Covariational Reasoning Among U.S. and South Korean

Secondary Mathematics Teachers[†,*]

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We investigated covariational reasoning among 487 secondary mathematics teachers in the United States and South Korea. We presented an animation showing values of two varying magnitudes (v and u) on axes in a Cartesian plane along with a request that they sketch a graph of the value of u in relation to the value of v. We classified teachers' sketches on two independent criteria: (1) where they placed their initial point, and (2) their graph's overall shape irrespective of initial point. There are distinct differences on both criteria between U.S. and South Korean teachers, suggesting that covariational reasoning is more prominent among South Korean secondary teachers than among U.S. secondary teachers. The results also suggest strongly that forming a multiplicative object that unites quantities' values is necessary to express covariation graphically.

Copur-Gencturk (2015), Zaslavsky (1994), and Thompson (2013) argued compellingly that how teachers understand a mathematical idea is an important factor in the mathematical understandings that students actually form. The more coherently teachers understand an idea they teach, the greater are students' opportunities to learn that idea coherently. Inversely, the less coherently teachers understand an idea they teach, the fewer are students' opportunities to learn that idea coherently.

A number of studies support the claim that reasoning covariationally is a powerful foundation for students' comprehension of many mathematical ideas. Students' ability to reason covariationally supports their understanding of:

- proportion (Karplus, Pulos, & Stage, 1979, 1983; Lobato & Siebert, 2002),
- rate of change and linearity (Adu-Gyamfi & Bossé, 2014; Castillo-Garsow, 2013; Confrey, 1994; Herbert & Pierce, 2012; Nunes, Desli, & Bell, 2003;

Thompson & Thompson, 1996; Thompson, 1994a, 1994c; Zaslavsky, Sela, & Leron, 2002),

- variable (Clement, 1989; Dogbey, 2015; Goldenberg, Lewis, & O'Keefe, 1992; Hitt & González-Martín, 2015; Montiel, Vidakovic, & Kabael, 2008; Schoenfeld & Arcavi, 1988; Thompson & Carlson, 2017; Trigueros & Jacobs, 2008; Trigueros & Ursini, 1999; Trigueros & Ursini, 2003; Yerushalmy, 1997),
- trigonometry (Moore, 2012, 2014; Thompson, Carlson, & Silverman, 2007)
- exponential growth (Castillo-Garsow, 2013; Confrey, 1991, 1994; Confrey & Smith, 1994, 1995; Ellis, Özgur, Kulow, Dogan, & Amidon, 2016; Ellis, Özgür, Kulow, Williams, & Amidon, 2012, 2015)
- functions of one and two variables (Boyer, 1946; Bridger, 1996; Carlson, 1998; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey, 1992; Hamley, 1934; Hitt & González-Martín, 2015; Kaput, 1994; Keene, 2007; Martínez-Planell & Gaisman, 2013; Nemirovsky, 1996; Thompson, 1994a, 1994b; Thompson & Carlson, 2017; Weber & Thompson, 2014; Yerushalmy, 1997).

These same bodies of literature, involving small numbers of subjects, suggest that reasoning covariationally is uncommon among students and teachers, at least in the U.S. Also, subjects in these studies were drawn from geographic locales. We know of no studies that investigate covariational reasoning either internationally or with a large, geographically diverse sample.

Theoretical Background

Our meaning of covariational reasoning is grounded in the mental operations described by Thompson's theory of quantitative reasoning (Thompson, 1993, 1994c, 2011). In this theory, a quantity exists only to the extent that someone conceives it, so the nature of any quantity is idiosyncratic to the individual conceiving it. That a person has conceived a quantity means that she has conceived an attribute of some object in a way that it is measurable. The person need not know an actual measure of the attribute, but she takes for granted that it has one and understands what it means.[1] Accordingly, a quantity's value varies when the person conceiving it envisions that the object's attribute varies and hence that the attribute's measure varies.

We characterize quantities as being idiosyncratic to the person conceiving them, for many reasons. One reason is that this removes the onus that we must describe quantities only in terms of the most sophisticated conceptions held by experts. We are free to characterize *learners'* quantities—their conceptions of objects' attributes and their quantification—as differing in principle from experts' conceptions. Most importantly, we are free to describe the quantities and relationships that individuals have conceived as opposed to describing what they have misconceived, or what an expert might say they do not conceive.

Saldanha and Thompson (1998) described mature covariational reasoning in terms of quantities whose values vary:

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[1] We speak of quantities' measures or values throughout this paper even though speaking of quantities' magnitudes often would be more appropriate. The distinction between magnitude and value or measure is important, but discussing it in here would not serve the paper's purpose. See (Thompson, Carlson, Byerley, & Hatfield, 2014) for a full discussion of measures and magnitudes.

> Our notion of covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image. (Saldanha & Thompson, 1998, p. 299)

Saldanha and Thompson's idea of multiplicative object derives from Piaget's notion of logical multiplication—an operation that Piaget and Inhelder described as underlying multiple classification and seriation, and more generally as underlying relationships of simultaneity (Inhelder & Piaget, 1964, p. 182). A person forms a multiplicative object from two quantities when she mentally unites their attributes to make a new attribute that is, simultaneously, one *and* the other. As noted by Thompson and Saldanha (2003), conceptualizing torque as a physical quantity is one example of forming a quantity by uniting attributes of an object (lever plus fulcrum in this case). The attribute "amount of twist" is conceived as being constituted simultaneously by a rotational force and a distance from the fulcrum at which the force is applied.

The ability to form multiplicative objects is at the heart of understanding mathematical ideas of function, rate of change, accumulation, vector space, and so on. Also, although the science education literature does not leverage the idea of forming a multiplicative object of two quantities' attributes, we see students' ability to form multiplicative objects as being central to their understandings of many physical quantities, e.g. force, work, momentum, energy, and so on.

Saldanha and Thompson's emphasis on the centrality of multiplicative objects in a person's ability to reason covariationally gains support from Stalvey and Vidakovic's (2015) investigation of 15 Calculus 2 students' attempts to envision height and volume of water simultaneously in each of two containers as they emptied. Stalvey and Vidakovic reported that a majority of students struggled to envision values of height and values of volume as varying simultaneously in order to sketch a graph of one in relation to the other. Students could envision general directions of the covariation (e.g., height decreases as volume decreases), which Thompson and Carlson (2017) termed *gross coordination of values*, but they could not reason about both height and volume varying simultaneously over small intervals of change. Stalvey and Vidakovic reported that students could attend to height sans volume or to volume sans height, but they struggled to attend to volume and height simultaneously. Carlson, Jacobs, Coe, Larsen, and Hsu (2002) reported similar results in their study of calculus students' abilities to reason covariationally about dynamic situations described textually.

Saldanha and Thompson (1998) did not mention graphs when they spoke of "the entailing correspondence being an emergent property of the image" of covariation. Rather, in line with Goldenberg and colleagues (Goldenberg, 1988, 1993; Goldenberg *et al.*, 1992), they spoke about a person's covariation scheme as entailing an overall image, in retrospect or anticipation, of the simultaneous states of two quantities as they vary. Graphs are a common way to represent this image.

Likewise, Kaput (1994) and Thompson and Carlson (2017) characterized mathematicians' early, pre-graphical conceptions of function relationships as entailing an image of covarying quantities. The genius of Descartes' method of graphs is that, for a person who reasons covariationally, a graph produced within the conventions of a coordinate system provides a visualization that captures what Saldanha and Thompson termed "the entailing correspondence" between values of two covarying quantities. Unfortunately, in school mathematics, Descartes' method of graphs has evolved into the practice among many students and teachers of static shape thinking (Moore & Thompson, 2015, under review)—thinking of graphs as if they are wire and associating their shapes with specific functions or properties of situations.

When someone conceives a generic point on a graph, either in retrospect or anticipation, so that its coordinates represent a state of two quantities' covariation, she has conceived the point as a multiplicative object and the graph containing it as a record of the quantities' covariation—what Moore and Thompson (2015, under review)[2] call emergent shape thinking. On the other hand, when someone understands the coordinates of a generic point on a graph as "over this much and up that much", he is conceiving the point's coordinates as a recipe for locating the point. He is not conceiving the point as a multiplicative object. Many of our calculus students manifest a recipe conception of coordinates when asked to represent $f(2)$ in a coordinate system. They do this by plotting the point $(2, f(2))$, as if the point is a value of the function. They do not think of $f(2)$ as a value – a magnitude – that if put anywhere in a coordinate system should be a length on the y-axis. Instead, they think of $f(2)$ as the "up" part of "over and up".

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[2] While the constructs of static and emergent shape thinking are highly related to covariational reasoning, we do not rely on them as explanatory constructs in this article or draw conclusions that inform static and emergent shape thinking. To do either would require interview data from each teacher that we do not have.

Research Questions

The study investigated three research questions regarding teachers' covariational reasoning.

1. To what extent do secondary school mathematics teachers in our samples reason covariationally about dynamic phenomena that they witness?
2. Are there differences between the teachers in the United States sample and South Korean sample in the prevalence of covariational reasoning?
3. To what extent is creating a multiplicative object of two quantities' attributes necessary to reason covariationally?

Method

As we already mentioned, prior research suggests that it is uncommon for school and university students in the U.S. to reason covariationally. Our experience in professional development projects is that it is also uncommon for U.S. high school mathematics teachers to reason covariationally. Our examinations of U.S., Japanese, South Korean, and Russian elementary textbooks suggested that while there is little attention to covariation in the U.S., there is explicit attention to covariation in Japan, South Korea, and Russia. For example, the 2008 Japanese Mathematics Course of Study Grade 4 standards contained the following statement under the heading *Quantitative Relationships*.

> Students will be able to represent and investigate the relationship between two quantities as they vary simultaneously. (Japan Ministry of Education, 2008, p. 11)

We therefore decided that it was prudent to include teachers from a country that addressed covariation explicitly in its school mathematics curricula. We did this to gain insight into whether U.S. teachers' difficulty with covariational reasoning is due to epistemological obstacles inherent in this way of reasoning (Bachelard, 2002; Brousseau, 1997), which would be suggested by equivalent levels of difficulty in both countries, or whether this difficulty is a cultural artifact of an educational system, which would be suggested by significantly different levels of difficulty.

Subjects

The study involved 487 secondary mathematics teachers: 366 mathematics teachers from South Korea (264 high school, 102 middle school) and 121 high school mathematics teachers from the United States (US). South Korean (SK) teachers were participants in mandatory "first class certificate" program that all SK teachers must take within 3-5 years of their initial placement. Testing in SK was done at four different sites—one in Seoul, one near Seoul, and two in smaller cities. We tested approximately 95% of all teachers taking the certificate exam in the summer of 2015.

US teachers were participants in Math/Science Partnership programs (state- or NSF-funded) in a Midwestern or Southwestern state in the US. All US teachers were tested as part of their participation in their respective MSP. SK teachers received payment for their participation in this study; US teachers received a stipend for participating in their respective programs.

Table 1 gives a breakdown of teachers' majors. All US teachers were currently teaching high school mathematics. We originally sought only South Korean high school (SKHS) teachers. In South Korea, however, middle school and high school mathematics teachers take the same credential program and sit for the same credential exam. We therefore accepted any SK teacher at the re-credential workshops who volunteered to participate. In assigning teachers with multiple degrees to degree categories, we classified a teacher under "math" if they had a math degree, under "math ed" if they had a math ed degree but no math degree, and "other" if they had neither.

Table 1. Degrees* held by study's teachers.

	Math	MathEd	Other	total
SKHS	81	175	7	263
SKMS	33	49	19	101
USHS	24	40	57	121

* "Math" means a bachelor's or masters degree in mathematics. "Math Ed" means a bachelors or masters degree in mathematics education. "Other" means any degree other than math or math ed. Two SK teachers did not report their degrees.

Table 2 presents teachers' highest course taught at the time of this study. The standard college-preparatory USHS mathematics curriculum includes two years of algebra separated by a year of geometry, then precalculus, possibly followed by calculus, then differential equations. The standard SKHS curriculum, which spans the last three years of students' schooling, is integrated. Derivatives and integrals are introduced in the second year. The SK middle school curriculum is also integrated. It includes what in the US is called Algebra 1, but also includes topics in geometry, statistics, and discrete mathematics.

Table 2. Highest course taught by country and level.*

	Highest US Course Taught					Highest SK Course Taught				
	Alg 1	Geom	Alg 2	Precalc	>Precalc	Math 1	Math 2	Calculus	Other	total
SKHS	0	0	0	0	0	11	33	211	8	263
SKMS	0	0	0	0	0	0	0	0	101	101
USHS	8	16	41	12	44	0	0	0	0	121

* ">Precalc" means College Algebra, Calculus, or Differential Equations.

SK teachers taught a mean of 3.99 years (s.d.=1.97); US teachers taught a mean of 4.35 years (s.d.=4.22). Figure 1 presents a histogram of the number of years that teachers had taught at the time of the study. The histogram for SK teachers is skewed because these teachers sat for their mandatory mathematics certification exam, which they must take within five years of their initial mathematics placement. SK teachers with more than five years teaching experience were switching from another subject into mathematics. We can only speculate as to why the distribution of US teachers' number of years teaching is highly skewed.

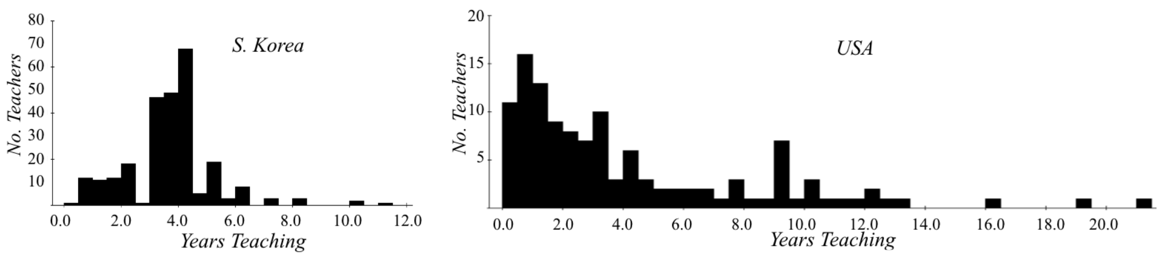


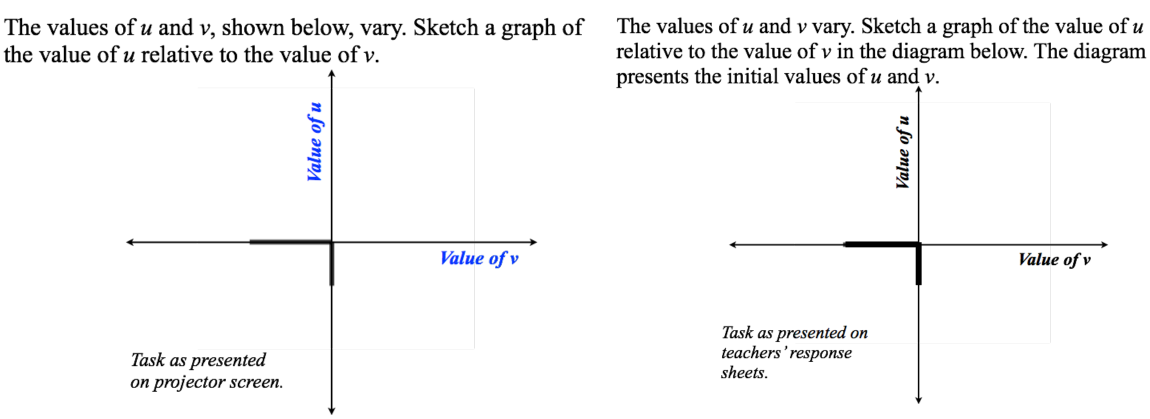
Figure 1. Teachers' number of years teaching

Task

Prior studies of covariational reasoning typically presented subjects with a textual or diagrammatic description of a dynamic scenario and asked them to construct a graph that represented the covariation of quantities in it. It was left to studies' subjects to envision the situation as dynamic, and what they actually envisioned was necessarily colored by their abilities to conceptualize the situation and quantities that researchers saw as within the situation. Our intent was to design a task that would penetrate cleanly to subjects' abilities to keep in mind two quantities' values simultaneously as they varied.

Another hurdle in investigating secondary mathematics teachers' covariational reasoning as expressed graphically is that they have a wide variety of experiences with graphs as students and as teachers. In order to gather data that would not be influenced by teachers' well-practiced graphing routines, we needed a task that they would not assimilate easily to those routines and for which a teacher's covariation scheme would be appropriate to the extent he or she has one.

The task consisted of an animation that presents two bars (labeled *v* and *u*) of varying length on the un-numbered axes of a Cartesian coordinate system, along with a request to sketch a graph that captures the values of *u* in relation to the values of *v* (Figure 2a.) The animation was projected onto a screen (approximately 3m wide by 2.3m high) and the text read aloud at the beginning of the animation. The animation played for two-minutes after the speech ended, repeating itself six times. Teachers sketched their graphs on a paper response sheet (Figure 2b). The response sheet contained the presented item's initial screen, the request to sketch a graph of the value of *u* relative to the value of *v*, along with the statement, "The diagram presents the initial values of *u* and *v"*. The task's text and speech were in Korean for SK teachers and in English for US teachers. The animation itself was the same in both versions. The English version of the animated task and teachers' response sheet can be seen at http://bit.ly/CovaryMagnitudes.



(a) (b)

Figure 2. Animated item to investigate teachers' covariational reasoning: (a) Initial frame of the animation as presented on a projector screen; (b) the task as it appeared on teachers' response sheets. © 2015 Arizona Board of Regents. Used with permission.

Figure 3 shows an accurate graph of the relationship between *v* and *u* as they varied. The graph was generated by uniting the values of *v* and *u* into a coordinate pair (making what we call a *correspondence point*) and by a trace of the correspondence point as the values of *v* and *u* varied simultaneously.

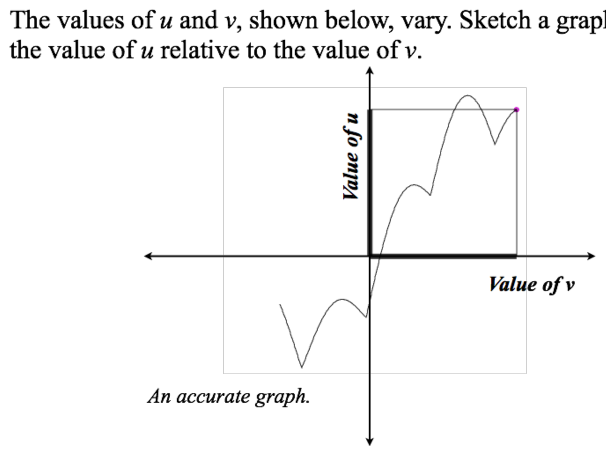


Figure 3. Accurate graph for animated task presented in Figure 2. © 2015 Arizona Board of Regents. Used with permission

The graph in Figure 3 captures the states of the correspondence point at each moment that *v* and *u* covaried in the animation. As the value of *v* increased steadily, the value of *u* first decreased, then increased, then decreased, and so on. To sketch an accurately shaped graph, a teacher must be cognizant of values of *v* when attending to values of *u*, and be cognizant of values of *u* when attending to values of *v*. Moreover, to sketch an accurate graph, a teacher must attend to variations in both values over small intervals of change. Otherwise they will lose track of the other value while focusing on one. These characteristics of successful tracking in similar tasks have been reported in (Castillo-Garsow, 2012; Frank, 2016b; Johnson, 2012a, 2012b; Saldanha & Thompson, 1998; Thompson, 1994a, 1994c).

We designed the seemingly unsystematic variation of *u* with respect to *v* purposely. We wanted to avoid the possibility that teachers would recognize a familiar graph, such as the graph of a quadratic function or a sinusoidal function. We chose to make the variation in *v* uniform because earlier trials convinced us that unsystematic variation in both variables is profitable only in the context of a teaching experiment (Moore, Paoletti, & Musgrave, 2014; Saldanha & Thompson, 1998). In a teaching experiment, a researcher first scaffolds subjects' activities through tasks like that in Figure 2 prior to asking them to deal with unsystematic variation in both variables.

Finally, we presented the animation on a large projector screen while asking teachers to sketch their graphs on paper for two reasons – one methodological and the other practical. Methodologically, we could not allow teachers to control the animation. Were they able to pause the animation they could choose to stop the animation, plot a point, and restart the animation—defeating our purpose of seeing the extent to which they could form a multiplicative object of the two variable's magnitudes that persisted under variation. The second reason was because of the logistics of asking 487 teachers to respond to an animation. We did not have the resources to present the animation to each teacher individually.

Scoring

Item and rubric construction followed the five-phase method inspired by Wilson and Draney at the UC Berkely Evaluation and Research Center (Kennedy & Wilson, 2007; Wilson & Sloane, 2000) and outlined in Thompson (2016). The phases are:

(1) Conduct interviews with item; pilot item repeatedly with small samples, interpreting teachers' responses, as best one can, according to the theory that underlies the item, remaining open to cases that the theory does not address, adjusting the item if necessary to eliminate unintended interpretations of it;

(2) Pilot the item with larger samples. Group responses according to common meanings and ways of thinking that they suggest;

(3) Codify criteria for grouping responses;

(4) Conduct small-scale inter-reliability scoring trials with samples of responses, adjusting the scoring rubrics where necessary, and repeating the scoring to test adjustments;

(5) Large-scale data collection. Score all responses and gather inter-scorer agreement data.

The rubric for this task focused on two features of teachers' graphs: their placement of the graph's initial point, and their graphs' shape (explained below). We separated these features because they convey different information about teachers' covariation schemes. Teachers' placement of their graph's initial point tells us about their construction of a point as a multiplicative object whose coordinates represented the initial values of *v* and *u* simultaneously. The shape of teachers' graphs served as a proxy for their abilities to form a multiplicative object of two quantities' values that persists as the values vary simultaneously. The two features were scored independently. It was possible for a teacher to have an accurate graph in terms of overall shape even though she placed the graph's initial point inaccurately, and it was possible for a teacher to place the initial point accurately even though her graph's overall shape departed significantly from the accurate shape.

Table 3 gives the final scoring rubric for teachers' placement of their initial point. Our focus on teachers' placement of their graph's initial point was to gain insight into whether they thought of a point on the graph as a multiplicative object—as a representation of the values of *u* and *v* so that the point's location represents the two values simultaneously. The initial values of *u* and *v* were highlighted in the coordinate system on which teachers were to sketch their graph. Thus, if they formed a multiplicative object of the values of *u* and *v*, the response sheet gave teachers all the information they needed to place their initial point accurately.

Table 3. Scoring rubric for Dimension A, teachers' placement of initial point (PIP).

Level A2:	Teacher placed initial point such that both horizontal and vertical coordinates are within 0.75 cm of accurate location. (See Figure 4.)
Level A1:	Teacher placed the initial point outside the region for Level A2, but either the horizontal or vertical coordinate is within 0.75 cm of the accurate value.
Level A0:	Initial point placement does not fit either of the above levels.

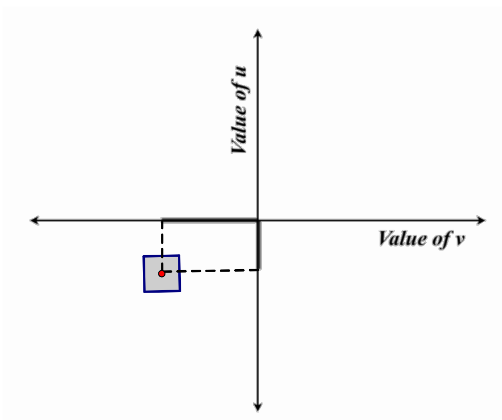


Figure 4. Level A2 for placement of initial point. The highlighted region contains all locations where both coordinates are within 0.75 cm of the accurate location.

We did not choose 1.5 cm sides for the square in Figure 4 arbitrarily. Rather, after collecting the data, we wanted to be generous regarding initial placement and at the same time choose a size that clearly distinguished between "close" and "not close" with no ambiguous cases.

We assumed that a more accurate shape signaled a greater attention by a teacher to the covariation of *v* and *u*. To score teachers' graphs for accuracy of shape we addressed a number of issues that arose because of graphs that we did not anticipate when initially drafting the item.

- The scoring rubric had to be based on features of a teacher's sketched graph. We could not define scoring categories theoretically for the simple reason that most scorers using the MMTsm would not have the theoretical background required to understand the rubric.

- Every local minimum in the accurate graph is a cusp. Two of 487 teachers drew a cusp. We therefore did not insist that graphs show cusps. Instead, we scored as if the accurate graph was smooth around its local minima.
- The most straightforward way to characterize the accurate graph is in terms of the number of local maxima and the number of local minima, with the proviso that local maxima occur in ascending order and local minima occur in ascending order.
- Some teachers' graphs had curvatures that resembled an accurate graph, suggesting they were tracking values simultaneously, but their graphs did not fit the "number of extrema" criteria. For these graphs, we counted inflection points, taking into consideration the concavity with which teachers started and ended their graphs. See Figure 5.

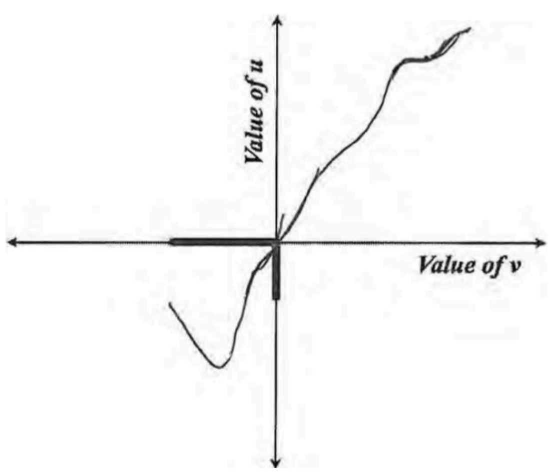


Figure 5. Example of a sketched graph that suggested covariational reasoning but which did not have correct number of extrema or did not have extrema in ascending order.

Table 4 gives the scoring rubric for the shape of teachers' graphs. It reflects our attempt to accommodate the ways described above that we could infer whether teachers' graphs were an expression of reasoning covariationally about the values of *v* and *u*.

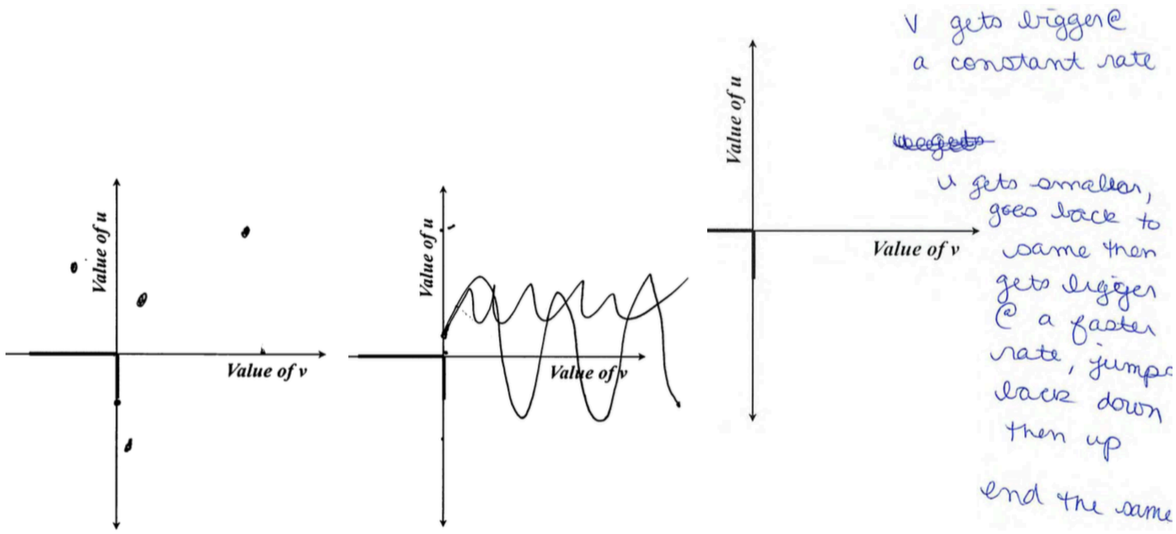
Table 4. Scoring rubric for Dimension B, shape of teachers' sketched graphs (SSG).

Level B4a:	The graph has four local minima in ascending order and three local maxima in ascending order*
Level B4b:	The graph begins decreasing, is generally increasing, has at least 2 local extrema, and has these points of inflection: – 6 if graph starts concave up and ends concave up – 7 if graph starts concave up (down) and ends concave down (up) – 8 if graph starts concave down and ends concave down
Level B3a:	The graph has 6 or 8 local extrema with minima in ascending order and maxima in ascending order.
Level B3b:	Same as B4b except that the graph has one too few or one too many points of inflection given the way the graph starts and ends.
Level B2:	The graph is generally increasing and has 2-5 or 9-12 local extrema, ignoring ascending order.
Level B1:	The graph has no more than 1 local minimum and is otherwise monotonically increasing.
Level B0:	The graph does not fit any of the above levels.

* By ascending order, we mean that from left to right each local minimum's *y*-coordinate was greater than the previous one, and likewise for each local maximum's *y*-coordinate.

Table 3 and Table 4 are summaries. The full rubric contains an explanation of the item, its rationale, the kinds of thinking it is meant to assess, the item's overall scoring strategy, examples of responses that fit each level on each dimension, and explanations for why an example was scored at that level. Levels B4b and B3b were added after collecting data and after initial scoring. Responses like that in Figure 5 would have been scored at Level B0 because they did not fit B4a, B3a, or any lower level. Teachers' responses were scored "IDK" if they indicated that they did not understand the question or did not know how to answer; they were scored "NR" if they gave no response. "NR" as a score meant that the teacher's page had no marks on it.

Given the high proportion of non-Math and non-Math Ed majors in the US sample, it is worthwhile to note that there were no significant statistical differences among US majors (Math, Math Ed, Other) in their patterns of responses. Also, there were no differences among US teachers' graphs according to highest course taught. The three graphs in Figure 6 are by US teachers of different majors.



Nation: US, Major: Math Education, Highest Course Taught: Differential Equations

Nation: US, Major: Mathematics, Highest Course Taught: Algebra 2

Nation: US, Major: Physics, Highest Course Taught: Differential Equations

Figure 6. Examples of US teachers' graphs that were scored at Levels A0 and B0.

Scorer Training

Members of our research team scored all US graphs. Korean scorers scored Korean graphs. Yoon (third author) conducted a three-day scoring workshop in South Korea for six candidate scorers, and then selected five Korean scorers for their English proficiency and for their performance during the workshop. English proficiency was a requirement of Korean scorers because all scoring rubrics were in English. Yoon monitored scoring in Korea, performing quality checks as scoring proceeded.

Inter-rater agreement scoring was conducted in the US by having members of the US team score 30-response subsets of both US-scored and SK-scored graphs. "Agree" meant a perfect match in scores. Inter-rater agreement for the item reported here was 90.0% for Point's Initial Placement (0.855 Cohen's Kappa) and 86.7% for Shape of Sketched Graph (0.831 Cohen's Kappa).

Results

We gave the animated task in Figure 2a to 487 secondary mathematics teachers from the US (121 high school) and South Korea (264 high school; 102 middle school). Teachers' response sheets presented the bars' initial states and the written reminder, "The diagram presents the initial values of *u* and *v*" (Figure 2b). We scored teachers' graphs on the dimensions Placement of Initial Point (PIP) and Shape of Sketched Graph (SSG). Table 5 presents results for Group (country and grade level) in relation to PIP.

Table 5. Group by Placement of Initial Point (PIP).*

Group	*Level A0*	*Level A1*	*Level A2*	*IDK*	*NR*	*total*
SKHS	83 31.4%	62 23.5%	111 42.0%	3 1.1%	5 1.9%	264 100.0%
SKMS	35 34.3%	30 29.4%	33 32.4%	0 0.0%	4 3.9%	102 100.0%
USHS	71 58.7%	20 16.5%	25 20.7%	5 4.1%	0 0.0%	121 100.0%

* Cells contain number of respondents and percent of row total. IDK = "I don't know"; "NR" = "No Response"

A larger percentage of teachers in the SKHS and SKMS groups gave Level A2 responses than USHS teachers. Further, more USHS teachers gave lower level response (Level A0, IDK, or NR) than SKHS and SKMS teachers. To conduct statistical tests of the relationship between Group and PIP we combined IDK and NR responses with Level

A0 in Table 5. The justification for this is that teachers had two (2) minutes to respond after the narrator read the question's text. We therefore interpreted no response or a response that conveyed "I don't know" as equivalent to a Level A0 response. Table 6 gives the Kruskal-Wallis scores for Group and PIP. The Kruskal-Wallis test showed a statistically significant relationship between Group and PIP ($H_{\text{ties}} = 27.23$, df = 2, $p < 0.0001$; average ranks used for ties) that has a small effect size ($\eta_H^2 = 0.05$) with a 95% confidence interval of (0.02, 0.09).

Table 6. Kruskal-Wallis Scores for Groups

Group	*N*	*Sum of Scores*	*Expected under H_0*	*Std Dev Under H_0*	*Mean Score*
SKHS	264	70426.5	64416.0	1443.52	266.77
SKMS	102	25210.5	24888.0	1178.96	247.16
USHS	121	23191.0	29524.0	1252.00	191.66

Table 7 presents post-hoc pairwise comparisons of Group by PIP using the Dwass, Steel, Critchlow-Fligner (DSCF) method, controlling the experiment-wise error rate at $\alpha = 0.15$ (Hollander, Wolfe, & Chicken, 2014). We found statistically significant differences between SKHS and USHS teachers' PIP (DSCF = 7.29, $p < 0.0001$), as well as between SKMS and USHS teachers' PIP (DSCF = 4.75, $p = 0.0023$). Each of these differences is in favor of the South Korean teachers. Table 7 also gives R^2 and probability of superiority (PS) scores, which we explain below.

Table 7. Dwass, Steel, Critchlow-Fligner post hoc comparisons between groups for placement of initial point †

Comparison	*N*	*DSCF*	*Wilcoxon Z*	*p-value*	R^2 *(95% CI)*	*PS*
SKHS vs. SKMS	366	1.91	1.35	0.3670	--	--
SKHS vs. USHS	385	7.30	5.16	< 0.0001	0.07 (0.03, 0.12)	0.65
SKMS vs. USHS	223	4.75	3.36	0.0023	0.05 (0.01, 0.12)	0.63

† Pairwise adjusted threshold using Šidák's method is $\alpha = 0.0527$.

The R^2 column in Table 7 provides effect sizes for the two significant differences in Table 6. We can interpret these values as representing the amount of the variation in the placement of the initial point that is explained by the teachers' group. According to Cohen (1988), R^2 and η_H^2 values between 0 and 0.01 are negligible, between 0.01 and 0.06 are "small", between 0.06 and 0.14 are "medium" and larger than 0.14 are "large". The effect size for SKHS versus USHS is "medium" and the effect size for SKMS versus USHS is "small". An additional way to think about effect size is by the probability of superiority (PS). PS is the long-run relative frequency that a randomly selected member of the "superior" group will have a higher score than a randomly selected member of the "subordinate" group (Fritz, Morris, & Richler, 2012). The PS would be 50% when there is no difference between the groups. For example, the PS for the SKHS and USHS comparison is 0.65. This means that 65% of the time that we select one teacher from each group at random repeatedly, the SKHS teacher's initial placement will have a higher score than the US teacher's initial placement.

Table 8 shows the classification of teachers' responses according to Group by Shape of Sketched Graph.

Table 8. Group by Shape of Sketched Graph (SSG)*

Group	*Level B0*	*Level B1*	*Level B2*	*Level B3b*	*Level B3a*	*Level B4b*	*Level B4a*	*IDK*	*NR*	*total*
SKHS	104	20	58	1	32	3	46	3	5	264
	36.4%	7.6%	22.0%	0.4%	12.1%	1.1%	17.4%	1.1%	1.9%	100.0%
SKMS	50	9	13	1	7	1	17	0	4	102
	49.0%	8.8%	12.7%	1.0%	6.9%	1.0%	16.7%	0.0%	3.9%	100.0%
USHS	60	22	11	3	11	2	7	5	0	121
	49.6%	18.2%	9.1%	2.5%	9.1%	1.7%	5.8%	4.1%	0.0%	100.0%
total	206	51	82	5	50	6	70	8	9	487
	42.3%	10.5%	16.8%	1.0%	10.3%	1.2%	14.4%	1.6%	1.9%	100.0%

* Cells contain number of respondents and percent of row total. "IDK" = "I don't know"; "NR" = "No response."

According to Table 8, almost one third (31%) of SKHS teachers sketched an accurate or semi-accurate graph (Level B3b to B4a) whereas only a quarter (25.6%) of SKMS teachers and one fifth (20.7%) of USHS teachers did so. As before, we combined IDK and NR with Level B0 to conduct statistical tests of relationships between Group and SSG. We also combined Levels B3a and B3b into Level B3 and combined Levels B4a and B4b into Level B4.

It is important to note that this was a challenging task for teachers at all levels and in both countries. Therefore, even though it was theoretically possible for a teacher to construct an accurate graph by noticing the uniform motion of v and then placing primary focus on u, this seems not to have been a prevalent strategy among teachers. It might be that just to think of this strategy requires a level of covariational reasoning that many teachers do not possess.

Table 9 presents Wilcoxon scores for Groups with B-levels combined as described above. It shows a statistically significant relationship between Group and SSG (H_{ties} = 13.81, df = 2, p = 0.001) that has a small effect size ($\eta_H^2 = 0.02$) with

(0.0047,0.0575) as the 95% confidence interval for η_H^2. Post-hoc pairwise comparisons (Table 10) revealed a statistically significant difference between SKHS teachers and the USHS teachers (DSCF = 5.09, p = 0.0009), with group membership accounting for 3% of the variation in the teachers' sketched graphs (R^2 = 0.03). From a probability of superiority standpoint, we could expect that when we randomly select a SKHS teacher and a USHS teacher, that 60% of the time the SKHS teacher's sketched graph would have a higher score than the US teacher's.

Table 9. Kruskal-Wallis Scores for Group and SSG

Group	*N*	*Sum of Scores*	*Expected under H_0*	*Std Dev Under H_0*	*Mean Score*
SKHS	264	69624.5	64416.0	1462.03	263.73
SKMS	102	23553.5	24888.0	1194.08	230.92
USHS	121	25650.0	29524.0	1268.05	211.98

Table 10. Dwass, Steel, Critchlow-Fligner post hoc comparisons between groups for SSG

Pairwise Comparison	*N*	*DSCF*	*Wilcoxon Z*	*p-value*	*R^2 (95% CI)*	*PS*
SKHS vs. SKMS	366	2.85	2.02	0.1077	--	--
SKHS vs. USHS	385	5.09	3.60	0.0009	0.03 (0.0072, 0.0759)	0.60
SKMS vs. USHS	223	1.31	0.93	0.6241	--	--

† Pairwise adjusted threshold using Šidák's method is α = 0.0527.

With regard to the relationship between PIP and SSG, we first examine it separately by group to highlight similarities across cultures in regard to the importance of constructing a multiplicative object in thinking covariationally. Table 11, Table 12, and Table 13 present cross-classifications of PIP and SSG for South Korean High School teachers, South Korean Middle School teachers, and U.S. High School teachers, respectively. We excluded IDK and NR responses from each table because IDK or NR on one part correlates automatically with IDK or NR on the other.

Table 11 shows that only 8.4% of SKHS graphs that had a badly misplaced initial point (Level A0) also had an accurate or semi-accurate shape (Levels B3b-B4a), while 53.1% of graphs that had a well-placed initial point (Level A2) had an accurate or semi-accurate shape (Levels B3b-B4a).

Table 11. Placement of Initial Point vs. Shape of Sketched Graph (SKHS)*

	Level B0	*Level B1*	*Level B2*	*Level B3b*	*Level B3a*	*Level B4b*	*Level B4a*	*total*
Level A0	56 67.5%	12 14.5%	8 9.6%	0 0.0%	3 3.6%	0 0.0%	4 4.8%	83 100.0%
Level A1	22 35.5%	6 9.7%	18 29.0%	0 0.0%	7 11.3%	1 1.6%	8 12.9%	62 100.0%
Level A2	18 16.2%	2 1.8%	32 28.8%	1 0.9%	22 19.8%	2 1.8%	34 30.6%	111 100.0%

* Cells contain number of responses and percent of row total. Five teachers did not respond and three wrote "I don't know". They are omitted from this table.

The results in Table 11 are conservative with regard to associations between A levels and B levels. All four graphs at (A0,B4a) were drawn by SKHS teachers who ignored the presented coordinate system and sketched their graphs below or beside it. We scored their PIP at Level A0 because we could not determine a relationship between their initial point's location and a point that reflected the initial values of v and u (see Figure 7). We suspect that all four teachers would have been scored at Level A2 had they used the provided axes.

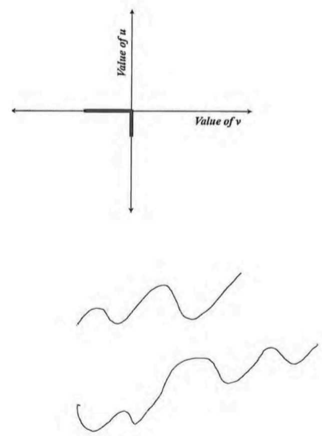


Figure 7. South Korean high school teacher's first attempt and second attempt to sketch a graph. The teacher sketched them in a way that we could not judge the accuracy of the graph's initial point. The second graph was scored at Level A0 and at Level B4a.

Table 12 shows that only 5.7% of SKMS graphs that had a badly misplaced initial point also had an accurate or semi-accurate shape, while 42.4% of graphs that had a well-placed initial point had an accurate or semi-accurate shape. Table 13 shows that only 4.2% of USHS graphs that had a badly misplaced initial point also had an accurate or semi-accurate shape while 60% of graphs that had a well-placed initial point also had an accurate or semi-accurate graph.

Table 12. Placement of Initial Point vs. Shape of Sketched Graph (SKMS)*

	Level B0	*Level B1*	*Level B2*	*Level B3b*	*Level B3a*	*Level B4b*	*Level B4a*	*total*
Level A0	28	2	3	0	2	0	0	35
	80.0%	5.7%	8.6%	0.0%	5.7%	0.0%	0.0%	100.0%
Level A1	12	6	2	1	3	0	6	30
	40.0%	20.0%	6.7%	3.3%	10.0%	0.0%	20.0%	100.0%
Level A2	10	1	8	0	2	1	11	33
	30.3%	3.0%	24.2%	0.0%	6.1%	3.0%	33.3%	100.0%

* Cells contain number of responses and percent of row total. Four teachers did not respond. They are omitted from this table.

Table 13. Placement of Initial Point vs Shape of Sketched Graph (USHS).*

	Level B0	*Level B1*	*Level B2*	*Level B3b*	*Level B3a*	*Level B4b*	*Level B4a*	*total*
Level A0	48 67.6%	15 21.1%	5 7.0%	0 0.0%	3 4.2%	0 0.0%	0 0.0%	71 100.0%
Level A1	9 45.0%	4 20.0%	2 10.0%	0 0.0%	4 20.0%	0 0.0%	1 5.0%	20 100.0%
Level A2	3 12.0%	3 12.0%	4 16.0%	3 12.0%	4 16.0%	2 8.0%	6 24.0%	25 100.0%

* Cells contain number of responses and percent of row total. Five "I don't know" responses excluded from this table.

Table 14 combines all groups into one to investigate the overall relationship between PIP and SSG regardless of teachers' group membership. We excluded IDK and No Responses and formed Levels B3 and B4 as explained earlier. Since both PIP and SSG are ordered, we used the Jonckheere-Terpstra (JT) test for ordered alternatives (Higgins, 2004; Hollander *et al.*, 2014), treating PIP as the independent variable and SSG as the dependent variable. Our alternative hypothesis was that SSG scores increase as PIP scores increase.

Table 14. Placement of Initial Point by Shape of Sketched Graph (US/SK combined). All IDK and NR responses are removed

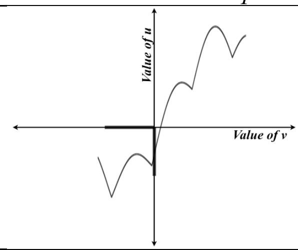
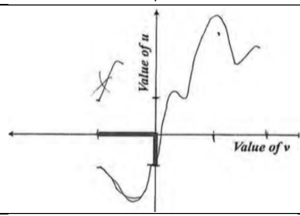
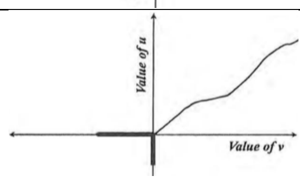
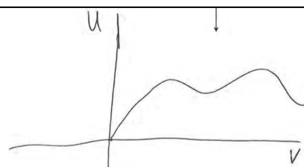
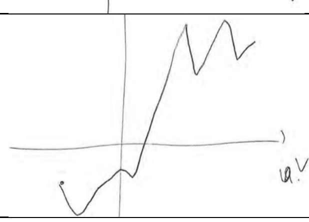
	Level B0	*Level B1*	*Level B2*	*Level B3*	*Level B4*	*total*
Level A0	132 69.8%	29 15.3%	16 8.5%	8 4.2%	4 2.1%	189 100.0%
Level A1	43 38.4%	16 14.3%	22 19.6%	15 13.4%	16 14.3%	112 100.0%
Level A2	31 18.3%	6 3.6%	44 26.0%	32 18.9%	56 33.1%	169 100.0%

* Nine (9) non-responses and 8 "I don't know" responses excluded from this table.

There is a statistically significant relationship between PIP and SSG (JT = 53940.5, $Z = 11.86$, $p < 0.0001$). The effect size is large, with the level of the teachers' initial point accounting for 30% of the variation in the level of shape of teachers' sketched graph, with a 95% confidence interval of (0.23, 0.36). The PS of 0.82 for this comparison shows that 82% of the time we randomly select two teachers with different classifications of their placement of initial point, the teacher with better initial placement will have a more accurately sketched graph. To measure the strength of association between PIP and SSG we calculated Kendall's τ_b for Table 14, getting $\tau_b = 0.47$ with a 95% confidence interval of (0.41, 0.53). This relationship echoes both the R^2 and PS effect sizes. Thus, there is strong evidence that the accuracy of teachers' placement of their initial point influenced how accurately their sketched graphs captured the covariation of quantities' values.

We interviewed four SK teachers about why they sketched the graphs they did (Table 15). Teachers viewed a replay of the animation before explaining their graphs.

Table 15. Teachers' explanations of the graphs they sketched.

Teacher, Degree, (Lev A, Lev B)	*Sketched Graph*	*Explanation*
Accurate Graph		
Teacher 1, Math Ed, (2,2)		The values of v are changing from negative to positive, so I focused on the values of u. I thought of (v,u), then collected points (v,u).
Teacher 2, Other, (0,1)		I thought the initial point is the origin, and (the graph is) increasing and stops increasing. When looking at values of x and y, increasing and not being as big and increasing and not being as big again, and increasing again. I looked at the two bars together. The horizontal bar is steadily increasing even when the vertical one is decreasing, so I found when the graph is slowing down. When I think of a graph's changes, I think of the origin because it cannot be started from any location.
Teacher 3, Math, (0,0)		I started the graph where they have the same amount or distance, so the initial point is zero. At zero [the origin], the values of v and u seem to be the same amount of distance. [while looking at the vertical bar] It is increasing and decreasing and increasing…
Teacher 4, Math Ed, (2,4a)		I started it from the point because it started there. "v" started here and "u" started here, so the initial point is here. When v is moving to the right, u is first decreasing, so I drew it this way. I focused on changes in u's values when following v's values.

Had we the resources to interview all teachers after they sketched their graphs we would have used the framework presented in Thompson and Carlson (2017) to categorize teachers' covariational reasoning as expressed in their explanations. Explanations by Teacher 1 and Teacher 4 clearly fit what Thompson and Carlson call "smooth continuous" covariational reasoning. Explanations of SSG by Teacher 2 and Teacher 3 fit what Thompson and Carlson called "gross coordination of values". They did not create an initial multiplicative object of the values of *v* and *u* and therefore they could not create a correspondence point that persisted as they imagined *v* and *u* vary.

Discussion of Results

Our research questions were:

1. To what extent do secondary school mathematics teachers in our samples reason covariationally about dynamic phenomena that they witness?
2. Are there differences between the teachers in the United States sample and South Korean sample in the prevalence of covariational reasoning?
3. To what extent is creating a multiplicative object of two quantities' attributes necessary to reason covariationally?

Our answer to Question 1 is that, as measured by this study's task, covariational reasoning at a mature level is present among both US and SK samples, but not to an extent that anyone should celebrate. Our answer to Question 2 must be conditioned by country and level. Results on Dimension B (shape of sketched graph) show that prevalence of covariational reasoning among the SKHS sample is significantly higher than in the USHS sample.

Regarding Question 3, it was not surprising to us that creating a multiplicative object of two quantities' attributes plays an essential role in successful covariational reasoning. Saldanha and Thompson (1998) predicted this. We were surprised, however, to find that the issue of creating a multiplicative object of quantities' values would present itself so vividly in the relationship between how accurately teachers plotted the initial points of their graphs and the accuracy of their graphs' shapes.

We now think we understand the relationship in this data between placement of initial point and accuracy of sketched graph. Teachers needed to unite quantities' values to place their initial point, and they needed to maintain that unity so that it persisted as they attempted to track the quantities' values as they varied simultaneously. Some teachers did not unite quantities' values at all, some teachers united quantities' values initially but could not maintain the unity, and some teachers united quantities' values as a multiplicative object that persisted as they tracked quantities' covariation.

When sharing the results for teachers' placement of their graph's initial point, we often hear the comment, "But this is just plotting a point. Surely there is something wrong with your task." We reply by pointing out that the common image of "plot a point" confounds three different scenarios, which provoke three different cognitive acts:

- A person is given the coordinate pair (2,3) and is asked to plot it in Cartesian coordinates.
- A person is given a specific point in a Cartesian coordinate plane and is asked, "What are its coordinates?"
- A person is given two quantities' values and is asked to represent them simultaneously.

The first two scenarios commonly trigger a convention one has learned – how to plot points when given a coordinate pair or how to estimate a point's coordinates within a coordinate system. The third scenario is quite different. The idea of coordinates or a coordinate system is not mentioned. The person must *decide* that placing a point in a coordinate system does what she desires – to represent the values of two quantities simultaneously.

We hasten to add that we do not think that the results of this study imply the need for an increased emphasis in teachers' professional development on plotting points or having an increased emphasis in instruction on having students plot points. Plotting points as a procedure already receives too much emphasis. Rather, we see this study's results as implying a need to increase teachers' capacity to support students' decisions about how to display the results of two measurements so that readers of them can see that they are intended to be taken simultaneously, in relation to one another. However, this emphasis cannot exist in isolation of a larger emphasis. We anticipate that it would require teachers to involve students in the general practice of creating representational infrastructures, such as described by Lehrer and colleagues (Lehrer, 1994; Lehrer & Schauble, 2000; Lehrer, Schauble, Carpenter, & Penner, 2000) and diSessa and colleagues (diSessa, 2004; diSessa, Hammer, & Sherin, 1991; diSessa & Sherin, 2000).

Our final comment addresses an issue we raised outside the research questions. It was whether teachers' difficulties with covariational reasoning are due to epistemological obstacles inherent in this way of reasoning or whether this difficulty is a cultural artifact of an educational system. That all groups had a minority of teachers scoring at A2 and B3/B4 suggests that difficulties are partly due to epistemological obstacles (e.g., creating a multiplicative object of two quantities' values that persists under variation). That SKHS teachers and SKMS teachers sketched semi-accurate or accurate graphs at a significantly higher rate than USHS teachers suggests that covariational reasoning is partially an artifact of an educational system. The outcomes reported here suggest that explicit, sustained attention to students' covariational reasoning is necessary from early grades on.

Limitations & Future Research

Limitations

An obvious limitation of this study is that teachers were not selected randomly and that it is not clear how representative they are of the respective teacher populations in the U.S. and South Korea. We are more confident of the representativeness of the South Korean sample than of the U.S. sample because all secondary math teachers in South Korea are required to take their first-class teacher exam, and we tested approximately 95% of all teachers taking this certificate exam in the summer of 2015.

The format of the task used here could itself be a limitation. Because the horizontal bar's length varies such that the unfixed end moves at a steady pace from left to right, teachers could notice that the value of v (horizontal bar) increases steadily, like experiential time. Once noticed, they can diminish their attention to v and use experiential time in place of the value of v, focusing entirely on the value of u. Thus, we could have gotten false positives on shape as a sign that they covaried u and v. We are piloting various alternatives that address this problem. On the other hand, substituting experiential time for the actual value of v is indeed a primitive form of multiplicative object, just one that allows a strategy which relies less on conscious, explicit attention to both quantities. That said, we feel that this limitation is also a strength of our task and our data, because *even with this avenue for false positives* 44% of Korean high school teachers, 57.8% of Korean middle school teachers, and 67.8% of US high school teachers did not give a response that attained a rubric score of B2 or higher. In other words, given a quite low bar of a B2 score and a strong possibility of false positives, half of Korean teachers and two-thirds of American teachers did not pass that bar. A more complicated task would not have revealed the widespread difficulty that teachers have with *seeing* covariation, and a more complicated task might not have revealed the differences we found between SK and US secondary school teachers.

Another potential limitation in this task is that teachers needed to look away from their paper to watch the animation, and then look away from the animation to sketch their graphs. While this gives greater assurance that teachers' sketches reflected their conceptualization of the covariation of *v* and *u* and assured us that they could not mimic the animation, we might have seen greater percentages of responses at Levels B3 and B4 had they been able to keep the animation within their perceptual field while sketching. On the other hand, Frank (2016a, in press) included the *u*-*v* task and several like it in her teaching experiments with three college precalculus students. Animations and sketching area were both within students' field of vision. The tasks were challenging for these students in the same ways reported here.

Another possible limitation of the task is that the animation itself might be improved to focus subjects' attention on the fact that it shows the variation in quantities' values repeatedly. Some teachers might have thought initially that they were to sketch a graph of the animation, in its entirety, and realized too late that the variation of *v* and *u* repeated itself. However, we saw no evidence of this in earlier trials, but we should nevertheless rule out this possibility by design. Future versions will show the animation once, then the voiceover will state what they just saw will be repeated six times. On the other hand, many teachers did interpret the task as we intended and sketched accurate or semi-accurate graphs. It might be that the rapidity with which a teacher understood the task in its current format is associated with how well the teacher reasoned covariationally.

Finally, this study focused on the nature of teachers' covariational schemes. It did not focus on the availability of those schemes to teachers as they reasoned about situations described in text that are appropriately conceived as involving covariation. Nor did this study focus on the presence of covariational reasoning in teachers' instructional discourse and the extent to which they intend that their students reason covariationally. We are currently investigating both of these connections and we urge others to investigate them as well. We also hope that future research on students' and teachers' covariational reasoning in applied contexts can incorporate this study's methodology to gain greater insight into sources of success and difficulty.

Future Research

The results presented here suggest strongly that one must construct a multiplicative object of quantities' attributes in order to reason about their values covarying smoothly and continuously. However, there is little research that gives insight into the sources of students' and teachers' capacities to create multiplicative objects. Our experience with school and college students is that it is more difficult for advanced high school students and college students to create covariational reasoning than it is for middle-school students. Our work with teachers in professional development settings gives the same experience. This is an empirical issue that needs to be researched.

If it is the case that adults indeed have greater difficulty than younger students in learning to reason covariationally, it might be that, like learning to read, there is a period of time in a person's neurological development (e.g., the onset of myelination in the prefrontal cortex) that students are most open to developing the capacity to create multiplicative objects, and that missing this window of opportunity makes the same development more difficult later in life. This question opens many lines of inquiry. One would be to examine students' development of covariational reasoning in conjunction with techniques from neurological research (De Smedt & Grabner, 2016; Norton & Deater-Deckard, 2014). Another line of research would examine whether greater instructional emphases on smooth, continuous covariational reasoning in earlier grades is productive for students' later abilities to reason covariationally at high levels when investigating and representing dynamic situations.

Yet another line of future research could hinge on using tasks like the one presented here as didactic objects—artifacts that are designed to support reflective conversations about mathematical ideas (Thompson, 2002)—in mathematics instruction. We have used tasks like the one in this study to spark discussions about graphs, quantities, covariation, and modeling with classes of middle school students, high school algebra students, mathematics majors, and mathematics education majors. One particularly powerful use of such an animations in instruction is to support students' conceptualization of graphs as emergent traces made by keeping track of two quantities values as they vary. The aim of the activity, first described in Thompson (2002), is for students to construct a correspondence point and to track its position while quantities' values vary. This activity unfolds as described below. The setting is that the teacher is projecting her computer screen so that the entire class can see it easily.

- Show a diagram like the initial frame of the animation used in this study.
- Show a correspondence point for the two quantities' values and discuss what it represents (the two values simultaneously). Hide the correspondence point.
- Repeat the first two steps with other values of the two quantities.

- Hide the correspondence point, but ask students to pretend it is still there. Change the quantities' values. Ask them to point a finger at the correspondence point's new position. Do this several times.
- Forewarn students that you will run the animation, and ask them to keep a finger pointed at their imagined correspondence point as the animation plays repeatedly.
- Repeat the prior step, but ask students to imagine that they have dipped their finger in a bowl of pixie dust so that, like Tinkerbell, it will leave a trail of dust particles everywhere it has been. Discuss what each particle of pixie dust represents relative to their finger movement and relative to the quantities they tracked.
- Ask students to sketch the graph that they have traced.
- If the quantities' initial values come from a context, ask students to interpret the graph they have traced in terms of the situation from which their graph arose. Guide the conversation so that it is about simultaneous values of the quantities and how they varied together.

We suspect that such uses of dynamic covariation tasks as didactic objects in the classroom have the potential to support students in conceiving relationships between varying quantities and their representations in graphs and formulas.

Finally, we did not investigate the issue of reasoning covariationally about quantities whose values are related by a formula. Frank (2016a, 2017, in press) is investigating students' difficulties in this area with precalculus and calculus students; Ellis and colleagues (Ellis *et al.*, 2016; Ellis *et al.*, 2012, 2015; Fonger, Ellis, & Dogan, 2016) are investigating the same with middle school students. We suspect that thinking covariationally about quantities in dynamic situations is foundational for reasoning covariationally about functions defined by a formula, but is not sufficient. We agree with Thompson and Carlson (2017) that explaining students' progress in this area will draw on theories of quantitative and covariational reasoning and on the APOS constructs of action and process conceptions of function (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & McDonald, 2001; Dubinsky & Wilson, 2013).

Conclusion

We were not interested just in whether teachers could construct accurate graphs of covarying quantities. Rather, we were interested in what teachers' responses would tell us about their covariational schemes as expressed in their graphing activities. We contend that past research has frequently over-interpreted students' and teachers' conceptual operations when they sketch a graph that researchers take as appropriate, or when students and teachers interpret a graph as reflecting a situation holistically. In this study we turned the question around. We asked whether teachers would see that they were being asked to use the conventions of graphing in the Cartesian plane to sketch a graph from quantities whose values vary and which are *presented* as the coordinates of points on the graph.

Our results highlight the importance for conceptualizing graphs emergently of creating a multiplicative object of two quantities' attributes and holding it in mind persistently. Table 5, however, suggests that many teachers did not create a point as a multiplicative object whose coordinates were the values of the animation's two quantities. Forty-two percent (42%) of SK high school teachers, 32% of SK middle school teachers, and 21% of US high school teachers had a well-placed initial point. In addition, 34% of

SK high school teachers, 29% of SK middle school teachers, and 23% of US high school teachers sketched an accurate or semi-accurate graph. This suggests, to us, that conceptualizing and representing the simultaneous variation in two quantities' values is nontrivial and merits greater attention in both research and instruction.

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