Animations: Windows to a Dynamic Mathematics

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Abstract. *Students have difficulty thinking of mathematics dynamically. Animations can be helpful in this regard, but only when the animations are designed to support teachers in holding conceptual conversations about important mathematical ideas.*

It is a longstanding problem that school students conceive mathematical ideas statically. We see this problem vividly in their understandings of variables. For example, students commonly think "*x*" in $3x^2 - 5 = 10 - x^2$ stands for "the answer" when it is more productive to think, "Out of all the values *x* can have, which one(s) makes this statement true?" Or, more precisely, to understand the equation as, "We're given $y_1 = 3x^2 - 5$ and $y_2 = 10 - x^2$. What value(s) of *x* make y_1 have the same value as y_2?" This second way of thinking is behind understanding equations graphically (see Figure 1).

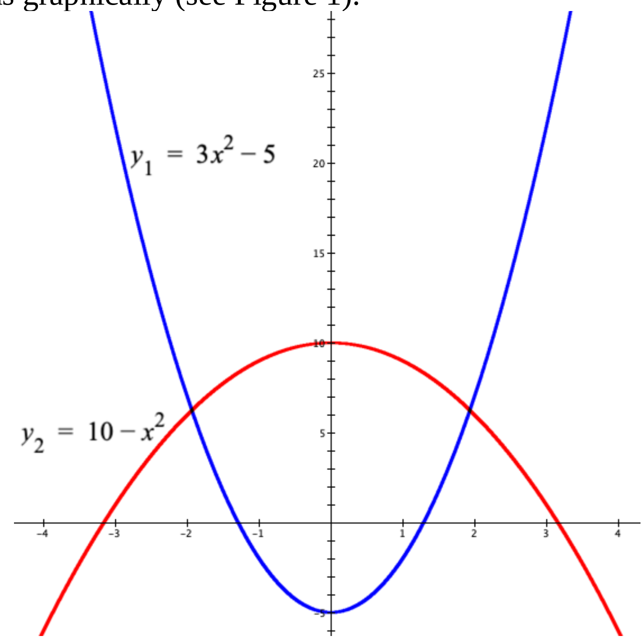

Figure 1. Graphical illustration of $3x^2 - 5 = 10 - x^2$.

As teachers, we see many things in Figure 1 that are not evident to students learning about depicting equations' solutions graphically. In particular,

- The variable *x* in $3x^2 - 5 = 10 - x^2$ *can have values that make the statement false*. That's how we get two graphs. If we limit ourselves only to values of *x* that make the statement true (what students often call "the answer"), our graph would be composed of two points.
- Both graphs are composed of points having coordinates (x, y_1) or (x, y_2). Points of intersection tell us about solutions to $3x^2 - 5 = 10 - x^2$. *But the intersection points are not solutions to the equation.* Values of *x* are on the *x*-axis. Values of y_1 and y_2 are on the *y*-axis. Intersection points have coordinates (x, y_1) and (x, y_2) so that $y_1 = y_2$ for the same value(s) of *x*.

We can help students see the nuances of Figure 1 by helping them see a graph as emerging from the covariation of two variables (Moore & Thompson, 2015; Thompson & Carlson, 2017). This means they see the value of *x* as varying, the value of y_1 as varying with the value of x, and the value of y_2 as varying with the value of x – and see that all can be taken as varying simultaneously.

However, it is difficult for students to see diagrams like Figure 1 depicting dynamic relationships. Instead, they commonly interpret graphs as if they are bent wire, associating their shapes with formulas having particular characteristics (e.g., “ x^2” means bent up). You could use the animations linked here to help students develop ways of seeing dynamic relationships in static diagrams.

I must quickly emphasize that what students understand from animations depends greatly on the conversation their teacher manages around the animations. Students cannot easily decide on what to focus when several things happen at once. The teacher must bring these things to their attention. For example, in the animation linked above, it is incumbent upon the teacher to point out, for example,

- “Notice that values of x are shown with a black bar and not a point. Why do you suppose the animator designed it like this?”
- “The value of x starts to the left of zero. Do you see the value of x getting larger or smaller as it varies toward zero?”
- “Notice the value of y_1 is shown with a bar along the y-axis. Why do you suppose it appears where it does? Did you expect it to appear somewhere else?”
- (After showing values of x and y_1 varying together.) “The animation’s title says, ‘The value of y_1 varies with the value of x.’ What does that mean? In what way does the value of y_1 vary with the value of x?”
- (Before showing the graph of y_1 versus x.) We see the values x and y_1 varying simultaneously. How could we anticipate the graph of y_1 versus x by watching the two varying together? (E.g., look in the plane using your peripheral vision to focus on a location of the correspondence point having values of x and y_1 as coordinates.)
- (After showing the graph of y_1 versus x being generated.) “What do you think is the purpose of those faint lines that meet where the graph appears?”
- (and more about the individual graphs)
- (Before showing the two graphs generated simultaneously.) “Can you imagine both graphs being generated at the same time? What will the display look like while they are being generated? What will the display look like after they’ve been generated?”
- “What will be true when the values of y_1 and y_2 are the same for a value of x?”
- (and many more)

The animations linked to this article and the questions above illustrate three important points.

(1) For animations to be effective, students must attempt to anticipate what they will see before they see it, then explain to themselves and to others what they have seen (Hegarty, Kriz, & Cate, 2003; Schnotz & Rasch, 2005).
(2) The animation must be designed to support reflective classroom conversations (Cobb, Boufi, McClain, & Whitenack, 1997), conversations that take students’ meanings and understandings as objects of discussion, as opposed to steps for getting answers.

(3) Teachers must conceptualize the classroom conversation they wish to have. This includes important points to raise if they do not arise naturally. Moreover, the conversation must be organized around the mathematical ideas teachers wish students to learn (Thompson, 2002).

Animations can be an important aid in your instruction when your goal is to foster productive imagery. Students always learn more powerfully when they have imagery that helps them organize their activity when learning new ideas or methods. However, for animations to provide such support you must think carefully about the mathematical thinking you hope to support.

Lastly, using animations productively in your instruction requires significant time and effort. Devote the energy to incorporate an animation into your instruction only if the mathematical ideas it supports are ones you anticipate students will use repeatedly in their future learning.

References

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