CHAPTER 18

RESEARCHING MATHEMATICAL MEANINGS FOR TEACHING

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Teachers' mathematical knowledge for teaching is a central focus of recent research on teaching mathematics. At the same time, research on mathematical knowledge for teaching has focused largely on teachers' declarative knowledge, which is difficult to link to teachers' decisions and actions in the context of teaching or planning to teach. Much of current research on assessing mathematical knowledge for teaching seeks correlates between declarative knowledge and successful teaching instead of looking for reasons for teachers' decisions and actions. This distinction is important when the goal is to use assessments diagnostically, to design interventions meant to affect teachers' instructional decisions and actions. In this chapter I argue in principle and by example that a focus on teachers' mathematical meanings for teaching mathematics is both important and potentially productive regarding the improvement of teachers' teaching, which I take as necessary for the improvement of students' mathematical learning. I also outline a method for developing items and instruments that focus on teachers' mathematical meanings for teaching mathematics. I then examine how we can link research and assessment more intimately than simply using the results of each in the activities of the other.

Research on mathematical knowledge for teaching (MKT) has the goal of trying to find relationships among the mathematics that teachers know, their instruction, and students' learning. The underlying assumption seems to be that if we find these relationships, and if we then help teachers obtain the appropriate knowledge, they will be positioned to teach better and support better student learning. This conceptualization of research on MKT seems quite plausible—until we ask, "What do we mean that a teacher *knows* something? How can a teacher knowing something help a student know it, too?" Answers to these questions will reveal the importance of being clear about what we presume we are assessing when we assess teachers' MKT and about why we even care to assess it.

What is *knowledge* and why do we care?

Mason and Spence (1999) analyzed historical uses of "knowledge" and quickly determined that *knowing* is much more useful for thinking about teaching and learning than is *knowledge*. *Knowing* connotes activities of a knower, while *knowledge* connotes facts—justified true beliefs. They argued that thinking about teachers' *knowledge* leads us to separate knowers from what they know, which has the consequence of separating, in our thinking, what teachers know from their thinking, deciding, and acting. Mason and Spence urged us to think instead about teachers' *acts of knowing*, which brings us closer to describing sources of teachers' actions and decisions. Within their acts of knowing

Mason and Spence distinguished two broad kinds: *knowing about* (knowing-that, knowing-how, knowing-why) and *knowing-to*—"active knowledge which is present in the moment when it is required" (Mason & Spence, 1999, p. 135).

Mason's and Spence's stance on knowledge versus knowing is in line with Glasersfeld's explanations that to interpret Piaget's concept of knowledge we need to think quite differently about knowledge than as justified true belief. He explained that we need to understand Piaget's use of knowledge as connoting the dynamic, adaptive, and organized functioning of an organism's neural system—and as having reference only within the organism's experience (Glasersfeld, 1978, 1981, 1985). Put another way, "knowledge" and "knowing" are the same concept in Piaget's genetic epistemology. As I will explain in later sections, *knowing-to*, as described by Mason and Spence, can be characterized more expansively by appealing to Piaget's notion that a scheme is a meaning—an organization of actions, images, and other meanings. Thus, one *knows-to* act in a particular way in a particular context because the actions implied by one's understanding of a context are in the scheme to which you assimilated the context. In this regard, I hasten to add that Piaget had an expansive meaning of action, as "all movement, all thought, all emotion that responds to a need" (Piaget, 1968, p. 6). Thus, when Piaget spoke of schemes, he had in mind organizations of mental and affective activity whose contents could be highly nuanced and could contain several layers of structure.

Hill and Ball's *Learning Mathematics for Teaching* (LMT) project demonstrated that there is a correlational link between K-8 mathematics teachers' mathematical knowledge for teaching as measured by their LMT instrument and the mathematical quality of teachers' instruction (MQI) as measured by their MQI instrument (Charalambous & Hill, 2012; Hill, 2011; Hill, Blunk et al., 2008; Hill & Charalambous, 2012). At the same time, Schilling, Blunk and Hill (2007) point out that they tacitly assumed in developing the LMT assessment that knowledge held by teachers, specifically their knowledge of content and students (KCS), was declarative.

> When we began developing items in this domain [KCS], we hypothesized that teachers' knowledge of students existed separately from their mathematical knowledge and reasoning ability. We thought of such knowledge as "declarative," or factual knowledge teachers have of student learning. Results from these validation studies, however, suggest that this "knowledge" may actually contain both elements of mathematical reasoning and knowledge of students and their mathematical trajectories. (Schilling, et al., 2007, p. 121)

The approach to investigating teachers' MKT that I describe in this chapter builds from Schilling et al.'s observation that our understanding of MKT can be broadened profitably by shifting our focus from teachers' (declarative) mathematical knowledge, to focus instead on teachers' mathematical meanings. This shift is essentially from a philosophically mainstream view of knowledge as justified, true belief and about things external to the knower to a Piagetian perspective in which meaning and knowledge are largely synonymous, and both are grounded in the knower's schemes.

This shift allows us to move, for example, from asking what teachers *know* about equations to what teachers *mean* by an equation. Compare Teacher 1's and Teacher 2's meanings of algebraic equations.

Teacher 1 thinks that any mathematical statement that contains an equal sign is an equation. Upon seeing an algebraic equation Teacher 1 thinks "do the same thing to both sides to keep the equation balanced", thinks that a solution is the number in the final step that produces a statement like "x = (number)", and has the goal of reaching the final step. Teacher 1 also feels confused about differences among equations and identities, and between equations and functions. Teacher 1 thinks they should all be called equations because they all contain an equal sign.

Teacher 2 thinks that an algebraic equation is a statement of equality together with the question, "For what values of the variable(s) will this statement be true?" Teacher 2 has the goal of answering that question, and thinks to put the equation into an equivalent form that she can solve by inspection. Teacher (2) thinks of a "step" as applying an equivalence-preserving transformation to one or both sides of the equation so that it is closer to being solvable by inspection.[3] Teacher 2 has no difficulty distinguishing between equations and functions and between equations and identities. To Teacher 2, a function is a statement about a relationship between two quantities' values. An identity is an equation that is true for all values in the equation's domain. Teacher 2 realizes that all statements with an equal sign *could* be called "equations". However, she realizes also that to do so, the general meaning of an equation would have to be that an equation represents its solution set and that she would therefore need to define functions as sets of ordered pairs—ideas that she feels are too general for her students.

Both teachers could exhibit similar performances in answering questions about equations and procedures for solving them. Their different meanings, however, would provide different potentialities regarding what they say to students *about* equations and equation solving.

The mathematical knowledge that matters most for teachers resides in the mathematical meanings they hold. Teachers' mathematical meanings constitute their images of the mathematics they teach and intend that students have. Teachers' mathematical meanings guide their instructional decisions and actions (Thompson, 2013). Dewey (1910) said as much when he elaborated the connection between thinking and meaning: "That thinking both employs and expands notions, conceptions, is then simply saying that in inference and judgment we use meanings, and that this use also corrects and widens them" (Dewey, 1910, p. 125). Dewey also alerted us to the dangers of being vague about our central constructs.

> Vagueness disguises the unconscious mixing together of different meanings, and facilitates the substitution of one meaning for another, and covers up the failure to have any precise meaning at all. (Dewey, 1910, p. 130)

Vagueness in our meaning of knowledge becomes especially problematic when we set out to assess it. We place ourselves in the uncomfortable position of defining knowledge, or types of it, in the same way that many psychologists use the idea of operational definition to define intelligence—intelligence is defined to be what intelligence tests assess. Bridgman, who originated the method of operational definition, roundly criticized using a measure of a construct to define that construct: "Without doubt it is possible to apply the procedure suggested here, but I believe that the situation seldom arises which one would be content to treat finally by any such method as this" (Bridgman, 1955, Chapter 1, Kindle Locations 507-508). Without explication, the word "knowledge" becomes a primitive term in research on mathematical knowledge for teaching, open to any interpretation that a person can pack into it.

I will not try to explicate what others might mean by knowledge in discussing what teachers know and how what they know is related to what they do. To do so would take us into a morass of philosophical disputes, such as knowledge versus belief (Thompson, 1992) and constructivism versus realism (Glasersfeld, 1992; Howe & Berv, 2000; Phillips, 2000; Suchting, 1992)—disputes that turn out to be immaterial for the purpose of improving mathematics teaching. Instead, I will argue here, as I have argued elsewhere (Thompson, 2013), that a focus on teachers' mathematical meanings, as opposed to their mathematical knowledge, offers a fruitful approach to uncovering important sources of teachers' instructional decisions and actions and provides useful guidance for designing teachers' professional development.

In this chapter I make two proposals. The first is that a focus on teachers' mathematical meanings, in line with Mason and Spence's focus on *knowing-to*, is more productive for understanding and improving teachers' instruction than is a focus on mathematical knowledge. With meaning defined appropriately, a focus on meanings positions us to help teachers focus on creating instruction that helps students develop productive meanings. The second proposal is a means to gain insight into mathematical meanings teachers have. To do this requires a theory of meaning as well as a set of techniques that can be used at scale for creating useful models of teachers' mathematical meanings—models that provide guidance in designing mathematics teachers' professional development that helps them to help their students create coherent mathematical meanings.

Finally, an example might give further clarity to the distinction between knowledge and meaning. Suppose a child lays three meter sticks end to end, and then is given a fourth meter stick to lay down. Upon putting it down, we ask, "How much did you add to the total length?" ("A meter.") The child *knows* that he added a meter. But what did it *mean* to him? Did he mean that he added one more stick called "a meter"? Or did he mean that he added a meter in length that is constituted by centimeters, which in turn are constituted by millimeters, which in turn are constituted by (and so on). Understanding what people mean gives more insight into their thinking than does understanding what they believe to be true.

EXAMPLES OF INVESTIGATING TEACHERS' MEANINGS

Two examples will set the stage for discussing the idea of meaning, mathematical meanings for teaching, and how one might assess them. They are drawn from *Mathematical Meanings for Teaching secondary mathematics* (MMTsm), a 43-item diagnostic instrument designed for use in mathematics professional development.[4,5]

Example 1: Meanings of "over"

We noticed that English speakers often speak of average speed as "Distance over time" and represent it with a symbolic expression like *d*/*t*. However, what "over" means to persons saying this was unclear to us. Are they speaking of a spatial arrangement of symbols? Or are they thinking that two things happened concurrently—something moved and some amount of time elapsed.

We designed an item (Figure 1) to see the extent to which teachers distinguish or confound "over" as meaning spatial arrangement of symbols versus "over" as meaning that two events happened or are happening concurrently. The difference in these meanings can reflect an importance difference regarding what teachers intend that students understand from their instruction of average rate of change. If teachers intend to convey only a spatial arrangement of symbols, then their utterances are about marks as written and are not about what students should understand about variables varying or about the meaning of quotient within a context.

A college science textbook contains this statement about a function *f* that gives a bacterial culture's mass at moments in time.

The change in the culture's mass over the time period Δx *is 4 grams.*

Part A. What does the word "over" mean in this statement?

Part B. Express the textbook's statement symbolically.

Figure 1. An item to investigate teachers' meanings of "over". © 2014 Arizona Board of Regents. Used with permission.

The purpose of Part A in Figure 1 was to have teachers commit to a meaning of "over" in a context where, when interpreted normatively, it means "during". The purpose of Part B was to give teachers an opportunity to show how they interpreted the context in which "over" occurred while expressing it symbolically. Since the statement is about a change in mass, the symbolic representation of it should reflect a change in mass that happened as time passed from one moment in time to another. Since the function *f* gives the culture's mass at moments in time, and since the change in time is represented by "Δx", one representation of the change in mass would be $f(x+\Delta x)-f(x)=4$ or $f(x_0+\Delta x)-f(x_0)=4$, where x_0 refers to a specific moment in time.

Figure 2 contains one teacher's response to Parts A and B. In Part A, the teacher said that "over" means "during", but went on to say that you also can think of "over" as meaning a ratio. This teacher's Part B response shows more than that "over" brings to mind a spatial arrangement of symbols. It reveals two additional issues: (1) the teacher defined "$f(x)$" in terms of an expression in which "x" does not appear, and (2) used "$f(x)$" to represent a rate of change even though the text stated that "the function *f* gives the culture's mass at moments in time".[6]

During or duration. You could also think of it as a ratio so $\frac{\Delta mass}{\Delta X} \leftarrow$ Δmass is over ΔX	$f(x) = \frac{\Delta mass}{\Delta time}$
One Teacher's Part A response	This teacher's Part B response

Figure 2. A teacher's responses to questions in Figure 1.

We designed a scoring rubric to capture the range of meanings we discerned from responses given in summer 2013 by 96 high school mathematics teachers in the midwest and southwest United States. Table 1 and Table 2 show gradations among levels of responses to Parts A and B, respectively. We put any response equivalent to "during" at the highest level for Part A of Figure 1, and any response like "$f(x+\Delta x)-f(x)=4$" or "$m_{t_2}-m_{t_1}=4$ where $t_2-t_1=\Delta x$" at the highest level for Part B. While a method for creating rubrics that focus on meanings is discussed in a later section, for now it is worth pointing out that though this item's design emerged from being attentive to teachers' and students' meanings in prior research and in our daily work with teachers, the rubrics for scoring responses to Figure 1 emerged from analyzing teachers' responses to the item itself. When distinguishing between levels of responses, we continually asked, "How might a student interpret what the teacher produced? How productive would it be for students' development of coherent meanings were a teacher to express what he or she did?"

Table 1. Rubric for scoring Figure 1, Part A. © 2014 Arizona Board of Regents. Used with permission.

A3:	Teacher said "during", or otherwise referred to the culture's mass in relation to a passage of time.
A2:	Any of the following: • The teacher conveyed "during" but represented the time interval using a symbol other than Δ*x*. • The teacher described "over" as meaning an amount of time.
A1:	The teacher conveyed that "over" means division, regardless of saying anything else.
A0:	Any of the following: • The teacher wrote "I don't know." • The scorer cannot interpret what the teacher meant by "over". • The teacher's response is not described by any of levels A1 to A3.

Table 2. Rubric for scoring Figure 1, Part B. © 2014 Arizona Board of Regents. Used with permission.

B4:	The teacher represented a difference in the mass of a culture at different moments in time, with the resulting difference being 4. If the teacher used a variable other than "*m*" or "*y*" to stand for mass instead of using *f*(*x*), or a variable other than *x* to represent elapsed time, the letter must be defined.
B3:	Any of the following: • The teacher wrote a statement like Δ*m*=4 or Δ*y*=4. • The teacher represented a difference of masses that equals 4 using a letter other than *m* or *y* to represent mass, ***and*** said that their variable represents mass. In addition, the teacher did not represent a time interval.
B2:	Teacher's response contains a combination of Δ*m*=4 and Δ*m*/Δ*x* = 4.
B1:	The teacher wrote a quotient showing the *change* in mass divided by the *change* in time is equal to 4 (with or without a unit), or some algebraically equivalent statement.
B0b:	The teacher's response conveys division but the response is not described by level B1.

B0a	Any of the following: • The teacher wrote "I don't know" • The scorer cannot interpret the teacher's response. • The teacher's response is not described by any of Levels B0b to B3.

In subsequent tables I share results of our scoring for Figure 1 to illustrate how a focus on teachers' meanings can provide useful information about their thinking. Table 3 summarizes teachers' responses to Part A. Sixty-one percent (61%) of responses were assigned Levels A3 or A2—teachers spoke of "over" in a way that suggested something happening during a passage of time. Thirty-two percent (32%) were assigned Level A1—they specifically mentioned that "over" meant division or ratio in the statement about a change of mass over a time interval. Table 3 also breaks down responses by teachers' undergraduate major. "Math" points to teachers who received a B.Sc. in mathematics; "Math Ed" points to teachers who received a B.Sc. or B.A. in secondary mathematics education; "Other" is any other undergraduate degree. Fifty-nine percent (59%) of Math majors, 53% of Math Ed majors, and 73% of Other majors answered at Levels A3 or A2.

Table 3. High school mathematics teachers' responses to Figure 1, Part A (n = 96).

A-Level	*Math*	*MathEd*	*Other*	*total*
A3	11	15	16	42
A2	5	4	8	17
A1	9	14	8	31
A0	1	3	1	5
No Ans	1	0	0	1
total	27	36	33	96

Table 4 shows the classification of teachers' symbolic representations of the entire statement. Three percent (3%) of responses were at Level B4; 5% of responses were at Level B3, 60% percent of responses were at Level 1 or Level B0b, and 26% were placed at Level B0a. Responses at levels B1 and B0b contained a quotient or the equivalent of a quotient (e.g., $\Delta m = 4\Delta x$). Figure 3 shows four examples of Level B0a responses. The first two responses in Figure 3 are by teachers holding a degree in mathematics; the second two are by teachers holding a degree in mathematics education.

Table 4. High school mathematics teachers' responses to Figure 1, Part B (n = 96).

B-Level	*Math*	*MathEd*	*Other*	*total*
B4	0	2	1	3
B3	1	2	2	5
B2	2	0	0	2
B1	10	13	11	34
B0b	7	11	6	24
B0a	5	8	12	25
No Ans	2	0	1	3
total	27	36	33	96

$f(t)=m$ $f(m)=4$	$M(\Delta x)=M+4$	$f(x)=4$	$f(x)=\Delta x\cdot 4$
B.Sc. Math	*B.Sc. Math*	*B.Sc. Math Ed*	*B.A. Math Ed*

Figure 3. Examples of Level B0a responses.

Table 3 shows the majority of teachers responding at a high level regarding a meaning of "over" while Table 4 shows the majority of teachers responding at a low level regarding a representation of the entire statement—representing it with a quotient or with a wholly inappropriate response. Table 5 examines the relationship between teachers' responses to Parts A and B. One-hundred percent (100%) of teachers who responded at a low level (A0 or A1) for Part A scored at a low level (B0a, B0b, B1, or B2) for Part B. Moreover, 51 of 59 teachers (86%) who scored at a high level for Part A (A2 or A3) scored at a low level for Part B (B0a, B0b, B1, or B2). This suggests strongly that the phrase "change in mass over change in time" triggered a variety of meanings among teachers, most of which were unlike the meaning that the culture's mass changed by some amount while elapsed time changed by some amount.

Table 5. Relationship between teachers' responses to Part A and to Part B of Figure 1 (n = 96).

	B4	*B3*	*B2*	*B1*	*B0b*	*B0a*	*No Ans*	*total*
A3	2	4	2	12	4	17	1	42
A2	1	1	0	4	3	7	1	17
A1	0	0	0	17	14	0	0	31
A0	0	0	0	1	3	1	0	5
No Ans	0	0	0	0	0	0	1	1
total	3	5	2	34	24	25	3	96

It is worthwhile to unpack two implications of what we learned from this item. The first regards mathematical modeling—teachers understanding situations described verbally and describing their understandings symbolically. The second regards what teachers might convey to students unthinkingly about fractions.[7]

Modeling. In this item, the textbook's statement is read normatively as being about a change in mass that happened during a change in time. That such a large percentage of teachers associated the statement fundamentally with a ratio suggests that they did not interpret the statement in terms of the quantities involved (a difference of masses and a difference of times) and a relationship between them (concurrence). This raises the possibility that teachers' meanings for the quantities and relationships that any rate of change entails are muddled when they teach the idea of rate of change or model situations that involve a rate of change. Put another way, "over" meaning a spatial arrangement of symbols is in line with "*more* means addition, *less* means subtraction*, of* means multiplication"—meanings that muddle young children's thinking when solving arithmetic word problems.

What teachers convey to students about fractions. Teachers should be alert to how students interpret "*a*/*b*". We would hope they attend to whether their students assimilate “*a*/*b*” just as two numbers separated by a slash mark or that they also assimilate it to a scheme that relates the value of *a*, the value of *b*, and their relative size. This second meaning can be expressed as “*a/b* is a number *m* which tells you that *a* is *m* times as large as *b*”.[8] The second meaning of “*a*/*b*” is the meaning of quotient, and is related tightly to past research on ratio-as-measure (Lobato & Thanheiser, 2000; Simon & Blume, 1994). Teachers who read “*a*/*b*” as “*a* over *b*” and for whom “over” implies a spatial arrangement seem unlikely to worry about whether students are developing the quotient meaning of “*a*/*b*”. Cameron Byerley is investigating the viability of this claim.

Example 2: An item to investigate teachers’ meanings for slope

We often see teachers teach the idea of slope simply as a computation, expressed as “rise over run” or “the change in *y* divided by the change in *x*.” We also often see it taught as a property of a triangle drawn against a graph. These are unproductive meanings for students to have. They only work to answer the question, “What is the slope?” It is important for students to understand that the idea of slope is tightly bound to the idea of relative size of changes in two quantities, which then ties the idea of slope to the idea of constant rate of change. It is also important for students to understand that a graph’s slope is independent of axes’ scales and independent of the coordinate system in which the graph is made.[9]

With the above in mind, we designed the item in Figure 4 to probe teachers’ meanings of slope in regard to relative size of changes and to issues of axes’ scales. Part A requests a *numerical* value for *m* even though the coordinate system’s axes are sans numbers. If you see the line segments as representing amounts of change in *x* and *y*, and if you understand the quotient $\Delta y/\Delta x$ as the measure of Δy in units of Δx, then you can decide to estimate the numerical value of *m* simply by physically measuring Δy using Δx as a unit. The change in *y* is 2.5 times as large as the change in *x*, so $m = 2.5$.[10]

Part A. There are two quantities *P* and *Q* whose values vary. The measure of *P* is *y* and the measure of *Q* is *x*. *y* and *x* are related so that $y = mx + b$. The graph of their relationship is given below, with *x* and *y* in the same scale. What is the numerical value of *m*?

Part B. What would be the numerical value of *m* if the *y*-axis were changed so that the distance between 0 and 1 is 2 times as large as the original?

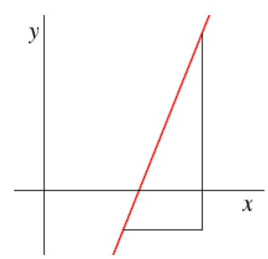


Figure 4. An item to investigate teachers’ meanings of slope. © 2014 Arizona Board of Regents. Used with permission.

We included Part B to see whether teachers thought of the line’s slope as being a property of the triangle *itself* instead of as a relative size of changes that the horizontal and vertical segments *represent*. When the *y*-axis is enlarged by a factor of 2, and remembering that the graph represents the relationship between values of *x* and values of *y*, the value of *m* remains the same because the change in *y* that the vertical segment represents does not change. Interviews with students and teachers convinced us that if someone’s meaning for slope is a property of a triangle, then he or she will say either that the slope will be twice as large (if imagining that the triangle stretches) or half as large (if imagining that the y-axis is rescaled but the triangle remains the same).

Table 6 shows responses from the group of 96 high school mathematics teachers mentioned in the prior example. Rows describe responses to Part A. *Close Num* means a numerical estimate from 2 to 3. *Far Num* means a numerical estimate less than 2 or greater than 3. *Diff Quot* means a response like "$\Delta y/\Delta x$" or "$(y_2 - y_1)/(x_2 - x_1)$". *Other* responses included "y/x", "P/Q", and "$m = (y-b)/x$". Columns describe responses to Part B.

Table 6. Teachers' responses to Part A and Part B of Figure 4. (n = 96)

Part A Response		***Part B Response***					
		Same	*Double*	*Half*	*Other*	*No Ans*	*total*
	Close Num	2	30	15	3	0	50
	Far Num	1	4	2	1	0	8
	Diff Quot	4	10	6	1	1	22
	Other	6	3	0	1	3	13
	No Ans	0	0	0	0	3	3
	total	13	47	23	6	7	96

Table 6 shows that 52% of the teachers gave a numerical approximation that was close to 2.5. Many of them showed work that suggested measuring Δy in units of Δx. However, 90% (45/50) of these teachers who gave a close approximation for Part A answered either "double" or "half" to Part B. This suggests that though they understood slope to be about relative size, they compared side-lengths of a triangle and not what those lengths represented. Teachers answering Part A with a number outside the "close" range seemed to give numerical values as an index of the line's perceptual "slantiness". These responses did not contain work to suggest that the teacher measured the length of one segment in terms of the length of the other. Teachers who gave a symbolic response (*Difference Quotient* or *Other*) to Part A had the highest percentage of "same" on Part B. We suspect that this was because they focused on symbolic formulas, which they could think of as remaining the same regardless of how, or whether, they interpreted the graph. The results in Table 6 suggest that a large percentage of these 96 teachers thought that "slope" meant a property of a triangle drawn against a graph that either reflects the relative size of its legs or that is associated with a computational formula.

TEACHERS' MATHEMATICAL MEANINGS

The items in the previous section share a trait: they were designed to elicit teachers' interpretations of a statement or setting about an idea that recurs in mathematics teaching, and then to elicit implications in teachers' thinking that their interpretations held. This design strategy is rooted in a theory of meaning that is based on Piaget's notion of *assimilation to a scheme*. In this section I will unpack the idea of assimilation and explain its connection to assessing teachers' mathematical meanings for teaching mathematics.

A sense of absorption is commonly associated with *assimilate*. Object A is assimilated to Object B when A is transformed to become part of B. As Piaget famously stated in elaborating his meaning of assimilation, “A rabbit that eats a cabbage doesn't become cabbage; it's the cabbage that becomes rabbit—that's assimilation. It's the same thing at the psychological level. Whatever a stimulus is, it is integrated with internal structures” (Bringuier, 1980, p. 42). By this Piaget meant that a person experiences the structures that are activated through assimilation, not the stimulus that an observer views as separate from the experiencer. To illustrate this distinction, suppose that a person, deep in thought, rounds a corner on the streets of Chicago and looks up at what we call the Willis Tower and sees the Empire State Building, recalling the dinner he had in it. This person *assimilated the Empire State Building*—despite it being 1286 kilometers away. His experience was that he saw the Empire State Building, even if he eventually corrected himself by realizing that he was in Chicago, not New York.[11]

Another way to understand assimilation in Piaget’s theory is to think of meanings that come to a person’s mind upon encountering a situation. What looks like absorption (taking in the situation) actually is the person’s imbuement of meaning to the situation.

> Assimilating an object to a scheme involves giving one or several meanings to this object, and it is this conferring of meanings that implies a more or less complete system of inferences, even when it is simply a question of verifying a fact. In short, we could say that an assimilation is an association accompanied by inference. (Johnckheere, Mandelbrot, & Piaget, 1958, p. 59) as quoted in (Montangero & Maurice-Naville, 1997, p. 72).

Johnckheere *et al.*’s reference to “a more or less compete system of inferences” was their way to talk about the implicative nature of meanings. A person’s meaning in a situation is what comes to mind immediately together with what is ready to come to mind *next*. The implicative nature of meanings is at the heart of Piaget’s notion of scheme (Piaget & Garcia, 1991). Thompson and Harel captured this in their system for differentiating among various forms of understanding (Table 7).

Table 7. Thompson and Harel’s definitions of understanding, meaning, and way of thinking. (Thompson, Carlson et al., 2014)

Construct	*Definition*
Understanding (in the moment)	Cognitive state resulting from an assimilation
Meaning (in the moment)	The space of implications existing at the moment of understanding
Understanding (stable)	Cognitive state resulting from an assimilation to a scheme
Meaning (stable)	The space of implications that results from having assimilated to a scheme. The scheme is the meaning. What Harel previously called *Way of Understanding*
Way of Thinking	Habitual anticipation of specific meanings or ways of thinking in reasoning

Understanding in the moment addresses what a person understands of something said, written, or done in the moment of understanding it. All understandings are understandings-in-the-moment. Some understandings might be a state that the person has struggled to attain at that moment through functional accommodations of existing schemes (Steffe, 1991). This is an understanding that can be easily lost once the person's attention moves on and is typical of when a person makes sense of an idea for the first time. The *meaning of an understanding* is the space of implications that the current understanding mobilizes—actions, operations, or schemes that the person's current understanding suggests.[12] An understanding is *stable* if it is the result of an assimilation to a scheme. A scheme, being stable, then constitutes the space of implications resulting from the person's assimilation of anything to it.[13] The scheme is the meaning of the understanding that the person constructs in the moment. As an aside, schemes provide the "way" in Harel's "way of understanding" (Harel, 2013). Finally, Harel and Thompson characterize "way of thinking" as a person having developed a pattern for utilizing specific meanings or ways of thinking in reasoning about particular ideas.

The previous section's examples were designed to gain insight into aspects of teachers' meanings. The first example examined their meanings of "over" as revealed in their linguistic and mathematical descriptions of two events' concurrence. The second example examined their meanings of slope. In neither example can we say that we determined teachers' meanings. The best we can say is that we gained insight into their meanings. Since meanings are schemes ("implications of an understanding"), their boundaries and connections with other meanings are often subtle and sensitive to context. Thus, diagnoses of teachers' meanings to support the design of professional development require a battery of items that reveal broader schemes, or clear indicators of limited meanings, among a body of related mathematical ideas. I address this issue in the next section.

ASSESSING TEACHERS' MATHEMATICAL MEANINGS FOR TEACHING

To assess teachers' mathematical meanings for teaching requires that the assessment designers have a theory of the meanings they intend to assess. Assessment designers must say what they will take to constitute productive and less productive meanings for students' learning regarding a mathematical idea—and an explanation of why one is more productive than another. Productive meanings are propaedeutic (preparing the student for future learning) and they lend coherence to the meanings students already have.

A theory of meanings-to-be-assessed should also draw from research on meanings that prove to be problematic when students have them. For example, research on students' understandings of fractions shows that "*a* out of *b*" as a meaning for *a*/*b* is highly detrimental for students' later mathematical learning (Carpenter, Coburn et al., 1976; Norton & Wilkins, 2009; Thompson & Saldanha, 2003; Torbeyns, Schneider et al., 2014; Vinner, Hershkowitz, & Bruckheimer, 1981). Teachers who have unproductive meanings can easily convey them to students unthinkingly (Izsák, 2012; Thompson, 2013), so it is essential to create assessment items that give teachers the opportunity to display unproductive meanings as well as productive ones. The next section illustrates the process for designing such items in the context of the concepts of variation and covariation.

Continuous Variation and Covariation

It is well established that students profit by thinking that values of variables vary continuously on a connected subset of the real numbers (Castillo-Garsow, 2010; Confrey, 1994; Confrey & Smith, 1995; Thompson, 1994a). The idea that variables' values vary continuously is the basis for thinking covariationally, which is an essential component of understanding functions, graphs, and relationships (Carlson, Jacobs et al., 2002; Thompson, 1994c).

To assess teachers' meanings for continuous variation and covariation, we must first say precisely what we mean by "understanding continuous variation" and "understanding continuous covariation" and how teachers' thinking might be at different levels regarding them. Table 8 and Table 9 are a culmination of prior research on students' understandings of variables, functions, and rate of change (Carlson, et al., 2002; Castillo-Garsow, 2010; Confrey, 1992; Confrey & Smith, 1995; Saldanha & Thompson, 1998; Thompson & Thompson, 1996; Thompson, 1994a, 2011; Thompson & Thompson, 1994).

Table 8 describes different levels at which someone could think of a variable's value varying. The two highest levels of thinking are about a meaning of variation that creates continuous intervals. The distinction between the two levels is that a person thinking at the highest level has a recursive anticipation that any variation can be refined (Thompson, 2011, p. 47). A person at the second level ("chunky") envisions variation over an interval in fixed increments without the accompanying image that variation happens within each increment, as if laying rulers end-to-end. The lower three levels capture thinking about variation as an act of replacement—the individual thinks of a variable's value as something that is substituted for the letter.

Table 8. Meanings of continuous variation behind the MMTsm.

Meanings of Continuous Variation	
Level	*Description*
Smooth Continuous Variation	The individual thinks of variation of a variable's value as the variable's magnitude increasing in bits while anticipating simultaneously that within each bit the variable's value varies smoothly.
Chunky Continuous Variation	The individual thinks of variation of a variable's magnitude as increasing by intervals of a fixed size. The individual imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 to (and so on). Values between 0 and 1, between 1 and 2, between 2 and 3 "come along" by virtue of each being part of a chunk, but the quantity does not have them as a value in the same way it has 0, 1, 2, etc. as values. Chunky continuous variation is *not* just thinking that increases happen in whole number amounts. Thinking of a variable's value going from 0 to 0.25, 0.25 to 0.5, 0.5 to 0.75, and so on (while thinking that entailed intervals "come along") is just as much thinking with chunky continuous variation as is thinking of increases from 0 to 1, 1 to 2, and so on.
Discrete Variation	The individual thinks of a variable as taking specific values. The individual sees the variable's value changing from a to b by taking values $a_1, a_2, \ldots, a_n$, but does not envision the variable taking any value between a_i and a_{i+1}.
No Variation (NV)	The individual envisions a variable as having a fixed value. It could have a different fixed value, but that would be simply to envision another scenario.
Variable as Letter (VL)	A variable is a letter. It has nothing to do with variation.

Table 9 imports the meanings of continous variation (Table 8) into meanings of covariation. Table 9 could be expanded to account for the possibility that an individual conceives of two variables having different kinds of variation, but in practice this has not been workable.

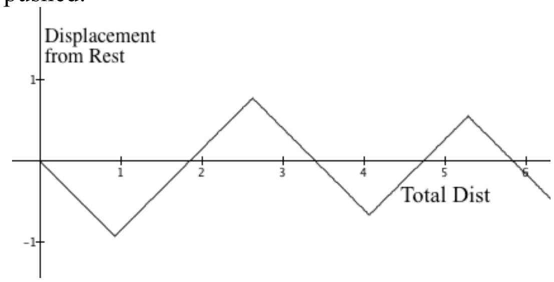
Table 9. Meanings of continuous covariation behind the MMTsm.

Meanings of Continuous Covariation	
Level	*Description*
Smooth Continuous Covariation (*SCC*)	The individual envisions changes in one variable's value in relation to changes in another variable's value where variables vary smoothly and continuously.
Chunky Continuous Covariation (*CCC*)	The individual envisions chunky continuous variation in one variable's value in relation to chunky continuous variation in another variable's value.
Coordination of Values (*CV*)	The individual coordinates the values of one variable with values of another variable with the anticipation of creating a discrete collection of pairs $(x, f(x))$.
Pre-coordination of Values (*PCV*)	The individual envisions two variables' values varying together but asynchronously (one variable changes, then the second variable changes, then the first, etc.). The individual does not anticipate creating pairs of values.
Variation but No Coordination (*VNC*)	The individual has no image of variables varying together. The individual focuses on one or another variable's variation with no coordination of values.

Figure 5 shows the fourth version of one of several items that we designed to assess teachers' meanings of covariation. The highest level of reasoning we anticipated is this: If you imagine the ball bobbing, and if you coordinate small changes in displacement in a direction with small changes in total distance, you will realize that the two changes are always the same magnitude but possibly in different directions. Thus, the graph's slope will be ±1 over *any* interval in which the ball's displacement changes without changing direction.

We designed the item in Figure 5 purposely to exclude considerations of the ball's speed and its elapsed time. Our intention was to create a situation that forced teachers to conceptualize both quantities in the covariation. Our reason for this was that teacher interviews in the early stages of the item's development convinced us, in line with prior research, that teachers could envision an event happening in time without actually conceptualizing time as a quantity.

A ball is hanging by a 10-foot rubber cord from a board that is 20 feet above the ground. The ball is given a hard push downward and left free to bob up and down. The following graph represents the ball's *displacement from its resting point* in relation to its *total distance traveled* after having been pushed.



Why is the graph of *Displacement from Rest* versus *Total Distance* made of straight segments instead of being a smooth curve?

Figure 5. Version 4 of an item to assess covariational reasoning. © 2014 Arizona Board of Regents. Used with permission.

Table 10 summarizes responses from 111 high school mathematics teachers in the midwest United States who took the Summer 2012 version of the MMTsm. It is split into two groups—responses that involved some description of variation (n = 17) and responses that contained no description of variation (n = 94). Only 10 responses (9%) spoke of displacement and total distance covarying.

Table 10. Responses by high school mathematics teachers in Summer 2012 to the item in Figure 5. (n = 111)

Response	*Freq*
1. Covaried distance and displacement	10
2. Coordinated distance and displacement	3
3. Varied total distance, mentioned displacement, but no covariation	3
4. Varied displacement, mentioned total distance, but no covariation	1
5. Used physical situation wrongly to explain the graph	22
6. Used graph to describe the ball's behavior	29
7. Measuring distance implies a linear graph	16
8. Other	27

The item in Figure 5 was extremely difficult to score, for two reasons. First, teachers often used poor grammar and vague language. Second, surprising to us, most teachers explained the graph in ways having nothing to do with the quantities involved and their variation. Instead, many teachers took the graph as given and tried to use its properties to explain the situation (Figure 6). Other teachers tried to explain the graph in terms of what happens when a ball bobs up and down (Figure 7). Yet other teachers tried to explain the graph in terms of an overarching theme (Figure 8). Finally, many teachers mentioned time in their responses to Figure 5.

the line represents how far the ball went until it rest and the sharp points represent the ~~highest~~ max and min Distance the ball reached.

Figure 6. Example of a response that explains the ball's movement based on properties of the graph.

The ball only pushes down + recoils up then down + up down + up then down

Figure 7. Example of response that uses physical situation to explain the graph in Figure 5.

You are measuring distance (which is a linear measurement) as opposed to motion. This would imply lines rather than curves.

Figure 8. Example of response that explains the graph in Figure 5 in terms of the theme that distance is a linear measurement.

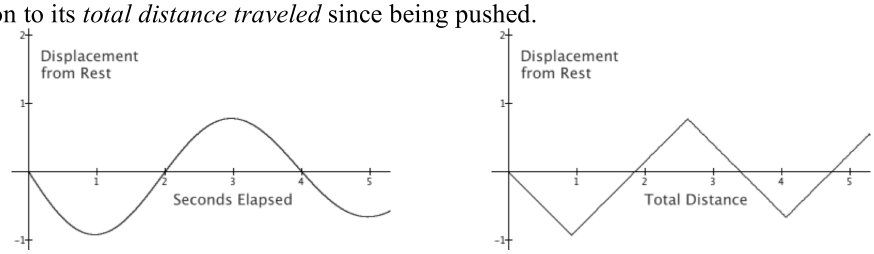
These non-covariational responses led us to add categories of "thematic thinking" (categories 5 and 7 in Table 10), "shape thinking" (category 6 in Table 10), and "non-responsive" to our theoretical framework. We added these categories in order to capture ways of thinking that teachers exhibited in response to an item that is ostensibly about covariation, but which elicits responses having nothing to do with covariation. We also modified the item to alert teachers explicitly that they should not incorporate time into their explanations.

Our difficulty in developing a scoring rubric for version 4 of this item (Figure 5) led us to create a multiple-choice version (Figure 9), where the options were worded to reflect the kinds of thinking we detected in teachers responses to Figure 5. We anticipated that we would see an increase in responses aligned with smooth continuous covariation (option *d*) simply because teachers might see that it is the most reasonable explanation even though it might not occur to them spontaneously. Indeed, we thought it entirely possible that most teachers would recognize option *d* as the response they should select.

Option *a* reflects thematic thinking (TT). Option *b* reflects chunky covariational thinking (CCC). Option *c* reflects coordination of variations (CV). Option *e* reflects shape thinking (ST)—using properties of the graph to describe properties of the situation. We were unable to include an option that would reflect pre-coordination ways of thinking, largely because our descriptions were too long. We included Part B as a second opportunity for teachers to express shape thinking, or to recognize the inappropriateness of the student's shape thinking, but we do not yet have data on Part B.

A ball is hanging by a 10-foot rubber cord from a board that is 20 feet above the ground. The ball is given a sharp push downward and is left free to bob up and down.

The graph on the left represents the ball's *displacement from its resting point* in relation to its *time elapsed* since being pushed. The graph on the right is the ball's *displacement from its resting point* in relation to its *total distance traveled* since being pushed.



Part A. Why is the graph of *Displacement from Rest* versus *Total Distance* made of straight segments?

Select the best answer.

a. Distance, which is a linear measurement, must be represented with a straight segment.
b. A change of one in displacement corresponds to a change of one in total distance.
c. The ball's displacement in either direction is correlated with changes in total distance.
d. Any small change in displacement in a direction is the same magnitude as the change in total distance.
e. The graph represents the motion of the ball. The graph is made of line segments, demonstrating that the ball travels in a linear fashion.

Part B. A student said, "The graph on the right shows that the ball's speed is constant between about 1 and 2.5 seconds." Is the student's statement true? Explain.

Figure 9. Version 10 of item in Figure 5. © 2014 Arizona Board of Regents. Used with permission.

Table 11 shows responses from 96 high school mathematics teachers who took the Summer 2013 version of the MMTsm (which did not include Part B). It shows that we were correct to anticipate a higher rate of responses aligned with smooth and chunky continuous covariation. But, since 67% of responses were other than *d*, our concern was unfounded that making the item multiple-choice would "give away" what we considered the most appropriate response. Also, Table 11 shows that no teacher selected option *e* (shape thinking). This was our motive for including Part B. We are collecting data on Part B with 200 high school mathematics teachers in Summer 2014. Interviews with teachers, and trials with senior mathematics majors, suggest to us that Part B will elicit shape thinking among teachers who are prone to think this way.

Table 11. Responses to Figure 9 (version 10 of the item in Figure 5) by 96 high school mathematics teachers in Summer 2013.

Response	*Math*	*MathEd*	*Other*	*total*
d (SCC)	7	12	11	30
b (CCC)	9	12	8	29
c (CV)	5	7	4	16
a (TT)	4	4	8	16
I Don't Know	0	1	0	1
No Ans	1	0	2	3
No Time	1	0	0	1
total	27	36	33	96

I mentioned that a teacher's response to a single item is insufficient to gain insight into the boundaries and connections within his or her meanings—that to do this requires multiple items that involve aspects of a meaning in different settings. My final example is of a variation item (Figure 10) given in a context where the question is not about variation or covariation directly. Rather, it addresses the seeming contradiction described by Chazan (1993) that many students worldwide experience while studying algebra: Why do we call "x" a variable in equations like $3x + 7 = 12$ when it stands for just one number?

> In a lesson on linear equations, Darren, a student in Mrs. Bryant's class, asked, "Mrs. Bryant, why do they call x a variable in $3x + 7 = 12$ when x can be only one number? Didn't you say that a letter that stands for just one number is a constant?"
>
> How would you respond?

Figure 10. Item to investigate teachers' conceptions of variation in equations.

The conundrum that this item raises is an artifact of the way many teachers and textbooks speak of equations. They convey to students that equations have a particular form (formula on one side, number on the other) and that they are triggers for a special collection of activities that should end when you get a letter on one side of an equal sign and a number on the other. Thus, students learn that the letter in the equation you start with stands for the number(s) your equation-solving activities end with. The meaning that a variable in an equation stands only for a solution is highly debilitating for students' making meaning of related ideas. For example, in $y = mx + b$ students must distinguish between x and y as variables (in the sense that we envision their values varying) and m and b as constants (meaning, they can have different values, but we envision any value of m and b being fixed while values of x and y vary). Also, the meaning that a variable in an equation stands just for a solution is incoherent with solving equations graphically. If a solution is the only possible value of x in $3x + 7 = 12$, then any graph we draw would have only the point $\left(5/3, 12\right)$ on it. Later in their study of mathematics, students must think of both m and x as variables in $y = mx + b$ and that the two together create a surface whose cross-sections are lines. Thus, our intention for the item in Figure 10 was to see the extent to which teachers hold meanings that would support students' thinking "in an equation, x stands for the equation's solutions".

Table 12 displays the rubric we developed for scoring responses to Darren's question. Level 4 responses conveyed the message that you can think of the value of x varying in magnitude just as in non-equation settings. Level 3 responses conveyed the message that x can have different values, but the values are substituted for x.[14] Level 2 responses separated the meaning of variables into two categories – a meaning for variables used in functions and a meaning for variables used in equations. Level 1 responses agreed with Darren's interpretation—that in equations, a letter in an equation stands only for its solutions.

Table 12. Rubric for scoring teachers' responses to Darren's question in Figure 10.

Level 4	The teacher conveyed that the value of x varies in the sense of varying values or of a magnitude growing larger, and conveyed that a solution to the equation is a value of x that makes the statement $3x + 7 = 12$ true.
Level 3	The teacher conveyed a sense that we substitute values for x (including "x can be any number" or "x can be many numbers"), and that we are looking for the number or numbers that make the statement $3x + 7 = 12$ true.
Level 2	The teacher conveyed that x could be used in more than one way. For instance, when x is in equations, it stands just for a solution, but when it is in something like $y = 3x + 7$, x can vary (or, it can stand for any number).
Level 1	A Level 1 response conveys that x represents a single value (possibly not until we solve for it). Responses at this level support the idea that x does not vary, or at most that the value of x changes depending on the value on the right hand side of the equal sign.
Level 0	Any of the following: • Scorer cannot interpret the teacher's response. • The teacher wrote "I don't know" or equivalent. • The teacher addressed the question with incoherent reasoning. • The teacher stated that a variable is just a letter, and makes no further statements that fit in a higher level. • The teacher's response does not fit into Levels 1 to 4.

Table 13 displays our scoring of 96 high school mathematics teachers in Summer 2013 using the rubric in Table 12. I include teachers' initial major to show that responses' levels were largely independent of major. It shows that only 13% of these teachers resolved the conundrum at the highest two levels, that 22% said that the meaning of x depends on whether it occurs in an equation or in a function, while 57% essentially agreed with Darren that x is a constant in the equation $3x + 7 = 12$. Level 1 included responses like, "Yes, x stands for only one number in $3x + 7 = 12$. But it would stand for a different number in $3x + 7 = 14$."

Table 13. High school mathematics teachers' responses to Figure 10 in Summer 2013. ($n = 96$)

	Math	*MathEd*	*Other*	*total*
Level 4	1	1	2	4
Level 3	2	3	3	8
Level 2	4	9	8	21

Level 1	18	20	17	55
Level 0	1	3	3	7
No Ans	1	0	0	1
total	27	36	33	96

Table 14 examines the relationship between teachers' responses to *Bouncing Ball* and to Darren's question. Responses to neither item are predictive of responses to the other. We see this as an artifact of two things: (1) We created an over-estimate of teachers' covariational thinking by making the Figure 9 item multiple-choice, and (2) the Figure 10 item, for many teachers, triggered an "equation" scheme in which the idea of variable is isolated from the idea of variation. In either case, our recommendation to these teachers' professional development leaders was that they work with teachers to build meanings of variation and covariation so that they are coherent across the topics of functions and equations.

Table 14. Relationship between responses to items in Figure 9 and Figure 10. (n = 96)

		Bouncing Ball					
		d (*SCC*)	*b* (*CCV*)	*c* (*CV*)	*a* (*TT*)	*Other*	*total*
	Level 4	1	0	2	0	1	4
	Level 3	5	2	1	0	0	8
x varies	*Level 2*	9	6	3	3	0	21
	Level 1	11	21	9	11	3	55
	Level 0	3	1	0	2	1	7
	No Ans	0	0	1	0	0	1
	total	29	28	16	16	5	96

An Item's Focus

Results from assessing teachers' meanings for "over" (Figure 1), their covariational reasoning (Figure 9), and their meanings for variation (Figure 10) highlight an issue that will be faced in designing any assessment of meaning. It is that a meaning can never be isolated in teachers' responses to an item—teachers often activate many meanings when interpreting an item. For example, in the "over" item (Figure 1), many teachers incorporated their meanings of function and their meanings for ratio into their responses. In the Bouncing Ball item (Figure 9), even though we designed it to focus on covariation, teachers saw two graphs, which certainly activated their schemes for graphs. Also, the item describes a ball bobbing on a rubber cord; teachers certainly envisioned its behavior idiosyncratically. Teachers also thought of a variety of quantities, some of which turned out to be immaterial to the question—such as that the ball bounces in time and with a velocity. In the "*x* varies" item (Figure 10), the question is about variables and constants, but in the context of discussing an equation. Teachers certainly activated their meanings for equations as well as their meanings for "constant" and "variable".[15]

Thus, in devising a scoring rubric one must identify the rubric's focus. In the "over" item, we chose to ignore aspects of responses that revealed problematic meanings for function notation and for rate of change—simply because not all teachers thought to use function notation or to mention rate of change in their responses. For most items, we chose to ignore arithmetic errors because the items' foci were on something that was immaterial to correct calculations. The idea of *focus* in designing and scoring an item is tantamount to deciding to which aspects of teachers responses you will attend, which also entails the willingness to ignore other aspects of teacher's thinking that are revealed in their responses, no matter how interesting those revelations are.

METHODOLOGICAL ISSUES IN ASSESSING TEACHERS' MATHEMATICAL MEANINGS FOR TEACHING MATHEMATICS

Our overall method for developing the MMTsm resembles the methods used by Hestenes and colleagues to create the *Force Concept Inventory* (FCI, Hestenes, Wells, & Swackhamer, 1992; Savinainen & Scott, 2002) and by Carlson and colleagues to create the *Precalculus Concept Assessment* (PCA, Carlson, Oehrtman, & Engelke, 2010): (1) Create a draft item, interview teachers (in-service and pre-service) using the draft item. A panel of four mathematicians and six mathematics educators also reviewed draft items at multiple stages of item development. In interviews, we looked for whether teachers interpret the item as being about what we intended. We also looked for whether the item elicits the genre of responses we hoped (e.g., we do not want teachers to think that we simply want them to produce an answer as if to a routine question); (2) Revise the item; interview again if the revision is significant; (3) Administer the collection of items to a large sample of teachers. Analyze teachers' responses in terms of the meanings and ways of thinking they reveal; (4) Retire unusable items; (5) Interview teachers regarding responses that are ambiguous with regard to meaning in cases where it is important to settle the ambiguity; (6) Revise remaining items according to what we learned from teachers' responses, being always alert to opportunities to make multiple-choice options that teachers are likely to find appealing according to the meaning they hold; (7) Administer the set of revised items to a large sample of teachers; (8) Devise scoring rubrics and training materials for scoring open-ended items; revise items only when absolutely necessary.

Though the overall method described above resembles the development of the FCI and PCA, our focus on assessing teachers' meanings rather than performance introduced many new issues. In this section I share issues to which we found ourselves attending methodically in our attempts to assess teachers' mathematical meanings for teaching. They can be grouped into three themes: (1) item design, (2) rubrics for scoring, and (3) aggregating data. Though it is these three themes I will expand, a general comment to frame them might be helpful. You will profit by approaching the task of designing items to assess mathematical meanings much like you would a design experiment (Brown, 1992; Cobb, Confrey et al., 2003). The difference between a design experiment and what I outline in this section is that within the cycle of design-evaluate-redesign you will have mini-cycles with the same structure, and you will maintain a more intimate dialectic between design and theory throughout the design and refinement process.

Designing items

The foremost characteristic of meanings is that they are invoked in an act of interpretation. An item to assess teachers' mathematical meanings must therefore be designed so that teachers reveal something about their interpretation of it. Ideally, it will invite teachers to think that the meanings that come to their minds in understanding the item are the ones that the item writers want them to share. Second, an item must have a focus—a meaning that you deem important enough to merit devoting one or more of a relatively small number of items to it. Third, the collection of meanings you assess must form a coherent background for the assessment itself. The collection must address the body of ideas that comprise the conceptual skeleton of the elementary, middle, or secondary curriculum for which teachers taking the assessment are responsible. These three considerations undergird the design of individual items.

Start with meanings that matter to students, both positively and negatively.

Research on teachers' and students' mathematical understandings and thinking often provides inspiration for items, as do ways of thinking by teachers and by students that you notice in your everyday interactions with them. Research on students' or teachers' performance that emphasizes correct responses is usually unhelpful. Research that reveals *sources* of students' productive and unproductive meanings and ways of thinking are more useful. For example, APOS theory (Arnon, Cottril et al., 2014; Dubinsky & McDonald, 2001) describes students' difficulties with the idea of function as a mapping of the function's domain onto its image. The idea that a function maps a set of values A to a set of values B entails thinking that the function is evaluated at *every* element of A, and that every value of the function is an element of the set B. One might get at teachers' thinking regarding the idea of mapping a set to a set by asking something like Figure

The graph of a function h is shown to the right. The expression $\left[0.5,1.5\right]$ represents all real numbers from 0.5 to 1.5, inclusive. (a) What might the expression $f\left(\left[0.5,1.5\right]\right)$ mean? (b) According to your meaning, what is $f\left(\left[0.5,1.5\right]\right)$?

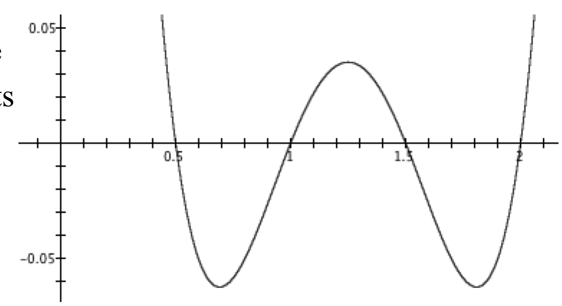


Figure 11. Potential item to probe teachers' meaning for function as mapping.

11.[16]

Part (a) asks what the expression $f\left(\left[0.5,1.5\right]\right)$ *might* mean, not what it *does* mean. It asks teachers to make a meaning for the expression even if they've never seen this notation. The purpose would be to see whether teachers could think about what it *might* mean to have an interval of numbers instead of a single number as input to a function. Part (b) gives teachers an opportunity to express whatever meaning they expressed in Part (a) more concretely. This could reveal that they have an intuitive scheme for mapping a set that they cannot articulate, or it could reveal that their verbal formulation of a meaning for $f\left(\left[0.5,1.5\right]\right)$ is not grounded in a scheme of actions and operations.

Leverage the implicative nature of meanings.

To leverage the implicative nature of meanings, one asks for a teacher's response and then follows up with a question to reveal what the teacher's response implies for him or her. Examples of this strategy appear earlier in the chapter.

Raise issues of conflicting meanings that often arise, or that a teacher should anticipate, in students' thinking.

The research literature on students' understandings of important mathematical ideas is a rich source for items that probe teachers' meanings. For example, it is well known that students often interpret graphs that show a function's rate of change as if the graph is of the function itself, such as mistaking a speed-time graph for a distance-time graph (Monk, 1992; Monk & Nemirovsky, 1994; Oehrtman, Carlson, & Thompson, 2008). A common explanation is that students did not pay attention to the axes' labels. We included an item on the MMTsm that showed a graph that represented the rate of change of a bacterial culture's mass relative to elapsed time since the measurements began, with axes clearly labeled "g/hr" and "hr". We asked, "Over what time intervals is the culture's mass increasing?" On the next page we presented the same graph with one point highlighted, and asked teachers to explain what the point represented. Then we asked teachers whether they would like to change their answer to the first part. Some teachers responded to the first part as if the graph showed the culture's mass, not its rate of change, interpreted the point correctly, and then changed their answer to Part A. These teachers had not attended to the axes labels and changed their response when their attention was drawn to them. Other teachers interpreted the graph as if about mass, interpreted the point as if about a rate of change, and left their answer to the first part the same. For these teachers, their original interpretation of the graph was stable. We interpret this as suggesting that, to these teachers, graphs are about amounts. It is also consistent with a form of reasoning about graphs that I mentioned earlier—shape thinking. We cannot be certain of either hypothesis without follow-up interviews, but these teachers' original interpretation was certainly resistant to perturbation.

Request symbolic responses sparingly.

Teachers often use symbolism idiomatically—the expression itself has meaning to them, but the meaning of the whole does not derive from meanings of its parts. Request symbolic responses only when it is important to see how teachers express their meanings symbolically and only in cases where you have another window to the meaning they think they are expressing.

Decide early on an item's focus.

Since meanings are schemes, an item can invoke multiple meanings in the person responding to it. See the discussion of *focus* on page 454.

Pilot items early and often

It is essential that you conduct interviews on as many items as your resources allow. Of course, we all operate with limited resources so you must decide strategically on which items you will interview. It is also useful to share items with mathematicians with whom you've established a working relationship. It is imperative that you also share your rationale for the item and what you hope teachers will reveal in their responses.

Scoring responses and refining items

Our method for developing scoring rubrics and for refining items is inspired by the work of Wilson and Draney at the UC Berkeley Evaluation and Research (BEAR) Center (Kennedy & Wilson, 2007; Wilson & Sloane, 2000). That work focuses on creating construct maps and progress variables that form the basis of instruments that use a developmental perspective on students' learning to assess students' progress—to evaluate learning progressions. BEAR assessments are given on multiple occasions to track students' progress over large periods of time in learning a body of interrelated ideas and ways of thinking.

MMTsm items, however, do not come with models for the development of meanings they assess. Teachers are not first-time learners of these ideas. Rather, in many instances they developed unproductive meanings as students, and then spent years learning to cope with mathematics instruction that they were unprepared to understand—developing ways to satisfy demands to perform without having a basis in meaning. Moreover, experienced teachers, over time, often develop curricular meanings that overlay meanings they already possessed. As a result, it is rare that we can place teachers' meanings on a developmental scale regarding progress in learning an idea. Our approach instead became to design rubrics to reflect levels of *productive* meaning—where "productive" is judged by the criterion of how useful a meaning would be for students' future mathematical learning were a teacher to convey it to them. Levels of productive meaning do not form a progression in the sense that we envision teachers going through lower levels to reach higher levels. But the levels do impose order on teachers' responses on a scale that is relevant to teaching the idea being assessed, and we may expect responses by teachers who are involved in professional development to attain higher levels when responding to an item on successive occasions.

It is essential that each free-response item have a scoring rubric. Your assessment will be unusable outside your project without scoring rubrics.

As mentioned above, our method for developing scoring rubrics derives from the work of Wilson and Draney. It has five phases that describe the scoring of items which have gone through the development phase of interviews and small-scale piloting and for which you now have a large number of responses.

Phase 1: Grounded theory. In the first phase you approach responses with the open attitude espoused by Strauss and Corbin (1998), but the "openness" is theoretical. Attend to how well you can make sense of teachers' responses using the theory of meaning behind the item. In this phase we sometimes made major changes to an item and placed the item in the pool awaiting further development without trying to score responses, or we discarded the item altogether. With some items we found that we could make sense of teachers' responses if we modified its theory of meaning. In other cases, we made minor changes to the item to use in later testing, but proceeded to analyze responses to the current item as given to teachers.

Phase 2: Group responses. When an item elicits responses that you feel give insights into teachers' meanings or ways of thinking, you then proceed to group responses by levels of productive meaning. In this case, "meaning" is *the researcher's* attribution of meanings and ways of thinking that would explain why a teacher wrote what he or she wrote. Our team found it useful to conduct these analyses individually with subsets of responses to an item and then report our analyses to the team for discussion. Lengthy discussions of individual responses, over time, will increase the overall coherence of the scoring scheme for all responses. I hasten to add that it will often happen that you cannot describe a group of responses in terms of meanings that teachers might have had. But you may still be able to describe a group of responses in terms of meanings that you envision students might construct were teachers to say in class what they said in their responses to the item.

Phase 3: Codify your criteria for grouping responses. It is in the third phase that you begin to have something that resembles a scoring rubric for an item. Your aim will be that the criteria's descriptions allow a scorer to place any response at a particular level. Sometimes a teacher's response will fit two levels. In this case the scoring rubric must say whether to place the response at the higher or lower level. The response in Figure 2 (meaning of "over") is an example of this. The teacher said that "over" means "during" and the teacher said that "over" means a ratio. We decided to state clearly that responses like this should go at the lower level.

Phase 4: Small scale inter-scorer trials. The rubric is ready to be "stress tested" after it has reached its first stable state. Select a small random sample of responses (we used samples of size 10) and have several team members score them independently *according to the rubric.* Team discussions of scoring discrepancies often lead to further revisions of the rubric during the discussion. When consensus has been reached on a rubric it is ready for inter-scorer agreement testing.

Phase 5: Inter-scorer agreement and scoring all responses. For efficiency we combined the scoring of all responses with scoring for inter-scorer agreement. We had 112 responses to score on each item of the MMTsm. Two team members each scored 66 responses, 46 that were unique to a scorer and 20 that were common to the two scorers.[17] Scorers scored their respective responses independently, entering scores in a spreadsheet. The team discussed discrepancies between scores on common responses. Some discrepancies were accepted as unavoidable error, which left the rubric intact. Other discrepancies pointed to problems with the rubric that, upon revision, eliminated the discrepancy. If we changed a level's description, each scorer revisited responses unique to his response set whose scores might be affected by the rubric's revisions.

It should be obvious that the method for developing a scoring rubric described here is an exercise in reflecting abstraction (Piaget, 2001). The shifts from Phase 1 to Phase 5 reflect the team members' progressive thematization of their thinking about the responses that an item elicits from teachers. A scoring rubric, in its final form, therefore reflects a scheme that the rubric designers built by continually reflecting on their actions of making sense of teachers' responses.

Aggregating data

Our goal for the MMTsm, from the outset, was to create an instrument whose results could inform teachers and their professional development leaders about areas in which teachers might work to strengthen their mathematical meanings for teaching and areas in which they have productive meanings. The issue of aggregating data, then, is really the issue of how to report results in a way that helps teachers improve their mathematics teaching and helps professional developers design an evidence-based intervention. Each item is rich in the information it provides about teachers' meanings. But results cannot be reported item-by-item. We also faced the problem that one mix of meanings on a set of related items can have very different implications for teaching and for professional development than another mix of meanings.

We are facing the issue of data aggregation as of this writing, and at this moment I cannot offer a solution. We hope that the data we've collected and scored are amenable to a variant of the BEAR assessment system, a system that was designed for tracking students' progress within a learning progression. But it is not clear to me that it even makes sense to look for structure among levels of productive mathematical meanings in responses to MMTsm items. Lower levels of a scoring rubric tend to be about unproductive meanings teachers might convey to students instead of about meanings they have.

For professional development projects currently using the MMTsm, we will report very simple profiles for individual teachers—scaled level scores for groups of items within the MMTsm item categories along with brief statements about what scores in different ranges mean regarding mathematical meanings for teaching that concept area. We have no illusions that this will be especially meaningful to teachers except to alert them that there might be something they should work on. However, we expect this information to be very helpful to leaders of the teachers' professional development projects. Project teams attend an 18-hour workshop on the MMTsm, its design, and on using the rubrics so score teachers' responses. The workshop, and the rubrics' supporting materials, also goes into great detail about implications for student learning that we foresee responses at the different levels having. It is this aspect of the MMTsm that we see having the greatest potential impact—alerting professional development leaders about the mathematical meanings they will be attempting to affect in their projects and why affecting them is important.

Two large projects are using the MMTsm as a measure of their yearlong projects' impact on teachers' mathematical meanings for teaching secondary mathematics. Their use of the MMTsm will allow us to inquire into two questions: (1) Can we be confident that gains on the MMTsm actually reflect higher quality and more coherent meanings? (2) Is it possible for teachers' scores to go *down* over the period of a year, and if so, why? Both questions require qualitative methods that we hope will yield results that triangulate with what we think teachers' responses to the MMTsm tell us.

CONNECTING ASSESSMENTS OF MATHEMATICAL MEANINGS TO CLASSROOM INSTRUCTION

I discuss the idea of teaching as a form of conversation in (Thompson, 2013), and explain how successful conversation relies in principle on participants' conscious attention to their and others' meanings. I cannot repeat that discussion here. One point, however, deserves repeating. Teachers convey meanings to students in the sense that students strive to understand what their teacher wants them to do or to understand, building meanings in the process. This happens regardless of whether teachers are aware of the meanings they possess, and it happens regardless of the coherence of the teacher's meanings. A teacher's aim should be that the meanings students build from instruction are meanings worth having for a lifetime. Teachers' instruction should support students in creating coherent meanings of the mathematics the teacher is teaching, and those meanings should lay a foundation for students' future learning. Attention to teachers' mathematical meanings for teaching mathematics will support this broad goal.

Having high quality, coherent meanings is an essential aspect of high quality instruction. But it is only a piece of high quality instruction. Other factors will affect whether teachers convey productive mathematical meanings to their students.

- The teacher has meanings. Is the teacher aware of them? Is the teacher oriented to conveying them to students?
- Does the teacher reflect on activities and problems that might give students an occasion to transform their current meanings into desired meanings?
- Does the teacher care about the meanings students construct from what he or she does and says, and convey that care to students? Is the teacher oriented to notice students' meanings, and adjust instruction accordingly?

CONCLUSION

In this chapter I argued in principle and by example that a focus on teachers' mathematical meanings for teaching mathematics is both important and potentially productive regarding the improvement of teachers' teaching, which I take as necessary for the improvement of students' mathematical learning. I also outlined a method for developing items and instruments that focus on teachers' mathematical meanings for teaching mathematics. In this concluding section, I will speak about how we can link research and assessment more intimately than simply using the results of each in the activities of the other.

Assessment as a context and source for research

While developing the MMTsm, we often found that existing research was inadequate to guide an item's design or to make sense of teachers' responses. For example, in designing items to investigate teachers' meanings for function we discovered that function notation is a far more complex notion for teachers than existing research on function suggests (Musgrave & Thompson, 2014; Yoon, Hatfield, & Thompson, 2014). This led to function notation becoming one of the identified areas in which we attended to teachers' meanings, which in turn has led to an expansion of our understanding of the conceptual requirements for students and teachers to use functions as models of dynamic situations. We anticipate that these developments will lead to new directions in research on students' and teachers' understandings of function.

More broadly, a focus on creating items that assess teachers' meanings will reveal lacunae in past research related to that those meanings, largely because you will need to address issues of teachers' and students' thinking that past research finessed. When this happens, you can both address the issue within your overall scheme for the assessment's design while simultaneously putting it on your and others' agendas for future research.

Finally, we are now planning proposals to use the MMTsm to draw national and international comparisons—not as a horse race among groups, but to investigate whether differences among teachers' mathematical meanings for teaching might be a partial explanation for differences among nations' mathematics education outcomes.

Research as source for assessment

I mentioned that research on students' and teachers' mathematical thinking often provides inspiration for deciding upon the areas that your assessment will cover and for items to include in it. There is, however, another important way in which we can leverage the conduct of research on mathematical thinking, especially qualitative research, to inform future assessments. In doing qualitative research on students' or teachers' mathematical thinking, one is attentive to nuances in individuals' thinking that point to understandings, meanings, or ways of thinking that might prove explanatory with regard to why it is reasonable, from the individuals' perspectives, that they do what they did. If researchers were to think also about the prevalence of any of these as possible explanations of phenomena that have already been witnessed broadly, they would have the beginning of an assessment focus. If in addition they were to fine-tune their tasks so that they could be given outside of interviews, and responses could be scored with that focus, they would have early drafts of assessment items.

Striving for common measures

While writing this chapter, the MMTsm team and I struggled with the question of what to include as examples of items that focus on teachers' mathematical meanings for teaching. We were reluctant to include actual MMTsm items, for the simple reason that if we amortize the amount spent by the National Science Foundation across items and scoring rubrics, each item and its rubric cost, on average, over $35,000. These items are not easy to create. I could have used discarded items, and they would have served the purpose of this chapter.

We decided collectively that I should use actual items and include our data on them, for three reasons. The first reason is that most people with whom we shared results are truly surprised by them—they were unaware of teachers' difficulties with mathematical meanings that they thought were largely unproblematic. We decided that it is important that the mathematics education community realize the depth of a problem that has gone largely unnoticed. The second reason that we decided to include actual items and data on them is to encourage others to use them, or the MMTsm in its entirety, in their research. It is only through the use of common measures that research results are comparable, and the use of common measures also supports the development of common conceptions of what is being measured. The third reason for including actual items and their data is that they are better than the discarded items, and we wanted to share the best items we could in hopes that they would inspire others to create even better ones.

NOTES

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[2] I thank John Mason, David Kirshner, Lyn English, Marilyn Carlson, Mark Wilson, Karen Draney, and Cameron Byerley for their thoughtful comments and suggestions on earlier drafts.

[3] This narrative also assumes that the teacher has a rich meaning of "equivalence-preserving transformation".

[4] MMTsm team members are Stacy Musgrave, Ioanna Mamona, Cameron Byerley, Neil Hatfield, Hyunkyoung Yoon, Surani Joshua, Ben Whitmire, Mark Wilson, Karen Draney, Perman Gochyyev, Diah Wihardini, Dong Hoon Lee, and JinHo Kim.

[5] The MMTsm assesses teachers' mathematical meanings in the areas of variation and covariation, function (definition, notation, and modeling), frames of reference, magnitude, proportionality, rate of change, and structure.

[6] A later section discusses general issues of method. One of the issues is that of "focus". Teachers' responses often tell you far more about their thinking than an item was designed to tap. What you decide to ignore in teachers' responses to a particular item is as important as what you decide to look for.

[7] My use of "convey" is not the same as "transmit". A meaning that a teacher conveys to a student is the meaning the student constructs in attempting to understand what the teacher meant.

[8] This is precisely the meaning of division stated in the Grade 5 mathematics textbook published by the Japan Ministry of Education (2008).

[9] The graph of $y = 3x + 2$ has a constant slope of 3 even when graphed in a polar coordinate system or in a log-log coordinate system—and its graph does not appear to be a line in either one.

[10] The graph in the actual item was much larger than it appears here, making physical measurement quite easy.

[11] I am indebted to Les Steffe for this example.

[12] I use the word "space" instead of "set" because meanings do have structure. Actions imply other actions by creating conditions for further action. The structure of a person's meaning arises from the structure of the interconnections among actions, images, and schemes that constitute it in that person's reality.

[13] One of Piaget's definitions of scheme was, "[Schemes are] organized totalities [of actions and operations] whose internal elements are mutually implied" (Piaget, 1952, p. 405)

[14] One reviewer of this chapter suggested that what we have as Level 3 in Table 12 should be the highest level. I respectfully disagree. Students and teachers who have the meaning of variables in equations as described in Level 4 see greater coherence among ideas of functions, graphs, equations, and solutions to an equation. In effect, they will see an equation's solution set S as $S = \left\{ x \in D \middle| f(x) = c \right\}$, where D is a function's domain and c is in the function's image. To think of values of x in D that make the statement true, one

must envision the possibility of all values of x in D, which is what the meaning of continous variation described here affords.

[15] For example, some teachers declared "x" is a variable by virtue of being a letter, and that only specific numbers, represented with numerals, are constants.

[16] This is item is not on the MMTsm.

[17] The solution to $2x + 20 = 112$ is 46.

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