Teachers' meanings for the substitution principle

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Structure sense is foundational to mathematical thinking. This report explores high school math teachers' meanings for the substitution principle, a sub-category of structure sense that research previously identified as sources of difficulty for students. A focus on meanings reflects our belief that teachers' meanings directly impact the mathematical meanings students develop. We suggest ways of thinking that could lead to various response types as a resource for teacher educators to design professional development targeting improved structure sense for teachers.

Keywords: Structure Sense, Representational Equivalence, High School Math Teacher, Substitution Principle, Mathematical Meanings

Structure is a foundational component of mathematics; one could (over) simplify the work of a mathematician as the study of the structure of objects and relationships between those objects. As such, developing structure sense is fundamental to the experience of math students. One powerful component of structure is representational equivalence, which splits into two categories: transformational equivalence and substitution equivalence. Transformational equivalence refers to the equivalence-preserving transformations one may perform on a mathematical object. For instance, while solving an equation, one ought to perform actions on the equation that do not alter the original relationship (e.g. multiply *both* sides of an equation by the same non-zero value). Substitution equivalence, called the substitution principle by other authors, points to the underlying structural "sameness" that holds when substituting a compound term for a variable or a variable for a compound term (Novotná & Hoch, 2008). For example, one might substitute u for $5x-1$ in the equation $(5x-1)^2-3(5x-1)=-2$ to highlight its quadratic nature. In this report, we discuss potential difficulties in applying the substitution principle in an abstract setting to manipulate an expression, as well as implications for this in teaching and learning mathematics.

While the focus of this report centers on the substitution principle, it hints at a broader issue of structure sense. The term "structure sense", coined by Linchevski and Livneh (1999) to signify the use of arithmetic structures in the transition to algebra, and broadened by Hoch (2003), references the "ability to recognize algebraic structure and to use the appropriate features of that structure in the given context as a guide for choosing which operations to perform " (p. 2). Note that the ability to recognize and utilize structure refers to actions that apply across all contexts of school mathematics. Hoch and Dreyfus (2006) demonstrated what this ability might look like in specific contexts, grounding this general definition in a way that could be useful for guiding student learning and curriculum design. The blanket term, however, captures the fact that the notion of structure *is* broad and spans every level of mathematics. Students conceive relationships between concepts, objects and techniques as they transition from course to course. Their development of these relationships supports students' awareness of structure (Mason, Stephens, & Watson, 2009).

Extant literature reveals that many students do not develop structure sense (Hoch, 2003; Hoch & Dreyfus, 2006; Linchevski & Livneh, 1999; Novotná & Hoch, 2008; Novotná, Stehlikova, & Hoch, 2006; Tall & Thomas, 1991). Student performance, documented at various stages of mathematical experience in the aforementioned studies, points to a lack of attention to and ability to employ structural qualities of mathematical objects. We suggest one explanation for this missing piece of students' mathematical development: teachers might not provide experiences that allow students to develop structure sense. We further suspect that this is not a conscious decision on the part of teachers, but rather a result of the fact they themselves do not possess robust structure sense.

This conjecture is problematic in light of the Common Core State Standards' call for students *to identify* structure, meaning a student must be aware that structure is something to look for in representations of mathematical objects, and for students *to act* in accordance with that structure. These are subtle, yet key, distinctions that teachers must be able to make. More often than not, student behavior leans towards acting rather than reflecting on actions. For instance, order of operations is frequently taught in the context of *calculating* values of expressions rather than identifying implicit structure. This compounds the difficulty Thompson and Thompson (1987) identified with regard to students' work with algebraic expressions. Expressions can be "structured explicitly by the use of parentheses, [or] implicitly by assuming conventions for the order in which we perform arithmetic operations" (p. 248). The standard practice of relying on order of operations to imply the structure of an expression means that students must first be aware that structure is something to which they should attend. Only then can they use their internalized conventions to determine the expression's structure. Tall and Thomas (1991) describe another student obstacle in determining structure as the *process-product obstacle*. The obstacle is that students must simultaneously view an algebraic expression as representing the process of a computation and the product of that process. Many students' difficulties stem from focusing on expressions as representing the process of computing rather than the reflecting on the expression as representing the result of computing.

In this report, we provide evidence that many in-service teachers have difficulty with structure sense. We focus specifically on the substitution principle (i.e. taking a complex expression as one object), a subcategory of structure sense that research points to as a common area of struggle for students (Hoch & Dreyfus, 2006; Novotná & Hoch, 2008). We believe that teachers cannot support students in developing richer meanings than the ones the teachers possess, making it imperative to understand the nature of teachers' mathematical meanings. With this understanding, teacher educators can devise ways to help teachers improve both their structure sense and their awareness of its importance for students' mathematical learning. By describing teachers' struggles with the substitution principle and possible sources of difficulty, we hope to identify not only task-specific difficulties, but also identify ways of thinking that lead to teachers' difficulties. Investigating teachers' meanings regarding structure will also allow us to identify potential sources of students' difficulties, thus giving a more comprehensive perspective on the issue of students' development of structure sense.

Theoretical Framework

We view an individual's meanings as her means to organize her experiences and, once formed, as organizers of her experience. Through repeated reasoning and reconstruction, an individual constructs schemes to organize experiences in an internally consistent way (Piaget & Garcia, 1991; Thompson, 2013; Thompson, Carlson, Byerley, & Hatfield, 2013). For example, part of an individual's meaning for a mathematical expression is how she sees its structure. One person might see "$x/2y$" as $(x/2)y$ while another might see it as $x/(2y)$. These two people hold different meanings for the given expression, and the consequences for such differences can be profound.

We take as given, subject to future investigation, that a teacher's meanings can be more or less productive in classroom instruction, with productive meanings supporting students' development of coherent mathematical meanings and ways of thinking. Investigations of teachers' mathematical meanings can inform professional development efforts to help teachers promote productive meanings and coherence in mathematics instruction (Musgrave & Thompson, 2014; Simon & Blume, 1994; Thompson, 2013).

Methodology

Our team of mathematics educators and mathematicians created a diagnostic tool called the Mathematical Meanings for Teaching Secondary Mathematics (MMTsm) in order to address this issue of investigating teachers' meanings. The MMTsm consists of tasks that provide teachers the opportunity to interpret the given scenario and respond according to their meanings. Our team designed, tested, interviewed, and refined items for approximately two years prior to giving these tasks in the summer of 2013. In this report, we concentrate on one item for which the substitution principle is foundational to reasoning about the problem.

We scored teachers' responses to each item in accordance to a scoring rubric. After collecting data in summer 2013 we developed an open coding scheme based on roughly 140 teachers' responses to categorize ways of thinking. We supplemented the coding process with teacher interviews during the yearlong process of developing scoring rubrics. We drew upon both the data and prior research related to how students and teachers understand the various ideas items were designed to tap. Once identified, we organized themes and ways of thinking into levels according to productivity for student learning to form an initial rubric. A group of 10 people from two institutions scored 10 responses to each item and discussed possible improvements to the rubric; we iterated the scoring-refining process until reaching a consensus. At this point, we tested the inter-rater reliability of each rubric with an external group and made adjustments to each rubric until we reached 100% agreement. The team then held a two-day scorer-training workshop on using the rubrics. Upon submission of scores, a team member verified each score for compliance to the appropriate rubric for each item. Any adjustments made followed the specifications of the rubrics. A more detailed description of the method for creating tasks and rubrics attentive to mathematical meanings can be found in Thompson (in press).

We administered the MMTsm to high school mathematics teachers involved in professional development programs from two states in the United States. Eighty-four (84) teachers took a version of the MMTsm containing the item discussed below. The teachers had varying backgrounds with regard to a number of demographic variables. Table 1 shows the distribution of teachers' highest degree obtained along with their major of study. Teachers under "STE" majored in science, technology or engineering. The "Other" major category includes all other majors, such as Business Administration and elementary education. Approximately two-thirds of the teachers majored in mathematics or mathematics education; with another 11% being other STEM related majors.

Table 1. Teachers' Highest Degree Obtained vs. Major.

	Math	MathEd	STE	Other	Total
Bachelor's	9	11	4	8	32
Master's	16	21	5	10	52
Total	25	32	9	18	84

The Task

Thompson and Thompson (1987) describe a common student difficulty in viewing a sub-expression as a single object. In their study, students used a computer program to manipulate expressions into equivalent forms (e.g. transform $(z-q)*u$ into $z*u-q*u$ by selecting and applying appropriate identities and transformations). Several tasks requiring students to view a sub-expression as a unit proved challenging for students, likely because that particular mental activity requires students to focus on the structure of an expression by mentally grouping parts of it as one object. Our research team adapted an item from Thompson and Thompson's 1987 study (Figure 1) for use on the MMTsm.

Δ is an operation with the following property
For all real numbers, a, b, and c, $(a \Delta b) \Delta c = a \Delta (b \Delta c)$.
Let u, v, w, and z be real numbers. Can this property of Δ be applied to the expression below? If yes, demonstrate. If no, explain.

$$(u \Delta v) \Delta (w \Delta z)$$

Figure 1. Associative Property Task. © 2014 Arizona Board of Regents. Used with permission.

Teachers could use tasks similar to the one in Figure 1 to guide classroom conversations about expressions, particularly focusing students' attention on structure while discussing how to view the expression in multiple equivalent ways. We thus administered the task in Figure 1 to teachers to gain insight on how they might respond in a situation necessitating the grouping of compound terms as one object. The task defines a property for an operation Δ in terms of three variables and asks if the property can be applied to an expression with four variables. In order to reason that the stated property of Δ applies to the expression, one must reason that $w\Delta z$ can be viewed as one object while viewing $u\Delta v$ as an operation on two objects, or similarly, view $u\Delta v$ as one object while considering $w\Delta z$ as an operation on two objects. We categorized teachers' responses according to a scoring rubric, which we will describe further in the next section.

Results and Discussion

Table 2 shows our classification of teachers' responses to Figure 1. Responses categorized in the top two levels include those that attend to structure in a way that shows that the property applies. The distinguishing trait between the top two levels is the quality of demonstration. Namely, responses at the highest level (Level 3) explicitly show how a sub-expression is treated as one object. We put at Level 2 responses that correctly applied the property without explicitly showing groupings (Type 1) because we believe that simply providing the answer without showing how it is achieved would be less supportive of students who are still developing the skill of identifying compound terms as one object. We also put at Level 2 any response that would have been put at Level 3, but which then showed further work that introduced ambiguity (Type 2). Level 1 captures two types of responses. The first is that the teacher claimed the property does not apply (13 of the 25 teachers at Level 1). The second contains responses that suggested that the teacher thought that the property meant they could move parentheses any way one wishes. Level 0 also captures a variety of responses. Specifically, if a teacher changed the order of the variables in the expression, stated the need to know the definition of Δ, claimed that Δ stood for addition or multiplication, substituted numbers for any of the variables, or wrote a final expression not containing exactly 3 "Δ" symbols, his or her response was scored at Level 0.

The Associative Property Task was atypical among items in our assessment with regard to results varying based on undergraduate major. Table 3 shows the distribution of teachers' responses by level and by undergraduate major. The distribution of responses by teachers with degrees in math, math education, and STE are similar distributions across levels. However, the distribution of responses by teachers with "Other" majors is noticeably different, with a disproportionately large number of teachers holding degrees in the "Other" category providing low-level responses. We suspect that the ways of thinking required to provide a high-level response are not regularly practiced outside of STEM fields. While teachers with "Other" majors have likely practiced grouping objects mentally, they may not have repeated experiences doing so in an abstract setting using symbolic manipulation.

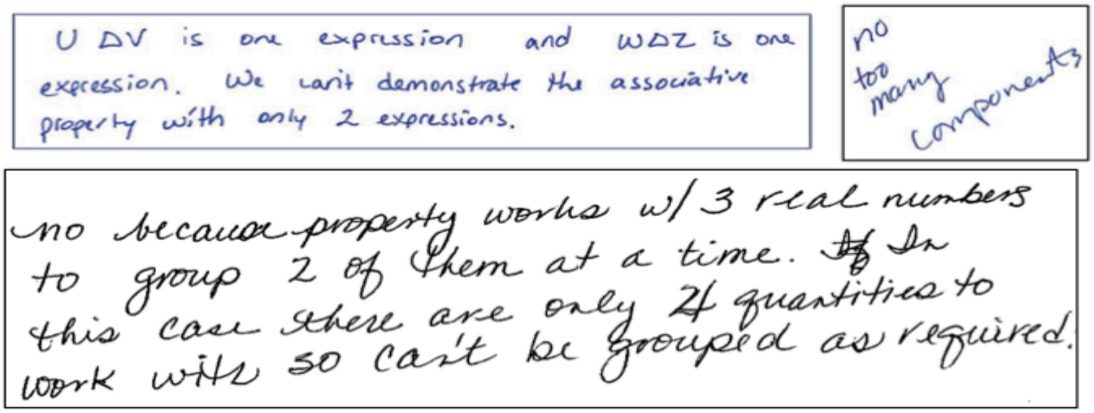
Table 2. Associative Property Task Sample Responses by Level

Response Level	Sample Responses	Number of Responses
Level 3	$(u \Delta v) \Delta \overset{c}{(w \Delta z)}$ $(u \Delta v) \Delta c = u \Delta (v \Delta c)$ $= u \Delta (v \Delta (w \Delta z))$	14
Level 2	Type 1: Correctly applied property without demonstrating how the property applies. Sure: $u \Delta (v \Delta (w \Delta z))$ Type 2: Contains elements of Level 3 response, but final answer does not serve the purpose of demonstrating how the property applies. $(u \Delta v) \Delta (w \Delta z)$ $(a \Delta b) \circ (c)$ let $u = a$ $v = b$ $w \Delta z = c$ $a \Delta (b \Delta c)$ $a \Delta (b \Delta w \Delta z)$ $\boxed{u \Delta (v \Delta w \Delta z)}$	15
Level 1	Type 1: Teacher said property does not apply $U \Delta V$ is one expression and $W \Delta Z$ is one expression. We can't demonstrate the associative property with only 2 expressions. Type 2: Placement of parentheses is inconsistent with the given property $u \Delta (v \Delta w) \Delta z$	25
Level 0	Substituting numbers or replacing Δ with a known arithmetic operation: associative prop $(a+b)+c = a+(b+c)$ $a=1$ $b=2$ $c=3$ $(u+v) + (w+z)$ $(1+2)+3 = 1+(2+3)$ $(u \Delta v) \Delta (w \Delta z) = u \Delta (v \Delta w) \Delta z$ Yes Changing order of variables and omitting Δ: $(u \Delta w)(v \Delta z)$	28

Table 3. Responses to Associative Property Task by Major

	Math	Math Ed	STE	Other	Total
Level 3	5	6	2	1	14
Level 2	5	6	2	2	15
Level 1	7	13	1	4	25
Level 0	6	7	4	11	28
No Response	2	0	0	0	2
Total	25	32	9	18	84

Our data suggests two common sources of struggle on this item. The first is the abstract nature of the problem itself; Δ is an unknown operator and variables are used instead of numbers. Indeed, 21 of the 28 Level 0 responses suggested the need to use numbers, or expressed a need to know what operation Δ represents, like multiplication or addition (see Table 2, Level 0, first response). The second source identified in the data is the difficulty in viewing similar structures $u\Delta v$ *and* $w\Delta z$ in two different ways *simultaneously*—one as two objects and the other as one object. In fact, some teachers responded in a way that showed they only saw two objects or that they saw four objects (Figure 2).



$U \Delta V$ is one expression and $W \Delta Z$ is one expression. We can't demonstrate the associative property with only 2 expressions.

no
too
many
components

no because property works w/ 3 real numbers
to group 2 of them at a time. ~~If~~ In
this case there are only 4 quantities to
work with so can't be grouped as required.

Figure 2. Sample Teacher Responses Demonstrating Conceiving Only of an Even Number of Terms. All teachers have a degree in Math Education.

Conclusions

In this report, we categorized teachers' responses on a task designed to reveal meanings for the substitution principle in the context of structuring an expression. We take teachers' responses as samples of their in-the-moment meanings in the given context with the hope of revealing ways of thinking that might be addressed profitably in professional development. In particular, by identifying teachers' ways of thinking and possible areas of difficulty, we are able to identify how teachers are positioned to support students develop meanings for structure sense. We warn the reader, however, that even if a teacher provided a high-level response, this does not mean the teacher will consciously support the development of structure sense in his or her students. As Novotná and Hoch (2008) and Mason et al. (2009) stated, it is not enough for teachers to just possess well-developed structure sense. They must also be *reflectively aware* of their structure sense, and make it a goal to foster its development in their students. Only then can teachers begin to make decisions about classroom activities and conversation that could support the development of structure sense in students.

The data shared here, along with data on five additional structure tasks from the MMTsm, suggest that low-level structure sense among teachers is commonplace. Such a statement is relatively unsurprising and extremely unhelpful, however. What is useful is that our data suggests that most teachers' responses appear to be driven by context rather than structural reasoning. Compartmentalized meanings could be part of the triggers that Hoch (2003) described as playing a role in how an individual classifies objects and properties into structures. Likewise, our findings support Novotná and Hoch's (2008) warning that "this lack of awareness [of structure sense among students] may return with [them as] teachers back to schools" (p. 102). Our research suggests that teachers operate on tasks based predominantly on contextual cues rather than structural awareness. We suggest future research explore *how* teachers rely on context instead of structure sense to approach problems. With such information, researchers and professional development professionals could begin creating tasks that would support teachers in developing structural awareness. In particular, future research needs to investigate how to generate attention to structure as something that is important and useful within the teaching community. Teachers with this belief will be better poised to support students in using the various facets of structure sense to guide decision-making processes in the act of solving problems.

Admittedly, our study is limited by the fact that our assessment was designed to explore teachers' meanings for a variety of content areas, so we only have six structure items to draw upon for analysis—and only one of which we shared here. Future research should extend the exploration of teachers' structure sense in a more focused fashion. For instance, one might use tasks designed specifically to distinguish between context specific reasoning and structural awareness, and conduct follow-up interviews of teachers aimed at eliciting their thinking. With insight into ways of thinking from such studies, researchers could then develop pedagogical items to be used in professional development and explore the effect of drawing teachers' attention to structure on those teachers' activity in their classrooms.

Our assessment and the corresponding analyses of responses aim to support professional developers in gauging the mathematical meanings by which teachers operate. We propose that classifying common ways of thinking, both productive and less productive in the normative sense, gives necessary information that professional developers need to support teachers in developing richer meanings and ways of thinking. We hope this approach of working with teachers' meanings for the substitution principle will increase the field's awareness of structure sense, thereby positioning mathematics educators to help teachers to better support students' development of structure sense.

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