then $x = 3$ means that Yolie is 3 meters to the east of her front door and $x = -5$ means that Yolie is -5 meters to the east of her front door (which could be interpreted as being 5 meters west of her front door if wanted, but also provides the same specific meaning without this reframing). No extra qualifiers are needed to make sense of the value, and there is a clear directionality of comparison: the value always says how much further in the eastern direction Yolie is than her front door.

In Thompson's 2011 paper he identified a number of dispositions that would aid students' construction of algebraic thinking from quantitative thinking, including a disposition to represent calculations in open form, propagate information, think with abstract units, and reason with magnitudes. To this list we can now add that a disposition to think about measures within a frame of reference, and specifically with a direction of comparison, aids students in algebraic thinking. In constructing formulas students are often perplexed as to how to choose between a –

b and $b - a$, or a/b and b/a. This confusion can now be explained by thinking about how students do or do not commit to a directionality of comparison. Let us think about a student that is comparing the heights of husbands and wives in a study of couples. If the student sometimes frames the results of the comparison as "the husband is 6 inches taller than the wife" and other times "the husband is 2 inches shorter than the wife" then he is internally switching between two quantitative operations, which have corresponding formulas of $h - w$ and $w - h$, where h represents the husband's height and w represents the wife's height, both in inches. Naturally such a student would have difficulty in developing a formula to compare heights. In contrast, another student may commit to a directionality of comparison by deciding the value of his measure will always describe 'how much taller the husband is than the wife'. Since such a commitment entails always using the same quantitative operation, such a student will have far less obstacles to describing his process in symbolic form as $h - w$.

Applications of the Frame of Reference Construct

In our description of a conceptualized frame of reference and reasoning with multiple frames of reference, we deliberately used simplified tasks to illustrate the mental actions a person would have to take. However, we feel that the power of these constructs lie in their explanatory power in far more complex tasks. Below we illustrate two such tasks in detail, as well as sample responses from high school math teachers.

The task in Figure **3** presents two functions with non-equivalent rules (i.e., $f(x) = 15x\text{-}50/3$ and $g(x) = 15x\text{-}65/3$) to represent the same quantity (i.e., the distance between the two men). The fact that these two different functions can both represent the same quantity as a function of time creates difficulties for students (and teachers) trying to understand the scenario.

Robin Banks ran out of a bank and jumped into his car, speeding away at a constant speed of 50 mi/hr. He passed a café in which officer Willie Katchim was eating a donut. Officer Katchim got an alert that Robin had robbed the bank, jumped into his patrol car, and chased Robin at a constant speed of 65 mi/hr. Willie started 10 minutes after Robin passed the café.

Here are two functions. They each represent distances between Willie and Robin.

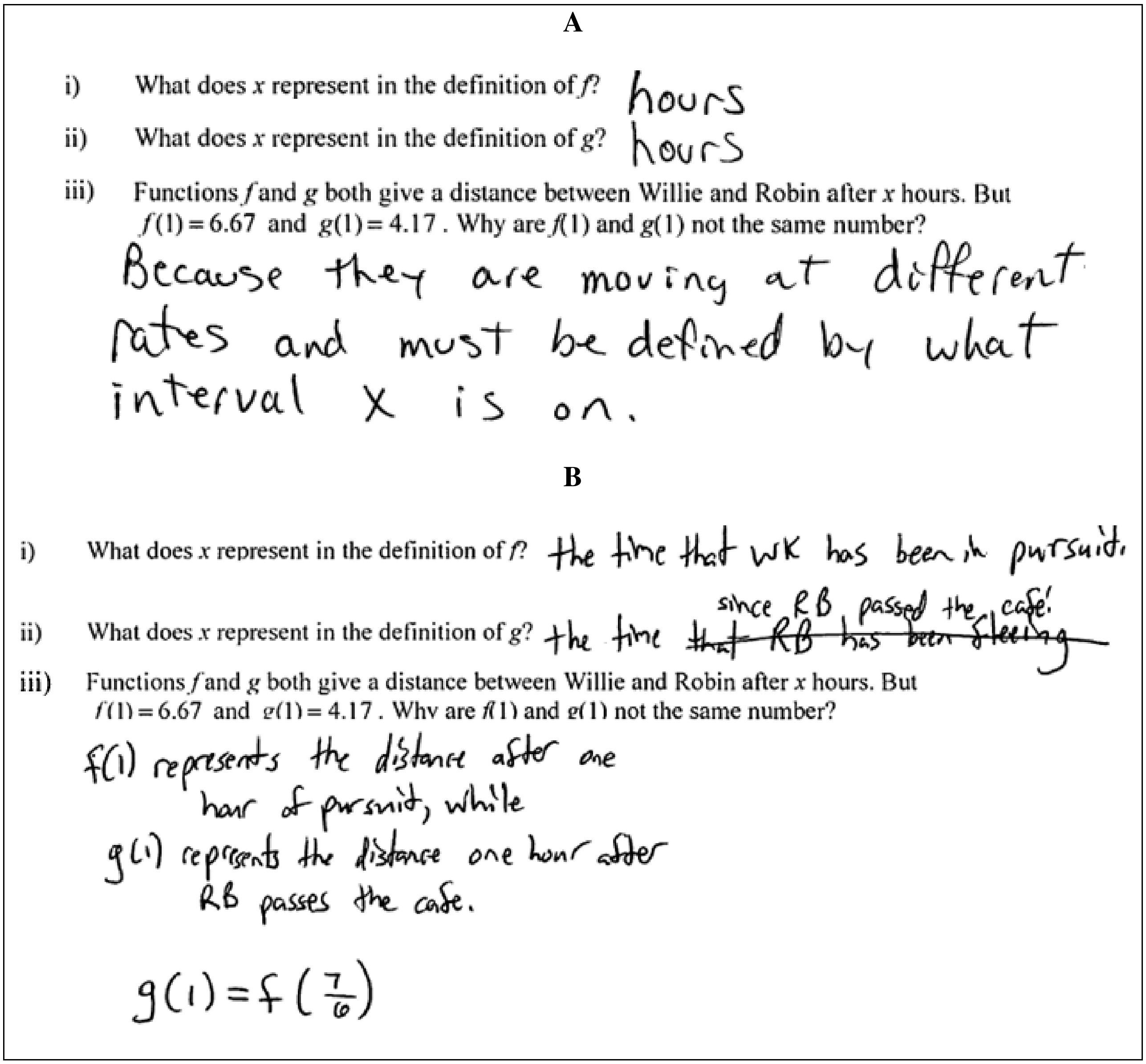
$f(x) = 65x - 50\left(x + \frac{1}{6}\right),\ x \geq 0.$ i) What does x represent in the definition of f?

$g(x) = 65\left(x - \frac{1}{6}\right) - 50x,\ x \geq 1/6.$ ii) What does x represent in the definition of g?

iii) Functions f and g both give a distance between Willie and Robin after x hours. But $f(1) = 6.67$ and $g(1) = 4.17$. Why are $f(1)$ and $g(1)$ not the same number?

Figure 3. Robin Banks Task. Adapted from Foerster, (2006). © 2014 Arizona Board of Regents. Used with permission.

A person who can both conceptualize and coordinate frames of references, however, can see that this seeming paradox is resolved when one acknowledges that all measurements are taken from some reference point. Willie's distance from the café is $65x$ miles where x is Willie's travel time in hours, and Robin's distance from the café is $50x$ miles where x is Robin's travel time in hours. However, the x's in these expressions have different meanings because they are measured from different reference points: the moment when Willie left the café and the moment when Robin left the café. To make a comparison of these two distances requires coordinating the two frames and re-expressing either measure in the other's frame. The distance between the men as described by $f(x)$ is the result of re-expressing Robin's distance from the café using Willie's "stopwatch", or frame, because at every point in time Robin has driven 1/6 hours more than Willie. Likewise, the distance between the men as described by $g(x)$ stems from re-expressing Willie's time using Robin's "stopwatch", or frame, because at every point in time Willie has driven 1/6 hours less than Robin.



A

i) What does x represent in the definition of f? hours

ii) What does x represent in the definition of g? hours

iii) Functions f and g both give a distance between Willie and Robin after x hours. But $f(1) = 6.67$ and $g(1) = 4.17$. Why are $f(1)$ and $g(1)$ not the same number?

Because they are moving at different rates and must be defined by what interval x is on.

B

i) What does x represent in the definition of f? the time that WK has been in pursuit.

ii) What does x represent in the definition of g? the time ~~that RB has been fleeing~~ since RB passed the café.

iii) Functions f and g both give a distance between Willie and Robin after x hours. But $f(1) = 6.67$ and $g(1) = 4.17$. Why are $f(1)$ and $g(1)$ not the same number?

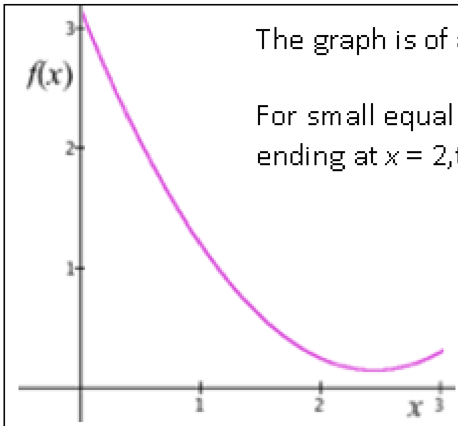
f(1) represents the distance after one hour of pursuit, while g(1) represents the distance one hour after RB passes the cafe.

$g(1) = f\left(\frac{7}{6}\right)$

Figure 4. Sample responses to Robin Banks task

Figure 4 displays two sample responses to the Robin Banks task given by high school math teachers. In Figure 4A parts i) and ii) the respondent did not think about the quantities with respect to a reference point, and so had no way to answer part iii) meaningfully. In contrast, in Figure 4B we see in parts i) and ii) that the respondent conceptualized the quantities with respect to specific reference points, and was also able to correctly coordinate the two frames in part iii).

Our frames of reference construct is also useful for examining individuals' struggles with situations devoid of motion. As an example, consider the task in Figure 5 that asks the reader to compare consecutive changes in the interval [1, 2].



The graph is of a function f over interval [0,3].

For small equal increases in the values of x starting at $x = 1$ and ending at $x = 2$, the corresponding changes in the value of f are...

a) positive and increasing
b) positive and decreasing
c) negative and increasing
d) negative and decreasing
e) I cannot tell

Figure 5. Comparing Changes Task. © 2014 Arizona Board of Regents. Used with permission.

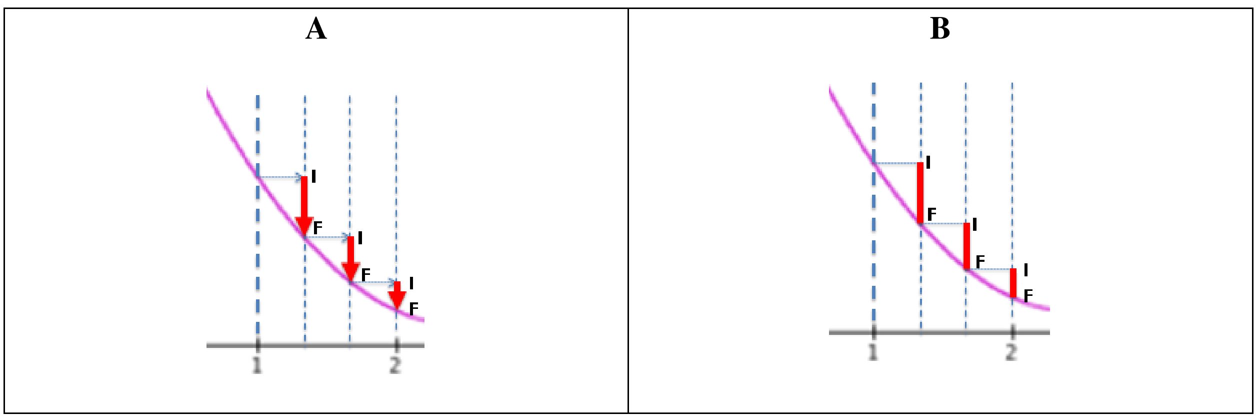


Figure 6. Different Visualizations of the Comparing Changes Task

This task proves challenging for people who do not think about changes within a frame of reference – specifically, people who do not maintain a directionality of comparison. Consider two hypothetical students: Dean who chooses option d) and Cathy who chooses option c). Assume both students understand the directionality of changes well enough to visualize changes as in Figure 6A.

Dean says that the changes are negative and decreasing because he has inadvertently switched the direction of his comparison between deciding "the changes are negative" and "the changes are decreasing." To determine that the changes are negative, he is engaging in a quantitative operation that we can formulize as [final y-value] – [initial y-value] and obtains a negative value for each. However, in deciding that the changes are decreasing, he is really only considering the magnitude of those changes, essentially switching his mental image to that shown in Figure 6B and engaging in a quantitative operation that we can formulize as [initial y-value] – [final y-value]. In comparison, Cathy says the changes are negative and increasing because she has maintained her directionality of comparison. For both her "changes are negative" and "changes are increasing" decisions, she engages in a quantitative operation that can be formulized as [final y-value] – [initial y-value]. We gain insight into individuals' difficulties with this task by noticing a lack of commitment to directionality of comparisons.

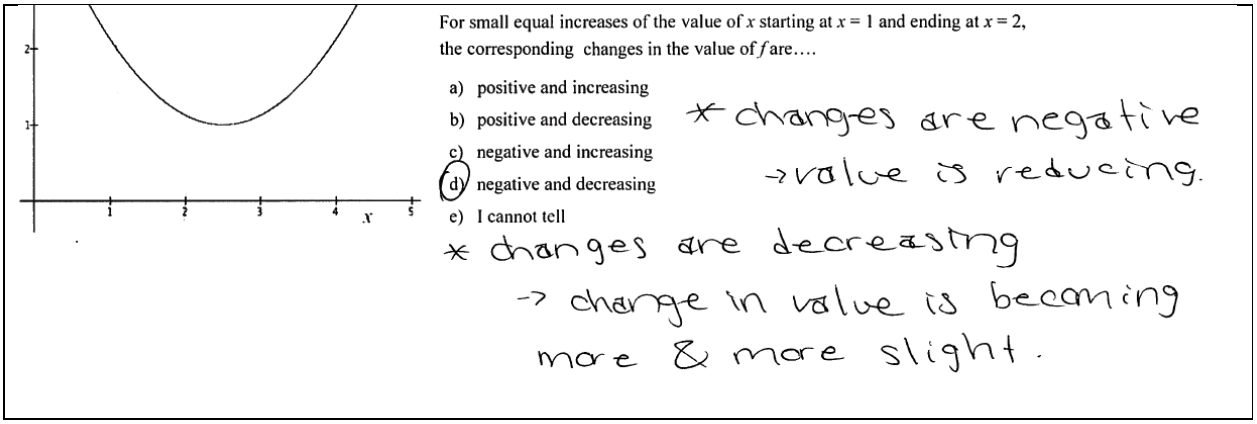


Figure 7. Sample response to Comparing Changes task

Figure 7 displays a sample response to the Comparing Changes task given by a high school math teacher. Note that the teacher's justification for his comment "changes are negative" refers to a directionality: "value is reducing." However, his comment "changes are decreasing" uses the language "changes in value are becoming more and more slight", which we see as a strong indication that the teacher suddenly switched to looking at magnitudes.

Discussion

The above are two examples where the constructs of a conceptualized frame of reference and reasoning with multiple frames of reference have explanatory power and potential for improving instruction. As we developed our descriptions of these constructs, we started to see applications in a variety of other domains. Below we give brief descriptions of some of these domains and where we see potential for future research and teaching.

Personal experiences in teaching pre-calculus and calculus had shown us that students frequently conflate the value of a quantity and a change in that quantity, which leads to difficulties in understanding the ideas of change, slope, constant rate of change, and rate (derivative) functions. This confusion may be explained by a lack of attention to reference point for each measure; if a student does not commit to a reference point when measuring a quantity, there is little meaningful difference between the measure of the total quantity and a change in that quantity over a given interval. On the other hand, developing the idea that the total quantity is really a change from (a reference point of) zero provides parallel ideas with which to distinguish the two. Highlighting reference point commitment in teaching and discussion may help to alleviate this confusion.

Students frequently categorize all motion within a false dichotomy of "real motion" vs. "imagined motion", where an object is only "really moving" if it is moving with respect to the surface of the Earth, and the measure of its speed or velocity is only "real" if measured with respect to the surface of the Earth (Panse et al. 1994). This hinders their ability to deal with relative motion tasks and has been a focus of study in physics education (Monaghan and Clement 1999). For example, students cannot accept that a bike moving 15mph towards a sign is also moving 5mph with respect to a walker and moving backwards with respect to a car. While Monaghan and Clement worked on developing their students' visual imagery, we believe that teaching students about conceptualizing all quantities as measured with specific reference points, and comparing quantities with specific directionalities of comparison, may prove beneficial.

This common student struggle with "real" versus "imagined" motion stems from a lack of understanding of the fundamental physics principle of relativity (Bandyopadhyay 2009) that states that there can be no way of verifying that any reference frame (or object) is at absolute rest, and therefore the entire notion of absolute rest should be abandoned. We believe emphasizing that a reference point is mandatory for any measure to be meaningful can provide a backdrop for students to also accept that what we talk about as motion measure in the real world always comes with its implicit assumption of a reference point (the surface of the Earth), and that if all reference points are arbitrary then the surface of the Earth is as well.

One of the most common struggles students have in physics is in understanding the concepts of velocity and acceleration. For example, researchers have found it extremely difficult to change the student perception that a positive acceleration means an object must be speeding up (when in fact it may be going from -5mph to -2mph, meaning it is slowing down but increasing in velocity). We have found in personal conversations that even professors who are known for their work in physics education have been teaching students that an object going from -10mph to -20mph means that "the velocity is increasing in the negative direction" probably to deal with these types of misunderstandings. But not only are such descriptions physically and mathematically inaccurate, they result in descriptions that are incompatible with observations about change and rate of change that can be derived from calculus. We believe that teaching students about a commitment to directionality of comparison is far more consistent and fruitful way to approach these concerns.

Panse et al. wrote a detailed description of seven alternative conceptions that students have about reference frames (Panse et al. 1994). Alternative conceptions 1, 2, 3, 4, and 6 are the consequences of seeing a reference frame as a physical object, while alternative conceptions 5 and 7 are the consequences of not fully understanding the principle of relativity. As we developed our constructs we identified an eighth alternative conception: the idea that a frame of reference is useful primarily (or only) for an observer that remains at the origin of the frame's coordinate system. We see the potential to reduce the number of students that develop all eight alternative conceptions in discussing frames of reference with students only in terms of three commitments on the part of the observer.

We are grateful to an audience member at our presentation of this paper at the RUME 18 conference, who offered the idea of electric potential as another concept we can reconceive through our constructs for frames of reference. It is true that students struggle with the idea of electric potential, and our minds immediately went to the struggles that physics and engineering students have with Kirchoff's second laws for circuits. Briefly stated, Kirchoff's second law states that the sum of the changes in electric potential around any loop in a closed circuit must be zero. Students often struggle with how to apply the rule because they feel a need to know where in the circuit the potential is "really zero" so that they can start their calculations there, not understanding that (like absolute rest) there is no such thing as absolute zero electric potential. These student difficulties may be alleviated by the same measures that help students to understand the principle of relativity in motion.

We believe that research on frames of reference and student thinking about frames of reference is warranted by the difficulties that students have with “typical” frames of reference problems. We think that the framework *conceptualized frame of reference* that we proposed offers new insight on student difficulties and contributes to a foundation for further research.

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