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A Coherent Approach to the Fundamental Theorem of Calculus Using Differentials

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We describe an approach to introductory Calculus that supports students in connecting their conceptions of derivatives and integrals by incorporating the FTC as a central idea from the first day of the course. To accomplish this goal we re-conceptualize the idea of differential, introducing it before the notion of derivative in the context of constant rate of change in linear variation. In doing so, we view changes in variables happening continuously, as opposed to happening in increments.

Several authors have built introductory calculus courses based on the concept of infinitesimal as introduced in Robinson's (1966) nonstandard analysis. Three prominent examples are Henle (1979), Rogers (2005), and Keisler (2012). They argued, and we agree to a certain extent, that an approach to calculus based on infinitesimals is more intuitive for students than is the more common approach that is based on limits.

Another point of entry into the calculus is through the use differentials *in place of* derivatives (e.g., Dray & Manogue, 2010; Rogers, 2005). Rogers' meaning of a differential seems, to us, to be very much like Robinson's infinitesimal. Dray and Manogue's use of differentials seems to be driven by notational simplicity that they provide. We cannot tell with certainty what Dray and Manogue mean by a differential, but it seems they meant *differential* to be a small change in a quantity. Regarding common meanings of differential in calculus textbooks, we surveyed 17 classic and contemporary calculus textbooks; most of them do not mention differentials at all for single variable calculus, and the few that do, define *differential* after having fully developed the derivative, and they define the differential dy as $dy=f'(x)dx$.

Existing approaches to calculus based on the ideas of infinitesimals, limits, or differentials fail to address an important common shortcoming in calculus students' thinking: students tend to think of variables statically. To them, variables do not vary. Calculus, to students who conceive variables statically, is divorced from ideas of variation, covariation, accumulation, and rate of change—the very ideas that the inventors of the calculus intended to address. White and Mitchelmore (1996), Jacobs (2002), Carlson et al. (2002), and Trigueros and Jacobs (2008) demonstrated the insidious effects on students' understandings of and ability to model dynamic situations and pointed to students' static conceptions of variables as being at the root of their difficulties.

Our final concern with approaches that support students' tendencies to think about variables statically is that the Fundamental Theorem of Calculus (FTC) is fundamental neither to students' understandings of derivatives nor to their understandings of integrals. Instead, derivatives are about slopes of tangents, integrals are about areas bounded by a curve, and the FTC, coming after both derivatives and definite integrals, is about neither slopes nor areas. There is nothing fundamental about the FTC in students' thinking.

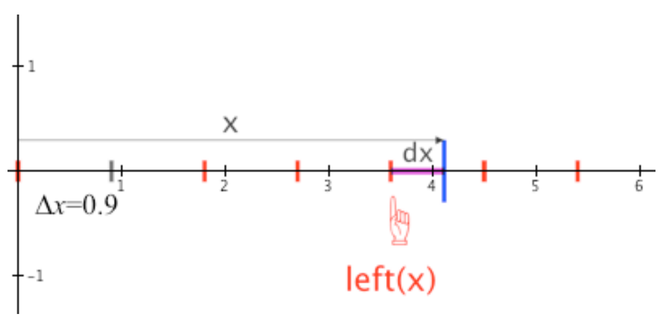
Here we outline our approach to developing the calculus so that it (i) explicitly addresses students' problematic, static meaning of variables, and (ii) supports students in connecting their conceptions of derivatives and integrals by incorporating the FTC as a central idea from the first day of the course. To accomplish this goal we needed to re-conceptualize the idea of differential.[1]

The fundamental theorem of calculus frames our entire course. We explain to students at the outset that the entirety of calculus addresses two foundational problems, namely:

1. You know how fast a quantity is changing at every moment; you want to know how much of it there is at every moment.
2. You know how much of a quantity there is at every moment; you want to know how fast it is changing at every moment.

We found that US college students and Israeli high school students are not prepared to think about these foundational questions profitably. Their image of function is typically a one-number-in one-number-out function machine, and they cannot use function notation representationally. Also, in line with earlier research, students think of variables statically. To them, a variable's value varies by substituting different numbers in its place—one number at a time. Accordingly, their understanding of the continuum (the real number line) is that it is composed of integers, a smattering of rational numbers, and 7-10 irrational numbers. Finally, their understandings of quantity are limited largely to lengths, areas, and volumes, where areas and volumes are conceptually one-dimensional (Thompson, 2000). As such, continuous variation is not part of their image of a real-valued variable and it requires a concerted effort on students' part to construct continuous variation as a way of thinking.

We address students' ill-preparedness in many ways, focusing on their conceptions of the continuum and on envisioning variables as varying continuously. The image of continuous variation also is an important part of our materials on the concept of function. We also develop the idea of constant rate of change in the guise of linear variation. It is in the context of linear variation that we introduce the idea of differential. When two quantities x and y change at a constant rate with respect to each other, then changes in y vary in proportion to changes in x. Or, dy=mdx. That is, we view changes in variables happening continuously, as opposed to changes in variables happening in increments. To this end, we talk about Δx as the length of intervals that partition the x-axis, but we speak of the value of x varying continuously through any Δx-interval interval. The value of dx is the difference between the "current" value of x and the beginning (denoted left(x)) of the Δx-interval that contains the current value of x.



[1] Background for this approach may be found in (Kouropatov & Dreyfus, 2013, 2014; Thompson, 1994; Thompson, Byerley, & Hatfield, 2013; Thompson & Silverman, 2008).

That is, a differential in x is a variable whose value varies through the interval $(0,\Delta x]$, repeatedly. Therefore, dy is a variable whose value varies through the interval $(0,m\mathrm{d}x]$, where m is the constant rate of change that relates changes in y with changes in x within the Δx-interval that contains the current value of x as it varies.

We hasten to point out that we introduce the idea of differential as soon we introduce linear variation. We *do not* base the idea of differential on the idea of derivative.

We then define the concept of a *moment* of a variable as a small interval containing a value of the variable. The idea of a *moment* is best illustrated by the case where the variable is time: taking a photo with the shutter being open for a small interval of time – a moment. Anything moving within the camera's range of view will create a small blur, and this will be true no matter the shutter's setting. The generalization to variables other than time is that *all variation is blurry.* Thus, a moment in a variable's variation is an interval.

We dwell on the idea of a moment in a variable's variation to introduce the idea of *rate of change at a moment*, meaning that a function has a rate of change that is essentially constant over a small interval of the function's independent variable. Since the rate of change is essentially constant over an interval, the change in the function over that interval is essentially equal to dy, where $\mathrm{d}y=m\mathrm{d}x$, as dx varies through that interval. It is with this image that we introduce the idea of a *rate of change function r_f for a function f,* meaning that every value of r_f gives the rate of change of f at a moment of f's independent variable. With the concept of rate of change functions, we are positioned to build a function whose values approximate values of f by accumulating changes in dy as x varies, starting from a reference point. We use the term *accumulation function* for functions that arise by their values having accumulated at some rate over small intervals of their independent variable.

It should be obvious that our approach entails developing integrals as accumulations from rate of change functions as the first major concept of the calculus. It is in this respect that we see the FTC as being at the core of the course from the outset. With this entry it is intuitively immediate that the rate of change of an accumulation function at any moment of its independent variable is the value at that moment of the rate of change function from which it is built.

The idea of integral becomes crystalized for students when we introduce the idea of a value of one function being *essentially equal to* the value of another—that making Δx so small that making it smaller produces no practical change in the estimate of the function's value. "Practical", of course, depends on context.

The second fundamental problem of calculus, knowing how much of a quantity you have at every moment and wanting to know how fast it is changing at every moment, entails reversing the process of creating accumulation functions from rate of change functions. The major insight that is required is to realize that any value of a function that gives an amount of a quantity at every moment must have accumulated at some rate over moments of the function's independent variable. That is, if $f(x)$ is an amount, then that amount accumulated from some reference point a, and therefore $f(x) = f(a) + \int_a^x r(t)\,dt$ for some rate of change function *r*. Put another way, the FTC becomes the *motive* for finding a method of deriving rate of change functions from accumulation functions.

This course evolved at Arizona State University over the past five years, and an electronic textbook for it now exists. The ideas have also been experimented with high school students in Israel. During the current academic year, a controlled experiment was carried out at ASU to compare students' learning in our and traditional approaches. In Fall 2015 two full-time faculty taught sections of Math 270T, traditional Calculus 1 (n=180, 68) while one full-time faculty and one graduate student taught two sections of Math 270R, our revised Calculus 1 (n=114, 35). The sections were undifferentiated in the schedule of courses so we believe that there was no selection bias among students. Thompson met with the instructors in summer 2015 to construct a 12-item pre-post test. All instructors agreed that the final set of questions addressed a broad spectrum of important understandings that students should have at the course's end. Students took the pretest in their first recitation meeting. The pretest was embedded in each instructor's final exam; thus, all students who took a 270 final exam took the pretest a second time.

Table 1 shows that there were no significant differences in pretest scores between students in 270R and 270T ($p<0.23$) and a highly significant difference in their posttest scores ($p<0.001$). Scheffe post-hoc tests showed no difference between traditional sections and no difference between revised sections, but each traditional-revised comparison showed a significant difference ($p<0.001$). There were no significant differences among sections in terms of percent of students who passed the derivatives mastery test.

Table 1. MAT 270 Pre-post Comparisons

	PreTest		PostTest	
	Mean	StdDev	Mean	StdDev
Traditional	3.18	1.53	4.89	2.45
Revised	2.98		7.90	

Individual interviews of students in both treatments also showed distinct differences in the quality of their understandings. Also, students who dropped 270R did so largely because its emphasis on meaning and meaningful reasoning did not fit their expectations of a mathematics class. Star and Smith (2006) reported a similar result in the University of Michigan's implementation of Harvard Calculus. Addressing students' expectations in 270R will be an important goal in the future.

We close by pointing out that our meaning of differentials *dy* and *dx*, as changes in quantities that are related linearly, is at the heart of our approach. It is by establishing powerful *meanings* of constant rate of change, linearity, and differentials that we incorporate the FTC in deriving accumulation from rate of change and in deriving rate of change from accumulation.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*, 352-378.

Dray, T., & Manogue, C. A. (2010). Putting differentials back into calculus. *The College Mathematics Journal, 41*, 90-100. doi: 10.4169/074683410x480195

Henle, J. M., & Kleinberg, E. M. (1979). *Infinitesimal calculus*. Cambridge, MA: MIT Press.

Jacobs, S. (2002). *Advanced placement BC calculus students' ways of thinking about variable.* Ph.D. dissertation, Arizona State University, Tempe, AZ.

Keisler, H. J. (2012). *Elementary calculus: An infinitesimal approach*. Retrieved from http://creativecommons.org/licenses/by-nc-sa/2.0/.

Kouropatov, A., & Dreyfus, T. (2013). Constructing the integral concept on the basis of the idea of accumulation: suggestion for a high school curriculum. *International Journal of Mathematical Education in Science and Technology, 44*, 641-651. doi: 10.1080/0020739x.2013.798875

Kouropatov, A., & Dreyfus, T. (2014). Learning the integral concept by constructing knowledge about accumulation. *Zdm, 46*. doi: 10.1007/s11858-014-0571-5

Robinson, A. (1966). *Non-standard analysis*, Studies in logic and the foundations of mathematics. Amsterdam,: North-Holland Pub. Co.

Rogers, R. (2005). Putting the differential back into differential calculus. In A. Shell-Gellasch & D. Jardine (Eds.), *From calculus to computers: Using the last 200 years of mathematics history in the classroom*, MAA Notes (Vol 68, pp. 9-20). Washington, DC: MAA.

Star, J. R., & Smith, J. P. (2006). An image of calculus reform: Students' experiences of Harvard Calculus. In F. Hitt, G. Harel & A. Selden (Eds.), *Research in Collegiate Mathematics Education, 6*, Issues in Mathematics Education (Vol 13, pp. 1–25). Providence, RI: American Mathematical Society.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*, 229–274.

Thompson, P. W. (2000). What is required to understand fractal dimension? *The Mathematics Educator, 10*, 33-35.

Thompson, P. W., Byerley, C., & Hatfield, N. (2013). A conceptual approach to calculus made possible by technology. *Computers in the Schools, 30*, 124-147. Retrieved from http://bit.ly/14vZjxC. doi: 10.1080/07380569.2013.768941

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics*, MAA Notes (Vol 73, pp. 43-52). Washington, DC: Mathematical Association of America.

Trigueros, M., & Jacobs, S. (2008). On developing a rich conception of variable. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education*, MAA Notes (Vol 73, pp. 3-13). Washington, DC: Mathematical Association of America.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education, 27*, 79-95.