
Calculus students' meanings for difference

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Students learn the mathematical operation of subtraction beginning in elementary school, along with key vocabulary to talk about that operation. However, the meanings that students develop for the word "difference" continue to play a role well into students' study of undergraduate mathematics. In particular, a meaning for "difference" as representing a change in a quantity is essential to understanding and communicating about foundational ideas in Calculus. In this preliminary report, we consider meanings about the word "difference" held by calculus students as revealed on a pre-test in an on-going study designed to explore Calculus students' structure sense. We further propose potential consequences for those meanings and describe methods to be used in data collection for the remainder of the Fall 2014 semester.

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Students learn about the mathematical operation of subtraction beginning in elementary school. Along with learning to perform subtraction to calculate values of expressions, students learn vocabulary words, such as "difference", "minuend" and "subtrahend" to refer to the structure of expressions involving subtraction, the quantity from which another is to be subtracted, and the quantity to be subtracted, respectively. Such words and attention to structure, however, often fall to the wayside in classroom conversation. Students also learn the other arithmetic operations of addition, multiplication and division, then proceed to learn the Order of Operations. Treatment of all these topics traditionally falls in the context of computing values of expressions, rather than identifying the structure of such expressions. For instance, when a student is given an expression like 3 – 5 + 6, more often than not, he is asked to simplify. If he follows the Order of Operations, he will first compute 3 – 5 to get -2 and write a new expression -2 + 6. Notice that as soon as he replaces 3 – 5 with -2, he loses the structural information of where the -2 came from. Likewise, when the student adds -2 and 6 to get 4, all information about where the value 4 came from is lost. Note that as this student engages in this type of computational activity repeatedly, he is likely to develop a meaning for the Order of Operations and the operations themselves that they are a call to *do* something. Exclusive engagement in computational activities will hinder the students' development of meanings for Order of Operations as a means to describe the structure of the expression (e.g. the above expression is a sum in which the first addend is the difference 3 – 5 and the second addend is 6).

The act of describing the structure of an expression can be thought of in terms of a larger area of study: students' structure sense. Extant literature shows that students have weak structure sense, both before and after completing coursework at the university level (Hoch & Dreyfus, 2006; Novotná & Hoch, 2008). We suspect that weak structure sense, particularly the awareness of and ability to identify structure of expressions, is a major contributor to the common struggle of Calculus students in applying appropriate techniques of differentiation and integration. After all, it is common to hear a student say, "I have memorized all the rules of differentiation, but when you give me those crazy functions, I don't know which one to use."

The broader purpose of our on-going study is to explore Calculus students' structure sense, particularly with regard to whether or not they attend to the structure of functions. Namely, when given a function, do students recognize the structure of the function rule as a sum, difference, product, or quotient? How do students' meanings for Order of Operations play a role in identifying structure? What activities can a teacher engage in to draw students' attention to structure? Does attention to structure alleviate typical student struggles with applying rules of differentiation and integration techniques?

In this report, we focus specifically on students' meanings for "difference" as revealed on a pre-test. We describe consequences for these meanings in the teaching and learning of Calculus ideas.

Theoretical Framework

We consider the meanings an individual develops as his means to organize his experiences, and once developed, those meanings serve as organizers of new experiences. Creating meanings entails constructing a scheme through repeated reasoning and reconstruction to organize experiences in a way that is internally consistent (Piaget & Garcia, 1991; Thompson, 2013; Thompson, Carlson, Byerley, & Hatfield, 2013). For instance, an individual's meaning for Order of Operations might be entirely situated in the context of computing the value of an expression and entail recollecting the acronym PEMDAS (Parentheses, Exponents, Multiplication, Division, Addition, Subtraction). Such a meaning might inhibit that individual's ability to make sense of the structure of an expression containing both numbers and variables.

Methods

At the time of writing this report, we are in the initial stages of data collection for a study that will conclude at the end of the Fall 2014 semester. We have collected data from 201 Calculus I students at a large research university in the southwestern United States. The design of the Calculus course is distinct from a conventional introductory Calculus course in that the curriculum is research-based and designed with the explicit intent of supporting students in developing rich meanings for the foundational ideas of calculus.

Participants. There are two sections of the specially designed Calculus I course. One section has 52 students and is taught by the lead author. The other section has 149 students and is taught by a senior instructor who has 3 semesters of experience teaching this particular course, and played an integral role in creating the student materials for the course. The students were unaware of the unique design and goals of the course when they registered for the class.

Data Collection. The students completed an 11-item pre-test that investigates their meanings for Order of Operations. Selected tasks from this pre-test and some preliminary results are discussed in the Preliminary Results section. During the remainder of the Fall 2014 semester, data will primarily be gathered from the 52-student section. The instructor of this section will record her lectures (capturing audio and the screen projected for students to see), which will be tailored to explicitly draw students' attention to structural qualities of expressions and functions. She will select 6 students from the pool of volunteers to conduct interviews to probe their thinking further and pilot tasks to be used during whole-class instruction. Student work on assignments related to supporting structure sense will be scanned as a reference to gauge students' tendency to employ structural awareness in their work. At the end of the semester, students from both sections will complete a post-test aimed to reassess students' meanings for order of operations and to see if there is a connection between those meanings and performance on differentiation and integration tasks. Data from interviews and the post-test will be presented in the event that this proposal is accepted.

Preliminary Results

In the discussion that follows, we describe 2 tasks on the pre-test that explicitly investigate students' meanings for difference and discuss preliminary results from 83 students' responses.

Task 1: What does it mean to say an expression is a difference?

We first asked students to answer the question, "What does it mean to say an expression is a difference?" Of the 83 responses coded so far, only 18 did not include the words "subtract" or "subtraction". For 6 students, "subtraction" or "to subtract" was the entire response. Formally, however, three pieces comprise a difference: a minuend, a minus sign and a subtrahend. In the context of Calculus, the minuend and subtrahend have significance; they represent values for measures of given quantities. For instance, one might be given a function f that describes the number of feet traveled by a rocket relative to the number of seconds elapsed since being launched. With the given information, one might symbolically represent the distance traveled by the rocket during the first 10 seconds of the launch by writing the difference $f(10) - f(0)$. It is necessary to imagine two quantities to produce this expression. We did not expect students to use language like "quantity" or "minuend", in their responses, but even after relaxing our criteria to determine how many objects (e.g. "numbers", "expressions", "something") mentioned in their response, 26% of responses contained a reference to fewer than 2 objects (Table 1).

Table 1. Number of objects described in a difference

Number of objects mentioned in describing a difference	Number of Responses (out of 83)
None	20 (24%)
Exactly 1	2 (2%)
"One or more" or "more than one" (as signaled by the use of pluralized words like "number*s*", "expression*s*")	19 (23%)
Exactly 2	31 (37%)
"Two or more"	10 (12%)
Unclear	1 (1%)

Task 2: Identifying differences in a mathematical sentence

In order to see how students operationalize the meaning they have for difference in the context of identifying structure, we asked students to identify differences in a mathematical sentence (Figure 1). We anticipated that students would at least rely on the number of subtraction symbols (4) to determine the number of differences they should list. However, 33% of students (27 of 83) only listed 3 differences. Most of these students listed (with some variation on which parts of the differences they identified, as discussed below) $f(x+h)-f(x)$, $(x+h)-x$ and $e^{7-x}-3\cos(2+x)$. Eight people did not respond or otherwise gave responses we could not interpret, four people listed more than 4 differences and 41 students listed exactly 4 differences.

List each difference that you see in the mathematical sentence given below.

$$d(x)=\frac{f(x+h)-f(x)}{(x+h)-x}+e^{7-x}-3\cos(2+x)$$

Figure 1. Identifying Differences Task

Only 5 students listed the four differences we identified, namely: $f(x+h)-f(x)$, $(x+h)-x$, $7-x$, and $\left(\frac{f(x+h)-f(x)}{(x+h)-x}+e^{7-x}\right)-3\cos(2+x)$.

Table 2. Types of responses to Identifying Differences Task

Characteristic of Response	Number of Responses (out of 83)
Listed 4 differences	41 (49%)
Listed 3 differences	27 (33%)
Listed more than 4 differences	4 (5%)
Wrote e^{7-x} instead of $7-x$	14 (17%)
Wrote an expression containing only the minus sign and the subtrahend (e.g. $-x$ instead of $7-x$)	12 (14%)
Pointed to or circled the minus signs	11 (13%)
No response/Researchers could not interpret response	8 (10%)

Note: students' responses may be listed in multiple categories; the counts (percentages) will not add to 83 (100%).

Table 2 summarizes a few interesting points regarding students' responses. Eleven of 83 students *only* pointed to, or circled, the subtraction symbols. For these 11 students, the subtraction symbol is the difference, rather than the whole expression comprised of the minuend, minus sign and subtrahend. Another 12 students wrote the minus symbol and the subtrahend without the minuend when listing the differences they identified. We suspect students in these two groups may struggle to make meaning of discussions held in class regarding changes in quantities. In particular, for a student who only circled the minus signs in Figure 1, thinking about a difference does not entail imagining a quantity, two values of that quantity and a comparison of those values. Yet holding all these things in mind is essential to reasoning about changes in quantities in Calculus, one of the foundational components to the idea of rate of change and, hence, the Fundamental Theorem of Calculus.

Cross-task Comparisons

An emphasis on the operation subtraction while discussing "differences" may explain the 11 students' responses that reference *only* the minus sign. Further, a meaning for subtraction as "take away" could explain the other 12 students' writing only the minus sign and the subtrahend. Table 3 on the next page shows several of these students' responses. We plan to conduct follow-up interviews to further probe these students' thinking.

Questions for the Audience

In Table 3, it appears that Emily used a dash as a bullet point to list differences. Other students did this as well. Ought we consider Brett's and Cindy's responses to also reflect using a dash as a bullet instead of a minus sign? Does this change the consequences for meanings?

While we only focused on the meaning for difference in this report, we also have data related to students' meanings for the other arithmetic operations and the Order of Operations. Many students tended to rearrange symbols to "show" structure via spatial arrangement. For instance, when presented with a single-line expression $x + 3 / 7 * y$, the student would rewrite the expression as a stacked fraction. Does this constrain students' meanings? Or is a reliance on visual cues an acceptable activity for students, since writing expressions on one line is typically reserved for typing mathematics (an activity most students never do)?

Table 3. Student responses to both tasks

Student	Response to Task 1	Response to Task 2
Ally	Subtraction	$d(x)=\frac{f(x+h)\textcircled{-}f(x)}{(x+h)\textcircled{-}x}+e^{7-x}\textcircled{-}3\cos(2+x)$
Brett	subtraction of given numbers or letters	$-f(x)$ $-3(\cos 2*x)$ $-x$
Cindy	total subtracted amount	$-f(x)$ e^{7-x} $-x$ $-3\cos$
Danny	This is a number that was previously subtracted from.	$d(x)=\frac{f(x+h)\overset{\downarrow}{-}f(x)}{(x+h)\underset{\uparrow}{-}x}+e^{7\overset{\downarrow}{-}x}\underset{\uparrow}{-}3\cos(2+x)$ all ~~arrows~~ arrows point to a difference
Emily	to say an expression is a difference? The expression is a subtraction equation	$-F(x+h)-F(x)$ $-(x+h)-x$ $-e^{7-x}$ $-e^{9-x}-3\cos(2+x)$

Note: All names are pseudonyms.

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