A THEORETICAL FRAMEWORK FOR UNDERSTANDING YOUNG CHILDREN'S CONCEPTS OF WHOLE NUMBER NUMERATION

by

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To the memory of my mother and father

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INTRODUCTION

It has long been regarded as a truism that effective mathematics teaching is most likely to occur when the teacher takes into consideration his or her children's current levels of thinking. Ausubel expressed this truism in terms of teaching using "anchoring ideas" (Ausubel, Novak, & Hanesian, 1968); Brownell expressed it in terms of his "meaning theory" (Brownell, 1935); Russian mathematics educators expressed it in terms of Vygotsky's "zone of proximal development" (El'konin & Davydov, 1966/1976; El'konin, 1966/1976); and Bruner expressed it in his "discovery learning" (Bruner, 1961, 1971). The author also accepts this truism. However, he sees a major difficulty in the approaches taken by those mentioned above: the theories of learning that they either proposed or adhered to lacked the specificity required to apply them to individual children in regard to a specific subject matter. No matter how well the theories might have described learning in general, they remained too far from the daily life of the classroom to assist teachers in dealing with the difficulties of individual children in learning the subject matter of elementary school mathematics. This remark applies to attempts to bring Piaget's theories into the classroom as well.

On the other hand, from recent work in information-processing psychology (e.g., Brown & Burton, 1978; Klahr & Wallace, 1976; Newell & Simon, 1972) have come formalisms powerful enough to give detailed models of individual children's thinking over extremely short periods of time. But here we see the other extreme--the formalisms are too powerful. We lose the subject matter of the elementary school curriculum--the concepts and their organization that children are to form as a result of teaching. Moreover, by the very nature of the computational metaphor through which information-processing psychologists view learning, we see an overemphasis on procedures and strategies for solving types of problems and little emphasis on children's understandings, meanings, and perspectives. There is a natural tendency to analyze children's behavior on limited sets of problems, and then to add the set of hypothesized routines which explain their behavior to the fast-growing pool of routines that, from a psychological viewpoint, children need to learn. To put it another way, whereas with general learning theories there was (and is) a lot of organization, but few details, with information-processing psychology we see many details, but little organization.

Clearly, what is called for is the development of what might be called "middle-level" theory. This would be theory which is broadly applicable to children, yet which is detailed enough to allow useful (to a teacher versed in the theory) diagnosis of individual children with regard to a specific subject-matter topic. Such a theory would be tied to more global anchoring posts (say, to Ausubel's theory of meaningful verbal learning), but would go beyond them in considering the requirements of children learning a specific topic (say, division of whole numbers) while addressing the conceptual context in which a child's learning takes place.

The theory would be composed of separate, yet integrated, sub-theories, such as for addition/subtraction/numeration and multiplication/division of whole numbers; fractions/rational numbers; and integers. The connections among sub-theories would be by way of the interactions among them in explaining both the growth of children's mathematical concepts and children's problem solving. Most assuredly, constructing such a middle-level theory is an adventurous task.

With regard to topics in mathematics education, the theory would not only have to allow descriptions of the growth and compositions of individual children's concepts and understandings, it would have to articulate with descriptions of the growth and composition of related concepts and understandings--preceding, concurrent with, and following it. For example, any theory that purports to explain children's understandings of addition and subtraction should articulate with the explanations it later allows of multiplication and division. Also, if the theory is to be applicable to individual children throughout their mathematical development, then it too has to be developmental.

Purpose of the Study

The aim of this investigation is to go a step in the direction of such a middle-level theory. The "step" is the construction of one frame (Minsky, 1968), or sub-theory.

The subject matter that the sub-theory addresses is whole number numeration, which the author sees as the turning point in children's mathematical careers. It will become apparent in the later chapters that by whole number numeration the author means far more than knowing the place value of the digits in a numeral. Rather, whole number numeration is meant also to subsume addition and subtraction of whole numbers as well as the germs of multiplication, division, and integers.

The anchoring-posts are provided by Piaget's genetic epistemology (Beth & Piaget, 1966; Piaget, 1951/1964, 1952, 1968a,b, 1970, 1976; Inhelder & Piaget, 1969; Piaget & Inhelder, 1973) and the very much overlapping radical constructivism of von Glasersfeld (1974, 1976, 1978a,b). Of course, if one is to "go a step," it must be from somewhere. The step the author is taking is from the largely unpublished work on counting carried out by Steffe and his collaborators (Steffe, Hirstein, & Spikes, 1976; Steffe, Richards, & von Glasersfeld, 1979; Steffe & Thompson, 1979; Steffe, Thompson, & Richards, 1981; von Glasersfeld, 1979, 1980, 1981). Aspects of Piaget's genetic epistemology and von Glasersfeld's radical constructivism that are crucial to the study will be reviewed in Chapter 2, as will the work of Steffe's project.[1]

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[1]The Learning and Teaching of Whole Numbers: An Interdisciplinary Study of an Experimental Model, National Science Foundation grant number NSF78-17365.

Chapter 2

BACKGROUND

This chapter presents discussions of two critical issues, and of theoretical constructs that will be essential to the reader's understanding of the later chapters. The issues are epistemological and methodological: what is the nature of mathematical knowledge and what are ways of describing it in children. The theoretical constructs are Piaget's notions of figural and operational thought, von Glasersfeld's model of the construction of units and number, and the notion of counting-types as developed by Steffe and his collaborators.

Epistemology and Methodology

The author has argued elsewhere (Thompson, in press) that there are, in principle, two dramatically opposed "world views" of research in mathematics education. These are called *environmentalism* and *constructivism*. Environmentalism is the view that mathematical knowledge arises by experience of one's environment--that mathematics *per se* exists independently of any one knower or community of knowers. Research in mathematics education carried out from an environmentalistic view emphasizes strict control of the subject's environment, such as by experimental design, and especially by controlling problem structure. Explanations in an environmentalist tradition emphasize the *effects* of environmental manipulations.

Constructivism is the view that mathematical knowledge arises by one's abstracting the structure of one's activity--possibly in interaction with an environment, but also from mental activity. In constructivism, there is no mathematics *per se*, there is only the mathematics that individuals construct. Research in mathematics education carried out from a constructivist view emphasizes the building of models that, were they substituted for the modeled subject, would reproduce the subject's behavior in the situation being investigated. The first concern, with respect to problem solving, in a constructivist view is to determine the problem that the subject created. A problem is never taken as a given (nor are treatments, etc.), for in the final analysis it is the subject that determines "the" problem that he or she solves.

The author would classify as "environmentalistic" most mathematics education research done from an information-processing paradigm, even though its focus has been on modeling. This is largely because of the stance implicit in that research that problems may be objectively analyzed independently of any particular problem solver--that a "problem" is an entity which exists independently of the solver, but which may be an item in the solver's environment that impinges directly upon her or him. The "objective" analysis of a problem is usually offered as a *task environment*: the set of objects and operators out of which a solution to the problem may be constructed (Newell & Simon, 1972). A *problem space* is the solver's representation of the task environment (Newell & Simon, 1972). The connection between a task environment and a problem space has not been made clear. That is to say, no one has yet proposed a way that a problem solver can gain access to a task environment in order to represent it.

From a constructivist view, and using the same terminology, a task environment is of necessity part of the researcher's model of the problem solver, as is a problem space. Of course, the researcher imputes each to the problem-solver, but only as a way to explain the solver's behavior. The connection between a task environment and a problem space, as imputed to the solver, is then clear: a problem space is the solver's understanding as constructed from his or her task environment.

The constructivist's relocation of task environments has two immediate methodological implications. First, a task environment is idiosyncratic to a problem solver, and hence the researcher can no longer rely on "objective" analyses to tell him or her what it is. The researcher must arrive at it by examining the solver's behavior and postulate it as a precursor to its behavioral manifestations. Second, if the researcher's aim is to be able to generalize, within the content area being investigated, to children as problem solvers, then the researcher must actually examine different children and their (imputed) task environments. The aim of looking across children (task environments) is to create a conceptual system that is flexible enough to capture individual task environments, while at the same time is general enough to be able to make theoretical statements about children's mathematical knowledge and understanding. The author has called such a system a theoretical framework (Thompson, in press). A theoretical framework is not as big as a theory of mathematical knowledge or problem solving, for it is limited to a particular content area. Nor is it as small as a model, for it is not offered as a description of any one child's task environment. It is a component of what was referred to in Chapter 1 as a "middle-level" theory. Figure 2.1 recaps the steps in constructing a theoretical framework.

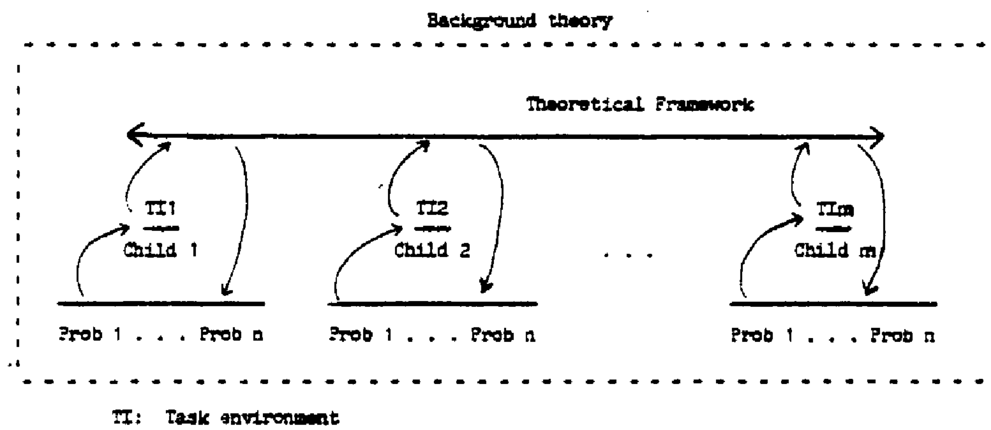

Figure 2.1. Dialectic construction of a theoretical framework.

A theoretical framework must emerge from somewhere. Figure 2.1 shows that, at all times, the researcher operates in the context of a background theory. This includes not only "theory" in the traditional sense, but epistemology and methodology. One always is directed by some form of expectation. The arrows in Figure 2.1 are meant to convey the dialectic and multileveled character of the construction. The researcher examines a child on a particular problem, and while doing that makes general hypotheses about the task environment of the child--which in turn may be examined in light of the child's behavior on other problems, or it may be analyzed in light of how it fits with the framework as it is at that time. Also, the current form of the framework influences the way that the researcher examines children's behavior on problems. The double direction associated with the theoretical framework indicates both the forward and retroactive influence that may be exerted. The task environment that the researcher imputes to a child may cause him or her to reevaluate the framework, and a change in the framework may cause the researcher to view differently behavior that has already been examined. When does one "really" have a theoretical framework? When the third level of analysis reaches a stable state. Then the researcher's aim is to refute it. That is to say, the dialectic moves to a new level. The researcher devises new tasks which may serve better to highlight ambiguities and contradictions within the framework, and tries them on a new set of children. The analysis proceeds as before--the framework must be made to fit the imputed task environments, and the task environments must account for the respective children's behaviors.

To place the author's framework in the above discussion, it has achieved its first stable state. Chapter 6 will discuss the next step.

Theoretical Constructs

Any theoretical framework (or theory, for that matter) that aims to be broadly applicable to children's mathematical knowledge and understanding must have a developmental aspect. There are two reasons for this. First, if the framework is to fit into a fully-developed middle-level theory, then it must encompass children's growth. Second, any time the framework is applied to a specific set of children, they are likely to be heterogeneous in their phase of development. Thus, as a tool for contriving explanations of children's behaviors it must take heterogeneity into account.

The author anchors his framework to Piaget's genetic epistemology. This is for two reasons. First, Piaget's is the only developmental theory which takes into account the special character of mathematical knowledge. Mathematical knowledge at all levels is far more than a storehouse of facts and procedures; it eventually becomes knowledge about the structure of knowledge. Second, Piaget's genetic epistemology provides a way of speaking about a child's construction of mathematical knowledge without making recourse to entities external to the child. Anything that a child comes to know must ultimately be accounted for in terms of what he or she knew before.

In giving a developmental aspect to the framework (Chapter 3), the constructs that have proved to be critical are figural and operational thought, and the allied construct of reflective abstraction. These are in turn used in von Glasersfeld's model of the construction of units and number, and in Steffe et al.'s notion of counting-types.

Figural and Operational Thought

Piaget (Beth & Piaget, 1966, p. 156) made a distinction between two aspects of thought: figural and operational. The figural aspect of thought is directed to perceptions and mental images (figural representations). That is to say, the figural aspect of thought deals with objects and their states. The operational aspect of thought is directed to transformations of objects from one state to another. Though Piaget's 1966 article is one of the few in which he actually defined figural and operational aspects of thought, it pervaded his work. In a later book summarizing his work, Piaget reiterated the distinction again:

> I shall begin by making a distinction between two aspects of thinking that are different, although complementary. One is the figurative aspect, and the other I call the operative aspect. The figurative aspect is an imitation of states taken as momentary and static. In the cognitive area the figurative functions are, above all, perception, imitation, and mental imagery, which is in fact interiorized imitation. The operative aspect of thought deals not with states but with transformations from one state to another. For instance, it includes actions themselves, which transform objects or states, and it also includes the intellectual operations, which essentially are systems of transformation. They are actions that are comparable to other actions but are reversible, that is they can be carried out in both directions (this means that the results of action A can be eliminated by another action B, its inverse, the product of A with B leading to the identity operation, leaving the state unchanged) and are capable of being interiorized; they can be carried out through representation and not through actually being acted out. (Piaget, 1970, p. 14)

To appreciate the full flavor of Piaget's distinction, we must note that he did not put objects (perceptions, etc.) into an ontological sphere independent of the child, but that objects themselves are constructed (Piaget, 1954, 1976).

Piaget's distinction was in large part due to his view of the development of intelligence as being achieved through progressively higher forms of equilibrium. Figural thinking achieves its highest form with what Piaget called intuition. A child's thinking is intuitive when he or she possesses complementary sensori-motor schemes which allow detours in action to achieve some desired goal. The reason that Piaget called this level "intuitive" is that the child cannot reconstruct his actions in the absence of performing them, such as reconstructing a route from home to school (Inhelder & Piaget, 1969).

In the above quotation, Piaget stated that operations are actions that can be "carried out through representation." However, he is also quite adamant in his position that operations are not representations of actions. Rather, they are actions constructed anew at a higher plane of thought. "Indeed, it should be well understood that an operation is not the representation of a transformation; it is, in itself, an object transformation, but one that can be done symbolically, which is by no means the same thing" (Piaget, 1976, pp. 76ff).

How do operations come into being? Again, Piaget was quite specific. It is through reflective abstraction. To understand what he means about reflective abstraction, however, it is worthwhile to examine its counterpart at the figural level--empirical abstraction.[1]

Empirical abstraction is abstraction from objects (again, as constructed from sensori data). It is the separating of the object

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[1]Piaget also at times called this "simple" and "generalizing" abstraction.

or object's composition into similarities and differences--what Piaget also called (schematic) differentiation (Piaget, 1951). An example:

> A child, for instance, can heft objects in his hands and realize that they have different weights--that usually big things weigh more than little ones, but that sometimes little things weigh more than big ones. All this he finds out experientially, and his knowledge is abstracted from the objects themselves. (Piaget, 1970, p. 16)

Reflective abstraction is knowledge abstracted from coordinated actions. The emphasis is on the transformations these actions bring about and that which remains constant when performing them. To continue Piaget's above example, it is through reflective abstraction that the child comes to know that whatever the weight of an object, it remains the same under transformations of elongation (if it is malleable) or other deformation (as long as nothing is added or removed). That is, the child's conservation of weight can only be abstracted as an invariant of his or her *actions* on objects, and not from objects *per se*.

What is the motor of reflective abstraction? Piaget was not completely clear on this, but it appears he saw reflective abstraction being carried out through internalized imitation of actions--thus thinking of doing without really doing. In this way, actions are constructed anew, but, being internalized, they act as transformations between states (figural representations) as opposed to transformations of objects.

As the child establishes systems of operations and coordinates them relationally in terms of inversions, reciprocities, and/or compositions he or she comes ever closer to a stable state within that system--a form of equilibrium. Equilibrium in this sense means that the operations, through their system of relationships, are capable of compensating perturbations of the system. This was Piaget's definition of conservation. The system of relationships is conserved--any state of the system is attainable from any other state (Inhelder & Piaget, 1969, p. 97). The closure of the system, however, is only with respect to the child's physical world. He or she can still encounter problems because of the assimilation of a scheme of operations to higher levels of thought. For example, a child who can conserve quantity may yet experience disequilibrium when attempting to conceive of derived units, such as specific gravity (Lunzer, 1969). This brings up a second way that Piaget used the terms "figural" and "operational" thought--operations at one level of thought are figural with respect to a higher level. Piaget exemplified his position in a discussion of the construction of formal operations:

> The child must not only apply operations to objects--in other words, mentally execute possible actions on them--he must also "reflect" these operations in the absence of the objects which are replaced by pure propositions. This "reflection" is thought raised to the second power. Concrete thinking is the representation of possible action, and formal thinking is the representation of a representation of possible action. (Piaget, 1968a, p. 63)

The importance to this study of figural and operational thought, and of empirical and reflective abstraction, is that of the mobility that operational thinking gives to children. Chapter 3 uses these distinctions in characterizing the growth of children's concepts of numeration, and Chapter 4 uses them in characterizing children's task environments for problems involving whole number numeration.

Units and Number

The notion of unit is critical to Piaget's theory of the child's construction of numbers. In his classical work (Piaget, 1952/1965), he argued that number is possible to the extent that a child can conceive of the elements of a collection as at once equivalent and non-equivalent--as units. In order for the elements of a collection to be equivalent, they must be stripped of all qualitative differences (i.e., to make units of them), but once devoid of qualitative differences they are indistinguishable--hence the need to differentiate them by putting them in some asymmetric order (Piaget, 1952/1965, pp. 189ff.)[2] In a later publication (Beth & Piaget, 1966, pp. 174 ff.) he characterized the emergence of number in children's thinking as the child's progressive synthesis of classification operations of inclusion and seriation operations which produce sequences of transitive asymmetric relationships. In short, Piaget characterized number as a unit of units, while his focus was upon how equivalent units could yet be distinguishable.

von Glasersfeld (1981) pointed out that, while Piaget's description of what a unit must be is good, his negative definition ("an element stripped of all qualities") does little to explain how it is possible for one to actually create units, let alone a number of them; von Glasersfeld approached the problem of specifying the means of

[2]Piaget was not referring to collections as if they existed on a table, which would make his statement nonsensical, but instead as constructions of an active mind.

creating units from the view of Cecato's (1966) conceptual semantics, expanding and modifying its central idea that many abstract concepts can be interpreted as "patterns of attention." "Attention," as used by von Glasersfeld, has a technical meaning that differs significantly from its common usage. As he puts it,

> I want to emphasize that "attention," in this context has a special meaning. Attention is not to be understood as a state that can be extended over longish periods. Instead, I intend a pulselike succession of moments of attention, each one of which may or may not be "focused" on some neural event in the organism. By "focused" I intend no more than that on attentional pulse is made to coincide with some other signal (from the multitude that more or less continuously pervades the organism's nervous system) and thus allows it to be registered. An "unfocused" pulse is one that registers no content. (von Glasersfeld, 1981, p. 85).

von Glasersfeld's notion of a moment of attention can be illustrated by an analogy. The brain is constantly active, and the activity is organized in pulselike rhythms. This activity can be compared, in principle, to the idle loop of a calculator. When a calculator is turned on, but is not used, it is nevertheless active--if it was not, then it could not assimilate the pressing of a key. The activity is called an idle loop; it is constantly checking the contents of its registers and input buffers. If, as it checks the contents of a buffer, it finds nothing (no key has been depressed), it goes on to the next buffer. This is a moment of unfocused attention. If it finds a key has been depressed, then it matches the signal and stores the pattern in an input register, and then continues its idle loop (only, however, if the pattern from the depressed key was not associated with an operation). This is a moment of focused attention.[3] Thus von Glasersfeld's notion of a moment of attention is offered as a building block of cognition.

von Glasersfeld's model of the construction of units and number begins with a characterization of the construction of a <u>sensori-motor item</u>. In his notation, it is characterized as

$$0 \; \overset{I}{\underset{a}{}} \overset{I}{\underset{b}{}} \cdot \cdot \cdot \overset{I}{\underset{n}{}} \; 0 \quad \text{(Sensori-motor item)}$$

where "0" stands for an unfocused moment of attention, "I" for a focused moment, and a, b, . . ., n for sensori-motor signals.

The unfocused moments surrounding focused ones are the experiential "bounds" of the item--that part of experience where the item is not. The heterogeneity of the signals is essential for the item to be considered (by the perceiver) as a "thing" (von Glasersfeld, 1981, p. 88). As an attentional pattern <u>qua</u> pattern, the child empirically abstracts (in the sense of Piaget) the structure of a <u>unitary item</u>--"the boundary of unfocused pulses around a focus on some sequence of sensory signals (n) that could be specified and that is now represented by one focused pulse, because the contained focused pulse is irrelevant for the conception of unity or wholeness" (von Glasersfeld, 1981, p. 89). He characterized this as:

$$0 \, \frac{I}{n} \, 0 \qquad \text{(Unitary item).}$$

When the child can form a succession of unitary items which "share a sensory feature that provides a basis for considering them

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[3]As an item of epistemology, it is interesting to note that were two keys depressed within the period of the loop, only the second would register. From the calculator's perspective, the first key stroke would not have existed.

equivalent in that respect" (p. 89), the child has constructed a plurality. This is denoted as

$$\ldots 0 \underset{a}{\overset{I}{\;}} 0\,0 \underset{a}{\overset{I}{\;}} 0\,0 \underset{a}{\overset{I}{\;}} 0 \ldots \qquad \text{(Plurality).}$$

A plurality seen against an experiential background is thus bounded in experience, and becomes a collection. von Glasersfeld gives this example:

> If you live downtown and a window on your ground-floor apartment looks out on the sidewalk of a busy street, you may see an endless plurality of people passing by. If, then, you consider the [people] you saw pass, say, between breakfast and lunch, you have a plurality of people that is framed between extraneous events that can be seen as part of your experiential background. At that point, the plurality turns into a collection. (von Glasersfeld, 1981, p. 89; emphasis in original)

A collection is denoted by:

$$(\ 0 \underset{a}{\overset{I}{\;}} 0\,0 \underset{a}{\overset{I}{\;}} 0 \ldots 0 \underset{a}{\overset{I}{\;}} 0\ ) \qquad \text{(Collection).}$$

The items in a collection must be sensori-motor--they must have originated in experience. As a result of empirical abstraction, the child may construct the items as figural representations of unitary items. This produces the attentional structure of what von Glasersfeld called a lot. A lot is denoted by:

$$(\ 0 \underset{n}{\overset{I}{\;}} 0\,0 \underset{n}{\overset{I}{\;}} 0 \ldots 0 \underset{n}{\overset{I}{\;}} 0\ ) \qquad \text{(Lot)}$$

where "n" serves the same role as for unitary item.

A unitary item is still not Piaget's "unit," nor is a lot a number. A unitary item must have sensori-motor content, either in the form of a sensory signal or a figural representation, and hence it still has sensory features (e.g., imagining a checker under a cloth). von Glasersfeld operationalizes Piaget's notion of "stripping an element of all qualities" by hypothesizing a re-processing of a unitary item. That is to say, a unitary item becomes the focus of a moment of attention. The result is an abstract unit structure without sensory features. Von Glasersfeld represents this by

$$\begin{array}{ccc} & I & \\ 0 & \overline{\;I\;} & 0 \\ & 0 \quad 0 & \\ & n & \end{array} ,$$

or, for brevity,

$$0 \; (\; 0 \; I \; 0 \;) \; 0 \; . \quad \text{(Abstract, or Arithmetic Unit)}$$

Similarly, a lot, when its items are reconstituted as abstract units, takes the form of an Arithmetic Lot, denoted by

$$(\; 0 \underset{(010)}{\overset{I}{}} 0 \underset{(010)}{\overset{I}{}} 0 \; . \; . \; . \; 0 \underset{(010)}{\overset{I}{}} 0 \;) ,$$

or, for brevity, by

$$(0 \; I \; 0 \; I \; 0 \; . \; . \; . \; I \; 0 \;) \quad \text{(Arithmetic Lot).}$$

An arithmetic lot is still not a number, for its boundaries are experientially derived from the boundaries of the reprocessed collection. This is indicated in von Glasersfeld's notation by the surrounding parentheses. For example, when a child represents a collection of counters under a card and then reflectively abstracts the unit structure of the items, but not the unit structure of the collection (whose boundaries are derived from his perception of the card), the result is an arithmetic lot, but not a number. To construct a (whole) number as an abstract unit of abstract units, an arithmetic lot must itself be the focus of a moment of attention. This is denoted by

```
         I
0 ( 0 I 0 I 0 . . . I 0 ) 0 ,
```

or, for brevity,

```
0 ( 0 I 0 I 0 . . . I 0 ) 0 .
```

A number, then, is entirely a creature of the child's mind. It has no sensory content, but may be given content by its application to sensory material. Moreover, it may be given content by a person's application of the unit structure to conceptual content, as when, say, thinking of a function as a unit--which is quite necessary when speaking of sets of functions (or, for that matter, sets of sets). An abstract unit is a unit that can be given meaning if the occasion arises where it is necessary to do so, and a number is a unit of abstract units. A summary of von Glasersfeld's model is given in Figure 2.2.

The importance of von Glasersfeld's model for this investigation is that it offers a way to characterize the quality of children's quantitative thinking. Phrases such as "arithmetical operations" and "numerical operations" abound in the literature on children's arithmetic, yet it is never made clear what these operations operate on. von Glasersfeld's model gives us an opportunity to be quite specific in that regard.

Sensory-Motor Item	$0 \underset{a}{\vert} \underset{b}{\vert} \cdots \underset{n}{\vert} 0$		Sequence of attentional pulses focused on sensory signals, bounded by unfocused pulses
(minimal representation)	$0 \underset{a}{\vert} 0$		Sensory-motor item characterized by specific sensory signal
Plurality		$\cdots 0 \underset{a}{\vert} 0\ 0 \underset{a}{\vert} 0\ 0 \underset{a}{\vert} 0 \cdots$	Open-ended sequence of sensory-motor items having a common sensory characteristic
Collection		$(0 \underset{a}{\vert} 0\ 0 \underset{a}{\vert} 0 \cdots 0 \underset{a}{\vert} 0)$	Experientially bounded plurality
1st (empirical) abstraction Unitary Item	$0 \underset{n}{\vert} 0$		Abstracted "figurative" pattern of sensory-motor item, still dependent on sensory material
Lot		$(0 \underset{n}{\vert} 0\ 0 \underset{n}{\vert} 0 \cdots 0 \underset{n}{\vert} 0)$	Abstracted "figurative" pattern of collection, still dependent on sensory material
Arithmetic Lot		$(0 \vert 0 \vert 0 \cdots \vert 0)$	Sequence of unitary items reduced to iterative attentional pattern
2nd (reflective) abstraction Arithmetic Unit	$0 \underset{(0 \vert 0)}{\vert} 0$		Attentional pattern of unitary item reprocessed to constitute focal item of unitary pattern
Number		$0 \underset{(0 \vert 0 \vert 0 \cdots \vert 0)}{\vert} 0$	Attentional pattern of arithmetic lot reprocessed to constitute focal item of unitary pattern

Figure 2.2. Schematic summary of steps in the construction of the abstract concepts of unit and number. (From von Glasersfeld, 1981.)

Counting Types

Von Glasersfeld developed his model while working in Steffe's project for research on young children's counting. The aim of the project at the time was to find a way to capture the qualities of children's counting that made some children much more flexible than others in its application. The result of that investigation was what is called "counting types" (Steffe, Thompson, & Richards, 1981; Steffe, Richards, von Glasersfeld, & Cobb, in preparation).

To define counting types, it is essential to make clear Steffe et al.'s view of counting. It is that when a child counts, he or she is counting something. Since children often count when there is apparently nothing in their physical environment toward which it might be directed, it is then problematic as to what it is that they are counting. Percy Bridgman (1959) addressed this very question-- "What is the thing that we count?"

> It is obviously not like the objects of common experience--the thing we count was not there before we counted it, but we create it as we go along. It is the acts of creation that we count. (p. 103)

The "acts of creation" that Bridgman spoke of eventually became operationalized in von Glasersfeld's model of units and number. The focus of Steffe et al.'s investigation, however, was how children manifested them and the implications of limitations in a child's "creativity."

The definition of counting that Steffe et al. arrived at was: the production of a sequence of number-names where each number-name is coordinated with the creation of a unit item. (The reader should note that, by von Glasersfeld's model, even perceptual unit items are created.) Children's advancement in counting is characterized as (1) the progressive differentiation (empirical abstraction) of the items involved in the acts of counting, and (2) the reflective abstraction of the unit structure of an act of counting. These will be elaborated further.

When a child coordinates a number-name with a perceptual unit item (performs an act of counting) there are several components of the act. There is the formation of the perceptual item, the formation of the verbal number-name, and the formation of the motor act which serves to coordinate the item with the number-name. From the beginning counter's perspective these form an undifferentiated, experiential whole, and all components must be present for him or her to count successfully. The aspect of the whole of which he is most aware is the perceptual item--he makes a unit-item of it by bounding it within the confines of the time of the counting act and the space of its location. This is called <u>counting with perceptual unit-items</u>.

As the child differentiates his scheme for counting and empirically abstracts the perceptual unit-item from it, he can <u>substitute</u> a figural representation of an item in its place. He again makes a unit-item of it by experientially bounding the formation of the representation within the confines of the time of the counting act and the location that he attributes to the represented item. This is called <u>counting with figural unit-items</u>.

The next level of differentiation is the elimination of the perceptual component altogether, so that only the verbal and motor component are required to implement a counting act. The motor component is made into a unit-item by its experiential boundaries of beginning and ending. This is called counting with motoric unit items. It is worth noting that, prior to the child's achievement of this level, the motor term was "there," but it is not until she makes a unit-item of it that she is explicitly aware of it as an item to be paired with a number-name.

The final level achieved through empirical abstraction is for the child to take the number-name itself as a unit-item. The boundaries of the item come from the motoric aspect of saying it or, if unspoken, the mental act of forming its sound image (sound images will be discussed more fully in Chapter 3). This is called counting with verbal unit-items.

The unit-items in counting with perceptual, figural, motoric, and verbal unit-items are each sensori-motor unit-items. The progression in abstractness comes by way of empirical abstraction, and is of this order because of the relatively increasing diffuseness of the sensory signals of which a unit-item is made. None is entirely abstract, however, because of the child's need for sensory features or figural representations to form them. The next level can only be acquired by reflective abstraction--the child makes an abstract unit-item of a counted item and pairs it with a number-name. This is called counting with abstract unit-items.

Behavioral features of the counting types are given in Figure 2.3. It should be noted that the indicated behaviors are not intended as univocal correspondents of the counting types, but as one way that they may be manifested.

Type	Manifestation
Counting with:	
Perceptual unit-items	Pointing to the objects in a collection in coordination with producing "one, two, ..."
Figural unit-items	Pointing to specific locations (as if objects were there) in coordination with producing "one, two, ..."
Motoric unit-items	Sequentially putting up fingers in coordination with producing "one, two, ..."
Verbal unit-items	Producing "one, two, ..."
Abstract unit-items	Counting with any type of unit-item while producing a coordinated count (e.g., "five (is one), six (is two), ..."

Figure 2.3. Counting types.

It is worth mentioning why the author offered the particular example of putting up fingers as a manifestation of counting with motoric unit-items. First, if the child counts in this way she cannot be counting the elements of a perceptual collection, for the collection quite literally does not exist until after the child stops counting. Second, the child cannot be counting figural representations of fingers, for otherwise there would be no need to put them up. Finally, the salient feature of the child's activity seems to be the motoric phenomenon of putting up a finger. This conclusion is also confirmed by the fact that children who do count in this way need not look at their fingers till <u>after</u> they have finished counting, and apparently do so only to see their record of having counted.

It is also worth mentioning that concluding that a child can count with verbal unit-items is a highly inferential task. In large part such inferences are made on the basis of what the child appears to make of his counting episode <u>after</u> he has completed it. If it appears that he has made a lot from his counts, then he counted with verbal unit-items. If it appears that the episode had no quantitative meaning for him, then he merely produced a sequence of number-names. The reader is referred to Steffe, Richards, von Glasersfeld, & Cobb (in preparation) for a more complete discussion of counting with verbal unit-items.

In any particular problem-solving situation where a child counts, she may count with a variety of unit-items. In fact, one can easily imagine situations where a research mathematician would be forced to count with perceptual unit-items (say, by asking her to count the letters on this page). Steffe et al. use counting types, however, in more than a descriptive sense. They use them to characterize children as types of counters. A <u>counter with perceptual unit-items</u> is a child who cannot count without perceptual unit-items. A <u>counter with motoric unit-items</u> is a child who cannot count with verbal or abstract unit-items, but can count with perceptual, figural, and motoric unit-items. And so on. In short, a counter of a particular type is a child who can count with unit-items up to and including that one, but not with more abstract unit-items.

The importance of counting-types lies in their usefulness to explain both potentialities and limitations in children's counting. For example, a counter with perceptual unit-items <u>requires</u> collections to count, so there are certain methods of solving addition and subtraction problems that are beyond his capability to spontaneously employ (e.g., counting-on and counting-back).

The close correspondence between counting types and the development of arithmetical structures also allows hypotheses that relate children's use of counting in solving arithmetical problems to their understandings of the problems. The constructs developed by Steffe et al. that account for connections between methods and understandings are called <u>operations involving counting</u>.

A discussion of operations involving counting will not be given here. This is because the author left the Steffe group at the time when the operations were in their initial formulation. The author's current formulation of their operations is an outgrowth of their existence at the time of his departure (late 1979), and since that time much work has gone on at Georgia in formulating and reformulating them. Rather than attempt to give an up-to-date account of the Steffe project's version of operations involving counting, the author will give his formulation with the understanding that it may differ in some respects to accounts given in future publications from the Steffe project. The discussion of operations involving counting (which will be called numerical operations) is given in Chapter 3 as part of the theoretical framework.

Chapter 3

THEORETICAL FRAMEWORK

In this chapter a theoretical framework for explaining children's knowledge and understandings of whole number numeration is presented. When the framework is applied with the aim of explaining a child's behaviors in the post-teaching experiment interviews (Chapter 4), the result is a model of his or her knowledge and understanding of numeration.

Three lines of development are taken in the presentation. The first is to discuss the framework from a general perspective. This discussion focusses upon children as problem solvers. The second line of development is a discussion of domains of knowledge that underlie a child's understanding of numeration. The third is a discussion of how the framework can be used to characterize children's concepts of numeration.

General Perspective

An important feature of the theoretical framework is that children are viewed as problem solvers. A great deal of their activity in interview sessions may be characterized as problem solving. A child may find a problem when attempting to understand what the interviewer is asking of him; one might arise when performing a well known procedure (say, through overassimilation); or when attempting to get the interviewer "off his back."

When one analyzes a child's problem-solving behavior, it proves useful to compartmentalize it into episodes of routine and nonroutine activity. "Routine" is intended to mean that the activity is automatic for the child; there is no need for the activity to be elaborated step-by-step. A routine (as a noun) is a procedure that may be called upon and executed largely in one step.[1] Analogous to "routine" are scheme (Piaget, 1951), compiled program (Baylor & Gascon, 1974), and uninterpreted ("assimilated") program (Newell & Simon, 1972). Nonroutine (heuristic) activity normally occurs when the means to an end is not readily apparent to the child, or when an end itself is not clear.

In many investigations of problem solving, the investigator first analyzes the task in order to ascertain its logical structure and the knowledge required to construct various solutions. In the language of Newell and Simon (1972), the investigator first delineates the task environment of the problem--the network of "objects" and "operators" from which a solution may be constructed. This approach has been applied in the areas of logic theorem proving (Newell & Shaw, 1957; Newell, Shaw, & Simon, 1957; Newell & Simon, 1972), medical diagnosis (Elstein, Kagan, & Shulman, 1972), and geometry theorem proving (Greeno, 1976a), to name but a few. This

[1]This is true, of course, only from the point of view of the calling program. From the point of view of the "hardware" that will execute the routine it may have many steps.

approach becomes less frequent in studies of children's problem solving, and the reason has not been made clear. In the author's opinion, a task environment can be delineated only in the case where the investigator can himself construct understandings of the problem, expressible in his formalism, which would generate behavior more or less isomorphic in its organization to the subject's. From these understandings, he may then move to a structural description of a framework that describes the understandings that he attributes to his subjects. Without first generating understandings the investigator is limited to descriptions of procedures that children use to solve the problems under study. Researchers who wish to analyze children's performance on interview tasks from a problem-solving point of view must first admit that the prime difficulty is to create an understanding of the problem as it might appear from the child's perspective.

The methodology of constructing understandings of children's problems and an encompassing framework to explain them is not as clear cut as depicted above. One does not construct understandings, and then encompass them within a framework. The process is a dialectic where understandings and framework exert a reciprocal influence upon each other. As one refines the understandings one has of children's problems, the appearance of the framework changes; as one refines the framework, the imputed understandings may be viewed in a new perspective. When one comes to analyze one's understanding of children's problems for presentation, the dialectic has come to a point where it no longer is apparent--either in the presentation of the framework or in the analysis of the understandings. What appears is the result of the cumulation of refinements up to the point of presentation.

Another confounding aspect of a presentation, for one wondering how a particular framework came about, is that it probably reverses the order of construction. Presented first are the pieces of the framework, and then comes the organization of the pieces into a whole. In actuality, the process of construction is never so coherent--most likely some sort of gestalt precedes any form of analysis.

With the above in mind, the reader should view the following characterization of routines and heuristics that children use in interview tasks as the end product of a dialectic between the author's attempts at understanding individual children's behaviors on a specified set of tasks and the construction of a framework that is intended to encompass each of them.

Routines

A basic tenet of this framework is that children construct their routines. For a particular child, what is now a routine was once an elaborated sequence of steps, each demanding that its successor be constructed. Constructing routines, however, is not held to be a process of accretion--adding one step at a time. Accretion, for the phenomena under study at least, provides only a weak and partial explanation. For instance, as children learn to recite number-names in conventional order, the initial routine for generating them is likely constructed by memorizing successors:

"one, two, buckle my shoe; etc." Memorization of this form, however, can only carry one so far.

At some point the child must abstract a criterion for the succession of steps. To continue the above example, a child might know the names "one" through "nineteen" in conventional order, and, when asked, continue the sequence with "tenteen, eleventeen, ..." A child producing this sequence would clearly have abstracted a rule for continuing past "twelve": continue counting from "three" and say "teen" after the word (with slight modifications in "three" and "five"). The set of successor relationships "two follows one," . . ., "twelve follows eleven" in combination with the rule "say next number-name and then 'teen'" constitutes a routine, albeit one that we would wish the child to modify eventually. Another abstraction that we would expect a child to make is that one continues after "nineteen" as follows: "twenty" (and then "one" through "nine"), "thirty" (and then "one" through "nine"), and so on. More will be said about routines for generating sequences of number-names in a later section ("Domains of knowledge").

Another sort of routine is one that allows a child to readily construct an understanding of an arithmetical situation or statement. For instance, one child reading "5 + 7 = __" may understand the sentence by way of a scheme of actions, e.g., one makes a pile named "five," makes a pile named "seven," and counts them all and reports the last number-name (Steffe, Thompson, & Richards, 1981). Another child may understand "5 + 7 = __" by way of a scheme of operations, e.g., prior to initiating a solution procedure he constructs a number (as yet unnamed) and equates it with another which comprises numbers named "five" and "seven."

A scheme of actions, by its very nature, is limited in its scope of application, whereas a scheme of operations is much more flexible. A child limited to understanding sentences through the scheme of actions mentioned above would likely produce the same behavior when solving "5 + __ = 9" as when solving "5 + 9 = __" (Steffe, Thompson, & Richards, 1981). A child that understands addition sentences through a scheme of operations would likely distinguish between the two. For him a sum necessarily comprises two numbers, so the problem is to supply the name missing from among the sum and the two addends which compose it. From the scheme which gives him an understanding of addition sentences comes a subproblem--finding, or constructing, an appropriate routine that will supply the missing name.

A scheme of actions will be called an <u>empirical</u> routine. It is abstracted from successive states that result from the performance of actions, and the actions serve merely to connect the states.[2] The empirical routine in the above example has as its states three piles, one of which is derived from the other two--the "addends" and the "sum." The actions connecting the states are counting with perceptual unit items.

A scheme of operations will be called an <u>operational</u> routine. The term "operation" is intended largely as Piaget (1951/1964,

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[2]The term "empirical" is not intended to mean that the objects acted upon are real-world structures. Rather, it means merely that the child acts upon a structure that, from his perspective, constitutes a bonafide object, such as a word or a mental image.

1952/1965, 1970) uses it. It is an interiorized action—an action that can be carried out in thought.[3] The operational routine in the above example has as operations what will later be called integration and separation. A number (sum) is separated (in thought) into two numbers, and they are in return integrated (in thought) into one which is semantically equivalent to the sum. More will be said about the operations of integration and separation in a later section ("Domains of knowledge").

So far it has been said that routines are abstractions from constructions and that they may be either empirical or operational. The question of how a process of abstraction may result in qualitatively different routine arises naturally. This is a critical question, and the explanation will be used in many contexts within the remainder of this chapter. Thus, a short digression seems warranted.

A common sense notion of abstraction, and probably the most widely held, is that one extracts commonalities from objects or situations. This notion of abstraction seems to have been the one held in mind by Newell and Simon when they modified their General

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[3]Piaget actually requires much more. An action must not only be interiorized, it must have an inverse operation and must exist within a system of related operations. The criterion of reversibility will not be used in this study, for it does not seem necessary that an operation and its inverse become interiorized at the same time. Nor does it seem necessary that an operation have an inverse. To turn a phrase by Piaget, I can smoke my pipe "in my mind," but I cannot imagine unsmoking it. The question of the prior existence of a system of operations poses a theoretical problem. Is there a "first" operation that becomes interiorized?

Problem Solver (GPS) so that it operated within an "abstract planning space." For instance, GPS, in solving logic problems, abstracted connectives, signs, and the order of letters from propositional statements. So $A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C)$ became, upon abstraction, $A(BC) \rightarrow (AB)(AC)$, while $A \supset B \leftrightarrow \sim A \vee B$ became $AB \leftrightarrow AB$ (Newell & Simon, 1972, pp. 428-435). Abstracting inessential differences and retaining commonalities allowed GPS to construct plans of action in the course of proving logic theorems.

Children use abstraction of this form to construct what could be called plans of action, but it would be more accurate to call them "patterns" of action. For instance, many children who cannot count by one hundred from a given number-name can continue the sequence once they have been supplied the first few terms. A plausible explanation of the way they are able to continue is provided by a mechanism similar to GPS's. Given the terms "thirty-eight, one hundred thirty-eight, two hundred thirty-eight," all they need to do is abstract the similarities ("thirty-eight") and the differences (nil, "one," "two"), and apply their routine for producing sequences by one to account for the differences between terms.

Abstraction as a process of weeding out similarities and differences accounts for the construction of empirical routines. It cannot, however, account for the construction of operational routines. The difficulty may be seen in Dienes' (1961) attempt to explain the construction of mathematical concepts by way of the common sense notion of abstraction. He offers the following as an example.

> Forming the concept of two is an abstraction process, as it consists mainly of experiences of pairs of objects of the greatest possible diversity . . . . The essential common property of all such pairs of objects is the natural number two. From all pairs of objects encountered (elements) we form the attribute (class) of two. (Dienes, 1961, p. 282)

Dienes clearly intended that the "thing" abstracted from the pairs (the property of "two-ness") is part of the objects of experience. If that is so, then how does it become internal to the experiencer? This is an epistemological difficulty, and one that pervades any attempt to explain the abstraction of mathematical concepts and schemes of operations in terms of properties of objects that are external to the abstractor.

Piaget overcame the difficulty seen in Dienes' characterization by distinguishing between two types of abstraction--what he calls "simple" (empirical) and "reflective" abstraction (Piaget, 1970, p. 69). Simple abstraction is abstraction from objects ("big things are heavier than small things"). This is the common sense version of abstraction. Reflective abstraction is abstraction from actions. More precisely, reflective abstraction is the separation of an operation from the states ("material") upon which it acts. Separating the operation of making a unit item from the sensory material out of which the item is composed (von Glasersfeld, 1981) is one example. A more mundane example is the abstraction of the operations of bisecting an angle with ruler and compass so that they may be applied to a straight line (which perceptually is quite dissimilar to an angle, for it has no vertex or sides)--resulting in the construction of a pair of perpendicular lines. Metaphorically speaking, one engages in reflective abstraction when asking oneself, "What is the character of what I did that gave me what I got?"

Empirical routines are constructed through empirical abstraction; operational routines are constructed through reflective abstraction. Empirical routines are always closely bound to situations similar to those from which they are abstracted, and a child has great difficulty applying them flexibly. Operational routines are, by the very nature in which children construct them, flexibly applied--the operations have been "removed" from a specific context.

The point of the above paragraph may be put another way. An empirical routine may be implemented only when its input requirements are currently present in the child's experience. They are data driven. An operational routine may be activated in the absence of any specific input conditions, though specific inputs may later be created to fully implement it. In other words, an operational routine may be an item in a plan, and is placed there because its product may be useful.

There are limits to the applicability of operational routines. A child, in order to apply one, must "insert" it into a context, and the context may overpower the operations. The following example, though beyond the level of elementary school mathematics, illustrates the effect of context.

Suppose a student has an operational understanding of simultaneous equations in n variables (both in terms of the meaning of a solution vector and in terms of a method of solution), and he is faced with the problem of constructing a polynomial function

$f(x) = a_n x^n + \ldots + a_1 x + a_0$ whose graph passes through the points $(x_1, y_2), \ldots, (x_n, y_n)$. Normally, one does not think of simultaneous equations in the context of polynomial functions--simultaneous equations involve linear functions (and, to many people, a particular figurative arrangement), not polynomial functions. However, if the student reconsiders the polynomial as a linear function of its coefficients $a_n, \ldots, a_0$, then he obtains the following set of n equations in n unknowns.[4]

$$\begin{aligned}
x_1^{n-1} a_{n-1} + \ldots + x_1 a_1 + a_0 &= y_1 \\
x_2^{n-1} a_{n-1} + \ldots + x_2 a_1 + a_0 &= y_2 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
x_n^{n-1} a_{n-1} + \ldots + x_n a_1 + a_0 &= y_n
\end{aligned}$$

Once the student changes the context, he may understand the problem through his routine for solving simultaneous equations. The original context, however, was not conducive for the student to understand the problem as one involving a family of linear equations, let alone to apply his scheme of operations for solving them simultaneously.

Problem Solving

Problem solving, from the perspective of the theoretical framework being presented, will be viewed in general as goal-directed cognitive activity. One might object that, by this definition, there

[4]Recall that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ have been given particular values.

is little cognitive activity that is not problem solving. However, as Greeno (1980) has argued, when one closely examines a person's behaviors on even the most routine (to the solver) tasks one finds all the ingredients of problem solving--goals, subgoals, heuristical reasoning, planning, and so on. The major consequence of viewing problem solving in such a broad manner is that it allows an investigator to compare a child's behavior in situations where he is "really" solving a problem to his behaviors in related situations where he appears to be handling the task more routinely. The differences between such sets of behaviors may then be contrasted with their commonalities in an attempt to characterize the quality and nature of the child's understanding.

For example, one child was given a written arithmetical statement (10 + 3 = __) and was asked to "find ten plus three." She immediately wrote "13," and justified her answer by saying that the "3" covered up the "0" in addition. When, later, asked to "find ten plus three," she counted on three from ten using her fingers. By viewing both sets of behaviors as manifestations of problem solving, we may infer that she did not see the two as the same problem, and hence that her understanding of the additive property of numeration is closely tied to specific empirical routines.

The remainder of this section will describe three major aspects of problem solving as they pertain to this framework under discussion: abstracting, planning, and heuristic reasoning. These compose the part of the framework which is used to characterize what Newell and Simon (1972) call unprogrammed activity--the operations and actions performed depend largely on the nature of the specific problem being solved. Routines compose the part used to characterize programmed activity. For example, when a child routinely counts by ten in the context of an addition problem, his production of the sequence of number-names is automatic, whereas the formation of the goal which calls for his counting by ten may have come about by way of his understanding of the specific problem he is solving.

Abstraction. Reflective and empirical abstraction lead to qualitatively different behaviors. Suppose that a child is asked to use the sentence (1) "9 - 5 = 4" to complete the sentence (2) "5 + __ = 9," and that "5 + 4 = 9" is not a fact for him. The child could use either empirical or reflective abstraction to complete it. If the child were to use empirical abstraction, he might see a chain of reasoning something like: "There's a '9,' a '5,' and a '4' in (1) and a '5' and a '9' in (2). So '4' is missing from (2). The answer is '4.'" The child disregarded the minus and equal signs and looked only for a way to account for the difference between the two collections of numerals. This sort of abstraction has nothing whatsoever to do with addition and subtraction.

Reflective abstraction would support a type of reasoning not allowed by empirical abstraction--the abstraction of the operations: "Let's see . . . nine take away five leaves four . . . and five and some more makes nine. Well, if nine take away five leaves four, then nine is four and five more . . . the answer is four." By focussing on his operation of taking away, a child can realize that what is taken away from a number to get a difference can also be put back with the difference to regain the number. This is reflective abstraction in the alternative sense of Piaget (1970). The child reflects the situation as figurally constructed to a scheme of operations.

Planning. It can easily be argued that, at one level at least, planning and understanding are two sides of the same coin. If understanding is taken as the assimilation of a situation or set of statements to a scheme of routines, then planning can be taken as an elaboration of an understanding with the aim of achieving some goal. As children's understandings may exist at various levels of operationality, so may their plans. Not every elaboration of an understanding will be considered as a plan, however. A distinction must be made between a sequence of steps where the sequence has been constructed prior to its execution and where the sequence results from hill climbing--choosing the next operation or action only after the execution of its predecessor (Simon & Newell, 1972). Only the former will be called a plan. Of course, plans may be modified during their implementation, which is a mark of a flexible problem solver.

While the notion of a plan is fairly unambiguous, it is not always easy to infer from specific behavior whether it is a manifestation of planning or hill climbing. As Greeno (1980) notes, planning may occur in quite different ways. Two that have received the greatest attention are means-end analysis (Newell & Simon, 1972) and top-down analysis (Sacerdoti, 1977). In means-end analysis the problem solver identifies differences between the current and goal states and selects a sequence of operations or actions to eliminate the differences. In top-down analysis, the problem solver <u>understands</u> the problem by assimilating it to a network of global operations or actions, each one of which comprises sets of specific operations or actions that have specific preconditions for its implementation and which produces a specific type of result. Planning in top-down analysis becomes a problem of arranging the global operations or actions in proper sequence. It might be said that means-end analysis occurs when one doesn't know what to do, and that top-down analysis occurs when one "has an idea" of what to do.[5] Hill climbing can be characterized as a primitive form of means-end analysis. Behaviors emanating from the two can only be distinguished by the rapidity with which successive steps of the solution are carried out and by fortuitous (from the point of view of the investigator) remarks given by the problem solver as he carries out the steps. Distinguishing between a plan arising from top-down analysis and from means-end analysis can be difficult, for the two may occur in the same solution episode. The distinction comes largely from the quality of the solution procedure (immediacy, apparent understanding, etc.).

Questions of plans and planning will arise mostly in Chapter 4 when interpreting children's behaviors during their interviews. The nature of plans is here made explicit so that they may be utilized

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[5]Means-end analysis is more empirical in nature than top-down analysis--the focus is on states and differences between or among them. Top-down analysis is more operational in nature--the focus is on the operations to be performed, while the states are implied by the preconditions for their implementations.

there when interpreting the qualities of understandings held by the children.

Heuristic Reasoning. Traditionally, heuristic reasoning has been thought of as reasoning that a problem solver employs to "chop down a problem space to manageable size" (Newell & Simon, 1972, p. 883). This view of heuristic reasoning has been employed largely in situations where the problem solver is operating in a fairly well defined context (from his or her point of view), in that the problem being solved is clear to the solver and (from the investigator's point of view) is the one intended by the person posing it. Looking back, working backward, and means-end analysis, for example, are each normally classified as heuristics. The situation often arises with a child, however, where the problem he or she has in mind is nothing like what the investigator intended, and the child's overall goal becomes to say or do something that will earn an "okay" from the problem-poser--even if the "something" said or done makes little sense from the child's perspective. It is this author's experience that many of children's behaviors in such situations may fruitfully be examined from a perspective of heuristic reasoning, and in this study the term will be used in this broader application.

A commonly employed heuristic may be characterized as "change the answer a little bit," as in:

Int: What is eight plus four?
Child: Eight--nine, ten, eleven, twelve, thirteen. Thirteen.
Int: Thirteen? Are you sure?
Child: Fourteen!

Another is "change the conditions of the problem so that they share a desired characteristic of the goal." Two examples of this are given below. The first is of a child who is attempting to give a number-name to a collection of multi-base blocks; the second is of a child trying to add one hundred to twenty.

Int: (Portion of dialogue omitted.) How many was over here?
Child 1: Twenty-two.
Int: Right. And how many is that (places hand on a flat)?
Child 1: One hundred.
Int: And so altogether?
Child 1: One hundred and . . . twenty-two.
Int: (Uncovers 2 unit cubes; covers them again.) What did we have?
Child 1: One hundred and twenty-two.
Int: (Uncovers the 2 unit cubes again.)
Child 1: Two (looking at the 2 unit cubes) . . . hundred and twenty-two.
Int: How many is that--just there (pointing to the 2 unit cubes)?
Child 1: Two . . . and this is two hundred. Two (pointing to the 2 unit cubes) hundred (pointing to the flat) . . . two hundred.

Child 1's behaviors in this example can be made understandable by thinking of him as operating by a heuristic: if you want to name a number in a collection that is made up of subcollections, then concatenate the subcollection names in a way that fits your grammar for number-names. His goal was to give a single name; the conditions were that he had several collections and several names. His application of the heuristic made one name of the several.

Int: (Places card with "20" written on it onto the table.) What number is one hundred more than this number?
Child 2: (Pause.) Two hundred.
Int: How did you get that?
Child 2: I worked it out.
Int: How did you work it out?
Child 2: (Pause.) I was thinking it in my head. And I said two hundred.

If we assume that Child 2 inferred that adding one hundred would produce a name involving "hundred," then we may explain his behavior as an attempt to change "20" (or "twenty") into a "hundred" name. Mentally tacking a "0" at the end of "20" does just that--hence "two hundred."

A final example of what will be taken as heuristic reasoning illustrates the term's scope of application in the theoretical framework. In this example, a child solved a subtraction problem using an empirical routine that she had abstracted during schoolwork.

Int: (Places card with "91 - 29 = __" onto the table.)
Child 3: Ninety-one take away twen . . . twenty-nine. (Pause.) Seventy-eight.
Int: How did you get seventy-eight?
Child 3: I counted . . . . I took one away from nine, and that left eight. Two take away nine is seven. (She goes on, at the interviewer's insistence, to use multi-base blocks to work the problem.)
Child 3: Sixty-two.
Int: Do you think what you were doing before--taking the one away from the nine and the two away from the nine--is a good way to do it? Does it always work?
Child 3: Sometimes.

Child 3's answer "sometimes" is the key to concluding that her subtracting routine was applied heuristically. It indicates that she was aware that it didn't always work, but that it worked often enough (drew enough "okays") to be useful in situations where she was expected to perform.

Domains of Knowledge

A working hypothesis of the framework is that children's understandings of whole number numeration can be characterized by examining their knowledge in qualitatively different areas, or domains, and specifying the relationships within and among the domains. The domains used in the framework are language, subitizing, numerical operations, and reading and writing numerals.

Language

The focus of the framework in this domain is twofold: individual number-names and arithmetic words, and the meanings children give to them; and children's productions of sequences of number-names. The first is upon the representational aspect of number-names, the second upon the procedural aspect of putting them in sequences.

The term "representation" is offered here in a wholly psychological sense--it does not refer to marks on a paper, nor to words in the normal sense of spoken or written words. Rather, it refers to words as "sound-images" (de Saussure, 1915/1977) and to non-linguistic representations of actions or operations. A representation is a psychological entity that can be used by a child in place of some set of coordinated actions or operations. When a child reflects, say, on what he has done after counting ten beyond 32, he must somehow re-present the episode to himself. The psychological substitutes of his manifested actions and operations constitute his <u>representation</u> of the event as it took place.

The above reference to sound-images needs to be elaborated. First, any system of signifiers and signifieds must, of necessity, reside entirely within the child who associates them. "Three" as a signifier within the child's linguistic system cannot refer to anything outside his cognition--it must refer to a meaning or meanings <u>within</u> the child. De Saussure (1915/1977) made this completely clear in his discussion of the objects of linguistic analysis--the collection of mental images of auditory impressions of spoken words ("sound-images") together with their conceptual associates. Second, it is assumed that a child's system of number-names (as opposed to written numerals) come to be his primary representational system in whole number numeration. Much attention will be given to the structure and organization of this system. Number-names will be characterized as having structure, being decomposed and concatenated, operated upon, and so on--and this can make sense only if they have first been characterized as mental objects.

Representations can be carried out at any one of three levels. The lowest level is representation of an action or scheme of actions by a part of it--such as representing a type of dance by a prominent feature of dancing it, as in representing the square dance by bringing to mind the exchange of hands in a "do-si-do." Mathematical examples would be a child representing to himself the actions of counting by rhythmically tapping his finger on a table, or taking a number-name as an index of counting because number-names occur as part of counting. An index of an action has the quality that it is linked to its referent by a direct inference, in the sense of a sensory part-whole relationship. However, for the child there is no signifier and signified--the two are undifferentiated.

The next level is representation through signs. The perception of a curved arrow ("↱") offered by a highway department signifies (to the experienced driver) that a bend in the road lies ahead. A curved arrow has nothing to do with one's experiences with roads as such, yet it suggests a property of the road. Similarly, the image of a blank in an open sentence has nothing to do with one's experiences in carrying out arithmetical operations, yet it suggests (to the "experienced" sentence reader) that something is missing. Signs are inferentially linked to their referents, but the inference is much less direct than is the case with indices. They are figurally similar to their referents (Piaget, 1968a). A word may also exist as a sign if the child who uses it does so as a univocal substitute for its meaning--that word <u>has</u> that meaning, and no other pairing is possible.

The highest level of representation comes through the use of symbols. A symbol is linked to its referent only by way of association. Symbols have the qualities of arbitrariness and, in the case of symbols which serve a communicatory function, conventionality (Hockett, 1960; von Glasersfeld, 1977). What was a sign for a child can later be a symbol. For example, the blank in a missing addend as a sign of a missing addend can later be a symbol for a child when he or she realizes that a "Δ" or an "x" could just as well be in its place without changing the meaning of the sentence.

It should be noted by the reader that a "symbol" is a symbol only insofar as its user assigns it a referent. An adult's perception of a squiggle on a piece of paper may be a mathematical symbol (having a mathematical referent), but for a child it may have an altogether different referent or none at all. In the latter case it is not a symbol for the child, it is merely a squiggle on a piece of paper. Similarly, when two adults converse, each can normally assume with a more or less fair degree of certainty that the other's referent of a word is conceptually similar to his own. When an adult and a child converse, however, such an assumption is unwarranted.

Though the children in this study primarily used words to communicate with the interviewers, there were infrequent indications of possible signs (one child brought his hands together in a sweeping motion while explaining how he mentally combined two numbers of tens), but any such inferences were situation specific. Indications of personal signs that are involved in a child's representation of aspects of numeration will be pointed out within the context of the case studies.

The role of number-names in children's understandings of numeration is, as noted previously, assumed to be that of a primary representational system. Perceptions of materials, such as a Dienes flat, and of numerals are assumed to be mapped into (associated with) items in the child's system of number-names, and hence are of secondary importance with respect to representation. In the present study it seemed apparent that, in almost any context, it was the number-name that children constructed from their perception of a numeral which was critical for further processing, as opposed to the perception of a numeral as such. In most situations where a child misread a numeral, he or she operated according to the "incorrect" number-name given to it instead of the numeral itself.

There are two aspects of number-names as they pertain to numeration. The first is the procedural aspect of constructing number-names and sequences of number-names, and the relationship among procedures; the second is the semiotic function served by individual number-names and sequences of number-names. Though related, in that a procedure can have an intrinsic semantics, these two aspects will be treated separately.

Number-name sequences. The discussion of number-name sequences will be given in the form of a hypothetical system capable of producing them. The system as presented illustrates the ways in which correct performance is accounted for within the framework, and not as a definitive statement of how every child produces such sequences. When the framework is applied to individual children, the system may be modified accordingly to account for their behavior.

The system addresses three fundamental questions: (1) How can "unending" sequences of number-names be produced? (2) How can individual number-names be appropriately constructed? and (3) How can judgments of relative order be made between pairs of number-names?

Several types of number-name sequences are relevant to the above three questions. These are forward and backward sequences produced in increments of one, ten, one hundred, etc. Only increments of one hundred or less will be discussed.

The parameters of the system for producing number-name sequences are the starting point of the sequence (for forward sequences, the starting value is assumed to be "one" unless otherwise specified; no value is assumed for backward ones), the increment (default is one), and the direction (default is "forward"). Other default values may be set, which would be appropriate if the subsystem were more sensitive to general context--such as, "We've been counting backward by tens all day, so anytime I'm asked to count it will probably be backward and by ten."

One essential feature of the system is that the rules called upon to produce the next number-name in a sequence are a function of the counting-goal currently active in working memory. That is, the goal includes as attributes the increment and direction of the sequence. Another feature of the system is that the "increment" is really not an increment at all, but a rule system for associating number-names or derivations of number-names by the relation NEXT. The power of the rule system comes from its ability to focus upon a part of a number-name--ultimately allowing it to manifest the behavior of skip-counting, or producing sequences with increments other than one.

Figure 3.1 gives a subsystem which produces forward number-name sequences with increments of one. There are several features that deserve attention, for they incorporate significant psychological assumptions about the nature of a counter's knowledge of number-names and sequences of number-names.

First, the NEXT relation between the twelve pairs of words ("one," "two"), . . ., ("twelve," "thirteen") is explicit. That is, the sequence of words from "one " to "thirteen" is memorized. After

Sequencing by One [a] [b]

```
NEXT of (ONE) is (TWO) . . . NEXT of (TWELVE) is ((THIR)TEEN)

NEXT of ((WORD)TEEN) is ((HOM1(NEXT of HOM2(WORD))TEEN)
NEXT of ((NINE)TEEN) is ((TWEN)TY)

NEXT of ((WORD)TY) is ((WORD)TY)(ONE)
NEXT of ((WORD1)TY)(WORD2) is ((WORD1)TY)(NEXT of (WORD2))
NEXT of ((WORD)TY)(NINE) is (HOM1(NEXT of HOM2(WORD))TY)

NEXT of ((NINE)TY)(NINE) is ((ONE)HUNDRED)
NEXT of ((WORD)HUNDRED) is ((WORD)HUNDRED)(ONE)
NEXT of ((WORD1)HUNDRED)(WORD2) is ((WORD1)HUNDRED)(NEXT of (WORD2))
NEXT of ((WORD1)HUNDRED)(((NINE)TY)(NINE)) is ((NEXT of (WORD1))HUNDRED)

HOM1(TWO) is (TWEN)    HOM1(THREE) is (THIR)    HOM1(FIVE) is (FIF)[c]
HOM2(TWEN) is (TWO)    HOM2(THIR) is (THREE)    HOM2(FIF) is (FIVE)[c]
```

Figure 3.1. Routine for sequencing by one.

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[a]Rules for manipulating goals (such as marking a goal as being satisfied or deferred) and rules sensitive to "start" and "stop" conditions are omitted as technicalities. See Klahr & Wallace (1976) for examples of goal manipulation and "stop" rules. What is presented is the part of the rule system that actually produces sequences.

[b]The NEXT relationships operate by the convention that if more than one left hand side is satisfied, the one most closely matching the current condition is selected for action. See Forgy (1979) or McDermott & Forgy (1978) for a rationale for this convention.

[c]The HOM transformations operate by the convention that if an argument is not one in its explicit definition, then it does nothing to it. So HOM1(SIX) is (SIX). This convention is found in most production systems (see Forgy, 1979 or Newell, 1973).

thirteen, the successor to a number-name is rule-determined, with some rules more specific than others--such as the rules for transcending decades. Second, the rules for sequencing in the teens, and for sequencing from one decade to another, indirectly rely on the NEXT relation on the words "one" through "nine" by way of a homonymic translation (HOM1 and HOM2). The "fif" of "fifty-nine" sounds like "five," and since "six" follows "five," the next number-name after "fifty-nine" is "sixty." Without the special case rule for transcending decades the subsystem would produce sequences like "nineteen, twenty, twenty-one, . . ., twenty-nine, twenty-ten." Third, when a hundred-name is active in memory, the "hundreds" part of it is held constant, and the subsystem calls upon itself to increment the remaining part. Last, the subsystem structures number-names between twelve and one hundred so that they are formed by adding a suffix to one of "two," . . ., "nine," or a homonym thereof, and, if not a decade, associating it with one of "one," . . ., "nine." Thus "seventy-two" is held by the subsystem as ((SEVEN)TY)(TWO). That is, number-names in the subsystem are built up as structured chunks.

The reason for choosing this particular structure for a number-name in the subsystem is that it allows for explanations of detailed operations upon parts of it. If number-names were held by the subsystem as, say, (FIFTYTWO), it would be difficult, if not impossible, to develop general rules for constructing its successor. If the subsystem imposed less structure upon the number-names, the NEXT relation would have to be held more explicitly in memory. Rote memorization is the most explicit form that the relation could take--as it presently does in the subsystem for the number-names "one" through "thirteen."

Additional structure for constructing sequences of number-names comes from organizing "one" through "thirteen" into chunks. McLean and Gregg (1969) found that adults, when memorizing randomly arranged lists of the letters of the alphabet, organized the lists into sublists, or chunks, of varying length and built up complex structures of chunks of chunks. It seems that chunking allowed the subjects to keep the number of items of information held actively in memory at a manageable level. One possible organization of "one" through "thirteen" is shown in Figure 3.2.

```
    ( C1               C2                C3                 C4 )
      |                |                 |                  |
(((ONE)(TWO)(THREE)) ((FOUR)(FIVE)(SIX)) ((SEVEN)(EIGHT)(NINE)) ((TEN)(ELEVEN)(TWELVE)))
```

Figure 3.2. Hypothetical structure for memorized number-names.

Although the subsystem for producing sequences of number-names, as presented so far, does not make explicit use of chunks C1 through C4, it could be modified to do so--say, by establishing the relation NEXT both among elements of a chunk and among chunks. Thus C2 immediately follows C1, C3 immediately follows C2, and C4 immediately follows C3; "two" immediately follows "one" in C1, and so on. Such a structure might seem to be a luxury, since the subsystem shows no need of it to produce forward number-name sequences. The necessity of some sort of additional structure becomes apparent, however, when it is modified to incorporate the ability to count backward and to make judgments of the relative order of nonadjacent number-names.

With the structure given in Figure 3.2, it can decide which of "nine" and "four" comes after the other by appealing to chunk membership. "Nine" comes after "four" because C4 comes after C3. Within chunks, a determination may be made without reference to membership--"three" comes after "one" because one comes after the other in C1.

The rules for sequencing within decades in Figure 3.1 give additional structure to sequences of number-names--we have the teens, twenties, thirties, and so on. Since sequences in the teens and "ty's" rely on the sequence "one" through "nine," they inherit the chunk structure of "one" through "nine"--resulting in a structure on top of a structure (Figure 3.3).

$$(((\mathrm{WORD}_1)(\mathrm{TY}))\ (\mathrm{WORD}_2))$$

$$((\ C_1\ C_2\ C_3\ )\ C_4\ )$$

Figure 3.3. Structure of number-names.

The power of the organization shown in Figure 3.3 becomes more apparent when describing how it might be possible to construct sequences of number-names in reversed order--i.e., count backward. A common ploy when constructing a reversed sequence, and when one cannot remember the term immediately preceding the current one (x_j), is to "drop back" to a point somewhere in the sequence prior to x_j, and then construct the forward sequence. When the construction "$x_{j-1} \rightarrow x_j$" is made, then x_{j-1} is taken as the next term in the reversed sequence (McLean & Gregg, 1969; Fuson, 1979). The reader may experience this ploy by attempting to recite the alphabet backward.

An immediate question is, "How can one start in the 'middle' of a sequence?" In the case of the alphabet, one does not normally have it written; nor does it seem possible to have the entire sequence of letters in mind. So a "scanning to the left" operation seems unfeasible. However, if sequences are indeed organized into chunks, then one may "drop back" by either starting with the first term in the chunk currently being processed or with the first term of the immediately preceding chunk. Figure 3.4 shows one way that the alphabet might be organized in memory; Figure 3.5 shows how the number-name sequence might be organized, and the way it is organized in the subsystem for sequencing.[6]

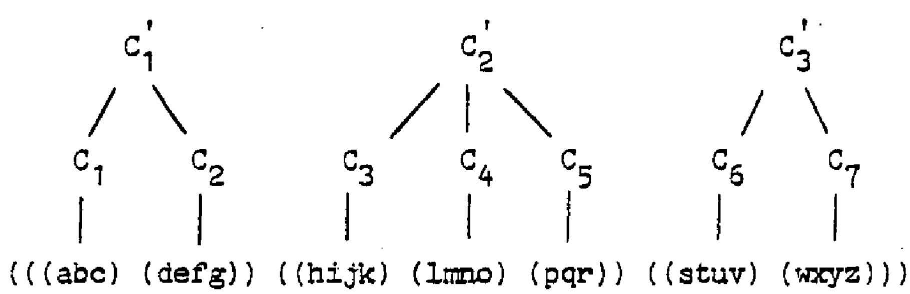

Figure 3.4. Hypothetical structure for the alphabet.

[6]Chunk structures can be highly idiosyncratic, depending on the individual's current cognitive structures and past experience. The chunk structure offered in Figure 3.4 is completely arbitrary, and is given only because the subsystem needs some structure. The point is that an individual holds some structure, and the subsystem could be modified to give a description of that individual.

Figure 3.4 depicts a multi-level organization for the letters of the alphabet. To produce the alphabet starting with "a," one exhausts C1, then moves to C2. When C2 is exhausted, one moves to C2', and hence C3, then C4, then C5. In producing the alphabet backward, if one forgets that, say "r" precedes "s," then the move will be from "s" to C6 to C3' to C2' to C3 to C4 to C5, and then to "p"--resulting in the behavior "s . . . s . . . p, q, r, s--Oh! r."

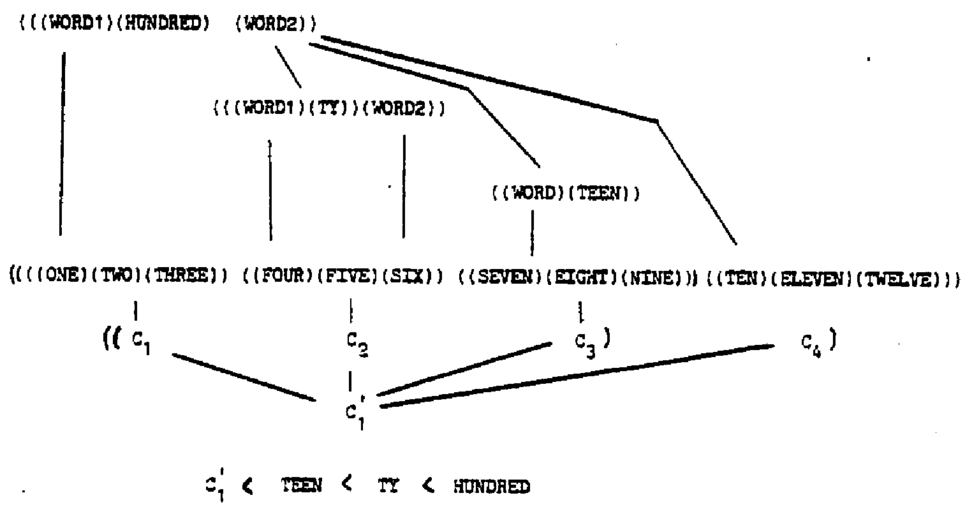

Figure 3.5. Structure of number-name system.

Figure 3.5 depicts a multi-level organization for the production of number-name sequences. The organization shown for number-name sequences differs significantly in character from that for the alphabet. Since words, or homonyms of words, that appear in the lowest level of the organization also appear in higher levels, the organizational structure of "one" through "nine" reappears in "twenty" through "ninety" and "one hundred" through "nine hundred." It must be stressed, however, that the subsystem employs its explicit structure on "one" through "nine" only when operating consistently within a level of the organization. Thus "two hundred" precedes "five hundred" because "two" preceded "five"; similarly, "one hundred ten" precedes "one hundred seventy" because "ten" precedes "seventy." But to compare "five hundred" with "seventy," an appeal must be made to the hierarchy--"ty" precedes "hundred."

It is worth noting that Figures 3.4 and 3.5 do <u>not</u> depict the structure of their respective sequences as entities having an ontological status independent of any knower. Rather, they depict the organizational structure of <u>producing</u> their respective sequences.

The next aspect of producing number-name sequences is "skip-counting" by ten and one hundred. Two modifications must be made to the subsystem. First, it must be sensitive to new goals, namely goals for sequencing in increments of ten and one hundred. Second, there must be rules which give successive number-names in the context of these goals. Figure 3.6 shows the augmentation to the subsystem that will give it the capability of incrementing by ten--a similar augmentation could be made for incrementing by one hundred. As it now stands, SKIP-TEN will count "nine hundred ninety-two, ten hundred two, . . ., ten hundred ninety-two, eleven hundred two," and so on. It lacks a rule for crossing into the thousands.

The rules in SKIP-TEN make up a fairly straightforward subsystem. When a "teen" word is active, ST4.1 produces a "twenty" word. Similarly, when, say, "thirty-seven" is active, ST5.2 produces "forty-seven." ST5.2, 6.3, and 6.4 are special case rules for crossing centuries. When "three hundred ninety-four" is active, ST6.4 produces "four hundred four."

SKIP-TEN [a]

ST1.1 NEXT-TEN of (ONE) is (ELEVEN)
ST1.2 NEXT-TEN of (TWO) is (TWELVE)

ST2.1 NEXT-TEN of (WORD) is (HOM1(WORD)TEEN)

ST3.1 NEXT-TEN of (TEN) is ((TWEN)TY)

ST4.1 NEXT-TEN of ((WORD)TEEN) is ((TWEN)TY)HOM2(WORD)

ST5.1 NEXT-TEN of ((WORD1)TY) is (HOM1(NEXT of HOM2(WORD1))TY)
ST5.2 NEXT-TEN of ((WORD1)TY)(WORD2) is (NEXT-TEN of ((WORD1)TY))(WORD2)
ST5.3 NEXT-TEN of ((NINE)TY) is ((ONE)HUNDRED)

ST6.1 NEXT-TEN of ((WORD)HUNDRED) is ((WORD)HUNDRED)(TEN)
ST6.2 NEXT-TEN of ((WORD1)HUNDRED)(WORD2) is ((WORD1)HUNDRED)(NEXT-TEN of (WORD2))
ST6.3 NEXT-TEN of ((WORD1)HUNDRED)((NINE)TY) is (NEXT of (WORD1)HUNDRED)
ST6.4 NEXT-TEN of ((WORD1)HUNDRED)((NINE)TY)(WORD2)) is (NEXT of (WORD1)HUNDRED)(WORD2)

Figure 3.6. Routine for sequencing by ten.

---

[a]The remarks given in the footnotes to Figure 3.1 apply here as well.

The basis of SKIP-TEN's ability to manifest incrementing by tens is its reliance on incrementing by ones, which appears in ST5.2 and ST6.3. Another modest sophistication is its reliance upon itself in ST5.2 and ST6.2. In effect, ST6.2 holds the hundred word constant while incrementing the remainder.

An example might be helpful for the reader to understand how this rule system actually produces sequences in increments of ten. Suppose the currently held goal is simply to sequence from "three." ST2.1 would produce ((THIR)TEEN). ST4.1 would then produce ((TWEN)TY)(THREE). ST5.2 would then produce ((THIR)TY)(THREE) and would continue operating upon its output till it produced ((NINE)TY)(THREE), each time calling upon ST5.1 to produce the "ty" part of the next name. At ((NINE)TY)(THREE), ST5.2 would call upon ST5.3 to produce ((ONE)HUNDRED)(THREE). After that, ST6.2 would take over and start the process anew. The sequence that SKIP-TEN would produce is "three, thirteen, twenty-three, . . ., ninety-three, one hundred three, one hundred thirteen, . . ., one hundred ninety-three, two hundred three," and so on.

The form SKIP-TEN takes here is certainly not the only possible one. The general rules ST5.2 and ST6.2 could be made less general by introducing explicit relationships, such as "NEXT-TEN of ((FIF)TY)(WORD) is ((SIX)TY)(WORD)." Modifications such as these would likely be necessary when describing a child who is in the process of forming the abstraction embodied in ST5.2.

The last aspect of producing sequences of number-names is the production of sequences within which the increment changes--counting by hundreds, tens, and ones. Here we face a problem that has been largely finessed so far--the problem of goal formation. Up to this point the focus has been upon building a system that produces the appropriate sequencing behavior once a goal has been formed. Now the question of why the system is behaving must be addressed--we must pay attention to its cognitive environment. Given that the system is capable of incrementing by ones, by tens, and by hundreds, the present task is to augment it so that, given a context, it can decide whether to increment by ones, tens, or hundreds. That is, there must be processes for forming goals.

Two ways that a sequencing goal may come into being are, first, as a result of a request for performance (a teacher saying "count by tens starting from eight") and, second, as a result of an "operator call" from another system. Behavior resulting from the former could easily be carried out without meaning (much as young children recite the Pledge of Allegiance). Performance resulting from the latter is necessarily meaningful, for an operator call implies that, from the point of view of the calling system, sequencing is a means to an end. The meaning that sequencing may take for a child, however, is a function of the nature of the systems that call upon it, which in turn depend upon the meaning the child gives to individual number-names. The discussion of the semantics of number-names will make this connection clearer.

It should be pointed out once more that the sequencing subsystems presented here are not offered as strict models of the way every child produces number-name sequences. Rather, they are given as one way that correct sequences may be produced. What is hypothesized is that every child who sequences number-names does so by some linguistic rule system, that it may be implemented rotely (executed with no significance), and that generalized rules for sequencing, and relationships among rules, are abstracted from sequencing in the context of its applications.

When the framework is applied to specific children, an attempt will be made to determine the level of operationality of their rule systems for generating sequences of number-names. This will be done by examining their ability to generate forward and backward sequences with various starting points and various increments (both homogeneous and mixed), and their ability to seriate collections of numerals. The former is aimed at characterizing the generality of the rule systems, while the latter is aimed at characterizing the relationships the child has established among rules for producing number-names (see Figure 3.5, page 58).

The reason for examining the level of operationality of a child's linguistic system for constructing number-names and sequences of number-names can be put quite simply. When a child reaches the level of operationality in his number-name system, he quite literally has a symbol system with which he can operate largely in place of numbers. The system then becomes <u>arbitrary</u>, in that any other linguistic system that exemplifies the same structure may be used in its place. Prior to operationality, this is not the case. Also, when concepts of numeration that are tied to the <u>structure</u> of the child's linguistic system (e.g., concepts of ten, one hundred, etcetera;

place value), the child then has a base system of whole number numeration.

The use of a numeral seriation task to assess the operationality of a child's rule systems for generating sequences of number-names requires that a mechanism for reading numerals be proposed. The hypothetical system under discussion has one, but it relies on the child's subitizing system. The discussion of reading (and writing) numerals will thus be postponed until after subitizing has been discussed.

Number-names. There are two types of referents, and hence of meanings, for number-names within the hypothetical system. The first is a collection, lot, or number, as von Glasersfeld (1981) uses the terms. They may be perceptual, figural, motoric, verbal, or abstract, depending upon the immediate situation and the child's capabilities. For the moment, the focus will be on the manner in which a child may construct a referent for number-names beyond "five" or "six" (e.g., "eighty-nine") without counting or intending a count.

"Subitizable" number-names, i.e., number-names that the child readily associates with a characteristic pattern of a specified numerosity, ("numerosity" from an observer's point of view), may be given substance by establishing such a connection--bringing the pattern to mind. A method for doing likewise for number-names that are not so readily associated with characteristic patterns is that the child generates an "uncharacteristic" pattern (say, a bounded, but intentionally open-ended, sequence of abstract unit-items)--one that is, from his or her point of view, indeterminate. From an observer's point of view the pattern certainly would have a definite numerosity, but it is not one that is identifiable by the child (as would be, say, a spontaneously constructed "dot-dot-dot"). From the child's point of view, there is merely a record of a construction of a lot, or number, which could have had any numerosity--including the one that would correspond to the number-name had a detailed construction been carried out. A collection, lot, or number referred to by a number-name will be called an <u>extensive</u> meaning of the number-name.

The second type of referent is an action-based or operational one. A number-name can refer to the scheme of actions or operations which would result in the construction of a sequence of labeled unit items that terminates with the number-name. The reference may be direct, in the sense that number-names occur in the context of sequencing, so the number-name is an "index" of a sequence, or it may be virtual, where the scheme itself is somehow labeled. The former may be inferred when a child seems to manifest a need to actually put out, one at a time, say, seven marbles to talk specifically about seven marbles (see Steffe, Thompson, & Richards, 1981). The latter may be inferred when a child gives some indication that he knows that, say, twenty-five objects could be counted, but there is no need to do so since he would merely verify that there are twenty-five by ending his sequence with "twenty-five." When a number-name is taken as referring to a scheme of actions or operations, it will be said to have (at that time) an <u>intensive</u> meaning.

The connection between number-names that is established by arithmetic words, such as "take away," "plus," etcetera, is characterized in the framework as coming about through reference to operations which operate on the referents of the number-names. The nature of the extensive or intensive meaning that a child gives to, say, "plus" may depend upon the immediate situation and is limited by the type of counter that the child is. More will be said about limitations in the discussion of numerical operations.

The last aspect of the hypothetical system to be discussed in this section is the construction of number-names *per se*. As previously noted (page 54), number-names come to have a structure, e.g., (((FIF)TY)(TWO)) for "fifty-two." One reason is that such an "encoding" seems necessary, theoretically, for detailed operations to take place upon parts of the number-name. Another is that children often write, say, "seventy-two" in numerals as "702"--"70" followed by a "2" (Ginsburg, 1977), and, moreover, consistently say number-names within a century by connecting the "hundred" part of the name with the remainder by "and"--five hundred AND one. Assuming that the conceptual basis for "and" is something like conjoining, it follows that concatenation is a syntactic operation on parts of a number-name. It must be said, however, that concatenating (putting "forty" and "seven" together to make "forty-seven") need not have a quantitative significance. It is, strictly speaking, an operation on indices, signs, or symbols. As concatenating becomes reversible, however, the child can use it to supplant quantitative operations on a special set of numbers--namely, centuries, decades, and units. Concatenating "eighty" and "two" can come to take on the significance of conceptually combining eighty and two. Likewise, "eighty-two" can come to take on the significance of the sum of eighty and two. The elaboration of these significations provides the basis for what will be called the additive property of numeration: that the sum of *x*ty and *y* (*x*,*y* digit-names) is named by "*x*ty-*y*," and the sum of *u* hundred and *v* (*u* a digit-name, *v* a name preceding "one hundred") is named by "*u* hundred *v*."

Subitizing

Psychologists have long been aware of the phenomenon of subitizing--what was originally viewed as the mind's "immediate apprehension of number" (Beckwith & Restle, 1966), or the spontaneous attribution of a number-name to a number of objects (Kaufman et al., 1949). Only recently has it been made clear that subitizing can be viewed as occurring in two ways, only one of which involves quantity. Klahr and Wallace (1976) presented an information-processing model of subitizing which characterizes it as a matching of "symbol structures"--essentially, attributes of a perception and relationships among attributes, one of which is held internally and associated with a number-name, the other of which is a percept. Von Glasersfeld (1979, 1980) presented a model that characterizes subitizing at two levels: the establishment of semantic links between number-names and recurrent figural patterns, and the establishment of semantic links between abstract attentional patterns constructed from, but not dependent upon, recurrent figural patterns.

von Glasersfeld's and Klahr and Wallace's models are quite similar in their description of subitizing as naming figural patterns qua patterns, but differ in the way they see children generalizing subitizing to number. Klahr and Wallace characterize generalization as an elimination of redundant or superfluous properties of the named symbol structure, while von Glasersfeld characterizes generalization as two independent processes. The first is merely the establishment of alternative semantic links (as when the word "dog" becomes linked to various figural representations of specific breeds). The second is the independent process of reflectively abstracting the attentional structure of constructing the figural image (see Chapter 2).

von Glasersfeld's model will be used within the present framework. Its division into figural and abstract components will allow for explanations of what otherwise might appear as bizarre behavior on the part of children. For instance, it is not uncommon for children to read numerals, name Dienes blocks and unifix cubes, and otherwise exhibit correct performance in providing number-names, yet show little ability to do anything which we would call "numerical." Similarly, von Glasersfeld's model also provides a way of explaining why some children can appear to be capable of numerical operations only when small numbers are involved--they may indeed by thinking numerically, but only with respect to abstract patterns that they associate with the number-names occurring in the problem. When they cannot associate a pattern to go with a number-name they have no basis for proceeding.

The definition of subitizing given by von Glasersfeld will hence be adopted for use within the framework under discussion: subitizing is the spontaneous attribution of a number-name to a figural or abstract attentional pattern. Thus, naming a single digit, attributing a number-name to a perceived object or configuration of objects, and creating a figural representation of an object or configuration to go with a number-name will each be considered as subitizing. The first two examples differ in kind from the third, in that the child is faced with an item to be named. In the third example, he has a number-name in mind and creates a figural associate.

The importance of subitizing within the framework is fourfold: (1) naming written digits and perceptual patterns; (2) what Fuson (1981) calls "tracking"--keeping track of one's counts; (3) the genesis of understandings of (meanings of) addition and subtraction; and (4) the convergence of subitizing and counting as part of the genesis of the meanings given to number-names.

Point (1) is simply the naming function that children acquire from age one year on. The figural patterns ".˙. ," "... ," and "3" become semantically equivalent by each being linked to "three." Similarly, "10," a bundle of sticks, a Dienes base-ten long, and both hands open become semantically equivalent by each being linked to "ten."

Point (2) addresses the common phenomenon of a child correctly counting-on or back a small number without apparently generating a record of his counts. One explanation for such behavior is that the child generates an attentional pattern as an associate of the number-name designating the amount to be counted on and uses it as a criterion for stopping (Steffe, Thompson, & Richards, 1981). As the child executes the count, he constructs successive attentional patterns from his counts up to that point; he stops when the pattern of counts matches the criterion.

Point (3) is, in essence, that perceptually joining and separating configurations of objects give a figural basis for later abstractions of numerical operations. Given that a child recognizes "⁙" and names it "five," and that he likewise knows "∴" and ".." as "three" and "two," respectively, it is a small step to the realization that a "five"-configuration can be created by perceptually joining a "three"- and a "two"-configuration. As von Glasersfeld (1980) points out, the completion of the "five" configuration is, in principle, no different from completing a face by drawing a mouth in a circle that already has two eyes and a nose. By reflectively abstracting the operational structure of perceptually joining and separating configurations to create others, children can give addition and subtraction extensive meaning at the level of mental operations. Point (3) will be revisited under the heading "Numerical operations."

Point (4) addresses the relationship of intensive and extensive meanings of number-names. A numerical structure (abstract attentional pattern taken as a unit) is constructed by the child by reflectively abstracting the unit structure of items in figural configurations, and by making a unit of the units. If a figural configuration is one that is recurrently associated with a particular number-name, then the abstract pattern may be associated with the same number-name, whence abstract, extensive meanings for number-names. Intensive meaning develops as the child comes to count with ever more abstract units and his production of number-name sequences becomes routinized and operational. These two meanings develop independently; however, the child is bound to notice that when he counts the items in what he recognizes as a recurrent configuration _called_ "five" (say, "⁙"), he ends up with "five." The equivalence of subitizing and counting, while initially only semantic (both result in the same number-name), provides one way for the child to reflectively abstract the logical equivalence of cardinal and ordinal number (of finite sets).

Numerical Operations

The title of this section is meant generically, in that the discussion will focus on actions and operations that children perform on their referents of number-names. Though the term "operation" is conventionally applied only to an action that can be carried out in thought, it will here be used to include actions and action schemes as well. Operations that are carried out in thought on numbers or arithmetic lots will be called _numerical_ operations. The reason for this somewhat unconventional use of terms is that a genetic view of numerical operations will be taken, and for clarity of presentation the names of the operations will be held constant. For example, when one speaks of actions at the sensori-motor level which constitute the figurative basis for what will later become the operation of separating (discussed below), they too will be called separating even though they are not operations in the strict sense.

Similarly, it will be useful to be able to refer generically to the attentional structures hypothesized in von Glasersfeld's model of units and number. The term proposed here is "amalgam," with the understanding that one item can also be an amalgam. Von Glasersfeld's schematic of amalgams is presented again (Figure 3.7) for the reader's convenience, as specific types will be referred to by name.

The operations discussed in this section are separating, integrating, extending, and declending; the operands of separating and integrating are amalgams, while extending and declending do not, strictly speaking, have operands. Each of the operations will be discussed below.

Separating. Separating is the operation of forming two amalgams from one. A separation is the result of separating. If the initial amalgam is a collection (perceptual), then separating would take the form of introducing an attentional marker in the construction or maintenance of the perception. A child might do this by physically displacing items in the collection (to aid in his maintenance of the separation between collections), or by perceptually focussing on a position within the spatial or temporal configuration of the collection.

If the initial amalgam is a lot (figural), then separating takes the form of the construction of two lots that together are tacitly equated with the initial one. The means of equating the

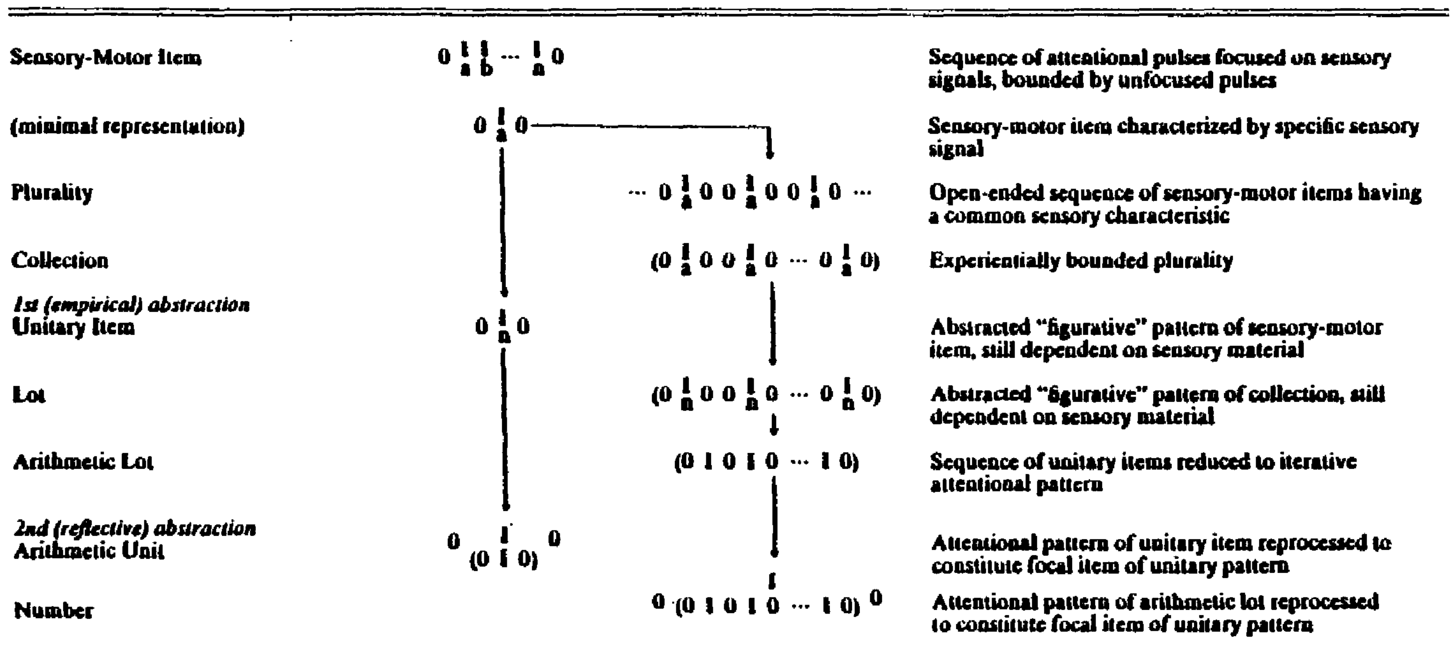

Figure 3.7. Schematic summary of steps in the construction of the abstract concepts of unit and number (From von Glasersfeld, 1981).

two comes from the child's having constructed the lots as figural representations of a collection, and the initial and separated lots each serve the representative function of substituting for a state of the collection. In other words, the child represents to himself the figural states of "before" and "after" separating. Separating lots is analogous to Piaget's (1970) notion of figural representation as deferred imitation.

Separating as applied to collections and lots constitutes a sensori-motor scheme, or empirical routine. It is not truly an operation, for it is a one-way connection between an initial and final state of perceptual or figural configurations. A child that is limited to separating collections can understand, say, subtracting three from seven in no other way than by acting-out the routine--put-out a collection of seven (marbles), separate it into two collections (one of which has three marbles), and count the other collection. It would also be impossible for such a child to conceptualize the initial collection as comprising its sub-collections. In a very real sense, once the child has separated a collection into two, the initial collection no longer exists. Prior to acting it out, the child has no <u>conceptual</u> understanding of the problem.

A child that can separate lots has achieved a level of flexibility far removed from a child limited to collections, largely because of the representative function that lots can serve. Whereas the child that creates collections in arithmetic does so with the aim of configuring countable items, the child that can create lots may do so with the aim of structuring his counting. Thus, following the ability to separate lots we see the emergence of primitive forms of counting-on and -back strategies in subtraction, as well as the emergence of the ability to conceptualize missing subtrahend and comparison problems in subtraction ("John has eight apples, Jane has five. How many apples do we have to take from John so they have the same number?"). However, a child who is limited to separating lots still cannot conceptualize the initial lot as comprising its sublots, for they are two state-representations that are equated only because the child attributes them to the same collection of objects. The relationship between them is functional, as opposed to relational.

If the amalgam to be separated is an arithmetic lot or number (abstract), then separating takes the form of creating two lots or numbers that together are explicitly equated with the initial one. The <u>means</u> of equating them comes by the child attributing to them the same numerosity--they are classified as having the same number-name. Separating at this level of thought is a numerical operation.

A child who can separate arithmetic lots has largely removed himself from the tyranny of his sensori-motor world (except for his need of experientially derived boundaries). He no longer needs to think representationally of collections, but may conjure numerosities, and hence structure his activity, at will. He is constrained only by his strategic capabilities (this is not a small constraint). A child that can separate numbers lacks even the need for experientially derived boundaries.

The relationship between the initial lot or number that a child at this level creates and its separation is still not one of inclusion, though it is close. The relationship is a semantic equivalence of the two--they are both, say, "twenty-five," with one preceding the other in time. The discussion of the relationship of inclusion will be addressed after discussing integrating as a numerical operation.

Integrating. Integrating is the operation of attentionally bounding into a composite whole--framing within parentheses, so to speak. An integration is the result of integrating. A sensori-motor item is the result of integrating sensori-motor signals; a collection is the result of integrating sensori-motor items; an arithmetic lot is the result of integrating attentionally reprocessed sensori-motor items (arithmetic units); a number is the result of integrating an arithmetic lot--taking it as one.

Integrating amalgams is the operation of forming one amalgam from two. If the initial amalgams are collections, then integrating takes the form of collecting the objects as one. That is, the child reconstructs the "collections" without separating. If the initial amalgams are lots, then the child creates another lot that is tacitly equated with the initial two together. The means of equating the two comes from the child's having constructed the lots as figural representations of a collection, and the initial and integrated lots each serve the representative function of substituting for a state of the collection. The child represents to himself the states of "before" and "after" integrating.

Integrating as applied to collections and lots constitute a sensori-motor schema, or empirical routine, for it is a one-way connection between an initial and final state of perceptual or figural configurations. A child that is limited to integrating collections can understand, say, adding three to five in no other way than acting-out the routine--putting out three (marbles), putting out five (marbles), integrating the marbles as one collection, and then counting to assign it a name. It would also be impossible for such a child to conceptualize the final collection as composed of the initial two. After the child integrates the two, they cease to exist.

A child that can integrate lots has achieved a level of flexibility far removed from the child limited to collections, largely because of the representational function that lots can serve. Whereas the child that creates collections in arithmetic does so with the aim of configuring countable items (so that later he may apply a number-name), the child that can create lots may do so with the aim of structuring his counting. Thus, following the ability to integrate lots we see the emergence of primitive forms of counting-on and -back strategies in addition, as well as the emergence of the ability to conceptualize missing addend and comparison problems in addition ("Henry has eight apples and John has five. How many apples does John need so that they have the same number?"). However, a child that is limited to integrating lots still cannot conceptualize the initial lots as composing the final lot, for they are two state-representations that are equated only because the child attributes them to the same collection of objects. The relationship between them is functional, as opposed to relational.

If the amalgams to be integrated are arithmetic lots or numbers, then integrating takes the form of creating an arithmetic lot or number that is equated with the two together. The means of equating them is by the child attributing to them the same numerosity--they are classified as having the same number-name. Integrating at this level of thought is a numerical operation.

A child who can integrate arithmetic lots has largely removed himself from the tyranny of his sensori-motor world (except for his need for experientially derived boundaries). He no longer needs to think representationally of collections, but may conjure numerosities, and hence structure his activities, at will. He is constrained only by his strategic capabilities (this is not a small constraint). A child who can integrate numbers lacks even the need for experientially derived boundaries.

The relationship between the initial lots or numbers that a child at this level creates and their integration is still not one of inclusion, though it is close. The relationship is a semantic equivalence of the two--they are both, say, "twenty-five." However, they would be logically equivalent only if a change in one would imply a change in the other. As semantic equivalents, this need not be the case, as a child may modify a lot in the structure (changing its numerosity) and yet still consider them as semantic equivalents. Several examples of this will be seen in the case studies.

Relationships. It should be apparent from the previous discussions that separating and integrating are (from our point of view) two sides of a coin. At the sensori-motor level, they are complementary routines. The final state of one may be taken as the initial state of the other. However, as empirical routines, the relationship can be no more than complementarity. This limitation will be made clearer below.

As numerical operations, separating and integrating are again (from our point of view) two sides of a coin, and their relationship is that of structural similarity. As semantic equivalence relationships, the child may view them as merely being reversals of one another--they each start at the other's end. As was pointed out earlier, however, the equivalence is only semantic, and it is one-way. A subsequent change in the end product of either operation, when the equivalence relationship is merely semantic, bears no necessary implication for the initial state. A classical example of this is when a child attempts to subtract, say, 29 from 71 using MAB longs and unit cubes, and does so by putting out seven longs and one unit, separating two longs and the unit from 71 and then adding eight more units so that there are 29. Assuming that the child could conceive of subtraction in terms of separating numbers (which would have to be established on the basis of tasks other than this one on this occasion), we can characterize the child as having separated 71 into two numbers (which together remained "seventy-one"), and then as having gone on to satisfy an additional requirement of the problem: one of the two numbers had to be named "twenty-nine." The adding of eight cubes to the subtrahend had no implication, to the child, for the initial number.

Separating and integrating become two sides of a coin <u>from the child's point of view</u> (in addition to ours) after he has constructed them as numerical operations <u>and</u> operationally related them as each being the inverse of the other. The sign of when a child has done this is when he does see a necessary implication for the initial state of the amalgams when a numerical change in the end state is made after performing the operation. The operations of integrating and separating, and their relationships are given schematically in Figure 3.8.

It is only after a child has operationally related integrating and separating as inverses of one another that he can truly conceive of one number comprising two or two composing one, for it is necessary that both aspects be present at once. What composes the whole is in return comprised by it, and vice versa.

The idea of collections "vanishing" and "appearing" may be difficult to come to grips with, so an analogy will be offered. The drawing in Figure 3.9 may be seen in two ways. You likely see either an old woman or a young beauty. If you see an old woman, focus your attention on her right cheek--you should see the nose and eyelash of a young beauty. If you see a young beauty, focus your attention on her necklace--you should see the mouth and chin of

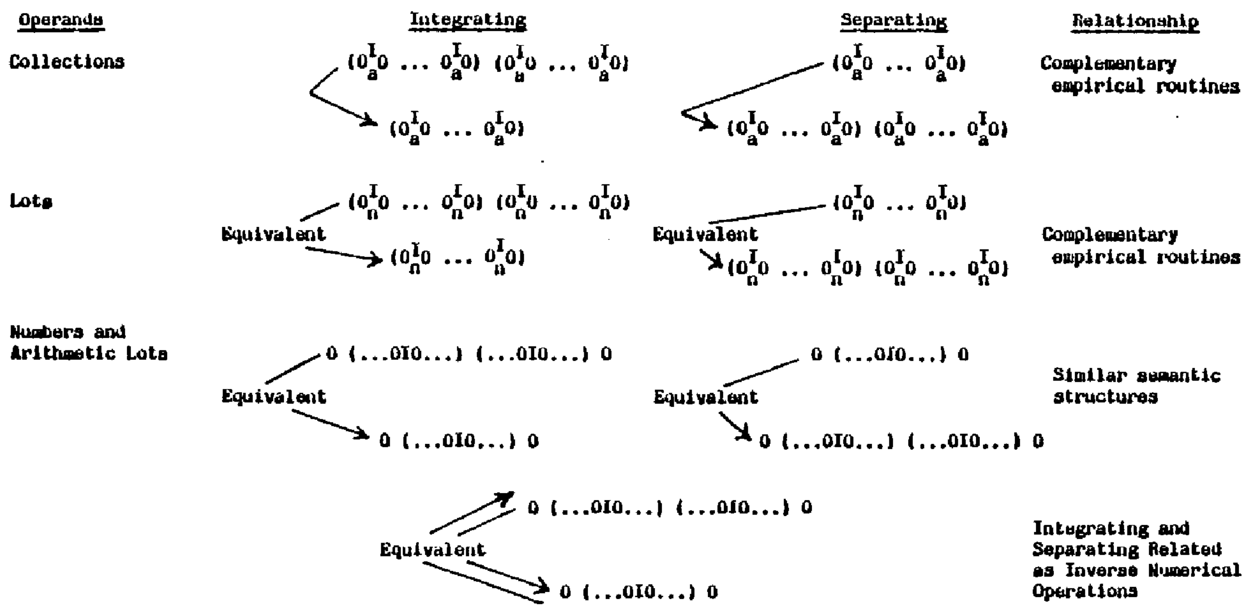

Figure 3.8. Schematic of the development of integrating and separating.

Figure 3.9. Old woman/young beauty.

an old woman (the young beauty's ear becomes the old woman's eye). Now look back and forth between the old woman and the young beauty. More than likely you can see both, but only one at a time. This is analogous to a child who can see two collections, or one, but not both at the same time. Each is a perceptual reorganization of the other, and once reorganized a perception disappears.

Now look away from Figure 3.9 and imagine both the old woman and the young beauty. Each mental image is a figural representation of the "object" to which it is attributed, but they are the same only in that regard. This is analogous to the child who can represent a collection as one or two lots, but the two are equated only because he attributes them to the same "object."

Knowing what you have to do, without having to do it, to go from the old woman to the young beauty and from the young beauty to the old woman is analogous to the child who has integrating and separating as numerical operations. One would have the old woman → young beauty and young beauty → old woman operations reversibly related if a modification in one (say, changing the arrangement of the old woman's scarf) would imply a necessary modification in the other. (The author has not so related them.) This is analogous to the child having related the numerical operations of integrating and separating as each being the inverse of the other.

Extending and Declending[7]

The implementations of separating and integrating result in perceptual and/or conceptual (numerical) structures. In the context of a child's understanding of an addition or subtraction problem, some parts of the structures may be named, others not. If the child's aim is to answer a "how many" question, then this can be characterized as having the goal of supplying the missing number-name or names in the structure of amalgams that constitute his understanding of the problem. Figure 3.10 characterizes a child's construction of an understanding of a missing addend problem as it is being presented to him. The next stage in the child's reasoning is to implement operations that will supply the missing name. Operations based on counting will be called extending (for forward sequences) and declending (for backward sequences).

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[7]An etymology of "declending": Steffe and Thompson (1979) were searching for words that would capture a child's senses of going beyond and retreating from a point in a counting sequence but which would not carry the connotation that the child is necessarily aware of a numerical increase or decrease of a quantity--whence "extension" and "declension." "Extension" has a conventional gerund form ("extending"), while "declension" does not. Thus, "declending" is here offered as the gerund form of "declension."

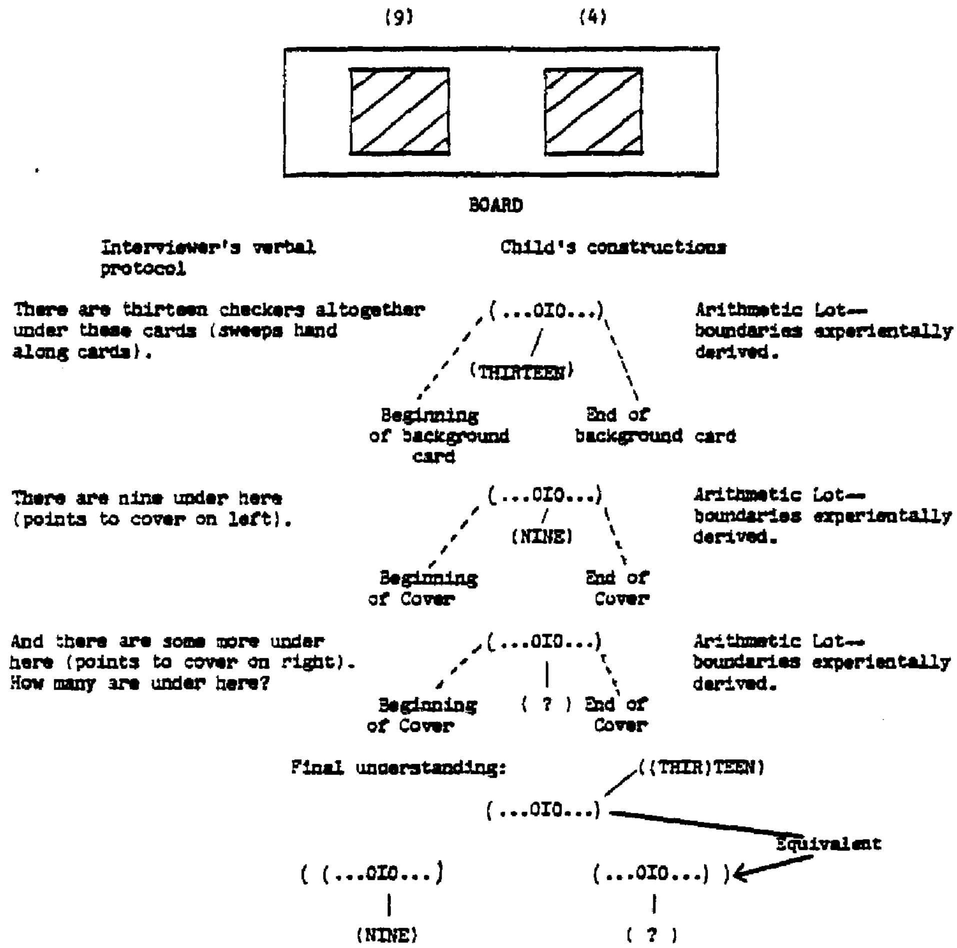

Figure 3.10. Child's construction of an understanding of a problem. (Task adapted from Steffe, Hirstein, & Spikes, 1976)

It may prove helpful if first we note the relationship, as viewed within this framework, between an understanding of an addition or subtraction problem and subsequent activity. In essence, it is that the child's understanding of the problem serves as a "guide" for solving the problem, in that what he subsequently does to solve the problem must culminate in a structure that conforms, more or less, to the guide. Of course, if a child is limited to separating and integrating collections, there is no conceptual guide. There are only the empirical routines.

Extending and declending as empirical routines serve the child as means to assign number-names to amalgams. They are counting routines with an associated direction. Their implementation is dependent primarily upon the child's first having a conceptualization of a problem in terms of amalgams, which in turn serves as a guidepost to structure his or her counting activity. When the child reconstructs extending and declending as numerical operations, however, a wholly new capability emerges. The child can conceptualize an arithmetic problem in terms of directed counting activities that he would perform <u>were he to actually carry them out</u>. Here we see the germ of directed numbers. The final understanding depicted in terms of arithmetic lots in Figure 3.10 would appear as depicted in Figure 3.11 in terms of extending and declending as numerical operations.

Extending and declending as numerical operations have two sources: intensive meanings of number-names and double-counting. Extending by, say, four [count (+4)] is the same operation regardless

Count (+13) ↘ *or* Count (−13) ↘

Equivalent Equivalent

Count (+9); Count (+?) ↙ Count (−?); Count (−9) ↙

Figure 3.11. Conceptualization of 9 + ? = 13 in terms of extending and declending.

of the starting point. The only criterion is that one actually extend the starting position by four. Similarly with declending by, say, seven [count (−7)]. In this sense, extending and declending are <u>operators</u> (in the mathematical sense). Mathematically defined, extending by x would be $f_x(y) = y + x$ where x and y are whole numbers. Similarly, declending by x would be defined by $j_x(y) = y - x$. Psychologically, extending and declending are the <u>mental</u> operations of forward and backward counting. It should be pointed out that, as a conceptual structure, an understanding of an arithmetic problem in terms of extending or declending does not dictate that the child actually count. If the child understands, say 10 + 7 in terms of extending, and if he knows that 10 extended by 5 is 15, and that 5 extended by 2 is 7, then he may draw upon this knowledge and solve it accordingly: $10 + 7 = 10 + (5 + 2) = (10 + 5) + 2 = 15 + 2 = 17$, without counting.

The direction of extending and declending as numerical operations is a formal attribute of the count. As such, for a child to relate extending and declending as inverse numerical operations, the child must <u>formally</u> compensate directions. For example, if the child (at the time of solving it) understands "13" of "50 - 13 = __" as 10 extended by 3 (10 + +3), but does not formally compensate directions, he will make the transformation 50 - (10 + +3) = 50 - 10 + +3. Similarly with 60 - 9 = 60 - (10 + -1) = 60 - 10 + -1. Several children in the investigation had constructed extending and declending as numerical operations. Further discussion of extending and declending will be deferred till the case studies where it will have a more meaningful context (see especially Case Studies 7 and 8).

One final note: the numerical operations of extending and declending are the germs of a later (possibly unfulfilled) concept of the additive group of integers. Once the child frees himself from thinking of extending and declending as operators on numbers, and begins thinking of them as operators on <u>operators</u> (through composition), he is well on his way to the additive group of integers. As an operator on numbers, extending produces another number; as an operator on operators, extending produces another <u>operator</u> (+10 + +3 = +13; extending by 10 followed by extending by 3 is the same as extending by 13). Similarly for declending. At this stage, the child needs only (only!) to map out the formal relationships within the system.

<u>Reading and Writing Numerals</u>

It was pointed out earlier (page 48) that number-names and arithmetic words are assumed to form children's primary representational system in their understandings of numeration, and that numerals are mapped into it (i.e., numerals are read). This position is analogous to that taken by reading comprehension researchers, who view reading as a process of translating visual images into sound images ("internally held words"). Questions of understanding become questions of what meanings the sound images have.

The preceding sections have focussed largely upon the comprehension side of reading and writing numerals. The present section focusses upon the mechanics of translating visual images of written numerals into number-names, and vice versa. It should be stressed that these processes in and of themselves are nonnumerical, insofar as their product is either a number-name or a numeral. Meanings for numerals are established via their connections with number-names.

Reading numerals. Within the context of this framework, it is assumed that children rely heavily upon subitizing and conventional cues when reading numerals. The reliance takes two forms: in naming individual digits and in applying position names ("thousand," "hundred," and "ty"). The second comes from a set of associations that the child has made: if the numeral has four digits or a comma, it will be a "thousands" name; if three digits, a "hundreds" name; if two digits, a "teen" or "ty" name, or "ten," "eleven," or "twelve"; if one digit, then its name. Once the child has established the initial unit-indicator, he then recursively applies the same strategy to groups of digits (if more than four) and individual digits within groups. When he encounters a zero, he says nothing, reads the next position name and proceeds to the next digit or group of digits to the right. A two-digit numeral beginning with a "1" must be taken as a special case. If it is followed by "0," "1," or "2," then the name of the entire numeral is "ten," "eleven," or "twelve." Otherwise, the numeral is (something) "teen," where (something) takes its value from the name of the rightmost digit in the numeral. A numeral of the form ##,### will be recognized, because of the comma, as (something) "thousand" where (something) takes its value from the application of the naming procedure to the digits to the left of the comma. The numeral "345" would first be recognized as a "hundred" numeral (3 digits). So it would be something "hundred" (three), something "ty" (four), something (five), or "three hundred forty-five."

Just as with any routine, children will misapply it or include faulty rules during its formation. Some are: transliteration (reading the numeral backward), especially with two-digit numerals ending with a "1"; not knowing what to do upon encountering a "0"; and failing to completely decompose the numeral into digits (e.g., "143" being read as "fourteen three") before attempting to apply position labels.

Writing numerals. Routines for writing numerals require that each digit name be abstracted from the position labels, and that the corresponding digit numeral be written. The child must process the number-name having in mind the succeeding position label (in temporal order), so that if it is missing from the name, the child writes a "0." Thus, to write "three hundred forty-five" as a numeral, the "three" is abstracted from its position label, and "3" is written, and so on. To write "four hundred five" as a numeral, "four" is abstracted from "four hundred," "4" is written, a check shows no "ty" part in the number-name, so "0" is written (thus "40"), and then "5" is written (thus, "405").

Common errors in routines for writing numerals include transliteration, again especially when a "1" is involved with the numeral. This suggests that there may eventually be some sort of expectation of what the numeral will look like once it is written. Another is writing the full numeral for each digit name and associated position label, e.g., "200402" for "two hundred forty-two."

Summary

Four domains of knowledge relevant to modeling young children's understandings of whole number numeration were presented. They were language, subitizing, numerical operations, and reading and writing numerals.

The linguistic domain contains routines for constructing number-names and for constructing sequences of number-names in various increments. The sequencing routines were characterized as rule systems for constructing a number-name to succeed a currently held one, and as abstractions from the activity of sequencing *per se*. Order relationships between number-names were described as coming by way of a chunk structure on the names "one" through "nine" and a hierarchy of relationships among rules.

Number-names were characterized as sound images, and were taken as children's primary representational system for number—superceding numerals in importance. Potential meanings for number-names were dichotomized into what were called *intensive* and *extensive* meanings. Intensive meanings involve the iterative construction of a sequence, either manifested or represented; extensive meanings involve figural or abstract attentional patterns. Once a child acquires the ability to create abstract unit items, he may semantically equate the two meanings, since a sequence of words may be taken as a basis for constructing an abstract pattern in the same way as can a perceptual collection. That is, a sequence of number-names has a number just as does a perceptual collection, but it is only after the child can make abstract unit-items that he can equate the two in terms of numerical structure.

Subitizing was defined as "the spontaneous attribution of a number-name to a figural or abstract attentional pattern." Two levels of subitizing were described. The first is the establishment of semantic connections between number-names and specific perceptual or figural patterns. The second is the connection of number-names with abstract attentional structures that have been reflectively abstracted from perceptual or figural patterns associated with a number-name. The role of the notion of subitizing in the framework is fourfold. It will be used in explanations of: (1) naming written digits and perceptual configurations; (2) "tracking"--keeping an internal record, without counting, of one's counts; (3) the genesis of meanings of addition and subtraction; and (4) the genesis of meanings given to number-names.

The domain of numerical operations includes the operations of separating, integrating, extending, and declending. Separating and integrating are operations on amalgams.

As empirical routines, the initial and end-products of integrating and separating are equated by the child's attributing them representationally to a single collection. The initial and end-products of separating and integrating are representations of states of collections.

As numerical operations, the initial and end-products of separating and integrating are equated by the child's attributing the same numerosity to them--they are given the same number-name. A child relates separating and integrating as inverse numerical operations when he or she sees a necessary implication in the initial number or numbers whenever a numerical change is made in the end-product of the operation.

Extending and declending, as empirical routines, are operations to name unnamed amalgams, and are based on counting. Conceptual structures constructed by separating and integrating provide a goal structure for naming unnamed amalgams, while extending and declending provide means for supplying missing names in the structure.

As numerical operations, extending and declending may be used by a child in conceptualizing arithmetic problems and not merely as operations to name amalgams. Extending and declending at this level of thought were characterized as operators which, when implemented, transform a number into another. As mental operations, however, they do not need a specific starting value. Extending and declending were characterized as the germ of the additive group of integers.

Routines for reading and writing base-ten numerals were presented. They were nonnumerical, in that they did not require an appeal to meanings of number-names. Rather, they depended only on subitizing and figural cues, such as commas to indicate groupings of digits. Number-names constructed through reading were considered to be of greater importance in children's understandings of numeration than numerals _per se_.

Concepts of Numeration

The previous section presented four domains of knowledge: language, subitizing, numerical operations, and reading and writing numerals. This section will show how these domains may be organized to characterize concepts of numeration. The concepts discussed are ten, one hundred, and place value.

It may prove worthwhile to address the nature of concepts that is assumed in this framework. In essence, by "concept" will be meant, as Piaget said, "a generalized scheme of thought" (Piaget, 1968, p. 46). Another way to put this is that a concept is a structured scheme of schemes or routines. Piaget would require that the routines in the scheme be operational, but this stance will not be taken here. Rather, a scheme of empirical routines will also be classified as a concept—a concept in action, so to speak. No child in this investigation will be approached with the question, "Does he or she have _the_ concept?" Rather, the question will be, "What is his or her concept like?"

Yet another way to characterize a concept is that it is a composite of its aspects. This may seem circular, but viewed from a particular perspective it is not. The aspects of a concept develop, possibly independently, and as the child creates problems (experiences disequilibria) in broader domains of experience, the aspects interact. Epigenetically, the child establishes relationships between and among the aspects, and they come to form a composite whole--a concept.

Speaking of a concept as a composite of aspects makes sense, however, only in the language of an observer. An actor focusses upon one aspect at a time, but with the knowledge that it has significance (is related to other aspects)--just as a problem embedded within a computer program may be "discernible" to an observer, yet from the computer's perspective it is always dealing at any one time with an aspect of the problem.

The concepts of numeration discussed here will be characterized in terms of relational networks. The nodes of the networks will be the aspects of the concept (routines, special words, meanings), and relationships among the aspects will be explicitly addressed. The aim of characterizing the concepts to be discussed in this manner will be to provide a way to explain, in terms of underlying structures, the manifold ways that they may be expressed in children's behaviors.

Concept of Ten

The relational network that constitutes a fully developed concept of ten, as taken in this investigation, is depicted in Figure 3.12. The three-dimensional aspect of the network is meant to convey the idea that the concept develops as its aspects develop and interact. The "nodes" of the network are operations or systems of operations (except for the word "ten"). Unlabeled connections between nodes connote signification; otherwise the nature of the connection is specified by a label.

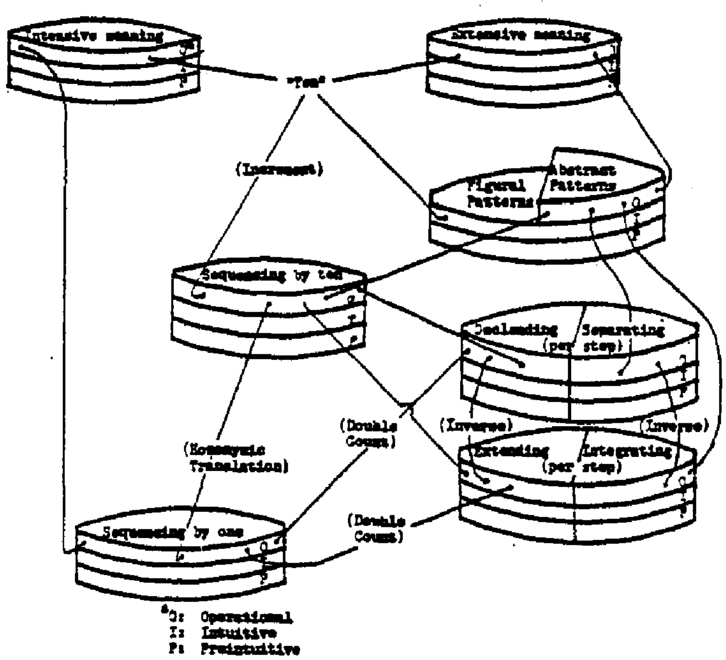

Figure 3.12. Concept of ten.

The word "ten" can take on two meanings--intensive or extensive. This is shown in the above figure by the unlabelled connections from "Ten" to "Intensive Meanings" and "Extensive Meanings." Intensive meaning is related to sequencing by one in that sequencing by one would be involved were one to elaborate a meaning of "ten" (construct a number that is composed of ten units). "Ten" is related to subitizing by way of semantic connections between the word and (possibly idiosyncratic) figural patterns. It could signify a bundle of sticks held together by a rubber band, a red chip, or a Dienes multi-base long. A red chip called "ten" would take on quantitative significance for a child by his giving "ten" either an intensive or extensive meaning. The word "ten" is also related to abstract patterns (through "Extensive Meaning") in that it could be a label for a number or arithmetical lot, or a label for the units in a number or lot. For instance, a child could create an abstract "two" pattern and attach "ten" as a label for the units--"I've got two things, and they're both ten." Finally, "ten" is related to the linguistic system for sequencing by its role as the increment when sequencing by ten.

Sequencing by ten is related to sequencing by one because of its reliance on the NEXT relation defined on "one, two, . . ., nine." It is also related to integrating and extending (when sequencing forward) and separating and declending (when sequencing backward) in that incrementing a number-name once by ten has the significance of extending ten times by one; incrementing backward has an analogous significance. Subitizing and sequencing by one are indirectly related, in that attentional patterns are associated with number-names, which in turn may be given intensive meaning.

The relationships between sequencing by ten, sequencing by one, extending, declending, and intensive and extensive meanings of "ten" provide the structure of ten as a unit of measure. In counting "ten (is one), twenty (is two), . . ." one may, in principle, construct the number of tens in a number, the number of tens between two numbers, and extend one number by another using the number of tens in the extension (when it is known or has been constructed).

When the number of tens in an extension or declension is small, one may subitize the number of increments made when sequencing by ten. The units in the subitized pattern each have the significance of extending ten times by one, since each is a unit made from an increment by "ten."

The knowledge that "x-ty y" denotes a number that has x tens and y ones comes by implication from the relationships between extending, sequencing by ten, and sequencing by one, but is, strictly speaking, not part of the concept. The "x-ty" of "x-ty y" maps into "x," which, when given meaning, signifies the number of increments, or extensions, by ten that would be made when constructing x-ty y. For instance, given "fifty-seven," applying HOM2 to "fifty" gives "five," and "five" can be given meaning, either extensive or intensive, as a number of units where each unit has the significance of extending by ten (via the connections between extending, sequencing by one, and sequencing by ten). Thus, one can know that "fifty-seven" denotes a number that has five tens in it without having to count them.

The concept of ten depicted in Figure 3.12 is one that would be held only at a late stage of development. At a point in the genesis of the concept in a particular child, we might find some of the nodes to be less than well-formed (empirical), some only beginning to form, and others not at all. Similarly, we might find less than well-formed relationships between nodes, or connections that exist as empirical abstractions. We will see many examples of this in the case studies.

Concept of One Hundred

The relational network comprising the concept of one hundred can be constructed largely from Figure 3.12 by substituting "hundred" for "ten" everywhere the latter occurs. The detail has been repeated in Figure 3.13.

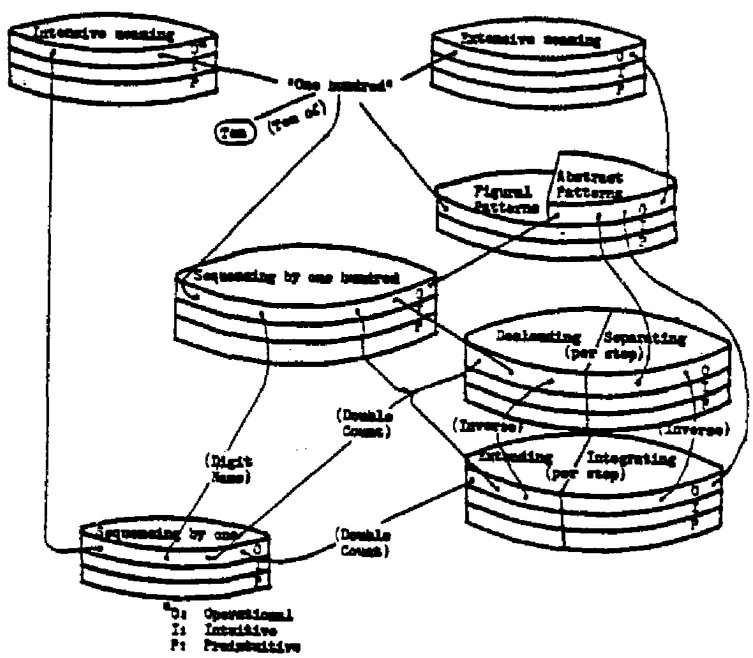

Figure 3.13. Concept of one hundred.

The major addition to the network is that "hundred" is connected to the concept of ten by the relation of "ten of." That is, one unit of one hundred is equivalent in meaning to ten units of ten. In a sense, the formalism makes the relationship between concepts appear simpler than it is, for the implications of the relationship show up anywhere the word "hundred" is involved. That is, the concepts of one hundred and ten must be coordinated. The development of a child's concept of one hundred is nowhere as simple as a substitution of words. A child must construct the aspects of the concept. The construction may parallel that of his concept of ten, but it is essentially a new construction.

Concept of Place Value

The concept of place value in numeration is the easiest of the three to schematize formally, yet is the most complex. It is formed by relationships among the concepts of one, ten, and one hundred, where the relationships are each "ten of," and the concept of position.

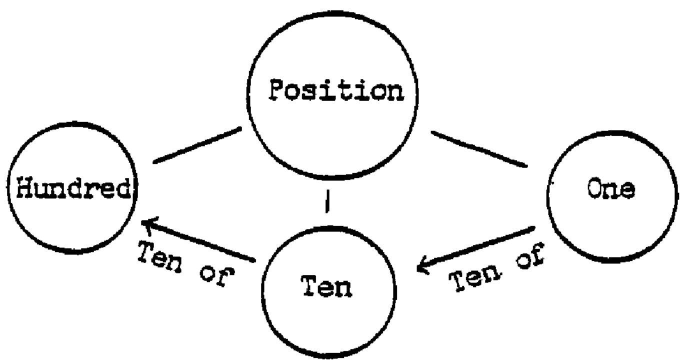

Figure 3.14. Concept of place value.

The concepts of ten and one hundred have already been discussed. The concept of one is essentially an Arithmetical Unit. The focus of the discussion, then, will be placed on the concept of position.

In its most fundamental form, the concept of position is tantamount to that of ordinal number. Position in a sequence (well-ordered set) is determined (1) by an order, and (2) by a cardinality. The criterion for an element in a well-ordered set to occupy, say, the third position is that there are two elements preceding it relative to the order. In the case of written numerals, the order relation comes from an analysis of the numeral into digits and a spatial right-to-left successor function. In the case of spoken number-names, the order relation comes from an analysis of the number-name into digit-names and a temporal order of reversed recitation. Of course, number-names are constructed so that the name of the "place" that the digit-name occupies is a suffix to the digit-name, which obviates any need for elaborate place value processing. In the concept of place value, however, the order is a result of recursion, in that the successor to a unit in the sequence of numeration units is constructed by taking ten of that unit as a unit.

The concept of position, together with the relation "ten of," forms the basis for place value in base-ten numeration. The nth numeration unit has a value that is equivalent to constructing ten of the (n-1)th numeration unit, or one hundred of the (n-2)th numeration unit, and so on. In order of increasing value, the first unit is one, the next is ten of one, the third is ten of (ten of one), and so on. Ten, as a base for a numeration system, is a rule for constructing the units in the system along with a rule for constructing numbers from those units.

Conclusion

To model a particular child's understandings of whole number numeration, it is necessary to (1) characterize the relevant routines he or she has, (2) assess the degree of operationality of these routines, and (3) characterize the organization that the child places on his or her routines. Each of these activities must be done in the context of attempting to explain the child's behaviors in problem-solving situations, paying particular attention to the child's difficulties.

The framework presented in this chapter provides an entry point into the task of building models of children's understandings of numeration. It is by way of the framework that the author explains the behaviors of the eight children of this study--the framework acts as a filter, guiding both observations and the construction of explanations. Just as with any filter, however, some things will not pass through it--they go unobserved and/or unaddressed. One class of activities largely unaddressed are those emanating from empirical routines for elaborate processing of numerals in the context of standard addition and subtraction algorithms, such as those proposed by Brown and his colleagues (Brown & Burton, 1977; Brown & van Lehn, 1979). The reason for this omission is that such explanations cannot provide insight into <u>understanding</u>. First, if a child fails, say, to align the numerals when performing the standard regrouping addition algorithm, then he certainly does not understand the <u>algorithm</u>, but such an error suggests nothing about his understanding of numeration. Second, routines for processing numerals may be learned altogether independently of any understanding of numeration aside from how to read them. A child may make connections between his understandings of numeration and routines for processing numerals, and the latter may serve as aids, but the importance of understanding lies in the child's conceptualizations.

Chapter 4

UNDERSTANDING CHILDREN'S UNDERSTANDING OF WHOLE NUMBER NUMERATION

Chapter 3 was devoted to presenting the framework used in this investigation. In this chapter, the framework will be applied to each of the eight children taking part in the study with the aim of understanding, and hence explaining, their understanding of whole number numeration.

Method

Each child was given three interviews (Appendix I) over a period of at most two weeks. Each interview was videotaped and a typed transcript was made from the tape, and then twice checked against the tape and edited. The application of the framework to a particular child's behaviors took the form of a case study. The method of preparing for the case study was to read the entire transcript in order to get a general sense of the child's abilities and characteristics, and to look for portions of the manuscript that lacked sufficient detail to picture what the child did. The entire transcript was read again while viewing the videotapes of the interviews. This was done for two reasons: to check the accuracy of the transcript, and to "get a feel" of the child. The manuscript was then read again, this time with the aim of partitioning the interviews into episodes that had particular salience from the perspective of the framework. (An example of a thrice-read transcript is given in Appendix II.) The method of making these determinations was to attempt to construct a model of the child's processes during his or her construction of a solution to an interview task. The components of the framework coming into play in a relatively complete model of the child <u>in that episode</u> and critical questions arising from interpretations of the episode were then summarized in a set of notes, and the episode was listed under the headings of the respective components. (An example of a set of notes and completed worksheet is given in Appendix III.)

The activity of modeling was carried out at two levels. First, an episode was examined from the perspective of communication between the interviewer and child. The aim at this level of analysis was to understand what the participants in the discussion understood each other to be saying. In more than several occasions, a child's behavior, and at times the interviewer's behavior, made sense only after one interpreted the situation as evolving through a process of <u>mis</u>communication. Second, the child's behavior was analyzed as cues to underlying cognitive processes--the problem that he or she had established: routines that were called; the qualitative nature of the routines; <u>meanings</u> that the child attributed to number-names, arithmetic words, and linguistic actions on words; <u>relationships</u> among routines, number-names, and meanings; and heuristics. These two levels of analyses were undertaken in any episode in support of one another--many times it was necessary to move back and forth between them in the course of a single exchange between the child and interviewer.

Before proceeding to the case studies it may be worthwhile first to establish a perspective. Put very loosely, the presentation in Chapter 3 was the author's attempt to communicate his "task" environment, where the task is to explain children's behavior in situations involving whole number numeration. It is the collection of "objects" and "operators" out of which he constructs a picture of a particular child solving a particular problem. A child's case study is an attempt by the author to paint a composite of these pictures--to unify them within an ideal child, so to speak, that behaves much like the child being studied. That is, the case studies are an attempt to explicate the child's task environment, and problem spaces that he or she constructed out of it.

Case Studies

The construction of the case studies took place by examining episodes listed under the headings on the worksheet (Appendix III), selecting those that best suggested the characteristics of the child's knowledge and understandings. The construction took the following order: domains of knowledge (reading and writing numerals; sequencing: by ones, tens, and hundreds; numerical operations: integrating, separating, and unit items upon which these operations took place, extending, and declending) and concepts of numeration (ten, one hundred, place value). The reason for this order of construction is twofold. First, concepts of numeration as depicted within the framework consist of relationships among routines, number-names, and meanings, and hence we must first examine the items we postulate will be related before attempting to relate them. Second, any anomalies in the framework will be more likely to show up by this order of construction--if the pieces do not fit with the puzzle, then either the pieces or the puzzle is awry.

The presentation of the case studies will take essentially the form of their construction. We will examine each child's characteristics within the categories of the framework (domains of knowledge) and then inspect the relationships the child has established among them.

The introduction to each case study will be composed of a discussion of the child's behaviors on a set of tasks presented to them in November of the school year. The discussions will not go into detailed interpretations. Their purpose will be only to give the reader a sense of the child's growth.

Warm-up

(8 green squares on a board) Count these squares.

Tasks

1. (3 squares visible; 5 covered) There are five squares under the cover. How many are there in all?
2. (7 squares visible; 3 squares covered) There are seven squares here. There are some more under here. There are ten altogether. How many are under here?
3. (5 squares covered; 4 squares covered) There are five squares under here, and four under here. How many are there altogether?

Figure 4.1. Counting tasks administered to the children at the beginning of the year.

Figure 4.1 presents a set of simple arithmetic problems. Their aim was to get an idea of the children's abilities to conceive of arithmetic in terms of counting.

Warm-up

(Places bundles of ten on the table.) Each of these bundles has ten sticks in it. Do you want to check to make sure they have ten?

Tasks

1. Count by tens as far as you can.
2. Start at two and count by tens.
3. Here's a bundle of ten sticks (place bundle on table), and here's four more. How many sticks are there on the table?
4. How many tens are there in thirty-two?
5. (Place card with "54" written on it onto the table.) Use the sticks to show me how many tens there are in this number.
6. (Place five single sticks and two bundles of ten on the table.) How many sticks are there here?
7. (Place 33 single sticks on the table.) Can you use tens to help you find how many are here?
8. (Place ten bundles of ten on the table.) Can you use tens to help you count all the sticks in these bundles?
9. (Place three bundles of ten on the table, then six single sticks next to the three bundles.) Start here (first single stick) and count to find out how many sticks there are.

Figure 4.2. Nine tasks administered to the children at the beginning of the year.

The problems in Figure 4.2 were given with the aim of gaining an idea of the meanings of ten to the children. The idea at the time of giving them was to base subsequent instruction on the interpretations of their behaviors. As it turned out, very little instruction was based thusly--it was too difficult to make sense of their behavior. In fact, the framework of this investigation (completed in 1981--a little late for its original purpose) is an outgrowth of that idea.

It is worth mentioning that the children's behaviors on the November interviews were in no way taken into account in constructing the case studies. The case studies are based entirely on the three interviews given each child in May of 1977. In fact, the introductions were written after all of the case studies had been completed.

The children in this investigation will be called Delta, Lambda, Kappa, Rho, Sigma, Gamma, Alpha, and Mu. The case studies will be presented in that order.

The reader is offered a word of advice: when reading the case studies pay special attention to the excerpts. They are offered not only to substantiate points made in the discussions, but also, and just as importantly, to give a sense of the child. Do as the author did--imitate the children; try to imagine that you are the child, and try to think in a way that the actions that you perform make sense from the perspective of your newly assumed identity. This is most critical--and most difficult--with respect to a child's errors. Many times what appears to us as an error is in fact a quite sensible thing to do given the way the child is thinking. The key is to imagine a way of thinking that makes the "error" a sensible way of acting.

Case Study 4.1: Delta

Delta was a second-grader (age 7 years at the beginning of the 1977 school year). In November of 1977, Delta could not solve any of the problems in Figure 4.1, nor did he attempt a solution to any. After all the problems had been posed, the interviewer asked him to put five and four fingers up and tell him how many altogether. Delta did so, and then was able to solve problem 1 by putting up five and three fingers and counting all eight while touching each finger against his lip. As for the problems in Figure 4.2, Delta could sequence "ten, twenty, . . ., ninety," but had, essentially, no concept of ten. He could not sequence by ten from two, answered that there were 20 sticks in all when a bundle of ten was combined with four more sticks, and that to make 54 using bundles of ten and single sticks he needed to put out 54 bundles, but count the individual sticks within the bundles. The final interviews were given to Delta on May 8, 11, and 15 of 1977.

Writing numerals. Delta had no difficulty writing numerals less than 100, nor for numerals within the first decade of each century (e.g., "101"; "209"). However, he systematically included a "0" following the hundreds digit in all other cases ("2019" for "two hundred nineteen"). As Delta wrote "101," "107," and "209," he said "hundred" as he wrote "0," and likewise for "2014," "2067," and "9034."

Reading numerals. Numerals less than 100 posed no difficulty for Delta. Numerals greater than 100 were more problematic, as seen in the following excerpt.

1 I: (Places card with "120" written on it.)
2 D: One hundred and . . . zero . . .
3 I: Would you say that one again for me?
4 D: One hundred and two . . . one hundred and twenty.

.
.
.

5 I: (Places card with "410" on it.)
6 D: Four . . . four hundred . . . (long pause) . . . and ten.
7 I: (Places card with "594" on it.)
8 D: Five hundred and nine . . . five hundred and nine . . . four . . . ninety-four.

Excerpt 4.1.1

If we assume that Delta's method of reading numerals was, in principle, that outlined in Chapter 3, then we may infer that he readily subitized the size of the numeral, since he showed no difficulty in establishing that "hundred" was to be said after saying the name of the left-most digit. His difficulty seemed to lie in structuring his analysis of the remaining portion of the digits--namely, taking the next two as <u>one</u> numeral to be named. Delta did, however, read "174," "201," and "311" correctly and without hesitation. It seems that it was his difficulty with "410" that caused him to have problems with "594"; Delta possibly focused on the ending "0" after saying "hundred" and, when attempting to analyze the digits following "4," found only one that was unused--forcing him to adopt a strategy of analyzing the numeral into single digits (this is more apparent in the way he read "120").

<u>Sequencing</u>. Seven episodes shed light on Delta's ability to sequence by ten and the rule system he used to do so. Illustrative of these is the following.

I: I want you to start at eight and count-on by tens. Pay attention now.
D: Eight . . . nine, ten, eleven, twelve, . . .
I: No--that isn't what I mean. I want you to count-on by tens.
D: Like eight, ten . . .
I: No--eight and ten more . . . what would that be?
D: Eighteen.
I: And ten more?
D: Seventeen . . . seven . . . seventeen.
I: No, eight and ten more you said was eighteen. Then ten more. Eighteen and ten more . . . What's eighteen and ten more?
D: Eighteen . . . (long pause).
I: Like this--eight, eighteen, twenty-eight.
D: Seventy-eight?
I: No--eight, eighteen, twenty-eight . . . .
D: Twenty-nine.
I: Are you listening? Eight, eighteen, twenty-eight, thirty-eight . . . .
D: Forty-eight, fifty-eight, seventy-eight, ninety-eight . . . ninety-eight . . . one hundred and eight . . . hundred and two . . . I mean . . . two hundred and eight, three hundred and eight, four hundred and eight, five hundred and eight, six hundred and eight, nine hundred and eight . . . nine hundred and eight . . . nine hundred and eight . . . .

Excerpt 4.1.2

Delta initially understood the interviewer to mean "start at eight and count" (3). Once he had empirically abstracted a pattern from the interviewer's examples, he continued the pattern (19-24). The rule that he apparently abstracted was to increment, through homonymic translation, the first-said part of the number-name--thus ". . ., eighty-eight, ninety-eight, one hundred eight, two hundred eight, . . ., nine hundred eight. Delta applied this same rule in two other episodes with the same overgeneralization.

Delta's reliance on homonymic translation (e.g., "thir" → "three") appears in another episode.

I: Start at the number ninety-seven and see if you can count-back by tens.
D: Ninety-seven . . . ninety-seven . . . ninety-seven . . . nine-- eighty-seven . . . seventy-seven . . . sixty-seven . . . fifty-seven . . . forty-seven . . . thirty-seven . . . (long pause) . . . forty--thir . . . twenty-seven . . . seventeen . . . seven . . . seven . . . that's all.

Excerpt 4.1.3

In two instances (3,6) Delta explicitly separated the first-said part of the number-name, apparently so that he could use his backward count by ones to construct the next term in the sequence, such as (((NINE)TY)(EIGHT) → ((NINE) → (EIGHT)) → (((EIGHT)TY)(EIGHT)).

Delta could not sequence by hundreds, except as an empirical abstraction to continue an example offered by the interviewer. The one time he did so was in the context of continuing the interviewer's sequence of "thirty, one hundred thirty, two hundred thirty" by saying "three hundred thirty, . . ., nine hundred thirty, ten hundred and thirty." It will be shown later that sequencing by one hundred was not a routine for Delta, and hence that his continuation in the above example was constructed only to satisfy the demands of his task.

From the preceding, it would appear that Delta's routines for generating number-names and sequences of number-names were empirically based on "one, two, . . ., nine." Were they operational, we would expect flexibility in transcending centuries while sequencing by ten, and a recognition that changing the position to be systematically incremented in the number-name corresponds to a change in increment. That Delta's sequencing routines failed to form an operational structure can be seen in his attempts to seriate numerals.

Cards: 16 17 18 19 20 21 22 23 (shuffled)

I: I've got eight cards with numbers on them and a board with eight places for the cards. Can you put the cards in order on the board so that each place has a card on it?
D: (Takes top card and places it.) (20)
I: Now, you're going to put them in order--right?
D: (Spreads rest of cards on table; pauses.)
I: Can you find the smallest one?
D: (20-21) Twenty-one. (20-21-22) Twenty-two. (Examines remaining cards.) Oh . . . (removes all cards from the board). (nil)
I: Now can you put them in order?
D: (16) Sixteen. (16-17) Seventeen. (16-17-19) Eighteen. (Examines remaining cards.) Ohhh! (16-17-18) Eighteen. (16-17-18-19-20) Twenty. (16-17-18-19-20-21) Twenty-one. (16-17-18-19-20-21-22) Twenty-two. (16-17-18-19-20-21-22-23) Twenty-three.
I: Which one is the smallest one?
D: (Points to "16.")
I: Which one is the biggest one?
D: (Points to "20.") Twenty.
I: Twenty is the biggest one there?
D: (Nods head yes.)
I: There isn't any number bigger than that, huh?
D: (Points to "23.") Twenty-three.
I: Is that bigger?
D: (Points to "19.")
I: Is nineteen biggest? Which would be the biggest one?
D: Twenty-one.
I: Why is that the biggest?
D: (Points to "21.")
I: Is it bigger than sixteen?
D: (Nods yes.)
I: Is it bigger than twenty?
D: (Shakes head no.)
I: Twenty is bigger, huh? Then which is the biggest one?
D: Twenty . . . (long pause). Twenty is the biggest one.
I: Twenty is the biggest, huh. Is nineteen bigger than twenty?
D: Nope.
I: Is twenty-one bigger than twenty?
D: Nope.
I: Is twenty-three bigger than twenty?
D: No . . . yep.
I: Why is it bigger?

D: Cause . . . it got the biggest number--three (pointing to the "3" of "23") is bigger than two (pointing to the "2" of "20").
I: Three is more than two? So that makes it bigger, huh.
D: (Nods head yes.)

Excerpt 4.1.4

Two aspects of Delta's attempts to seriate the numerals stand out: first, it appears that he assimilated the task to his routine for sequencing by one, rather than into a seriation structure. That is, instead of examining each card for the next largest numeral, he looked for the numeral corresponding to the number-name following the name of the numeral last placed. This appears to have been the case in (12), where he placed "19" and yet said "eighteen"--changing his mind only after he found a card with "18" on it. Second, his method of determining the relative pair-wise order of numerals, whatever it may have been, certainly did not depend on a hierarchical structure for generating number-names and sequences of number-names (See Chapter 3, Figure 3.5, p. 58). It appears that Delta's method was to appeal to individual digits within the numeral, as suggested by his comparison of "20" and "23" (41-45). Even then, Delta's choice of digits from the respective numerals seems to have been fortuitous, as suggested in (33-34), where he maintained that 20 is bigger than 21 (possibly comparing the "1" of "21" with the "2" of "20"). In several later seriation tasks, Delta employed the same strategy--saying that "48" comes after "61" because "the eight is more big than the one," and that "103" comes after "124" because "the three is bigger than the two."

Another episode suggests again that Delta ordered numerals by assimilating them to a routine for sequencing. In this task, he was asked to place the numerals 20, 30, 60, 70, 90, 100, 110, and 120 into order from smallest to largest.[1]

D: (Spreads cards on the table.) (70) Seventy . . . seventy . . . seventy. Oh . . . (nil). (60) Sixty. (60-70) Seventy. (nil) (20) Twenty. (Pauses; removes "20"; places "60.") (60) Sixty before twenty. Hmmm-mmm . . . (20) twenty. (20-30) Thirty. Forty . . . forty . . . forty . . . (examines backs of cards).
I: What's going to come next then? (Pause.) Do you have a forty?
D: Nope.
I: So you can't put that one down, can you? Put them in order from smallest to biggest.
D: (Begins to remove "30.")
I: Were you going to take that one off?
D: (Nods head yes.)
I: You were? What were you going to do with it?
D: I wanted forty, but I couldn't find it.

Excerpt 4.1.5

It seems clear that Delta examined the remaining cards for the numeral corresponding to the number-name that would be next in his sequence "twenty, thirty, forty, ..." Hence his determined effort to locate a card with "40" on it.

If we assume, as argued above, that Delta ordered numerals by assimilating them to a routine for sequencing number-names, then we have a strong explanation for what otherwise might appear to be bizarre behavior. In the following episode, Delta was asked to place 11, 21, 31, 51, 81, 91, 101, and 111 in order from smallest to largest.

Cards: 11 21 31 51 81 91 101 111 (shuffled)

I: Let's try doing the same thing with these cards.
D: (Spreads cards on table.) (11) Eleven. Twelve . . . twelve . . . twelve . . . I need twelve. (11-31) Eleven, thirteen. Fourteen . . . fifteen (11-31-51). Fifteen. Sixteen . . . sixteen . . . seventeen . . . (11-31-51-81). Eighteen . . . (11-31-51-81-91) nineteen . . . nineteen. (11-31-51-81-91-21). Oh! (11-31-51-81) (11-31-51-81-91-21) (11-31-51-81-91-21-101) One hundred ten. (11-31-51-81-91-21-101-111) One hundred and eleven.
I: Want to say them one last time for me?

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[1]The numerals in parentheses are those which Delta placed on the board. "(60-70)" means that the numerals "60" and "70" had been placed, "60" in the left-most position and "70" in the next; "nil" means that the board was empty.

D: Eleven, thirteen, fifteen, eighteen . . . eighty-one . . . eighty-one, ninety-one, twenty-one, one hundred ten, one hundred and eleven (while pointing to each card).

Excerpt 4.1.6

Given that his method of ordering was to sequence by ones, it seems clear that Delta forced the numerals into the pattern he expected. Notice that in (11), when he was merely reading the numerals, he switched back to an appropriate routine for constructing number-names from numerals, and that he apparently felt no conflict when "twenty-one" immediately followed "ninety-one."

In summary, Delta's routine for sequencing by ten was empirical, in that it relied explicitly, as opposed to relationally, on his routine for sequencing by one, and he had no routine for sequencing by hundreds. His system for generating number-names was pre-operational, since he had yet to relate components in his system in an hierarchical structure which would allow him to make judgments of the relative order of number-names. Finally, much of Delta's behavior may be classified as heuristical, in that the routines he called upon were a result of a "guessing game"--his goal often appeared to be to satisfy the interviewer, as opposed to his calling a routine because it was a sensible thing to do relative to the problem he had constructed.

Numerical operations. The episodes in which Delta sequenced number-names by increments of ten (Excerpts 4.1.2 and 4.1.3) suggest that he structured individual number-names into major parts, such as (((NINE)TY)(EIGHT)) and (((SIX)HUNDRED)((THIR)TY)), and that he could analyze a number-name into at least two major parts, such as ((NINE)TY) and (EIGHT). Another episode suggests that Delta constructed compound names by concatenating--forming one number-name by linking two. In this task, Delta was to count as the interviewer uncovered Dienes base-ten blocks that were glued to a board. We join the interview after 2 longs, 2 unit cubes, and a flat had been uncovered (in that order).

I: (Uncovers 2 unit cubes; recovers them.) What did we have before we got that?
D: One hundred and twenty-two.
I: (Uncovers the 2 unit cubes again.)
D: Two (looking at the 2 unit cubes) . . . hundred and twenty-two (looking at the previously uncovered wood).
I: How many is that just there (pointing to the 2 unit cubes)?
D: Two . . . two and this is two hundred (pointing to the 2 unit cubes). Two (pointing to the 2 unit cubes) hundred (pointing to the flat) . . . two hundred . . .
I: We have two hundred of these little tiny blocks altogether?
D: Yes. I counted it as two (moving hand from the 2 unit cubes to the flat) . . .
I: So there's one hundred twenty-two and now altogether there . . .
D: Two hundred and twenty-two.

Excerpt 4.1.7

If we ignore, for the moment, the incorrectness of Delta's answer and focus on his means of constructing a number-name for the blocks in front of him, it becomes apparent that his method is simply to concatenate number-names within the constraints of "good" grammar (he would not say, for instance, "one hundred and twenty-two two"). The meaning that Delta gave to number-names and concatenating number-names, however, is not as apparent. To examine Delta's semantics of number-names we must look to episodes that suggest the types of unit items with which he was able to count and the nature of intensive and extensive meanings. What significance did concatenation of number-names hold for Delta? The episode in Excerpt 4.1.7 suggests that, at least for number-names beyond "one hundred," it was at most the integration of perceptual collections. Two other episodes (Excerpts 4.1.8 and 4.1.9 below) lead to the same conclusion for number-names preceding "one hundred."

I: (Places card with "40 + __ = 46" onto the table.)
D: (Begins to extend fingers.) I don't know how to do that.
I: Can I read it for you? Forty plus something we don't know yet . . .
D: (With the interviewer.) Something we don't know yet.
I: . . . is equal to . . .
D: Forty-six. You do it for me.
I: Forty plus something is equal to forty-six. Would it be forty plus one is equal to forty-six?
D: Uh-huh (yes).
I: Would it be forty plus one equals forty-six?
D: Yep.
I: What is forty plus one?
D: I don't know.
I: You do.
D: I don't.

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I: Let's do this one with the wood. Can you make forty with the wood?
D: (Picks up 5 longs; looks at them; tosses 1 long back into the box.)
I: What have you got?
D: Ten, twenty, thirty, forty (touching each of the longs; places them on the table).
I: Can you make forty-six?
D: Ten, twenty, thirty, forty (touching each long) . . . forty-five (placing 1 unit cube on the table).
I: Wait, wait. Forty (placing a hand on the longs), forty-one (pointing to the unit cube), forty-two (pointing to another unit cube).
D: Forty-two. Forty-three, forty-four, forty-five, forty-six (while placing 4 more unit cubes).
I: Aha, look. Forty (placing hand on the longs) and this many together (placing hand on the 6 unit cubes) make what?
D: Forty-six.
I: Yeah! And what's this many here (placing hand on the unit cubes)?
D: (Pause.) Forty.
I: This many (indicates unit cubes).
D: (Subvocally utters "1, 2, 3, 4, 5"; pauses.) Ten . . . five.
I: Count them.
D: I counted them. One, two, three, four, five, six.
I: Look at this problem (points to card). That's a hint (places hand on the blocks). Can you tell me this problem? Read it for me. Forty plus something equals . . .
D: Forty plus something equals forty-six.
I: Do you know what that something is?
D: Huh-uh (no).
I: There it is there (places hand over the blocks). Forty (sliding the 4 longs to D's right) plus . . . six (placing hand on the unit cubes) . . . six more is equal to forty-six altogether.
D: (No response.)
I: You haven't done any like that before?
D: Umm-hmmm.
I: You have?
D: Yeah.

Excerpt 4.1.8

Delta's initial attempt to solve the problem on his fingers is likely a carry-over from his immediately prior solution procedure for "10 + __ = 13," where he held up ten fingers and counted-on from thirteen (this task will be discussed below). Since he couldn't put forty fingers up, he had no way to add 40 to 46, and hence "I don't know how to do that" (2). Delta's later behavior sheds light on the significance, to him, of concatenating number-names, but in a negative sense. Delta never did name the four longs "forty," though he did count them "ten, twenty, thirty, forty" (22). His later naming of the unit cubes as "forty" (37) suggests that he knew that there was a "forty" and a "six" involved in the problem, but that he had not separated the blocks into two perceptual collections, one named "forty" and the other named "six," which could then be integrated into a perceptual collection named "forty-six"--concatenating then being the linguistic correspondent of integrating. Moreover, had Delta known at the level of arithmetic operations (operations on Arithmetic Lots) that concatenating "forty" and the missing number-name referred to integrating lots, then it would have sufficed to separate "six" from "forty-six" to arrive at an answer.

A positive argument for Delta's reliance on perception can be made from the episode in Excerpt 4.1.7 and the episode given below.

I: (Places card with "__ + 9 = 79" onto the table.)
D: Nine . . . nine . . . something plus nine equals seventy-nine. One, two, . . ., nine (while placing 9 unit cubes onto the table). Got a nine (places the unit cubes into a pile). Ten, twenty, . . ., seventy (while placing 7 longs onto the table; the longs are at D's extreme right, the unit cubes at his extreme left). Seventy . . . seventy. Seventy . . . nine. I don't know how to do this.
I: Count what you have there.
D: I have seventy (placing hand on the longs).
I: How many do you have altogether?
D: Nine (placing a hand over the unit cubes).
I: No, how many ALL together?
D: Seventy.
I: Is that seventy-nine (pointing to the longs).
D: No, seventy.
I: Where's the seventy-nine?
D: (Slides the 9 unit cubes next to the 7 longs.)

Excerpt 4.1.9

After Delta had made 79 with seven longs and 9 unit cubes (3-7), he seemed to have difficulty in naming what he had just constructed as "seventy-nine"--most likely because of the relatively large distance between the two perceptual collections, which made it difficult for him to integrate them (7-8). It was only after he lessened the distance between the collections that he could construct "seventy-nine"; the smaller distance allowed him to integrate the two as perceptual collections. Thus it appears that for Delta to give significance to concatenating number-names, he had to first experience integrating two perceptual collections each named in a way that the concatenation fit his grammar for number-names.

Implicit in the above discussions is the claim that Delta could construct at most perceptual collections, and hence perceptual unit items. Many episodes, one of which follows, suggest that this claim is viable.

I: (Places card with "10 + __ = 13" onto the table.)
D: I don't know how to do that.
I: Would you like me to help you? Read it for me first.
D: Ten plus nothing equals thirteen.
I: Ten plus what equals thirteen?
D: Ten plus nothing equals thirteen.
I: It's not a nothing, now. That's not nothing. It's a space—a blank. That means we don't know what goes there. We have to find out. We've got thirteen altogether. So we already know the answer. So we've got to find out what number will go there (points to the blank). (Pause.) Let's think about it.
D: (Long pause; extends all fingers on both hands; pauses.)
I: We've got thirteen altogether, right?
D: Yep.
I: But we know we've got ten (points to "10"). What else goes up to make thirteen (points to the blank)?
D: (Extends all 10 fingers.) Thirteen. Fourteen, fifteen, . . ., twenty-three (pointing to each finger). Twenty-three goes there.

Excerpt 4.1.10

Two inferences can be readily made. First, "thirteen" did not refer to a number as composed of two numbers, as shown by his solution procedure (adding 10 to 13). Second, the manner in which Delta "added ten" is significant. By at once putting up the fingers of both hands, it appears he was satisfying the condition that he have a perceptual collection (in this case, a collection of fingers) prior to counting. One might argue that in the problem Delta did solve (13 + 10), since he counted-on from 13 he must have created 13 as an abstract unit item, and hence that he was a counter with abstract unit items. On the basis of the interviews alone, this would be a viable conjecture. However, very early in his instruction it was suggested, in an attempt to make him reflect upon his actions of counting, that Delta "put the first number in his head." Rather than reflect, he took it as a recipe to follow. In the task prior to "10 + __ = 13" ("10 + 7 = __") Delta actually spontaneously said "I put the biggest number in my head" and then at once put up seven fingers and counted-on from 10. (This aspect of Delta's manner and the appeal to Delta's history--that is, going beyond the interview transcript--will be commented upon at the end of the case study.) In short, Delta seems to have been a counter with perceptual unit items, and integrating and separating existed as at most sensori-motor schemas.

Three episodes suggest that Delta had at least a sense of extending and declending as relevant routines for situations in which he understood something being added or taken away, but only in contexts where there was little complexity. In two related episodes, Delta extended by incrementing "twenty-four" once by ten (after a long was placed) and "thirty-four" three times by one (after three unit cubes were placed).

I: You can see we have twenty-four little blocks under this screen (lifts screen to show 2 longs and 4 unit cubes; advances screen so that all blocks are covered). How many under here?
D: Twenty-four.
I: (Places MAB long adjacent to screen.) How many little blocks are there altogether now?
D: Thirty-four (writes "34").
I: What did you do to get thirty-four?
D: Add another block.
I: How did you do that? I added the block. What did you do?
D: Wrote the number.
I: (Advances screen so that all blocks are covered; places 3 unit cubes next to screen). How many little blocks are there altogether now?
D: Thirty-four (points to screen) . . . thirty-four--thirty-five,

thirty-six, thirty-seven (pointing to each cube; writes "37").
I: (Advances screen so that all blocks are covered; places 2 longs next to screen). How many now?
D: Twenty-seven.
I: How many did we have before?
D: Twenty-four. (Interviewer points to "37" on D's paper.) Thirty-seven.
I: Tell me out loud what you're thinking.
D: (Pause.) It's twenty-sssss . . . ten, twenty, twenty-four (points to "24"). Thirty-seven (points to "37"), twenty . . . twenty-four.
I: Twenty-four altogether now? Are you sure?
D: (Nods head yes.)

Excerpt 4.1.11

Though Delta did extend on two separate, but related, occasions (5-7, 12-16), the significance of his behavior in this episode is that it suggests that the product of extending did not refer to a number. That is, "37" (and hence "thirty-seven") did not refer to the number of blocks under the screen. Rather, it was the last thing he had said. It may be argued that at the time of saying "thirty-four" Delta had constructed a number, or arithmetic lot, since he took it as a starting point at which to begin extending, and since he pointed to the screen while saying it. This certainly would be contrary to the claim that integrating was a sensori-motor schema for him. However, interpreting "thirty-four" as referring to the product of an action schema for extending once by ten would be more consistent with his later behavior. In lines 24-26 we see Delta attempting to reconstruct his actions (perceiving 2 longs and 4 unit cubes, extending once by ten, extending three times by one) in order to reconcile his remembering 24 being under the cover and "37" being written on his paper. Moreover, given that he had a recipe for "adding" (putting the biggest in his head), it is a small step from "putting a 'number' in my head" to "putting a 'number' under the cover."

A subsequent episode in this task sequence illustrates the characteristics of Delta's routine for declending. The interviewer had 57 (five longs and seven unit cubes) under the cover (the 57 that Delta eventually counted up to subsequent to Excerpt 4.1.11), and began removing longs one at a time from underneath. Delta counted "forty-seven, thirty-seven," and then "twenty-seven" as three longs were successively removed from underneath. When the interviewer removed four unit cubes, Delta counted all those that had been removed from underneath, answering "thirty-four" left under the screen. In short, it appears that as long as Delta had to implement only a single routine, then he could "forget" about the reason he was declending and simply continue the pattern. When the four unit cubes appeared, the pattern no longer applied--but without having constructed a conceptualization of the task (57 being successively partitioned into a number of blocks under cover and a number outside) he was forced to do what seemed most reasonable with what was in front of him: count the blocks.

Concept of ten. Several points can be made about Delta's concept of ten before examining episodes with an eye toward relationships he had established among its components. First, since Delta was a counter with perceptual unit items, "ten" had at most figural meaning--two open hands, an MAB long, "one, two, . . ., ten," and so on. It did not refer to an arithmetical lot or number. Second, Delta's routine for sequencing by ten was still empirically based on his routine for sequencing by one. Sequencing by ten was a pattern--it was not a curtailment of sequencing by one. Third, integrating and separating were sensori-motor schemas for Delta; number-names did not refer to a conceptual integration of arithmetic lots or numbers. Rather, Delta had to construct a reference for a number-name through integrating perceptual collections. Finally, since Delta was a counter with perceptual unit items, we would not expect him to understand ten as a unit of measure---a numeration unit.

Delta did have at least some components of a concept of ten, but only in action. For instance, he knew that to make 46 and 79, he could quickly do so by first counting out a collection of MAB longs and then counting out a collection of unit cubes. In each of these cases a numeral was available to Delta from which he possibly extracted a cue. In one episode where he did not have a numeral, Delta responded that there were six tens in 67. These instances seem to run contrary to the expectation, stated above, that Delta did not understand ten as a numeration unit. There are several episodes that help to resolve this conflict.

I: How many tens are there in thirteen?
D: Three.
I: Three tens. How did you figure that?
D: Cause thirteen is three. They match.
I: And that's how many tens there are?
D: (Nods head yes.)
I: If I gave you thirteen blocks, you could make how many piles of ten?
D: Three.
I: How many tens are there in sixty-seven?
D: Six.
I: How did you know that?
D: Cause six and uh . . . six is the biggest number and six match . . . go together.

Excerpt 4.1.12

Delta's error in saying that there are three tens in 13, and his reason for saying so (4), and his reason for saying that there are six tens in 67 (13), suggest two things. First, he apparently answered by focusing upon the first-said part of the number-name and homonymically translating it. Second, he had constructed this routine expressly for answering questions such as "how many tens are in . . . ." A final episode lends support to the latter inference.

I: (Places pile of sticks on the table; covers pile with his hand.) Let's imagine that this pile has seventy-two sticks in it. If you took all the tens, how many sticks would be left?
D: None.
I: Nine? How did you get that?
D: I said zero, none.
I: How did you get zero?
D: Because if I take off all the tens, there would be none left.
I: How many altogether to start with?
D: Ten.
I: Altogether, there were seventy-two. And I'm going to let you take out all the tens. Seventy-two. How many sticks would be left here?
D: Zero.

Excerpt 4.1.13

If Delta's answers in Excerpt 4.1.12 had meant what they would appear to mean on the surface, then "seventy-two" would be a number of tens <u>and</u> a number of ones. Taking out all the tens would leave only the ones. In Excerpt 4.1.13 Delta appears to have equated "tens" with "sticks," and hence taking out all the tens would leave no sticks. Whatever he did, Delta certainly did not understand "ten" as referring to a numeration unit.

Two final episodes help to complete our picture of Delta's concept of ten. In the first, Delta was shown "50" and was asked what number is three tens more than it.

I: What number is three tens more than this number (places card with "50" written on it onto the table).
D: Five . . . sixty.
I: Three tens more than that number.
D: (Long pause.) Seventy.
I: Seventy? How did you get that? Count out loud so that I can hear you.
D: I thought it--seventy's more than fifty.
I: How many more?
D: Three more.
I: Three tens, three tens more. And you get what?
D: Seventy.

Excerpt 4.1.14

At first, Delta understood the question as "What number is ten more than fifty?" (3) Again, his explicit reliance on sequencing by one is shown when he said "Five . . . sixty." However, it remains to be explained how Delta could come to understand the interviewer's reminder (4) if his concept of ten was as weak as has been depicted so far. To do this, we only need to recall that Delta possessed a routine for sequencing by ten (which seemed especially strong for multiples of 10) and to assume that he could subitize a rhythmic "three" pattern. Thus, once he determined that sequencing by ten was a relevant routine (which he appeared to do in line 3), he sequenced "fifty, sixty, seventy," stopping because the anticipated pattern had been completed. In principle, Delta gave no more significance to this task than he would have to the request to say three letters starting with "f."

The second episode, which will be examined again under "Concept of one hundred," gives us an idea of the limitations of Delta's concept.

I: I want you to do some counting by tens (picks up 16 longs; begins to place them onto the table).
D: Ten, twenty, . . ., fifty, seventy (interviewer pauses). Sixty. Seventy, eight, a hundred (interviewer pauses). Ninety. A hundred (as the interviewer places the 10th long onto the table), one hundred and one (as the interviewer places the 11th long).
I: One hundred ten.
D: One hundred ten. Hundred and twenty, . . ., hundred and sixty (as the interviewer places 5 more longs onto the table).
I: That means there would be one hundred sixty of these little blocks in there (points to a unit cube). How many hundreds would there be?
D: Hmmm. About . . . a hundred and . . . a hundred and nineteen.
I: Okay, let's keep counting (picks up 4 more longs; points to longs already placed). How many there?
D: A hundred and sixty.
I: One hundred sixty. Ready?
D: Hundred and seventy, hundred and eighty, hundred and ninety . . . hundred and . . . hundred and . . . hundred and . . . two . . . no. Hundred and twenty (as the interviewer places the 4 longs onto the table).
I: One hundred ninety and ten more.
D: One hundred ten . . . I don't know.
I: Two hundred.
D: Two hundred . . . three hundred (while sliding 1 more long next

to the pile).
I: Is that three hundred?
D: I mean two hundred and one. Two hundred and two, two hundred and three, two hundred and four, two hundred and five, two hundred and six (while sliding 5 more longs next to the pile).
I: Let's go back to two hundred (removes 6 longs). How many tens are there?
D: Two hundred.
I: Can you tell me how many tens are in that pile of wood, without counting?
D: Six hundred.
I: There's two hundred of these little guys (holds up a unit cube).
D: Two hundred . . . six hundred.
I: Six hundred tens? Let's count the tens (slides a long to D's right).
D: Ten . . .
I: One ten.
D: Two tens, three tens (while the interviewer slides 2 more to D's right).
I: Can you tell me how many tens altogether?
D: Three tens.
I: Altogether now (pauses). Keep going.
D: Three tens. Four tens, . . ., nine tens, hundred tens (while the interviewer slides the next 7 longs to D's right).
I: Wait a minute. Nine tens (slides 2 longs back to D's left, then 1 back to his right).
D: One hundred ten.
I: That's one hundred now (placing finger on 10th long). That's ten tens. Okay (slides 10th long to the right). Ten tens. Keep going.
D: A hundred and ten tens (placing a finger on the 11th long).

Excerpt 4.1.15

Several aspects of Delta's behavior in the above episode are significant. First, he readily assimilated the task to sequencing by ten. Delta's transition into the hundreds (5-6), however, suggests that the meaning that he gave to each increment was that of another long being placed on the table, as opposed to a quantity of ten being added to the quantity already made. At the point where Delta attempted to make the transition from "one hundred ninety" to "two hundred," and in the subsequent portion of the episode (19-31), he experienced exactly the difficulty he encountered in situations without physical materials--not knowing which part of the number-name to increment, and finally incrementing the first-said part. These two observations together suggest that Delta took the placement of a long merely as a cue for sequencing by ten. Second, when Delta was asked to count the longs as <u>units</u> of ten (40-57), he apparently accommodated to the task by thinking, say, "eighty," but saying "eight tens." This interpretation explains why he said "nine tens, hundred tens" as the ninth and tenth longs were placed (49). Of course, it is possible that Delta knew that 10 tens is 100 and made that association after the 10th long had been placed. It will become apparent in the next section that 10 tens had no special significance for Delta.

Concept of one hundred. Throughout the interviews, Delta showed no concept of one hundred beyond "hundred" as a name for an MAB flat or as a label that suddenly appears if one sequences far enough. The only question that will be addressed is what relationship Delta had made between "ten" and "one hundred," as that entered into the discussion of his concept of ten.

Two episodes dealt directly with Delta's understanding of the relationship between ten and one hundred as numeration units. The first arose fortuitously in the course of a task on sequencing by hundreds; the second arose in a task that was aimed at revealing the relationships that Delta had made.

I: Do you know another name for ten hundred?
D: Four hundred?
I: Is that another name for ten hundred? Have you ever heard of a thousand?
D: Ten hundred?
I: Do you know another name for ten tens?
D: Eight tens . . . nine tens.
I: What's the other name for ten tens? You know it.
D: I forgot.
I: One hundred.
D: Oh.

Excerpt 4.1.16

The above episode may be taken as weak evidence of Delta's lack of relationship between ten and one hundred--weak because of the fact that Delta may not have made a shift of context from thinking about "hundred" to thinking about "ten." The following episode gives stronger support to the contention being made.

I: (Places MAB flat on table; holds MAB long in his hand.) How many of these (indicates long) could we make from this piece of wood (indicates flat)?
D: (Softly) One, two, three, four, . . . (while pointing to individual cubes in the flat).
I: What are you doing?
D: Counting them.
I: Counting what?
D: These blocks (places hand on flat). I don't know how many we can make.
I: Is there a quicker way than by counting the little ones?
D: Ten, twenty (pointing to the longs of the flat) . . .
I: (Interrupting.) I want to know how many of this size block (holds up a long).
D: Four hundred pieces of wood.
I: Four hundred of these (holds up a long)?
D: Make four hundred pieces of wood.

Excerpt 4.1.17

Delta understood that a flat is called "one hundred" (Excerpt 4.1.7) and that it is (literally) composed of longs, as seen in (12), where he began counting the cubes along one edge while sequencing by ten. Given this, it is possible that Delta originally intended to count the longs, as opposed to individual cubes (4) and became side-tracked by the interviewer's interruption. Even if we suppose this, however, the main point is still viable--that Delta appeared not to know how many longs comprised a flat, and hence that ten tens comprise one hundred.

Concept of place value. Since Delta had an ill-formed concept of ten, and essentially no concept of one hundred, there is little need to argue that he had no concept of place value. Instead of arguing the contention that place value did not exist for Delta, we shall examine instances of difficulties that he encountered as a result of having no concept. These occur in the context of counting Dienes base ten blocks that were glued to a board.

Board: 10 2 10 100 1 10 100

I: I want you to count these pieces of wood as I show them. (Uncovers MAB long.)
D: Ten.
I: (Uncovers 2 unit cubes.)
D: Ten. Eleven, twelve (pointing to each cube).

I: (Uncovers MAB long.)
D: Twelve . . . thirteen. Oh, no--twenty.
I: Is that how many altogether? Check.
D: Ten, eleven, twelve, thirteen (pointing to each block in the order they were uncovered). Oh, I don't know what that is (pointing to the last long).
I: That's ten, isn't it?
D: Yeah.
I: Let's count again.
D: Ten, twenty . . . ten, eleven, twelve . . . (points to last long) . . . .
I: What's ten more than twelve?
D: Mmm-mmm (shrugs shoulders).
I: You know that.
D: (Shakes head no.)
I: Twenty-two?
D: Yeah.

Excerpt 4.1.18

In several places Delta failed to switch from sequencing by ten and sequencing by one and vice versa (7, 9, 15), even though the blocks he was counting changed in quantity. This appears to have resulted from Delta's tendency, as noted before, to remain within a linguistic pattern once he had called upon it as a relevant routine. In (9-15), we see this happening twice--he counted "ten, eleven, twelve, thirteen" while pointing to a long, two unit cubes, and then another long (9), and "ten, twenty" (15) while pointing to a long and then a unit cube. Delta attempted to assimilate the situation to a single routine for sequencing, and encountered difficulty when it instead required a coordination of sequencing routines. Without a network in which his routines were operationally related to one another, Delta was forced to accommodate to a change in the type of block by ignoring the quantitative aspect of the problem and focusing instead on a block as a condition for implementing a particular routine for sequencing. When he had to make two accommodations of this type (change from one sequencing routine to another), Delta would lose track of the number-name from which he was to start (Excerpt 4.1.11).

Comment. An accurate characterization of Delta is that he was a counter with perceptual unit items, had little understanding of what was being asked of him, and had developed a method of dealing heuristically with situations in which he was being asked to do things which, to him, made little sense. The heuristic method he used was much like the means-end analysis described by Newell and Simon (1972): search for a relevant routine that will eliminate differences between a current state and a goal state. The difference between the means-end analysis employed by Newell and Simon's subjects and that employed by Delta is twofold: Delta could not anticipate the use of more than one routine at a time (i.e., he could not plan to use, say, integrating and then concatenating), and Delta's goals were figural in nature--construct a number-name corresponding to the integration of perceptual collections, abstract a criterion for sequencing in order to continue an example offered by the interviewer, and so on.

Delta used another type of heuristic--a "test taking" heuristic, if you will--that did not serve to satisfy a goal in the normal sense of the word. It was a way of handling the interview situation where he knew he was supposed to give an answer, but had no idea of what was being asked of him. The heuristic was to search for a routine that had input conditions closely approximating the situation as he had constructed it so far. If there were blocks in front of him, he counted. If a block was a long, he sequenced by ten; if a block was a unit cube, he sequenced by one. If an arithmetical sentence was presented to him, he "put the biggest number in his head." But the routines he had available were data driven--Delta called them only when the input conditions were present--as opposed to hypothesis driven, where a routine would be called because its product may serve to satisfy a condition of a goal-state constructed as part of a problem.

A final remark about going beyond the interview data to understand Delta's behavior. If the author had not known of Delta's instructional history, the fact that he would count-on could have posed a problem: he behaved much like a counter with perceptual unit items except when it came to extending beyond a "hidden" quantity, which is normally taken as grounds for inferring that a child can construct arithmetic lots. The author would not have made this inference even without his historical knowledge. Given the nature of Delta's behavior outside of the instances of counting-on, especially his lack of relationships among domains of knowledge in numeration, and given that many children are amazingly adept at abstracting figural routines for dealing with school mathematics (Erlwanger, 1973), it would have been supposed that Delta was in fact dealing heuristically with the situations in question. Had Delta shown operational knowledge of numeration and yet acted as if he were a counter with perceptual unit items, then it would have been impossible to explain his behavior within the constraints of the framework used in this study.

Case Study 4.2: Lambda

Lambda was a second-grader (age 9 years at the beginning of the 1977 school year). In November of 1977 she correctly solved Problem 1 of Figure 4.1 (p. 105), after several false starts, by counting "one, two, three, four, five," sequentially putting up fingers as she counted, and then continued "six, seven, eight" again putting up fingers. She did not attempt a solution to Problem 2, and solved Problem 3 using the same method as for Problem 1. Lambda could sequence "ten, twenty, . . . a hundred," but in continuing she sequenced by one. She could not sequence by ten from "two." She had essentially no notion of ten as a number: when asked to make 54 sticks using the bundles, she counted by one while putting out four single sticks and continued counting by one as she placed (at the interviewer's insistence) bundles of ten. She also thought that two bundles of ten and five single sticks made nine sticks in total. The final interviews were given to Lambda on May 5, 10, and 16 of 1977.

Some portions of Lambda's behavior in the interviews was difficult to model--she would respond with number-names that seemed to have no relation to the task the interviewer had in mind, nor to a task that she might reasonably have constructed from the interviewer's actions and remarks. Enough instances of this sort occurred that they will be specifically commented upon following the completion of Lambda's case study.

Writing numerals. Lambda would occasionally reverse the digits of a numeral, such as "81" for "eighteen." Also, she would include a zero following the hundreds digit, such as "1078" for "one hundred and seventy-eight." Digit reversals occurred most often with number-names involving "one" or "teen." However, Lambda correctly wrote "13," "41," "37," "61," and "72" on other occasions. If Lambda's routine for writing numerals was to write a digit for distinctive parts of the number-names (as it appears), then perhaps her errors for "eighteen" ("81") and "seventy-two" ("702") arose from a <u>syllabic</u> elaboration of the number-names--"eight-teen" and "seven-ty-two." Why Lambda would sometimes completely elaborate by syllables and sometimes not, however, is not clear.

<u>Reading numerals</u>. In Excerpt 4.2.1, the interviewer placed cards with numerals written on them, and asked Lambda to read them as they were placed.

		Card	Lambda
1		"18"	Eighteen.
2		"26"	Twenty-six.
3		"73"	Thirty . . . seventy-three.
4		"120"	A hundred and twenty.
5		"174"	One hundred and seventy-four.
6		"201"	Twen . . . twenty-one.
7		"311"	Thirteen . . . thir . . . thirty-one.
8		"410"	Fortesee.
9		"549"	Twen . . . fifty-nine.
10		"936"	Ninety . . . three.
11	I:	What about the six there?	
12	L:	Ninety-three . . . ninety-three six. Ninety-three to six. I	
13		don't know how to do that.	

Excerpt 4.2.1

The above episode is especially interesting. Except for beginning to read "73" in reverse (3) Lambda seemed not to have any special difficulty, even with "120" and "174." However, in (6), where she eventually read "201" as "twenty-one," we see that she <u>changed</u> her method of partitioning the numeral (*##-#*, as opposed to *#-##*). Her hesitation possibly stemmed from a global sense that she was reading differently than she had been (which cannot be modeled), or from reading "20" as "twenty," and not knowing what to do with the remaining digit. From the ways she read the subsequent numerals (7-10), it appears that she accommodated to the conflict by ignoring the middle digit and forming a number-name from the two end ones. Even then, Lambda read "936" as "ninety-three," which suggests that her accommodated routine was itself unreliably applied. Possibly, she was still trying to accommodate to the fact that the numerals had three digits and that she was using only two.

Sequencing. Lambda showed in several episodes that she could sequence by one and by ten with little difficulty--as long as she stayed within a century. She had tremendous difficulty making a transition from one century to another, whether sequencing by one or by ten. Lambda's attempts to make such a transition suggest both the nature of her sequencing routines and the structure of her number-names. The following episode took place in the context of Lambda's counting MAB longs. She had already counted up to 160.

I: Let's keep counting. One hundred sixty, right?
L: Hundred and seventy, hundred and eighty, hundred and ninety, hundred and . . . (while the interviewer places 17th, 18th, 19th and 20th longs)
I: What comes after one hundred and ninety? Ten more than one hundred and ninety?
L: One hundred and thirty! Is that right?
I: You think it's right?
L: (Pause.) Mmm-mmmm (shakes head no).
I: We already counted one hundred thirty. So it's not one hundred thirty. One hundred and ninety and ten more.
L: One hundred and ninety . . . one hundred and ninety . . . one hundred and ninety . . .
I: Do you want to count them all? One hundred ninety-one (pointing to a unit of the long) . . .
L: Hundred and ninety-one, hundred and ninety-two, . . ., hundred and ninety-nine . . . hundred and two (while the interviewer points to each unit of the long).

Excerpt 4.2.2

The explanation of Lambda's behavior in the above episode is quite elaborate. We shall use lines 16-18 and two other episodes to explain why she said "one hundred and thirty" as ten more than "one hundred ninety."

In (16-18) we see that Lambda went from "hundred and ninety-nine" to "hundred and two." In another episode where Lambda was counting MAB flats, she counted "hundred and three, hundred and four" as she pointed to two flats that were uncovered after she had already counted two others. These two observations suggest that Lambda was not merely suppressing "one" when saying "hundred ninety-nine," but that she did not include it as part of the number-name at all. When she came to a point where she had to transcend a century, Lambda had no digit-name to increment. Hence, what would conventionally have been "two hundred" came out as "hundred, hundred → hundred and two." If Lambda had had a history of generally dropping the digit-name preceding "hundred" when learning to sequence beyond "one hundred," then her method of transcending centuries would likely have become routinized to the point where we see the inconsistencies above regardless of whether she said a digit-name prior to saying "hundred" or not. Finally, Lambda knew that two tens is twenty (shown later in Excerpt 4.2.14). So in (5-7), where Lambda said "one hundred and thirty" as ten more than 190, she perhaps thought "one hundred ninety → hundred and two," and being in the context of both counting longs and sequencing by ten, made the association "two → twenty." Then she answered the question of ten more, thinking "hundred and twenty → hundred and thirty"--saying "<u>one</u> hundred and thirty" to match the grammar the interviewer had been using.

On the only three occasions where Lambda had the opportunity to sequence by ten either from a digit name to the teens or the teens to the twenties, she instead sequenced by one. This suggests that she had not constructed the special case productions that this requires (see Figure 3.6 , page 60). It does suggest, however, that she had at least functionally related the two sequencing routines, in that whenever she couldn't sequence by ten, Lambda could at least fall back on sequencing by one. Lambda's relationship between sequencing by ten and by one will be addressed again under "Concept of ten."

The following episode suggests Lambda's reliance on sequencing by one when sequencing by ten.

I: Start at ninety-seven and count-back by tens.
L: Ninety-seven, eighty-seven . . . seventy-seven . . . sixty-seven . . . sixty-seven . . . sixty-seven . . . thirty-seven--forty-seven--fifty-seven, forty-seven . . . thirty-seven . . . thirty-seven . . . twenty-seven . . . eleven, and that's all.

Excerpt 4.2.3

Two aspects of Lambda's behavior are significant. First, she appeared to be checking for a reciprocal relationship between the next-back and next-forward number-names. That is, "sixty-seven" is next-back from "seventy-seven" when sequencing backward by ten because "seventy-seven is next-forward from "sixty-seven" when sequencing forward by ten. The reason for postulating this "check" is that otherwise the fact that she caught her error (3-4) would be unexplainable. Second, the way in which Lambda terminated her sequence (5) suggests that she was explicitly relying on sequencing backward by one to generate her backward sequence by ten. By saying "twenty-seven . . . eleven, and that's all" Lambda may have been saying "there's nothing backward from one to make a ten-word out of," where "eleven" was her ten-word for "one."

It might seem that Lambda's routines for sequencing were at least close to being operational, since she had constructed a reciprocal relationship between next-back and next-forward—a classical criterion used by Piaget to assess concrete operationality (Piaget, 1965). However, since the relationship she used was, in effect, between terms of the sequence "ten, nine, . . ., one" this says little about how Lambda had related number-names in general. We can get a better idea of the relationships she had established by looking at episodes from tasks in which she was asked to seriate numerals.

The following two seriation episodes suggest that Lambda's routines for constructing and sequencing number-names had yet to be operationally related among themselves. The first (Excerpt 4.2.4) shows Lambda's lack of transitivity, while the second (Excerpt 4.2.5) shows that she did not have operational reversibility between ordering forward and backward.

Cards: 8 12 13 17 19 21 31 102 (shuffled)

I: Here are some more cards. Can you put these in order?
L: (Places cards one at a time onto the table.) (13)
I: Is that the smallest one?
L: (Nods head yes.) (13-17) (13-17-19)
I: Say the numbers aloud as you place them down.
L: Thirteen, seventeen, nineteen. (13-17-19-21) Twenty-one. Oh, I forgot. (12-13).
I: What's the number you just put down?
L: Twelve. (Pause.)
I: What's this number (points to "13")?
L: Twe . . . thirteen. (12-13-17) Seventeen. (12-13-17-19) Nineteen. (12-13-17-19-21) Twenty-one. (Removes all cards.) (8).
I: What's that card?
L: Eight. (8-12) Twelve. (8-12-13) Thirteen. (8-12-13-17-19) Nineteen. (8-12-13-17-19-21) Twenty-one.

Excerpt 4.2.4

In lines (6-7) and (12-13) we see that when Lambda found a numeral that she should have placed earlier in the sequence she accommodated by removing <u>all</u> the cards placed so far, placing the new card, and then reconstructing the sequence anew. Had she operationally seriated the numerals, she would have known, by transitivity of "after" within her number-name sequence, that it was necessary only to move each card one position to the right.

Cards: 30 47 48 49 52 61 67 76 (shuffled)

I: This time I want you to place these cards on the board starting with the biggest number in the first place (indicates) and the smallest number in the last place (indicates).
L: (Spreads cards on the table.) (48) Forty-eight is the largest number . . . no. (49)
I: Forty-nine is the largest number?
L: (49-48) (49-48-47) (49-48-47-61)
I: Say the numbers as you put them down.
L: Forty-nine, forty-eight, forty-seven, sixty-one.
I: Is sixty-one smaller than forty-seven?
L: (61) (61-49) (61-49-48) (61-49-48-47) (61-49-48-47-52) Fifty-two.
I: Is fifty-two smaller than forty-seven?
L: (61-49-48-52-47; switched "47" and "52").
I: Is fifty-two smaller than forty-eight?
L: Yeah. (61-49-48-52-47-67) And sixty-seven.
I: Is sixty-seven smaller than forty-seven?
L: (61-49-48-52-67-47; switched "67" and "47") (67-49-48-52-61-47); switched "57" and "61").
I: Now what have we got?
L: Sixty-seven, forty-nine, forty-eight . . . wait (67-48-49-52-61-47). Fifty, sixty-one, forty-seven (while the interviewer points to each).
I: Is sixty-one smaller than fifty-two?
L: No. (67-48-49-61-52-47; switched "61" and "52").
I: Is sixty-one smaller than forty-nine?
L: Yeah. (67-48-49-61-52-47-76) (67-48-49-61-52-76-47).
I: Why did you change those around?
L: Isn't this one bigger (holding up "76" card)?

Excerpt 4.2.5

Lambda initially only locally seriated backward, constructing the sequence "49-48-47," and then placing "61." When asked to reconsider the placement of "61," she destroyed the sequence in order to re-place "61" (again, showing lack of transitivity). It is difficult to tell what Lambda might have placed after "61" in (7) had the interviewer not interrupted, but in (27) we see that she was still aware of the aim to put the cards in backward order, and that she did so only in subsequences. Also, in (21) it seems that Lambda failed to fully coordinate sequencing backward with sequencing forward, as she switched "48" and "49" so that she had a forward subsequence of "48-49-52" implanted within the larger (backward) sequence.

Numerical operations. Three episodes together suggest that Lambda was at least a counter with motoric unit items. The first two are only indicative of this, while the last is strongly suggestive of it.

I: (Places card with "10 + 7 = ___" onto the table.)
L: Ten plus seven. (Places both hands flat on the table; moves the 5 fingers of her right hand one at a time, then 2 of her left.) Sixteen.
I: How did you get that?
L: Counted on my fingers.
I: Show me.
L: Ten. Eleven, twelve, . . ., fifteen (while tapping each finger of her right hand on the table). Sixteen, seventeen (while tapping 2 fingers of her left hand on the table). Oops! Seventeen.

Excerpt 4.2.6

This episode in itself could not be taken in support of a claim for Lambda having been a counter with motoric unit items. The fact that she placed both hands open on the table could be taken as suggestive that she required a perceptual collection prior to counting. However, since she did not form a "seven" pattern, and since she moved her fingers singly as she counted, the question remains open. The next episode gives more positive support.

I: (Places card with "10 + ___ = 13" onto the table.)
L: Ten plus blank equals thirteen. (Moves 3 fingers of her left hand.) Three more equals thirteen.
I: How did you get three?
L: Eleven, twelve, thirteen (while tapping each of 3 fingers of her right hand against the table).

Excerpt 4.2.7

The manner in which Lambda solved "10 + ___ = 13" suggests two things: first, that she conceived of "ten" and "thirteen" at least as being connected within a counting sequence, and second, that she could anticipate a count without having a perceptual collection of a specified size. We see this again in the following episode.

I: (Places card with "40 + ___ = 46" onto the table.)
L: Forty plus blank equals forty-six. (Holds out her right hand held in a fist; uses left hand to individually put up each finger of her left hand; puts up 1 finger of her left hand.) Six more equals forty-six.

Excerpt 4.2.8

In this episode it is clear that Lambda did not require a perceptual collection prior to counting, and that she had anticipated connecting "forty" with "forty-six" through counting. Thus Lambda was at least a counter with motoric unit items, and possibly a counter with abstract unit items. These episodes also suggest that, for Lambda, number-names (less than 100) were at least signs of a scheme for counting (as opposed to indices for sequencing--cf. Delta), and possibly as symbols for counting (being conventionally arbitrary in nature in that she could have used, with the same logic, any other memorized, linearly-ordered collection).

The possibility of her number-names being symbolic of Lambda's counting scheme is made implausible when we recall that her sequencing routines were still preoperational. Operationality of sequencing would seem to be necessary if her number-names existed as symbols, for the arbitrariness of the system of base-ten number-names comes from the realization that any linguistic system with the same structure could be used in its place. Since Lambda did not have a base-ten structure for her system of number-names, she could not have made this realization.

We have yet to see any counter-evidence to the possibility of Lambda's having been a counter with abstract unit items. The fact that she counted-on in each of Excerpts 4.2.6-8 suggests the possibility that Lambda constructed a number or arithmetical lot as the first addend. There is also the possibility, however, that counting-on was either a functional curtailment of counting-all or a result of instruction (as was Delta's counting-on strategy). This will be investigated in the subsequent episodes.

I: (Places card with "70 - 92" onto the table.) How many is it from seventy up to ninety-two?
(Picks up some MAB blocks.)
I: Can you do it without using the wood?
L: (Begins counting on her fingers.)
I: Say out loud what you're doing. Start again.
L: Seventy, seventy-one, seventy-two, . . ., ninety-one, ninety-two (hitting her left hand on her right with each utterance).
I: How many is that?
L: Three . . . three.

Excerpt 4.2.9

Lambda's behavior in (7-8) suggests again that she was at least a counter with motoric unit items, for she could anticipate connecting "seventy" with "ninety-two" through counting, and the only functional prerequisite for her counting was motoric activity. However, the fact that she did not quantify her counting suggests that she was not a counter with abstract unit items, and could quantify an extension only when she could construct a subitizable record of her counts (e.g., finger patterns--cf. Excerpts 4.2.7 and 8). This appears to have been her intention when she initially began putting up fingers (5).

Additional support for the inference that Lambda was at most a counter with motoric unit items comes from an inspection of episodes involving integrating, separating, and the relationship she had established between the two.

I: How much have we got here (places 2 longs and 4 unit cubes onto the table)?
L: Two tens and . . . twenty. Twenty-one, twenty-two, twenty-three, twenty-four (as the interviewer places 4 unit cubes next to the 2 longs).
I: (Advances screen so that all blocks are covered; places MAB long adjacent to screen.) How many little blocks are there altogether now?
L: Thirty-four (writes "34"). Ooops. (Erases "4" and writes "7" in its place—"37.")
I: (Advances screen so that all blocks are covered; places 2 MAB longs.) How many now?
L: (Pause; speaks very softly.) Thirty-seven . . . thirty . . . seven . . . forty . . . fifty. Fifty (writes "50").
I: Are you sure now? How many did we have here? Thirty-seven? And how many did we put down?
L: Twenty.
I: Thirty-seven and twenty?
L: It's twenty.
I: It's all that's under here. Thirty-seven and this many. Are you happy that it's fifty?
L: Fifty.
I: (Advances screen so that all blocks are covered; places 1 long and 1 unit cube.)
L: Sixty-one (writes "61").
I: (Advances screen so that all blocks are covered; places 2 unit cubes and 2 longs.) Write down how many little blocks I have altogether now.
L: Seventy-two.
I: (Advances screen so that all blocks are covered.) How many blocks have I got under here altogether now?
L: Seventy-two.
I: Write that in the little box now (points to box at the of Lambda's paper).
L: (Writes "72" in the box.)
I: I'm going to take out some pieces of wood. I want you to tell me how many are left after I take out these pieces of wood (removes 2 longs).
L: That's twenty.
I: How many's left?
L: (Long pause.)
I: What did we start off with?
L: I don't know.

Excerpt 4.2.10

It is possible that, at each stage in the development of the task, Lambda had integrated the collections of blocks as abstract units, and that the number-names she constructed referred to numbers (ignoring, for the moment, the incorrectness of the constructions). Lambda's behavior in (26-43) suggests that she did not construct numbers. Even though she said, and reiterated, that there were 72 blocks underneath the cover (29-35), when it came to removing blocks from beneath the cover Lambda did not know where to start--instead naming only the amount removed. That is, the number-names that Lambda had constructed as additional blocks were placed were kept in mind only as an input condition for continuing her forward counting sequence. The entire episode was <u>one</u> counting sequence for Lambda, with long pauses as the interviewer placed more blocks. She did not construct numbers from the successive additions of blocks.

Another episode suggests that Lambda could not conceive of integrating <u>anticipated</u> counting actions, and hence that her conceptualization of missing addend problems (Excerpts 4.2.7 and 8) was one of connecting number-names through counting while constructing a subitizable record of her collection of counts.

L: (Posing a problem for the interviewer--first part of dialogue is missing.) . . . twenty. How more than . . . how more . . . higher . . . to get to two hundred?
I: How many more do I need to get to two hundred? Oh, that's a hard question. Would you let me think about that for awhile?
L: And no guessing. You can use the blocks.

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I: Okay, I'll do your question. What was it? How many more to make two hundred? Okay, we've got twenty.
L: Right.
I: I'll show you how I'd do it. Twenty. Thirty, forty, . . ., seventy (placing 5 longs on the table). Am I right so far?
L: Kind of.
I: Eighty, ninety . . . ninety . . . ninety. What comes next after ninety? Could you help me?
L: No (giggles).
I: Ninety . . . hundred and one?
L: No! Ninety. Ninety-one, ninety-two, . . ., ninety-nine (stops; looks at interviewer with a smile).
I: Ninety-nine . . . ahhh . . . a hundred.
L: Nooooo! Two hundred!

Excerpt 4.2.11

Though the entire episode is interesting for the insights it provides about Lambda's conception of interviewing, its significance for the current investigation lies in the clues it gives to the problems she could construct. In (1-2), where Lambda is trying to pose her question, we see that she had trouble expressing the idea of *quantifying* an extension of 20--that is, of how *many* more. Also, her phrase ". . . to *get* to two hundred" (3) suggests that Lambda was in fact talking about the dynamics of getting from 20 to 200--counting; her rejoinder to the interviewer (20) suggests that the goal that Lambda conceived of was that of *arriving* at "two hundred," as opposed to quantifying the itinerary. That is, Lambda could not construct a number referred to by "two hundred" that was the integration of one number named "twenty" and another that extended it. What she could conceive of was extending from "twenty" to "two hundred"--continuing a counting sequence.

Several episodes suggest the nature of separating for Lambda, and the relationship she had constructed between separating and integrating. One that is particularly illustrative is given below.

I: (Places card with "70 - 31 = __" onto the table.)
L: Seventy take away thirty-one. (Lays both hands on the table with palms up; begins moving her fingers.)
I: If you want, you can use some of these (points to box of MAB blocks--which also had some popsicle sticks in it).
L: (Places 7 popsicle sticks on the table one at a time.) Eight, nine, ten (while placing three more popsicle sticks).
I: Tell me what you're doing.
L: Counting to seventy. Eleven, twelve, . . ., twenty-four (while placing more sticks onto the table). Wait a minute, I have an easier way. (Places all sticks back into the box.) Ten, twenty, thirty, . . ., seventy (while placing 7 longs onto the table). One, two, . . ., thirteen (while placing 13 unit cubes onto the table). Here's thirteen. I take away this (handing the unit cubes to the interviewer) and I have seven more.
I: Now tell me, what did you do?
L: Seven . . . seventy take away thirteen equals seven because . . . (writes "7" on the card).
I: You have seventy there (pointing to the 7 longs) . . .
L: (Interrupting.) And I took away thirteen and I have seven more left (erases "7"; writes "70").
I: You have seventy more left?
L: Yeah.
I: Just a minute now.
L: What?
I: You started off with seventy. And then you took away . . .
L: Thirteen. I have seventy more left. Like this (writes "70 - 13 = 70" on her paper). Seventy take away thirty-one equals seventy.

Excerpt 4.2.12

When Lambda constructed the piles for the minuend and subtrahend (9-13), she may have been acting out an action schema for subtraction that she had constructed as a result of instruction--make a "source" pile, make a "take-away" pile, remove the take-away pile, count what is left. Even if this is the case, we see that separating was an action schema for Lambda--an empirical routine. "Seventy" did not refer to a composite which contained 13; "thirteen" did not refer to a component of 70. That is, Lambda had at most only functionally related integrating and separating. Integrating produced larger collections while separating produced smaller collections, but Lambda related the two only in the sense of opposites--just as up is the opposite of down. If she had operationally related the two, then 70 would have been conserved because of the necessity inherent in an operational structure.

A final episode (a continuation of the one given in Excerpt 4.2.9) suggests that Lambda had also functionally related extending and declending.

I: So how many is it from seventy to ninety-two?
L: Three. Cause I had three hops.
I: I'm going to tell you. From seventy to ninety-two, it's twenty-two. Watch. Seventy to eighty is ten. Ten to eighty. Another ten to ninety.
L: Right.
I: Two tens.
L: Right.
I: And to ninety-two . . .
L: Right.
I: . . . is ninety-one, ninety-two. So that's two tens and two. Twenty-two. Okay?
L: (No response.)
I: (Places card with "92 - 70" onto the table.) How many is it from ninety-two down to seventy?
L: It's still seventy . . . twenty-two.
I: How do you know that?
L: Because you go backwards seventy (pointing to the card) . . . something seventy-two . . . you counted backwards, it's still twenty-two.

Excerpt 4.2.13

In Excerpt 4.2.9 we saw that Lambda could not quantify the extension of 70 to 92. In the above excerpt, however, we see that once the interviewer had told her that it was 22 and had exemplified the construction of it (3-12), Lambda was apparently able to reconceptualize the problem as an extension of 22 beyond 70 having taken place, and knew that what could be counted forward was the same even if counted backward. The reason that an inference of Lambda's operational reversibility between extending and declending is unwarranted is twofold. First, when claiming that it was still 22, only backward, she already understood that 22 was the answer to "70 → 92," and she referred to the card when saying "you go backward." So it is quite possible that the "backward" that Lambda had in mind was, essentially that "92 → 70" is backward from "70 → 92." Second, if we inferred that Lambda had operationally related extending and declending on the basis of this one episode, then we would have to explain why she didn't show operational reversibility in any other situation where she had an opportunity to do so.

Concept of ten. So far we have concluded that Lambda's routine for sequencing by one was fairly well established except for transitions between centuries, that her routine for sequencing by ten was established at least between the teens from century to century, that her number-names were at least (but likely no more than) signs of her scheme for counting, and that she was a counter with motoric unit items who could functionally relate integrating and separating and extending and declending by one. We have yet to examine the relationships that she had established among these components.

In Excerpt 4.2.12, lines (9-13), we see that Lambda changed from counting by ones to counting by tens while making a collection of 70. This suggests at least that Lambda had established a functional relationship between counting by ten and by one. From that episode alone it is difficult to infer more than a functional relationship, for it could have been that the "easier way" Lambda spoke of was to count longs instead of sticks, as opposed to count by ten instead of one. Another episode suggests that Lambda meant more than merely counting longs as her easier way.

I: Can you make me a pile with a hundred in it?
L: One, two, . . ., twenty (sliding a unit cube with each utterance; scoops the 20 cubes into a pile). Twenty-one, . . ., thirty (sliding a unit cube with each utterance; scoops all cubes into a single pile). Thirty-one, . . ., forty (sliding a unit cube alongside the pile with each utterance; pushes all together). Forty-one, forty-two, . . ., sixty (sliding a unit cube into the pile with each utterance).
I: How many more blocks would I need to make a hundred now? (Pause.) We're up to sixty now.
L: You have . . .
I: (Interrupting.) How many more do I need to make a hundred?
L: (Pause.) Three more. Sixty--seventy, eighty, ninety.
I: All right, keep going.
L: (Counts out 20 more blocks.) Sixty-one, . . ., eighty.
I: All right. How many more blocks will we need now?
L: One more. Ninety.
I: Okay, go on.
L: (Counts out 10 blocks.) Eighty--eighty-one, . . ., ninety.
I: Ninety. How many do you think we need now to make a hundred?
L: One.
I: Just one more?
L: (Nods head yes.) One.
I: Okay, keep going . . . ninety.
L: (Counts out 10 blocks.) Ninety-one, ninety-two, . . ., hundred.

Excerpt 4.2.14

The significance of Lambda's behavior in this episode comes from the fact that even though she had been counting single cubes, when the interviewer asked her how many more to make one hundred (12), she responded in terms of a number of tens (13). It appears that she conceived of an increment by ten as meaning extending ten times by one--a "cycle" of ten.[2] Reflecting on Lambda's "easier way" in Excerpt 4.2.12 (9-12), it appears that she meant that counting by ten could

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[2]The term "cycle," as used in this context, was offered by Deborah Wolpow, a student at San Diego State University.

be used in place of counting by one to achieve the same result--"seventy" as the name of a collection that, were it counted by one, would result in a sequence ending with "seventy."

The above discussion allows us to make at least some sense of Lambda's remarks at the end of Excerpt 4.2.9 and the beginning of Excerpt 4.2.13, where she said it was three from 70 to 92--because she made "three hops." It is possible that Lambda reconceptualized (after having counted) her count from "seventy" to "ninety-two" as taking place in stages--one from "seventy" to "eighty," one from "eighty" to "ninety," and another from "ninety" to "ninety-two." Thus, a cycle of ten appears to have been a "hop" for Lambda. This does not tell us whether Lambda thought of each cycle of ten as a unit, however. Three "hops" is most likely within her subitizing range, so Lambda might easily have thought of them quite literally as hops, and not as <u>countable</u> items. This inference is consistent with our earlier conclusion that Lambda was a counter with motoric unit items.

Since we have concluded that Lambda was a counter with motoric unit items, it would be difficult to imagine that when she said "Three more. Sixty--seventy, eighty, ninety" (Excerpt 4.2.14, line 13), it would be difficult to imagine that she meant each increment of ten to signify a <u>number</u> (of ones). That she did not mean this is suggested by the following episode (the continuation of Excerpt 4.2.2).

I: One hundred and two? Two hundred. Okay, so that's hundred and seventy, hundred and eighty, hundred and ninety (pointing to the 17th, 18th, and 19th longs; points to 20th long) . . .
L: Two hundred.

I: Good. Okay, how many tens are there in two hundred, Lambda? Do you know that without counting these tens?
L: Two hundred.
I: There's two hundred little tiny blocks. How many tens would there be?
L: Two hundred.
I: Let's count them. Count how many tens. There's one (sliding 1st long to his right).
L: Two, three, . . ., twenty (sliding each of the remaining longs to her left).
I: How many tens?
L: Twenty.
I: Twenty tens?
L: Mmm-hmmm (yes).
I: If I put this many more down (places 4 longs on the table), how many tens will there be now, altogether?
L: (Pauses; slides, one at a time, each long over to the pile of 20 longs.) Sixty!

Excerpt 4.2.15

It appears that Lambda's answer "sixty" (21-22) came from counting: "twenty--thirty (one long), forty (another long), fifty (another long), sixty (last long)." If this is the case, then when Lambda counted the longs as units, she did so without the significance that each was a unit of units. Thus, when she came to extending beyond 20, she conceptualized "twenty" as referring to the end of a counting sequence, and then took each long as an input condition (index) for sequencing by ten--where each increment by ten had the significance of a cycle of ten. In the end, she produced an implicit counting sequence to "sixty." It appears, then, that when Lambda explicitly counted by one (11-14), the significant aspect of each count was her motoric action, while when explicitly counting by ten it was her linguistic action of incrementing by ten.

Though Lambda had the basic conceptualizations necessary to give at least figural meaning to questions such as "how many tens in . . ." (e.g., cycles of ten), she apparently had not constructed the linguistic productions necessary to readily answer them (as had Delta, but without meaning). This can be seen in the following episodes.

I: How many tens are there in thirteen?
L: Forty.
I: Can you tell me how many tens there are in thirteen? (Pause.) What did you just tell me?
L: Ten.
I: How many?
L: Forty!
I: Mmm-hmmm.
L: It is right.
I: How many tens are there in sixty-seven?
L: Fifty . . . fifty.
I: How do you know that?
L: Cause it's more than sixty-seven.
I: How many tens are there in sixty-seven?
L: Eighteen.
I: How many? Eighteen or eighty?
L: Eighty.
I: Are you sure now? Why do you say that?
L: Because it is that.

Excerpt 4.2.16

Whatever Lambda's method of arriving at her answer in (1-7), it certainly was not by way of conceptualizing 13 as containing a number of tens. Possibly, Lambda assimilated "thirteen" as "thirty" and thought to form the next ten-word. Her method of arriving at her answers in (10-19) is not apparent at all.

The final episode examined in this section suggests once again Lambda's inability to conceptually coordinate ten and one as units.

I: (Places pile of sticks on table; covers pile with his hand.) Let's imagine that this pile has seventy-two sticks in it.
L: (Writes "702" on a piece of paper.)
I: If you took all the tens out of seventy-two, how many sticks would be left?
L: None. (She wants to count the sticks, but the interviewer will not let her.) None.

Excerpt 4.2.17

It seems that Lambda equated "tens" with "sticks"--if you take out all the tens, then there would be no sticks. This is a weak interpretation, however, since she may have understood the question as "If you took out all the <u>sticks</u>, then how many sticks would be left?" Of course, this would still suggest that even if she was able to understand the question as intended by the interviewer, her conceptualizations were not strong enough to make the assimilation process automatic. Since the interviewer failed to probe her answer, we cannot tell which of the two interpretations is more viable.

<u>Concept of one hundred</u>. Throughout the interviews, Lambda had considerable difficulty with any task involving hundred. As pointed out earlier, transitions across centuries while sequencing were almost always problematic for Lambda, as was sequencing by hundred.

I: Start at thirty and count-on by hundreds.
L: Thirty, forty, fifty, sixty . . .
I: (Interrupting.) Counting by hundreds--not tens.
L: Thirty . . . (pause).
I: Let me give you the next one. Thirty, one hundred and thirty . . .
L: One hundred and forty, one hundred and fifty, . . ., one hundred and ninety . . . one hundred and . . .
I: That'll do.

Excerpt 4.2.18

Board: 100 10 10 100 10 10 10 100 100

I: What's this (pointing to 1st flat)?
L: A hundred (pointing to the flat), a hundred and ten (pointing to long), a hundred and twenty (pointing to long) . . .
I: What did we say this one was (pointing to flat)?
L: A hundred and twenty . . . I don't know.
I: A hundred, a hundred and ten (Lambda joins in), a hundred and twenty (pointing to each of first 3 pieces on the board; points to flat). And a hundred more . . . two hundred and twenty. (Points to next long; pauses.)
L: Two hundred and ten . . . two hundred and twenty . . . two hundred.
I: Do you know what a hundred more than two hundred thirty is?
L: Huh-uh (no; pauses). Hundred and three, hundred and four (as the interviewer points to each of the last 2 flats).

Excerpt 4.2.19

In the first of the above excerpts we see that Lambda assimilated the task to sequencing by ten--possibly choosing it over sequencing by one because the increment was closer to "hundred." Had the interviewer exemplified another term in the sequence, Lambda might have abstracted a pattern.

The second excerpt suggests a basic difficulty in Lambda's ability to structure number-names involving "hundred" (addressed earlier under "Sequencing"). It seems that in her analysis of the number-name "two hundred and twenty," she suppressed "twenty" and transformed "two hundred" into "hundred and two" so as to fit her routine for sequencing by hundred (13-14).

Lambda did have some aspects of a concept of one hundred. She knew that an MAB flat was called "a hundred" and that "hundred" was a number-name that could be given intensive meaning. The latter is suggested, in part, by the episode below (the continuation of Excerpt 4.2.14).

I: Okay. Let's have a look at them. One hundred little blocks.
L: Just like that (reaches for an MAB flat).
I: Do you think that's a hundred little blocks there (points to the flat)?
L: It is. Because if I do it (while placing the unit cubes on top of the flat), it will be like that. Well, it would be like that (sweeping the cubes off the flat).

Excerpt 4.2.20

The inference that Lambda gave intensive meaning to "one hundred" is viable, but somewhat weak. It is viable since (a) Lambda demonstrated in other contexts that she was capable of attributing intensive meaning to number-names, (b) she had just counted the 100 unit cubes, and (c) because she said "If I do it . . . it would be like that"--that is, she knew that there would be a figural correspondence between 100 counted blocks and a flat called "one hundred." The inference is weak because Lambda could have meant that a flat is literally composed of unit cubes, and that since both the flat and the perceptual collection of cubes were named "hundred," they were semantically equivalent. Unfortunately, there are no other episodes that help us to choose between the two interpretations, except for those taken as suggesting that Lambda <u>could</u> attribute intensive meaning to number-names.

The final episode examined in this section shows that Lambda had not constructed a relationship between ten and one hundred.

I: (Places an MAB flat on the table; holds up an MAB long.) How many of these . . .
L: (Interrupting.) Ten.
I: Yes . . . are there in that piece of wood? (Pauses.) How many pieces this big? How many tens?
L: (Picks up flat.) Ten, twenty, . . ., ninety, one hundred and two, one hundred and three (pointing to units along an edge of the flat--miscounts?).
I: One hundred and three. If we could cut this (flat) up into pieces this big (unit cube), how many little tiny pieces would we get?
L: Ten, eleven, . . ., nineteen (pointing to each unit along an edge of the flat). Nineteen.
I: You think that there would be nineteen of these little guys?
L: Yeah . . . yeah.

Excerpt 4.2.21

First, it should be pointed out that Lambda was very familiar with Dienes' base-ten blocks, and that she knew that a flat was (literally) composed of longs. Given this, it appears that when Lambda's intention was to count longs in the flat (6-8), she took a column of the flat (i.e., a long) as an impetus to sequence by tens. That is, when Lambda's intention was to count tens, her meaning was to count in sequences of ten--not units of ten. Nevertheless, Lambda apparently did not conceive of one hundred being composed of units of ten, let alone a number of units of ten.

<u>Concept of place value</u>. Lambda did not have a concept of place value beyond her figural relationship between ten and one--ten as a cycle of ones--and one hundred as a number-name that could be given intensive meaning. Her linguistic systems for sequencing were preoperational, and she was unable to coordinate them beyond

situations in which she called upon her meaning for an increment by ten.

Lambda's inability to coordinate sequencing systems, her inability to sequence by hundred, and her limited relationships among one, ten, and one hundred caused her to experience difficulties in shifting from one concept to another, as shown below.

Board: 10 2 10 100 2 100

I: I want you to count these pieces of wood as I show them. (Uncovers long.)
L: Ten.
I: (Uncovers 2 unit cubes.)
L: Eleven, twelve.
I: (Uncovers long.)
L: (Pauses; points to each unit cube of the long.) . . . seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two.
I: (Uncovers flat.) Now how many altogether? (Pause.) Oh-oh, that's a big fellah!
L: (Pauses; places hand on top of the flat.) One hundred (places hand on top of the 2 unit cubes) and two.
I: (Uncovers 2 unit cubes.) Keep going.
L: Hundred and three, hundred and four (pointing to each newly uncovered unit cube).
I: (Uncovers long.)
L: (Points to each unit cube in the long.) Hundred twenty-four.
I: (Uncovers flat.)
L: (Looks around.) Fifty.

Excerpt 4.2.22

If we suppose that Lambda had to change from an anticipation of counting longs (and sequencing by ten) to counting unit cubes (and sequencing by one), then we may infer that she had difficulty changing <u>back</u> to her routine for sequencing by ten (3-8). That is, Lambda had difficulty coordinating her concepts of ten and one. Of course, it is possible that her difficulty arose from not having constructed the special case production for sequencing by ten from "twelve" to "twenty-two." Even if she did not have it, Lambda's behavior in (11-12) suggests that this was not the cause of her difficulty. By choosing to start with the flat and then extending with the unit cubes, Lambda was apparently showing a preference for staying with the single cubes after having had conceptualized (intensively) "hundred" as a collection of ones. Thus, in (7-8) and (16-17), when she counted the single cubes in the long, Lambda apparently did so because of not being able to coordinate her concepts of ten and one.

Comment. On a number of occasions, Lambda behaved in ways that were unexplainable within the constraints of the framework--she guessed. It should be noted very quickly, however, that "guess" is a theory-laden term--most often it means that the person classifying a response as a guess does so largely because it cannot be explained by his or her theory. The cases in question regarding Lambda quite often involved her giving "fifty" as an answer (see Excerpt 4.2.22, line 19). Possibly, Lambda had a "stock" answer to give in situations for which there seemed, from her perspective, no reasonable alternative.

Case Study 4.3: Kappa

Kappa was a second-grader (age 7 years at the beginning of the 1977 school year). In November of 1977 Kappa correctly solved Problem 1 of Figure 4.1 (p. 105), answered "fourteen" and "eleven" to Problem 2 (with no apparent process), and correctly solved Problem 3. His solution procedures to Problems 1 and 3 were to first form collections with his fingers and then count-all. Kappa could not sequence by ten in any form. When asked, he sequenced by one starting with ten, and could not continue the interviewer's example of "ten, twenty, thirty." When asked to "count by tens" starting at "two," he sequenced by two. He did know that 12 is ten more than two, but when asked for ten more again he said "thirteen." When given a bundle of ten sticks and four singles, he counted them all to find the total. Kappa answered "a hundred" to "How many tens in thirty-two?" To make 32 using bundles of ten, Kappa put out nine bundles and counted them by five. The final interviews were given to Kappa on May 5, 12, and 17 of 1977.

Kappa's case study was especially difficult to construct—the framework was stretched to its limits. The difficulty stemmed from inconsistencies in Kappa's behavior. For example, several instances suggested that Kappa had constructed ten as an arithmetical unit, while others suggested just the opposite. The task, then, was to account for these disparities within the constraints of the framework.

The format of Kappa's case study will differ slightly from the previous two. The discussion will at times get ahead of itself, especially in "Sequencing" and "Numerical operations," by comparing and contrasting episodes that normally would be discussed under separate section headings. In this way conflicts and questions about Kappa's understandings that remain unresolved can be made to stand out.

Writing numerals. Kappa's routine for writing numerals had two major deficiencies: digit reversals and the inclusion of extra zeroes. This can be seen in the following excerpt.

	Interviewer	Kappa
1	Thirteen.	"30"
2	Forty-one.	"41"
3	Eighty-four.	"84"
4	Eighteen.	"18"
5	One hundred one.	"1001"
6	One hundred seven.	"17"
7	One hundred fourteen.	"114"
8	One hundred twenty-one.	"120"; "121"
9	One hundred seventy-eight.	"178"
10	Two hundred nine.	"2009"
11	Two hundred nineteen.	"291"
12	Two hundred sixty-seven.	"267"
13	Nine hundred thirty-four.	"90034" I don't know that one.

Excerpt 4.3.1

In (1), we see that Kappa heard "thirteen" as "thirty"--this will be referred to again when discussing an episode in the section "Concept of ten." Otherwise, Kappa's errors were either reversing digits or mistakes in the number of zeroes in the numerals. Kappa's mistakes in the number of zeroes occurred, with one exception, with numbers where a zero in the middle position would have been proper--suggesting that Kappa was in the process of refining his routine so that zeroes for "hundred" were suppressed. After writing "1001" for "one hundred one" (5), he wrote "17" for "one hundred seven"--perhaps overcompensating for the "funny look" of "1001." As an aside, the interviewer should have asked Kappa to read the numerals he had written.

The mistake involving digit reversal (11) is more problematic. Though he only once reversed digits in a numeral ("291" for "two hundred nineteen") in this task, this was the more frequent mistake in the rest of the interviews, and seems to have been related to a fundamental difficulty in maintaining the order of a pair of digit names. We see this more clearly in the next and subsequent sections.

Reading numerals. In the standard task for reading numerals (see Appendix I, or Excerpt 4.2.1, page 132), Kappa made only two digit reversals out of ten numerals, reading "174" as "one hundred 'n forty-seven" and "201" as "two hundred 'n ten." However, in the majority of the remaining tasks in which Kappa had to read a numeral, he reversed the digits at least once. The following is an excerpt from a numeral seriating task. Kappa had reversed the digits in a numeral on so many occasions that the interviewer intervened and asked him to read each before attempting to put them in order.

Cards: 8 12 13 17 19 21 31 102 (shuffled)

I: Here are some more cards. Can you put these in order?
K: (Reads numerals as he places the cards on the table--reading from top to bottom from the "deck.") Twelve, nineteen, eight, thirteen, seventeen, thirteen.
I: What's this number ("8")?
K: Eight.
I: What's this number ("102")?
K: One hundred 'n twenty.
I: What's this number ("17")?
K: Seventeen.
I: What's this number ("21")?
K: Twelve . . . twelve.
I: What's this number ("12")?
K: Twelve.
I: Is that twelve (points to "21")?
K: No. It's twenty. It's twenty-one.
I: What's this one ("19")?
K: Nineteen.
I: What's this one ("13")?
K: Thirteen.
I: What's this one ("31")?
K: Thirteen.
I: (Points to and from "13" and "31.") Which is thirteen?
K: Thirty-one.
I: Point to thirteen.
K: Thirteen (points to "13").
I: Okay, point to thirty-one.
K: (Points to "31.")

Excerpt 4.3.2

Two hypotheses appear to account for the inconsistencies in the number-names that Kappa constructed from two-digit numerals. The first is that he had not established a convention of reading from left to right. The second is that he had two conflicting routines--one for constructing a number-name for two-digit numerals beginning with a "1," and one for otherwise.

The first hypothesis is weak, for Kappa rarely misread a three-digit numeral (he did so only once in all the interviews, reading "102" as "two hundred ten"). If he had not established a left-right convention for two-digit numerals, we would not expect one for three digits.

The second hypothesis is stronger. To name a two-digit numeral beginning with a "1," the reader must first read the <u>second</u> digit before assigning a name. To name a two-digit numeral that does not begin with a "1," the reader says the name of the digit followed by "ty," and then looks to the second digit--naming the digit if it is not a "0," and saying nothing if it is. If Kappa were to generally look back-and-forth between the digits to check for a "1," in the absence of his anticipating what the number-name will be, we would expect to see Kappa making reversals of "21," "31," . . ., and "91," but not "10," "12," "13," . . ., or "19." This is, in fact the case. Kappa never misread any of the latter numerals.

The reason for saying ". . . in the absence of anticipating what the number-name will be" in the above paragraph is that in several episodes Kappa correctly read two-digit numerals ending in a "1"--apparently because he expected that name or one like it. For example, Kappa correctly read the numeral "21" in a seriating task, but he had just placed "19" and "20"--so he expected the name of the next numeral to be "twenty-one."

What the above does not explain about Kappa's errors in reading numerals is why he also misread numerals such as "73" ("thirty-seven"). If we assume, as hypothesized above, that Kappa first looked for a "1" in the numeral, and that he began constructing the number-name with the digit he happened to be focusing upon after determining that there was no "1," then the digit with which he started would be largely fortuitous--depending only on where he "gave up" looking for a "1."

The reason for this elaborate analysis of Kappa's errors in reading and writing numerals is that they occur quite frequently in the interviews, and that each error had ramifications in Kappa's subsequent behavior. Rather than discussing each error individually, reference will be made to the previous discussion. Moreover, it seems that what mattered in Kappa's behavior was the number-name that he constructed from a numeral, and not the actual numeral. Thus, it will be noted that he misread a numeral, but the analysis will use the number-names he constructed.

Sequencing. Kappa rarely chose to sequence by one, and on those occasions where he did he showed no special difficulty. Since there were no tasks that aimed explicitly at uncovering Kappa's ability to sequence by one, and since there was no indirect evidence of difficulty, it will be assumed that he was capable of doing so.

There are several episodes suggesting the nature of Kappa's routine for sequencing by ten--it seems to have been based around the sequence "ten, twenty, . . . ninety." This can be seen in the following episode.

I: Start at ninety-seven and count-back by tens.
K: Ninety-seven?
I: What's ten less than ninety-seven? (Pause.) Eighty-seven . . .
K: Eighty-seven . . . sixty-seven--seventy-seven . . . sixty-seven, fifty-seven . . . forty-seven . . . thirty-seven . . . twenty-seven . . . ten sev . . .
I: Twenty-seven, tenny-seven?
K: Seventeen and seven.

Excerpt 4.3.3

The key to understanding Kappa's construction of his sequence is the way he constructed the term after "twenty-seven"--"ten seven." This suggests quite strongly that Kappa's routine was to construct the sequence "eighty, seventy, . . . twenty, ten" and append "seven" to each term. This may not have been the only way that Kappa could construct number-name sequences in increments of ten, but it does suggest that sequencing "ten, twenty, . . ., ninety" was a well-formed and automatic routine for him. If this is so, and it will be argued later that it is, then Kappa's solution procedure appears to have been heuristic--search for a routine that will account for the difference between terms in the interviewer's sequence "ninety-seven, eighty-seven" (means-end analysis). Sequencing backward by ten ("ninety, eighty, . . . ten") accounted for the difference.

The next episode shows Kappa's application of the same heuristic, and suggests the nature of his routine for sequencing "ten, twenty, . . . ninety."

I: Can you start at eight and count-on by tens? Go as far as you can.
K: Eight . . .
I: What's ten after eight?

K: Eighteen.
I: Good. What's ten after eighteen? Do you know that?
K: Uh-uh (no).
I: Twenty-eight.
K: Twenty-eight . . . thirty-eight, . . ., seventy-eight, ninety . . . ninety-eight . . . one hundred and eight, two hundred 'n eight, three hundred 'n eight, four hundred 'n eight . . .
I: That'll do.

Excerpt 4.3.4

In (9), we see that it was not until the interviewer said "eighteen twenty-eight" that Kappa was able to continue. This fits well with the above interpretation of Kappa's use of means-end analysis--"eight, eighteen" is not assimilable to "ten, twenty, . . ., ninety," whereas "eighteen, twenty-eight" is. "Eight" was not a ten-word to Kappa.

It at first seems odd that Kappa could answer the interviewer's question "What is ten after eight?" and yet did not understand what the interviewer meant by "Start at eight and count-on by tens." Three episodes suggest that he had a special routine for constructing a number-name from "ten" and a digit name. In (8) of Excerpt 4.3.3 we see that after beginning to say "ten seven," Kappa corrected himself, saying "seventeen." In two episodes where he was asked "What is ten plus (seven, nine)," he responded quickly, saying "seventeen" and "nineteen." It appears that when Kappa understood the context to be <u>adding</u>, he would concatenate "ten" and the digit name, but with the aim that the result fit his grammar for "teen" number-names, such as ((TEN)(SEVEN)) → ((SEVEN)(TEN)) → ((SEVEN)TEEN)--the word "ten" being transformed into the label "teen." To make the forward transition from a digit name to the next-ten name, Kappa needed "ten" to be explicitly in his understanding.

Lines (10-11) of Excerpt 4.3.4 suggest the nature of Kappa's routine for sequencing by ten. It was to increment the first-said part of the number-name, as opposed to incrementing the "ten" part. Of course, if this is the case, then he must have placed grammatical constraints on the result of incrementing, for otherwise he would have sequenced "one hundred 'n eight, twenty hundred 'n eight, thirty hundred 'n eight, ..."

When Kappa sequenced by ten without having to carry along a "ones" name ("ten, twenty, . . .") he did not experience the difficulty seen above in making transitions between centuries. This seems to contradict the hypothesis that Kappa's routine for sequencing by ten was based on the rule "increment the first-said part of the number-name." An inspection of those instances where Kappa successfully transited from one century to another suggests a reason: he was counting MAB longs. On one occasion, he counted longs by ten to 250. This suggests two alternatives for Kappa's difficulties in Excerpt 4.3.4: (a) he could not operate on the "interior" of a number-name (e.g., "one hundred TWENTY-eight"), or (b) he did not give sequencing by ten the significance of repeatedly <u>adding</u> ten, and hence that when sequencing by ten without a numerical context, transitions between centuries posed a problem unrelated to "ten more" and the constraints of distance between number-names related by ten. It will later be argued that the latter is the more viable interpretation.

The above discussion addresses a fundamental conflict: it was difficult for Kappa to <u>sequence</u> by ten, yet he had no trouble counting by ten. The same conflict appears when comparing his ability to sequence by one hundred and count by one hundred.

I: Can you start at thirty and count-on by hundreds. (Pause.) I'll help you. Thirty, one hundred thirty . . .
K: One hundred 'n forty.
I: No, two hundred thirty . . . three hundred thirty . . .
K: Four hundred 'n thirty, five hundred 'n thirty, . . ., nine hundred 'n thirty . . .
I: What comes next?
K: Ten hundred 'n thirty, forty hundred 'n thirty . . .
I: What's that one?
K: Forty . . .
I: Fourteen hundred and thirty.
K: Fourteen hundred 'n thirty, fifteen hundred 'n thirty . . .
I: You can stop there.

Excerpt 4.3.5

It appears that Kappa again applied means-end analysis to continue the interviewer's example (4-6), and that his pause in (6) was caused because of a lack of fit between "ten hundred 'n thirty" and his grammar for constructing number-names--one does not say "ten . . ." in the first-said part of a number-name when you expect to say more after it. Note also that Kappa appeared to increment the "thirty" of "ten hundred 'n thirty," next saying "forty hundred and thirty."

The following episode suggests Kappa's ability to count by hundred.

I: Let's do this once more. Start again (covers entire board).

Board: 100 3 10 10 100 1 10 100

I: (Uncovers MAB flat.)
K: One hundred.
I: (Uncovers 3 unit cubes; covers previous wood.)
K: One hundred three.
I: (Uncovers MAB long; covers previous wood.)
K: One hundred 'n thirteen.
I: One hundred and what?
K: Thirteen.
I: (Uncovers MAB long; covers previous wood.)
K: One hundred 'n . . . twenty-three.
I: (Uncovers MAB flat; covers previous wood.)
K: Two hundred 'n twenty-three.
I: (Uncovers 1 unit cube; covers previous wood.)
K: (Pause.) Two hundred . . . and . . . two hundred and . . . two hundred and twenty . . . four.
I: (Uncovers MAB long; covers previous wood.)
K: One hundred and . . .

I: What was it before? (Covers last uncovered long.) Two hundred and twenty what?
K: Three.
I: Two hundred and twenty-four.
K: Two hundred 'n thirty-four.
I: (Uncovers MAB flat; covers previous wood.)
K: (Pause.) Three hundred 'n thirty-four.

Excerpt 4.3.6

The episode is taken out of context, for he did experience some difficulty prior to its occurrence (his difficulties in this and related episodes will be addressed under the heading "Concept of place value"). Nevertheless, it does suggest that sequencing by hundred and counting by hundred were largely unrelated for Kappa. When counting by hundred he appeared to consider the position of the part of the number-name to be changed.

If Kappa's routines for sequencing were as ill-formed as suggested above, then we would not expect to see him behave at the level of operationality when seriating numerals. For the most part this is the case, but several episodes suggest that he had some aspects of operationality in <u>seriating</u>, but, strategically, could not apply his knowledge to numerals.

Before examining episodes from the seriating tasks, it should be pointed out that there were several major obstacles to forming hypotheses about Kappa's thinking. First, in several instances, it seemed that he understood that he was to put the cards in order, but did not understand that their spatial placement should reflect the order in which they were placed. Second, the interviewer did a particularly poor job in several critical episodes--almost directing Kappa to behave in a certain manner, and intervening with a question before Kappa had a chance to get far enough into a task to allow interpretations of his actions. Nevertheless, there are two episodes which suggest something of what Kappa was capable.

Cards: 20 30 60 70 90 100 110 120 (shuffled)

I: Here are some more cards. Can you put these cards in order on the board?
K: Twenty (________________ 20). (20; moves "20" to left end).
I: Have a look at them all first so you can see what you've got.
K: (Spreads cards on table; counts the cards while pointing to each; groups them back together; holds stack in hand; takes top card and places it on the board.) (30)
I: What number is that (pointing to "30")?
K: (30-70)
I: What number is this (pointing to "70")?
K: Seven. (30-90) (90-30) Eighty, ninety, thirty (spreads cards on the table). You got me mixed up. I'm missing forty.
I: Well, forty is not there, so you have to put them in order without forty. Can you do it without forty?
K: (90-30-20).
I: Which is the smallest number?
K: (30-20) Twenty.
I: Do you want to put twenty first?
K: (20-30) Thirty. (20-30-90) I'll leave that for now (pointing to "90").
I: Is there a number between this one (points to "30") and this one (points to "90")? That's what you want. If there isn't, then that'll be the next one in order, won't it?
K: (20-30-100)
I: What number is that (pointing to "100")?
K: One hundred. (20-30-60)
I: What number is this (pointing to "60")?
K: Sixty. (20-30-90)
I: What number is that ("90")?
K: Ninety . . . ninety. (20-30-90-100) Hundred. (20-30-90-100-120) Hundred 'n twenty. (20-30-90-100-120-70) Seventy. No, that's wrong. (20-30-90-70-100-120) (20-30-90-70-100-120-110) That's one hundred 'n ten. (20-30-90-70-100-120-110-60) One hundred 'n sixty.

Excerpt 4.3.7

It is not clear what Kappa had in mind at the outset of the task (1-9). In (11) it appears that Kappa attempted to apply his routine for sequencing by ten. In (19-20), what Kappa appeared to have in mind was that "90"'s position was only temporary, and that other terms in the anticipated sequence could be "slipped" in as they arose. Unfortunately, the interviewer interrupted him before he could proceed. The clearest indicator of Kappa's thinking comes in (31-34), where upon placing "70" he decided "that's wrong," and moved each of "100" and "120" one space to the right to make room for "70." Apparently, Kappa knew that a "ty" numeral precedes a "hundred" numeral, and that it was unnecessary to destroy the sequence in order to rectify the misplacement. That is, Kappa had some notion of transitivity. However, it seems that he did not compare "70" with "90," nor did he compare "110" with "120" after re-placing "70." This suggests that Kappa was not considering the entire sequence as one composed of terms connected pairwise by an order relation.

The next episode shows that Kappa could anticipate the possibility that terms within the sequence could be both related yet separated by others. However, actually filling in the "slots" with numerals was a problem for Kappa.

Cards: 11 21 31 51 81 91 101 111 (shuffled)

I: Let's try doing the same thing with these cards.
K: (Reads numerals as he places the cards on the table—numerals not visible to camera.) Thirteen, eleven, twelve, fifteen, nineteen, eighteen. What's that (holding up "111")?
I: What do you think it is?
K: A hundred eleven; (points to "101") is ten hundred and . . .
I: What do you think it is?
K: A hundred ten. (Scoops cards together; places them in his hand; places 1st card in hand in 2nd position from the left; places 2nd card in hand in 8th position; spreads remaining cards on the table.) (__ 91 __ __ __ __ __ 11) (__ 91 __ __ __ __ 21 11) (__ 91 __ __ __ 31 21 11) (__ 91 __ __ 81 31 21 11) (__ 91 81 __ __ 31 21 11).
I: I want you to tell me the numbers as you go along.
K: Eleven, twelve, thirteen, eighteen, nineteen (reading from right to left). (__ 91 81 __ 51 31 21 11) Fifteen. (__ 91 81 101 51 31 21 11) One hundred ten. (111 91 81 101 51 31 21 11)
I: Satisfied?
K: Yeah.
I: Tell me, why did you have this number (pointing to "101") between these (pointing to "81" and "51")? What's this number again (pointing to "81")?
K: Eighteen.
I: And . . . (pointing to "51").
K: Fifteen.
I: What's this number (pointing to "101")?
K: One hundred and ten.
I: Why is a hundred ten between those two?
K: Those two are lower, so I want to put this one right here (switches position of "91" and "101"; places "101" to far left). I know what they're doing except these two (points to "111" and "101").
I: You can't tell which is biggest?
K: Yes, I can tell which one is the biggest.
I: What's this number (pointing to "111")?
K: One hundred 'n eleven.
I: And which one is this one ("101")?
K: One hundred 'n ten.
I: So which one is the biggest?
K: (Points to "111.")

Excerpt 4.3.8

From the way he read the numerals at the beginning of the task (2-4), it is likely that when placing "19" and "11" (11) that he expected to end up with a sequence in the "teens"--a hypothesis supported by his remarks in (31-32). So in placing "19" near the left end and "11" at the right, Kappa likely anticipated a sequence in which "19" followed "11" (in right-left order), but would be separated from "11" by other terms. When he came to place "111" and "101" (16-17), Kappa had only two open positions, and did in fact place the cards in appropriate right-left order ("101" to the right of "111"), but he did not consider the position of "101" relative to those numerals already placed. It appears, then, that Kappa could anticipate a sequence, but in constructing it could order its terms only in subsequences. The implication of Kappa's behavior in these episodes was that his linguistic system for constructing number-names and sequencing number-names was preoperational, but that he had a limited sense of transitivity of "after."

<u>Numerical operations</u>. It is not clear what type of counter Kappa was, for he seemed to prefer not to count if it was at all possible. Several episodes do suggest, however, that Kappa was capable of constructing abstract unit items.

I: (Places bag with 235 unit cubes in it onto the table.) Guess how many are in there.
K: Two hundred.
I: That's pretty close. You know how many are actually in there? Two hundred and thirty-five. Cause I counted them. If I were to take those blocks out of the bag and make piles of one hundred little blocks, how many piles could I make?
K: Two.
I: How do you know that?
K: You said there were two hundred and . . .
I: Could I make three piles?
K: No.
I: Why not?
K: You could make three piles, but not three hundred piles.
I: Could I make three piles with exactly a hundred little blocks in each pile?
K: No! You won't have enough.

Excerpt 4.3.9

It would be easy to credit Kappa only with the linguistic transformations necessary to answer questions like "how many hundreds . . ." with no understanding, except that his remarks in (14) suggest that he was actually thinking of units of 100--mentally partitioning the blocks into piles of 100. Of course, this is based on the assumption that Kappa meant "three 'hundred' piles" as opposed to 300 piles, which seems reasonable given his reaction to the interviewer's follow-up question.

Another episode suggests again that Kappa could construct abstract unit items.

I: (Places card with "70 + 92" onto the table.) How many is it from seventy up to ninety-two?
K: (Pause.) It's two tens and two.
I: How many's that?
K: It's . . . twenty-two.

Excerpt 4.3.10

There was no evidence that Kappa used his fingers, but even if he had the point could still be argued that he had constructed abstract unit items. Supposing that he counted "eighty, ninety, ninety-one, ninety-two," or even "eighty, ninety, ninety-two," we would have to infer that he constructed ten as a unit and one as a unit, differentiating them by the labels "ten" and "one." His mechanism for quantifying his count appears to have been to subitize the increments by ten and one--"eighty, ninety (two tens), and two to make ninety-two." Of course, it is possible that Kappa operated solely on the numerals (9-7 & 2-0 = 2 tens and 2), but this would have been extraordinary for him, and we would have to explain how he came to understand "how far is it . . ." as "what is 92 - 70?"

Each episode that could be taken as suggestive that Kappa could construct abstract unit items (there were five in all) involved subitizing. Any time that Kappa attempted to deal with numbers beyond his subitizing range he employed strategies reminiscent of a counter with perceptual unit items.

Eight episodes suggest the nature of integrating for Kappa, and the common theme among six of them is his reliance upon subitizing. The episode below shows that Kappa could consider 13 as an extension of 10, and suggests his use of subitizing to quantify the extension.

I: (Places card with "10 + ___ = 13" onto the table.)
K: Ten plus blank equals thirteen. You only get three (holds up 3 fingers) to get thirteen.

Excerpt 4.3.11

It might be argued that Kappa either used a figural strategy, looking for the missing digit that, when "placed over the zero," would produce "13," or that he used a linguistic strategy of searching for the digit-name that needed to be supplied to his routine for producing "teen" names from "ten" and a digit-name (p. 161). Neither of these would account for Kappa's putting up three fingers, nor would they explain his failure in the next task.

I: (Places card with "40 + ___ = 46" onto the table.)
K: Fourteen . . .
I: What's this number again (pointing to "40")?
K: Four . . . forty plus . . . equals sixteen.
I: Forty plus blank equals . . .
K: Hmmm . . . forty-six. (Long pause.)
I: What are you thinking about?
K: I don't know it.
I: I think you do. Is it like this one here (pointing to "10 + ___ = 13")? It's like that one, isn't it?
K: You have to get six (points to the "6" of "46").
I: Yeah, the numbers are different.

Excerpt 4.3.12

If we suppose that Kappa indeed subitized an extension of ten to solve "10 + __ = 13," and that he attempted to apply the same strategy to "40 + __ = 46," then it is quite reasonable that he was not able to solve the latter--an extension of 40 to 46 was beyond his subitizing capability. Kappa's behavior after the interviewer asked him to compare the two sentences (9-10) suggests that it was then (and not in "10 + __ = 13") that he abstracted a pattern to answer by: search among the digits on the right-hand side for the digit missing on the left-hand side of the equation. His behavior on the next two tasks confirms this--and suggests moreover that his strategy was to answer with the digit-name of the right-most digit of the "answer."

I: (Places card with "__ + 20 = 25" onto the table.)
K: That's easy. You get five (points to the "5" of "25").
I: So what number would go in the blank?
K: Five.

- - - - - - - - - - - - - - -

I: (Places card with "__ + 9 = 79" onto the table.)
K: A nine would go right there (points to the "9" of "79").
I: What would go in the blank?
K: Nine.
I: Nine? Why? Why do you say that?
K: Cause blank plus nine equals seventy-nine (while pointing to each member of the equation). Cause I know it's a nine (pointing to the blank) that goes right there.

Excerpt 4.3.13

Our conclusion that Kappa considered 13 as an extension of 10 and that he could construct abstract unit-items suggests that he at least saw 13 as including 10 in a counting sequence. It does not tell us whether he could integrate the product of extending (a number or lot and the amount it was extended by) into a number or lot, nor whether he could conceptualize a number or lot as being separated into the integration of two others prior to extending or declending. From the following episode we will see that the answer to the former is yes, while the answer to the latter is if he can subitize it.

I: You can see we have twenty-four little blocks under this screen (lifts screen to show blocks; advances screen so that all blocks are covered; places MAB long adjacent to screen.) How many little blocks are there altogether now?

K: (Writes "34" on his paper.) Thirty-four.

I: (Advances screen so that all blocks are covered; places three unit cubes next to screen.) How many little blocks are there altogether now?

K: (Subvocally utters "35-36-37" while pointing to each block; writes "37" on his paper.) Thirty-seven.

I: (Advances screen so that all blocks are covered; places two MAB longs.) How many now?

K: (Subvocally utters "37-47-57" while pointing to each long; writes "57" on his paper.) Fifty-seven.

I: (Advances screen so that all blocks are covered; places 1 long and 1 unit cube next to screen.) How many now?

K: Fifty-seven . . . (points to blocks; writes "81" on his paper.) Eighty-one.

I: How did you get that?

K: See, it's sixty-seven (pointing to the long and unit cube).

I: (Interrupting.) How many? You wrote it down (pointing to "57"). Is that sixty-seven?

K: Yeah.

I: Is that sixty-seven (pointing to "57")?

K: No. Sixty-five.

I: Come on K! Fifty-seven. Okay . . . fifty-seven and this much (pointing to the long and unit cube).

K: Sixty-seven.

I: What's that number (points to "57" on K's paper)?

K: Fifty-seven. That's wrong (erases "81"; writes "61").

I: (Advances screen so that all blocks are covered; places two unit cubes and two MAB longs.) Write down how many little blocks I have altogether now.

K: Sixty-one . . . the answer is eighty-two.

I: (Advances screen so that all blocks are covered.) How many little blocks are under here altogether?

K: Hmmm . . . forty-seven . . . forty-seven (pointing at something on his paper).

I: Which number tells you underneath altogether?

K: I don't know (points to "82").

I: Right. How many is that?

K: Eighty-two.

I: Write eighty-two in there (points to box at bottom of K's paper).

K: (Writes "82" in box.)

- - - - - - - - - - - - - - -

I: I'm going to take some wood from under here, like this (removes 2 MAB longs). This is the first amount. I want you to tell me how many is left under here. (Pause.) How many did we start off with from behind here?

K: Two tens.

I: But how many altogether under here (replaces the 2 longs under the cover)?

K: Eighty-two.

Excerpt 4.3.14

Before focusing on the central point--Kappa's ability to integrate the product of extending--let us first convince ourselves that his errors and the miscommunications are irrelevant to it. In (17-18), Kappa appears to have counted-on from "seventy," possibly picking up on the "seven" of "fifty-seven" (17) and transforming it to "seventy." This is at least plausible, since he made similar errors on several other occasions (see Excerpt 4.3.5).

Of course, this would mean that he was redoing the extension in (20) and was not explaining how he got 81. Similarly, in (34) it appears that Kappa extended from "sixty" and not "sixty-one." Finally, lines (20-30) are a result of a basic miscommunication between Kappa and the interviewer. When Kappa said "See, it's sixty-seven," (20) referring to the long, the interviewer thought he was referring to the amount under the cover. When the interviewer said ". . . fifty-seven and this much" he intended that Kappa look at both the long and unit cube, while Kappa took the interviewer's "point" as referring to the long.

Putting Kappa's errors aside, we can infer that he did in fact integrate his products of extending into at least an arithmetical lot. Whether he maintained the structure of the components of the lot is not clear. The inference that he integrated is based on two aspects of the episode: first, he always extended from where he left off, and second (and more importantly), he used the product of extending in a different problem—one involving separating and declending.

The following episode is the continuation of that in the above excerpt.

I: I'm going to take some wood from under here, like this (removes 2 MAB longs). This is the first amount. I want you to tell me how many is left under here. (Pause.) How many did we start off with from behind here?
K: Two tens.
I: But how many altogether under here (replaces the 2 longs under the cover)?
K: Eighty-two.
I: So, now how many would be under here (removes the 2 longs again)?
K: That's easy. Sixty-two (writes "62").
I: (Removes 1 long and 1 unit cube). How many is left?
K: Fifty-one (writes "51").
I: (Removes 1 long.)
K: (Writes "41.")

I: What's that number?
K: Forty-one.
I: (Removes 6 unit cubes.)
K: (Subvocally utters "42-43-44-45-46-47" while pointing to each cube.)
I: How many?
K: Forty-seven.
I: Show me what you did. We had forty-one behind here and I took out this many (points to 6 unit cubes).
K: You had forty-one. You took out them. Forty-two, forty-three, forty-four, forty-five, forty-six, forty-seven (while sliding each cube toward himself).

Excerpt 4.3.15

It appears that during the first part of the task (1-17), Kappa conceptualized 82 as being separated into two tens and an unknown amount, 62 as one ten, one one, and an unknown amount, and 51 as 1 ten and an unknown amount. When the interviewer removed six unit cubes, however, Kappa could not subitize the declension and hence could not create an abstract structure of 41 separated into six ones and an unknown amount. It appears that in the process of accommodating to this development Kappa reconstructed the problem as there being 41 outside the cover plus the number of cubes that the interviewer had removed ("*You* had forty-one . . ."; 2 5-27). That is, once Kappa went beyond his capability to subitize the amount to be separated from the minuend, he lost the structure of the problem.

From the above discussions we concluded that Kappa could construct abstract unit items and that he could integrate abstract unit items. We have not discussed the nature of Kappa's product of integrating--whether or not he could construct numbers. His solution to "10 + __ = 13" and his behavior shown in Excerpt 4.3.15 suggests that he might have constructed numbers at least up to three (an arithmetical unit composed of three arithmetical units). That is, Kappa could give abstract extensive meaning to each of "one," "two," and "three." Beyond that we cannot tell. There remains the possibility, however, that Kappa had constructed arithmetical lots rather than numbers—the difference lying in whether or not the boundaries of the lots were conceptually or experientially derived. In solving "10 + __ = 13," it would have sufficed for Kappa to have temporarily bounded the extension; in the tasks of Excerpts 4.3.14 and 15, Kappa could have derived boundaries for the components of the operands from his perception of the cover and the spatial separation between the cover and the added or removed blocks. The structure of Kappa's problems in Excerpt 4.3.15 would then have taken the form shown in Figure 4.3.1.

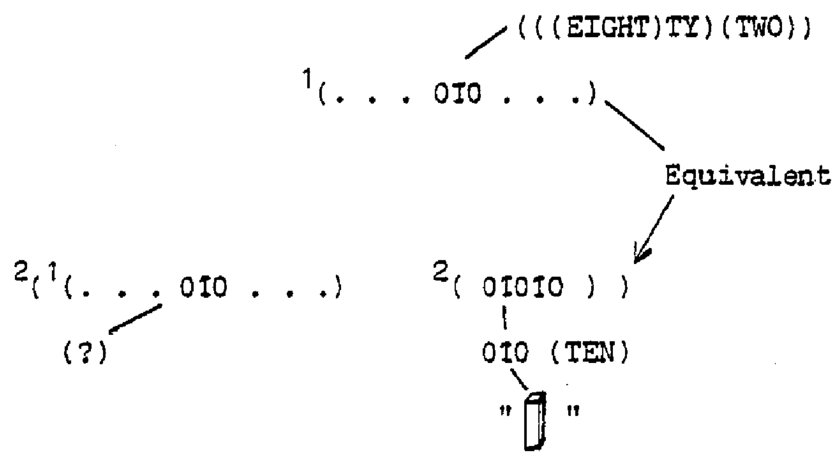

Figure 4.3. Kappa's construction of an understanding of removing two MAB longs from 82 blocks.

[1]Boundaries derived from cover.

[2]Boundaries derived from spatial separation.

There were no episodes in any of his interviews which would force us to impute any greater abstractness to Kappa's understanding of the task in Excerpt 4.3.15. We shall conclude, then, that Kappa could construct abstract unit items and arithmetic lots, and could integrate and separate arithmetic lots, but only when one of the operands consisted of at most three abstract unit items. It may also have been the case that Kappa could only construct arithmetic lots in a reconstructive manner--requiring that he had first constructed abstract unit items from sensori-motor items. Finally, Kappa's operations of extending and declending appear to have been conceptual only within his capability to subitize a count. Otherwise, he relied upon sensori-motor schemas for counting. This suggests an anomolous picture of Kappa--he could attribute extensive meaning to number-names at the level of arithmetical lots, but could not attribute intensive meaning to them beyond the level of signs for counting.

Concept of ten. It seems clear from Excerpt 4.3.10 that Kappa could construct ten as an abstract unit item, but the question still remains as to what internal structure it might have had. From the concluding discussion of the previous section, we would naturally suppose that it had little beyond the linguistic significance of it being labelled "ten." That is, that ten as an abstract unit, for Kappa, was not a unit that could be composed of ten units. This conjecture is supported by the following episode.

I: (Places card with "70 - 31 = __" onto the table.)
K: Seventy take away thirteen . . . thirty-one, I mean.
I: Do you know what that would be?
K: (Hands not visible.) Thirty-one . . . thirty, I mean.
I: Seventy take away thirty-one is thirty?
K: (Nods head yes.)
I: How do you know that?
K: I got seventy (extends 5 fingers of left hand and 2 of right) and I'm going to take away thirty-one (folds down little, ring, and middle finger of left hand; folds down index finger of left hand). So, it was thirty left.

Excerpt 4.3.16

Since Kappa could not construct a subitizable pattern for 31, he reverted to a more primitive strategy of putting everything on his fingers--the new twist being that each finger stood for ten. (Kappa attempted to solve "84 - 30 + __," "74 -   = 70," and "47 - 21 = __" in the same way: by trying to put everything on his fingers, failing to distinguish between ten and one.) That Kappa did not relate ten and one by the relation "ten of" is shown in (9-11) by his taking away "one" (ten)--to satisfy the requirement that he put down a finger for each <u>unit</u> to be taken away, not distinguishing between a unit of ten and a unit of one. This is shown again in the sequel to the above episode.

I: Do you want to check it with the wood?
K: Ten, twenty, . . ., seventy (while removing 7 MAB longs from the box; places the longs on the table). Take away thirty-one (removes 1 unit cube from the box; removes 3 longs from the 7 longs).
I: Now what do you have left?
K: Ten, twenty, thirty, forty (while sliding each remaining long toward himself). Ten, . . ., forty (counting remaining longs again). That was the same answer (pointing to something on the interviewer's record sheet).
I: But you said thirty before. Have you changed your mind?
K: Uh-huh (yes). If I would have took away theeeese many (removes 1 more long) I know what the answer would have been. I took away forty.
I: Put the wood down that you took away. Let's count it again. Count it all. How much have we got?
K: (Subvocally utters "10 - . . . - 70" while pointing to each long.) Seventy-one.
I: Seventy-one! But we don't want to have seventy-one. We want to start off with seventy.
K: (Picks up the unit cube.)
I: So what do we have to do? We have to put that one away.
K: Hm-hmmm (no). Take one of those away (places one of the longs into the box; counts out 7 unit cubes from the box).
I: What are you trying to do? You put one of these long ones down, and what are you getting?
K: Then I got seven of them.
I: Why did you do that?
K: Cause I needed . . .
I: (Interrupting.) You're trading, were you?
K: (Picks up 1 long from the box.) I needed thirty-one of those (points to "31" on the card) so I take away thirty-one (begins to remove 3 longs and 1 unit from the pile).
I: Now wait. When you gave me this one (holds up long) did you get ten little ones back?
K: No.
I: Don't you think you should? That's not a fair trade if you give a long and don't get ten back.
K: (Places 3 more unit cubes in the pile.)
I: So what have you got now? Do you have seventy there?
K: Yeah. How am I going to take away thirty-one? I can do it like that (removes 3 longs and 1 unit cube from the pile).
I: And what have you got left?
K: It's thirty . . . (counts the 9 unit cubes) thirty-nine.

Excerpt 4.3.17

First, note that in (2-4) Kappa took a unit cube from the box so that he had 31 in his "take-away" pile, which makes even more viable the imputation of unidirectional equivalence in Figure 4.3.1. His addition of a cube to the amount taken away (4) bore no implication, for Kappa, for the quantity that he initially intended to separate. Second, Kappa was quite happy to trade a long for seven unit cubes (22-27), again suggesting that ten and one were not conceptually related by "ten of." Kappa's overriding concern in his problem was to make his "take away" pile contain 31 (28-33). It was not to separate 70 into two parts that, in turn, composed 70.

Though Kappa did not have ten as a conceptual structure, he did have a functional knowledge of the base-ten structure of number-names, at least of those less than "one hundred."

I: How many tens are there in thirteen?
K: Three.

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I: How many tens are there in sixty-seven?
K: Six tens and seven ones.

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I: (Places pile of sticks on table; covers pile with hand.)
Let's imagine that this pile has seventy-two sticks in it.
If you took all the tens out of seventy-two, how many sticks
would be left?
K: Two.

Excerpt 4.3.18

Before proceeding, note that in Excerpt 4.3.1 Kappa wrote "30" in response to the interviewer saying "thirteen." It seems reasonable, then, to suppose that Kappa understood the interviewer as saying "thirty" in (1-2) above. Given that we have concluded that "ten," to Kappa, did not refer to a structure of ten ones, we must ask ourselves what "ten" could have meant to him in the above tasks. On the several occasions that Kappa constructed a collection of MAB blocks to be called by a given number-name, he consistently first counted by ten the appropriate number of longs and then counted by one the appropriate number of unit cubes. It seems reasonable to assume that he had constructed linguistic transformations which would <u>mirror</u> the result of this activity--to make "sixty-seven" one would put out six longs (or piles, or whatever, as long as they are called "ten") and seven unit cubes (or sticks, or whatever, as long as they are called "one"). Kappa did not need to have constructed the conceptual relationship "ten of," nor would he have had to understand that "ten" can refer to a unit of arithmetic units. Kappa's linguistic transformations did not carry the significance of numerical equivalence between, say, seven tens and two ones and 72 (ones).

It is only by way of the above argument that we may guess what meaning Kappa gave to an increment when counting by ten (as in Excerpts 4.3.10, 4.3.14, 4.3.15). In situations where he counted something like an MAB long, it seems reasonable to suppose that he constructed an arithmetical lot, since he knew that a long is composed of units. Otherwise, we are forced to assume that he constructed no more than an arithmetical unit labelled "ten." The latter claim is made viable when we observe that in the absence of objects labelled "ten," Kappa counted by ten only within his subitizing range. An increment by ten seemed not to carry the meaning of ten increments by one for Kappa.

Concept of one hundred. We have already established that Kappa could not sequence by one hundred, but that, with some difficulty, he could count by one hundred (Excerpts 4.3.5 and 4.3.6). Also, Kappa could create abstract unit items labelled "hundred" (Excerpt 4.3.9). The remaining questions to be addressed concern the relationships Kappa had established among one hundred, ten, and one. The next episode shows that Kappa had established at least figural relationships among them.

I: How many exactly like this (holds up unit cube) could we get out of a piece like that (indicates flat)?
K: One . . .
I: Out of a piece like that (indicates flat)?
K: A hundred.
I: How many exactly like this could we get (holds up long)?
K: Ten.
I: How do you know that?
K: Cause . . .
I: Could you prove to me why you get ten of these (indicates long)?
K: Cause these (indicates flat) has got ten bundles of those (draws finger horizontally across one row of the flat).

Excerpt 4.3.19

Kappa's language in (12-13) suggests that the relationship among one, ten, and one hundred as arithmetic units is at most figural. A ten is (literally) composed of ten unit cubes, and a flat is (literally) composed of ten "bundles" of ten (long). The following episode suggests again that the distinction that Kappa made among numeration units was largely figural.

I: I want you to count by tens for me (picks up a handful of MAB longs; places the 1st on the table).
K: Ten. Twenty, thirty, . . ., ninety, one hundred, . . ., one hundred 'n sixty (as the interviewer places 15 more longs on the table).
I: How many is there?
K: One hundred sixty.
I: How many hundreds are there?
K: One.
I: How many hundreds are there?
K: One.
I: How do you know that?
K: Cause I just now counted them.
I: Let's keep counting now (picks up 4 more longs). One hundred sixty.
K: One hundred seventy, . . ., two hundred (as the interviewer

places the 4 longs on the table with the others).

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I: (Screens (20) MAB longs from view; places 4 longs next to screen.) How many's there (lifts cover)?
K: Two hundred. Two hundred 'n ten, two hundred 'n twenty, two hundred 'n thirty, two hundred 'n forty (as the interviewer places the 4 longs on the screen).
I: How many tens are there altogether underneath and on top?
K: Two hundred 'n forty.
I: How many tens?
K: (Pause.) Two hundred 'n forty.
I: Two hundred forty tens? How do you know that there are two hundred forty tens?
K: Cause when you put these 4 tens up (places hand on the 4 longs), I counted them.
I: Okay. Let's put this out (places 1 more long on the cover). Now how many little blocks are there?
K: Easy. Two hundred 'n fifty.
I: Two hundred and fifty little blocks. How many tens are there?
K: Two hundred blocks up under there (points to cover).
I: Two hundred little blocks under here and how many on top?
K: Fifty.
I: How many tens are there?
K: Two hundred 'n fifty.
I: But you told me there were two hundred fifty little blocks. Like this (holds up 1 unit cube).
K: There's two hundred 'n fifty tens.
I: And there's two hundred fifty tens too?
K: Yeah.

Excerpt 4.3.20

The mechanism of Kappa's answer in (8-13)is not clear, but it is entirely possible that by "one" he meant that he only got to "one hundred" once--he had not yet gotten to "two hundred." Lines (24-27) and (35-45) indicate that the relation "ten of" between ten and one hundred that Kappa spoke of in Excerpt 4.3.19 was indeed figural. If we assume that the meaning Kappa attributed to "two hundred forty (fifty)" was that of arithmetical lots, which would be consistent with his remarks in (36-38), then we can see that Kappa's relationships among numeration units was not conceptual. "Two hundred forty (fifty)" had meaning as a lot of units, but without a conceptual relationship between ten and one, and one hundred and ten, a unit is a unit is a unit. Thus, for Kappa, there could just as well have been 240 tens as 240 ones under the cover.

<u>Concept of place value</u>. Although Kappa had not created ten and one hundred as conceptual structures, he did have at least an action-based scheme of place value. When counting base-ten blocks, he would count them in order of decreasing size--first the "hundreds," then the "tens," and then the "ones."

Board: 10 10 4 100 2 10

I: (Uncovers MAB long.)
K: Ten.
I: (Uncovers MAB long.)
K: Twenty.
I: (Uncovers 4 unit cubes.)
K: (Subvocally utters "1-2-3-4" while pointing to each unit cube.) Twenty-four?
I: (Uncovers MAB flat.)
K: One hundred . . . and . . . (looks away from board) twenty-four.
I: (Uncovers 2 unit cubes.)
K: (Pause.) One hundred and . . . (looks back) twenty-six.
I: (Uncovers MAB long.)
K: (Pause.) One hundred and . . . (looks back) thirty-six.

Excerpt 4.3.21

Kappa's scheme for counting base-ten blocks appears to have supplied a foundation for him to abstract the literal (as opposed to ordinal) position of the number-name to increment when he was unable to look over the entire collection of blocks, as shown in the next excerpt.

Board: 100 3 10 10 100 1 10 100

I: (Uncovers MAB flat.)
K: One hundred.
I: (Uncovers 3 unit cubes.)
K: One hundred . . . and three.
I: (Uncovers MAB long; covers previous wood.)
K: One hundred and . . . thirteen.
I: (Uncovers MAB long; covers previous wood.)
K: One hundred and four . . . three.
I: What's that one now?
K: One hundred and four three.
I: One hundred and forty-three (uncovers MAB flat; covers previous wood.)
K: (Pause.) Two hundred and four three.
I: (Uncovers 1 unit cube; covers previous wood.)
K: Two hundred 'n four . . . two hundred 'n forty-four.
I: (Uncovers MAB long; covers previous wood.)
K: Two hundred and . . . fifty . . .
I: What?
K: Two hundred and sixty-four.
I: (Uncovers MAB flat; covers previous wood.)
K: (Pause.) Three hundred and . . .three hundred and sixty-four.

Excerpt 4.3.22

Kappa's first mistake (8) provides support for the inference that it was literally the "ty" position of the number-name that he was incrementing. He had previously said "one hundred thirteen"; with the placement of the long he incremented "thir" (the part normally in the "ty" position) and then searched the original number-name for the digit-name to end with--"three." Why he didn't say "ty" after "four" is not clear. Perhaps he was thrown off by his unusual (for him) double use of "thir" in "thirteen."

The next two episodes show that Kappa needed a figural impetus for applying the abstractions he had made from his scheme for counting base-ten blocks.

I: (Writes "570" on a piece of paper.) What number is one hundred less than five hundred seventy?
K: (Pause.) Six hundred.
I: Six hundred is one hundred less than five hundred seventy?
K: Yeah.
I: How do you know that?
K: Cause you got five hundred seventy (points to "570") and I said six hundred. Cause six hundred is more than five (points to "570") . . . hundred.
I: But I want to know what is one hundred less than five hundred seventy.
K: (Pause.) I don't know the answer to it.
I: I think you do.
K: Five . . . it's six hundred and . . . eighty-six.
I: What would be one hundred more than five hundred seventy?
K: (Long pause.)
I: Do you know that?
K: Uh-uh (no).

Excerpt 4.3.23

I: (Places card with "20" written on it onto table.) What number is one hundred more than this number?
K: (Pause.) Two hundred!
I: How did you get that?
K: (Pause.) I worked it out.
I: How did you work it out?
K: (Pause.) I was thinking it in my head. And I said two hundred.
I: That's it, huh? Just like that?
K: (Nods head yes.)

Excerpt 4.3.24

These two excerpts show that Kappa's scheme for counting base-ten blocks was not conceptual. If it had been, then 20 and "one hundred more" would have been as easy as two longs and a flat;

100 less than 570 would have been one fewer unit of one hundred--4 hundreds. In a sense, the structure of Kappa's scheme for counting base-ten blocks, and the linguistic abstractions he had made from it, made him appear to know more than he did with respect to place value.

Case Study 4.4: Rho

Rho was a first-grader (age 6 years at the beginning of the 1977 school year). In November of 1977 she correctly solved each of Problems 1, 2, and 3 of Figure 4.1 (page 105). Her methods of solution to each involved pointing to "covered" squares in either domino or geometric patterns. For example, she solved Problem 2 by counting the visible squares (to "seven"), and then continued "eight, nine, ten" while pointing in a triangular pattern to the cover--saying "three" as her answer. In Problem 3, she used square and "domino five" patterns. Rho produced the sequence "ten, twenty, . . ., ninety, twenty" when asked to "count by ten," and could not sequence by ten from two. When given a bundle of ten and four single sticks, she counted the sticks individually to find the total, and similarly with two bundles and five single sticks. When asked how many tens in 32, Rho counted her fingers (to ten) and stopped. The interviewer asked her to continue, so she did it again. After being asked how many tens she had counted, and answering "two," Rho continued by opening and closing both hands while counting "three, four, . . ., sixteen." In using tens to find out how many sticks in a pile (33), she (correctly) made three bundles of ten, and when asked how many sticks she had "bundled," counted the sticks singly--answering "twenty-nine." The final interviews were given to Rho on May 8, 11, and 16 of 1977.

Rho's concepts of numeration were quite advanced, as we shall see. Hers is the first case study in which we shall see operational knowledge of integrating and separating and fairly complete concepts of ten and one hundred.

Writing and reading numerals. What would have been two sections has been combined into one for Rho's case study. She did not have any difficulty writing numerals and only occasionally misread one--usually soon catching her error. On only one occasion was she not able to read a numeral, and the ensuing discussion provides insight into her difficulty.

I: "201"
R: Two hundred and . . . wait. Two hundred and . . . hmmm . . . (long pause).
I: "311" Three hundred and eleven.
R: (Reaches for the "201" card.)
I: Do you want to go back to that one?
R: (Picks up "201" card.) Hmmm . . .
I: Do you want me to tell you?
R: Yeah.
I: Two hundred and one.
R: I was thinking when I saw the two "oh" and one (draws figures in the air) it was twenty-one.

Excerpt 4.4.1

Rho's remark in (11-12) that she was thinking that "201" was "twenty-one" suggests that, for some reason, she perceptually partitioned the numeral into "20" and "1" ("two 'oh' and one"; (11)). Since she did not say "twenty-one" at the time of reading "201," we may assume that she knew it was wrong--possibly by way of conditions she had associated with "twenty-one," such as that its numeral should have only two digits.

Another indication of Rho's association between number-name and digit size is the placement of pauses in her construction while reading a numeral. With each numeral beyond "100," if she paused at all, it was after she had said "hundred." This suggests that she had associated three digits with "hundred," and that she would perceptually partition the numeral into the pattern "# ##," saying the name of the left-most digit, and then constructing a name for the two right-most digits as a single numeral.

Sequencing. Sequencing by ten posed no difficulty for Rho. She successfully produced sequences by ten from eight to 178 and from 340 to 550, ending both times because the interviewer asked her to stop. Also, she correctly sequenced by ten from 97 to seven. Since Rho made no errors, the final-stage model of sequencing by ten fits her behavior.

Sequencing by hundred was more problematic for Rho. It appears that she did not have the special case production necessary for beginning the sequence with a name preceding "one hundred."

R: Two hundred and thirty, one hundred and thirty, four hundred and thirty, five hundred and thirty, . . ., nine hundred and thirty, ten hundred and thirty (pause) eleven hundred and thirty (pause) twelve hundred and thirty, . . ., nineteen hundred and thirty . . . .
I: Any more?
R: Huh-uh (no).

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.
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I: Begin at seventy-three and count-on by hundreds.
R: Huh-uh (no).
I: (Pauses.) Seventy-three, one hundred seventy-three . . . What's a hundred more than one hundred seventy-three?
R: Two hundred seventy-three.
I: Keep going.
R: Three hundred seventy-three, four hundred seventy-three, . . ., ten hundred seventy-three, eleven hundred and seventy-three, twelve hundred and seventy-three . . . .
I: Okay, that'll do.
R: What do you mean, that'll do?
I: Well, I don't need you to do any more. You could do more couldn't you?
R: Yeah, but I forgot which one I was on.
I: Twelve hundred and seventy-three. What's next?
R: Thirty hundred and seventy three . . . fourteen hundred and seventy-three.
I: You know the next one, don't you?
R: Uh-huh (yes).

Excerpt 4.4.2

In both cases, Rho needed only the first two terms to continue: "thirty, one hundred thirty" and "seventy-three, one hundred seventy-three." By the way she continued beyond "nine hundred and . . .," it seems clear that her routine was to sequence by one and concatenate "hundred (whatever)" to each term in the sequence.

Rho's lack of a production to begin the sequence, which might take the form "NEXT-HUNDRED OF (WORD) is (((ONE)HUNDRED)(WORD))," is suggested again by the following episode.

I: (Places card with "20" written on it onto table.) What number is one hundred more than this number?
R: (Long pause.) Is it in the hundreds?
I: Do you think it will be in the hundreds?
R: (Pauses.) Yeah.
I: How would you find that out? One hundred more than twenty.
R: (Pauses.) Hmmm . . . I don't know.
I: Twenty plus a hundred more.
R: That's too hard.
I: Is it?
R: A hundred and twenty.
I: How did you get that?
R: Instead of going twenty and counting up a hundred tens, I said a hundred, a hundred ten, a hundred twenty.

Excerpt 4.4.3

Though Rho's context in this episode was not one of strictly sequencing, it does highlight that changing the order in which she considered number-names, one of which was "hundred," was not automatic for her. She had to first assimilate the task to a scheme for addition--in which she could change the order of the <u>numbers</u>, taking the largest (determined by the relative order of their respective names) to begin with.

Rho's linguistic system for producing number-names and sequences of number-names was very close to being operational. Her method of seriating numerals in ascending order was, at each step, to search the remaining cards for the smallest numeral among them. She gave no indication that she was assimilating any of the tasks to a sequencing routine, as had Delta and Lambda. In only one task did she misplace a card ("17" following "31"), and it was because she read it as "seventy-one." The conclusion to be drawn, then, is that Rho implicitly compared the smallest of the remaining numerals at any step with the entire sequence placed up to that time--her reasoning being that the smallest of the remaining would next be the largest numeral placed. That is, Rho had operational transitivity of "after" within her linguistic system for producing number-names.

The reason for saying at the beginning of the previous paragraph that Rho's system was "close" to being operational is that she had not operationally related "before" and "after." This can be seen in her attempt to seriate numerals in descending order.

Cards: 30 47 48 49 52 61 67 76 (shuffled)

I: This time I want you to place these cards on the board starting with the biggest number in the first place (indicates) and the smallest number in the last place (indicates).

R: (Spreads cards on the table.)

I: Now the largest number starts here (points to first place) and the smallest number goes here (points to last place).

R: (76) Seventy-six. (Pauses; 76-48) Is that right?

I: Keep on going--you can change them if you need to.

R: (76-48-49) Hold it. (76-30-48-49) Right?

I: Where's the smallest number going to go? Which end? Point to the end.

R: (76-48-49; points to the end on her right; 76-61) Sixty-one. (76-61-67) Sixty-seven. (76-61-67-52) Then fifty. (76-61-67-52-48) Forty. (76-61-67-52-48-49) Forty-nine. (76-61-67-52-48-49-30) Thirty. Wait. (76-61-67-52-48-49-47-30).

I: The largest number is at this end (points to "76") and the smallest number is at this end (points to "30") and they're going in order, right?

R: See--seventy-six, sixty-one, sixty-seven, fifty-two, forty-eight, forty-nine, forty-seven, thirty (pointing to each card). Forty-nine, seventy-three.

Excerpt 4.4.4

Though Rho did end with the largest and smallest numerals properly placed, the sequence itself was only locally descending--no subsequence exceeding three terms in length. The subsequences were (76-61) (67-52-48) (49-47-30). Thus it appears that Rho could apply transitivity of "before" to at most three terms. That is, "before" and "after" were not operationally reciprocal to one another.

The task subsequent to the one above suggests that Rho was very close to relating the two, however.

Cards: 97 103 107 113 117 124 134 143 (shuffled)

I: I want you to do this just like the last one. Put the cards across the board from the biggest to the smallest. In order, okay?
R: (Spreads cards on the table.) (117) (143) (143-107) (143-117) (143-134) (143-134-124) (143-134-124-117) (143-134-124-117-107) (143-134-124-117-107-113) (143-134-124-117-107-113-103) (143-134-124-117-107-113-103-97) One hundred forty-three, one hundred thirty-four, . . ., ninety-seven (pointing to each card).

Excerpt 4.4.5

Rho appears to have had reflected on the method she used in Excerpt 4.4.4, and changed it so that she searched the unplaced cards for the largest remaining numeral. The two excerpts together suggest that though "before" and "after" were not operationally reciprocal, the connection was such that, upon reflection, she could construct a strategy to handle a novel situation.

<u>Numerical operations</u>. Rho was a counter with abstract unit items. The discussion will not focus on this, for the episodes leading to the conclusion tell us much more--that Rho had operationally related integrating and separating. The following episode is one that suggests this most clearly.

I: (Places card with "70 - 31 = __" onto the table.)
R: Seventy take away thirty-one. (Pauses.) Forty . . . one.
I: Forty-one?
R: No.
I: Do you want to work it out loud?
R: Uh-uh (no).
I: Do you want to use these things (indicates box of MAB's)?
R: Seventy (takes 7 longs from the box) take away thirty (slides three longs to her right). I need that (removes 1 unit cube from box; places it with the 3 longs).
I: So if you had seventy and take away thirty, how many do you have left?
R: (Looks at the 4 longs.) Wait. (Places one of the 4 longs back in the box; counts out 9 unit cubes; places them on the table; counts them.) I have seventy, including these (places unit cubes with the longs) and I take away thirty-one (removes 3 longs and a unit cube) and I have (counts the remaining unit cubes) thirty-nine.
I: I thought you said forty-one before?
R: I did, but I messed up.

Excerpt 4.4.6

Rho's initial answer of "forty . . . one" could be taken as indicative of her use of a figural strategy: subtract smallest "tens" digit from the largest; subtract smallest "ones" digit from largest; name accordingly. Later episodes will suggest this was not the case. The significance of Rho's behaviors in this task comes from the episode in which she used blocks to solve the problem (8-18). After she added a unit cube to satisfy the functional requirement that there be one in order to "take away" thirty-one (9-10), Rho appeared to realize that adding one to the amount taken away <u>had an implication for the amount she began with</u> (13-16). This can only be explained by supposing that she conceptualized the problem as a number being separated into two, which when integrated produced the original number. This would formally be depicted by Figure 4.4.1.

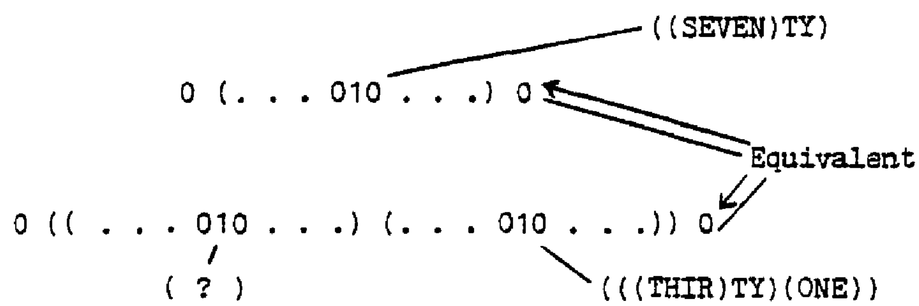

Figure 4.4.1. Operational relationship between integrating and separating in Rho's understanding of whole number numeration.

Whether or not Rho actually performed the computation 40 + 31 (4 longs plus 3 longs and a unit cube) and compared the result with 70 is irrelevant. If she did, we would still have to suppose she had operationally related integrating and separating to explain why she might have thought to do so in the first place. If she didn't, then we would have to suppose not only that she originally conceived of the problem as depicted in Figure 4.4.1, but that she subsequently constructed the conceptualization depicted in Figure 4.4.2.

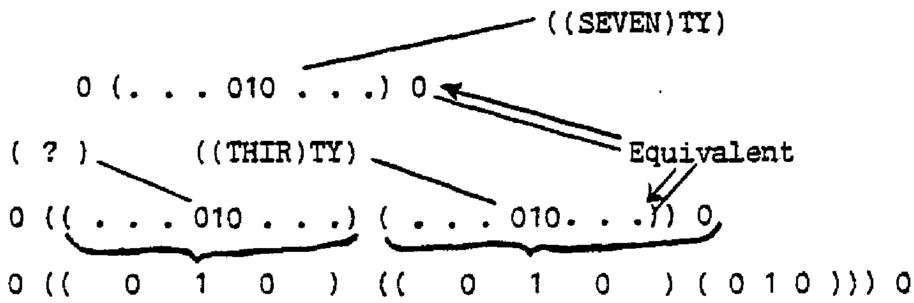

Figure 4.4.2. Rho's understanding of the implication for the initial number when adding one to its separation.

Figure 4.4.2, read from top to bottom, depicts Rho as thinking of "seventy" as a number, separating it into two--one of which was thirty--which when integrated produced 70, and then integrating the number named "thirty" with one named "one." Rho's introduction of the unit cube (9-10) created the nonequivalence between her final and initial numerical structures--they could not be reversibly related by integrating and separating.

Several further episodes give support to the contention that Rho had operationally related integrating and separating, and there were no instances that suggested otherwise. The discussion will then turn to Rho's operations of extending and declending, and the relationship she had established between them.

Rho had apparently established extending and declending as means of creating numerical structures that corresponded to her conceptualizations of addition and subtraction problems. The connection between her conceptualizations and extending/declending were intensive meanings attributed to number-names. This can be seen in the following episode.

I: (Places card with "84 - 30 = __" onto the table.)
R: Eighty-four take away thirty. (Pauses.) Fifty . . . fifty . . . I don't know.
I: How would you do that?
R: I'd have four in my head . . . huh-uh, I can't do it that way.
I: What do you have in your head?
R: Four.
I: Four? Not eighty-four?
R: Yeah, eighty-four. And I take away thirty.
I: Do you have to take away thirty all at once?
R: No.
I: How else could you do it?
R: I could say eighty-four take away thirty is seventy-four, sixty-four, fifty-four.
I: Oh, did you take away thirty?
R: Yeah.

Excerpt 4.4.7

The origin of Rho's difficulty at the outset is unclear. Perhaps she considered subtracting three tens from eight, and then mistakenly focused on the "4" of "84" while still thinking of "taking away." Rho's behavior in (9-16) is significant, however, of the main point. She understood that taking away 30 "all at once" (the interviewer's phrase) and taking away 30 ten at a time produced the same result—a number separated into a subtrahend and a difference. The <u>function</u> of declending was to name the difference. Several episodes suggest that intensive meaning also served as a connection between her initial conceptualizations and extending.

Extending and declending, however, were not reversible operations for Rho. The following episodes provide grounds for concluding this. The episode in Excerpt 4.4.8 is provided only because reference to it is made in that of Excerpt 4.4.9. The discussion will focus upon the latter excerpt.

I: (Places card with "47 - 21 = __" onto the table.)
R: Forty-seven take away twenty-one. Hmmmm (long pause).
I: Are you having trouble with that one?
R: Huh-uh (no; pauses). It's got to be twenty-six.
I: Got to be twenty-six, huh. How did you get that?
R: If I take away twenty-one from forty . . . I said like forty-seven, then I took away thirty-seven, and then twenty-seven, then I took away one more from seven, and it was six.

Excerpt 4.4.8

I: (Places card with "91 - 29 = __" onto the table.)
R: Ninety-one take away twenty-nine. Hmmmm . . . ninety-one . . . I can't do that one.
I: Do you want to use the blocks?
R: Yeah. (Reaches for box of MAB's.)
I: Can you do it the same way you did the other one? (Pauses.) Remember how you did this one (shows R the card with "47 - 21 = __")?
R: (Nods head yes.)
I: Can you do this problem the same way?
R: (Pauses; stares at card with "91 - 29 = __.") Maybe. Let me think. (Stares at card; pauses.) It's got to be sixty. It might be sixty.
I: It might be sixty, huh? Sixty is close?
R: See, ninety-one, eighty-one, seventy-one (sequentially putting up 3 fingers), and I can't take away nine from seventy-one.
I: Can you take away ten?
R: No.
I: Can you take away ten from seventy-one?
R: Yeah.
I: What would that be?
R: Sixty-one.
I: Did you take away too many when you took away ten?
R: Huh-uh (no).
I: How many was left to take away when you got down to seventy-one? (Pause.) Go ahead and take away the tens again.
R: Ninety-one, eighty-one, seventy-one.
I: Okay, how many more do you have to take away?
R: One . . . no, nine.
I: Okay, nine. Now if you take away ten, what will you have?
R: Six . . . seven . . . sixty.
I: Sixty . . .
R: Sixty-one.
I: Sixty-one. Now you took away ten. How many were you supposed to take away?
R: Nine.
I: So how many did you take away that were too many?
R: One more than nine.
I: Okay. So you have sixty-one, and you took away one too many. What should the answer be?
R: Sixty.
I: Sixty?
R: No . . . yeah . . . sixty.
I: Okay, try it with the blocks now.

Excerpt 4.4.9

The portion of Excerpt 4.4.9 that is important for the present point is in lines (25-43). The problem that the interviewer attempted to get Rho to construct was $91 - 29 = 91 - (30 - 1) = \_\_$. It appears that the one she did construct was $71 - 9 = 71 - (10 - 1) = \_\_$ in lines (34-41). For Rho to have successfully solved it, she would have had to make the further transformation $71 - (10 - 1) = 71 - 10 + 1$, which would entail the necessity of having operationally related extending and declending--counting nine backward is the same as counting ten backward and one forward. Instead, Rho made the following transformation: $71 - (10 - 1) = 71 - 10 - 1$.

Another set of episodes shows that once Rho had extended she could reversibly relate it to declending, but that she could not do so in the absence of having already extended.

I: (Places card with "92 → 70" onto the table.) How far is it from ninety-two down to seventy?
R: Twenty-two.
I: You didn't even count!
R: I know because it's just the same as that one. See, that one is just turned around (holds up "92 → 72"). Cause that one goes to that one (points from "92" of one card to "92" of the other) and that one goes to that one (points to "70" of one card, then "70" of the other).

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I: How far is it from this number to that number (places card with "64 → 40" onto the table)?
R: (Pauses.) I don't know.
I: Can you think of it a different way so that it would be easier?
R: (Pauses.) Huh-uh (no).

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I: (Places card with "40 → 64" onto the table.)
R: Same one.
I: Same one?
R: Yeah. But different.
I: Can you do that one?
R: (Pauses; sequentially puts up four fingers.) Twenty-four.
I: Twenty-four? So you just counted on. Can you tell me the answer to this one (refers to "64 → 40")?
R: Twenty-four. Cause it's still the same (holding the 2 cards together). It's just turned around.
I: Let's pretend like we're starting all over again. If I give you this one (places "64 → 40" onto the table) and ask how far it is from this number ("64") to this number ("40"), how can you find the answer?
R: Start at this one (points to "40") and go back to that one (points to "64").

Excerpt 4.4.10

It is not clear on what basis Rho concluded that "70 → 92" and "92 → 70" had the same answer (5-9). It is quite possible that she related the two at a purely figural level: everything about the cards was the same except for the relative placement of "70" and "90." It is the way she related "64 → 40" with "40 → 64" (25-30) that suggests that Rho could reversibly relate extending and declending once she had extended.

One final remark: in situations where Rho understood a problem as requiring either extending or declending, and the extension or declension exceeded her ability to subitize, she would either say something to the effect that she couldn't do it or she would use base-ten blocks. This in itself suggests that even though extending and declending may have been mental operations for Rho, she was constrained by her overreliance on subitizing when trying to implement them while keeping track of her count.

Concept of ten. As we have already seen, Rho could sequence by ten both forward and backward, and her linguistic system for producing number-names and sequences of number-names was quite close to being operational. Moreover, number-names referred to numbers for Rho, and she had operationally related integrating and separating. All this would suggest that Rho was quite capable of having formed a base-ten structure for her number-names. We have already seen indications of this in Excerpts 4.4.7 and 4.4.8, where Rho attributed to "31" the meaning of three tens and one, and to "21" the meaning of two tens and one--and used these meanings in solving problems. The following episodes are consistent with these observations.

I: How many tens are there in thirteen?
R: (Holds up 1 finger.) Give me a piece of paper. I want to write them down (writes "1" on a piece of paper). One.

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I: How many tens are there in sixty-seven?
R: (Writes "6" on the paper.) One six.

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I: (Places pile of sticks on table; covers pile with hand.) Let's imagine that this pile has seventy-two sticks in it. If you took all the tens out of seventy-two, how many would be left?
R: (Holds up 2 fingers.)
I: How many is that?
R: Two.

Excerpt 4.4.11

Rho could also impose the base-ten structure of her number-names onto numerals. The following episodes suggest not only that Rho could analyze a numeral into its digits, assigning respective meanings, but that concatenating number-names was the linguistic correspondent of integrating numbers.

I: (Places card with "40 + ___ = 46" onto the table.)
R: Forty plus blank equals forty-six. Six.
I: How do you know six goes there?
R: I can tell by that (points to the "6" of "46").
I: By that, huh. You mean the six over here in the answer?
R: Uh-huh (yes). If you take out that zero (places hand over "0" of "40") and put six right there (points to the blank).

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I: (Places card with "___ + 20 = 25" onto the table.)
R: Blank plus twenty equals twenty-five. Five.
I: How do you know?
R: I just know them.

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I: (Places card with "___ + 9 = 79" onto the table.)
R: Blank plus nine equals seventy-nine. (Pauses.) Seven . . . seventy goes right there.
I: Seventy?
R: Seven.
I: Seven plus nine is seventy-nine?
R: No. Seventy goes there.
I: How do you know that one?
R: I can tell by that seven (points to the "7" of "79" and that nine (points to the "9") that seventy has got to go there to make seventy-nine.

Excerpt 4.4.12

The first and second of the above episodes (1-11) shows that Rho had developed figural strategies for completing addition sentences involving a digit and a multiple of ten, but little more. There is no indication that she attributed numerical significance either to the digits or their concatenation. We do see evidence of this, however, in the third episode. When she said "seven . . . seventy goes right there" (13-14), Rho seemed to have analyzed "79" into "7" and "9," matched "9" of "79" with "9" of the left hand side, and had "7" of "79" as the missing part of the sentence. There are two possibilities for the way Rho quickly transformed "seven" to "seventy." First, she could have moved to a conceptual level and attributed to "seven" the meaning of seven tens. Second, she could have remained at linguistic level and matched "seven" and "nine" with the name "seventy-nine," corresponding "+" with linguistic concatenation. Rho's remarks in (20-22), and the following three episodes suggest that she did the latter, but with the significance of the former.

I: I have some number problems on these cards. I want you to think out loud while you do these so that I can hear what you're doing. Are you ready? Here's the first one (places card with "10 + 7 = __" onto the table).
R: Ten plus seven. Seventeen.
I: How did you know that?
R: We're having a test on these things.
I: If you had to work this out in your head real fast, how would you do it? Do you just know it, or could you work it?
R: I'd say seven more in my head, and I'd say ten right here (extends both hands). I'd say (folds all fingers) ten . . . eleven, twelve, . . ., seventeen (while sequentially putting up 7 fingers). I'd have seven in my head and have ten right here (extends both hands). Then I would take the seven out of my head and put it on my fingers. Then I'd go (folding both hands) ten . . . eleven, twelve, . . ., seventeen (while extending 7 fingers). I'd take the seven out of my head and put it on my fingers.

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I: (Places card with "10 + __ = 13" onto the table.)
R: Ten plus . . .
I: Blank.
R: Equals thirteen. Three.
I: How do you know that so fast?
R: I just know them.
I: If you had to work that one, how would you do it?
R: I'd have ten right here (extends fingers on both hands), just like I did seven. And I'd put three in my head.
I: How do you know to put three in your head?
R: (Pauses; laughs.)
I: You just know, huh?
R: Yeah. I put three in my head and take out three. And go ten . . . eleven, twelve, thirteen (while extending 3 fingers).

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I: (Places card with "60 - 20 = __" onto the table.)
R: Sixty take away twenty. Forty.
I: How did you do that?
R: I'd say sixty. I know that five take away two is four. I have sixty take away forty; it's got to be forty.

Excerpt 4.4.13

Rho's rapid initial responses in the first two episodes suggest that they were linguistic computations. Her clarifying remarks show, however, that they were carried out with the significance of numerical operations. The third episode shows, aside from her inattention to her reconstruction, that the "6" and "2" of "60" and "20" each held the significance of a number of tens: (((SIX)TY) take away ((TWEN)TY))

→ (((SIX)TENS) take away ((TWO)TENS)) → (((SIX) take away (TWO))TENS) → ((FOUR)TENS) → ((FOR)TY).

Thus far we have shown that Rho's linguistic system had a base-ten structure, that her linguistic operations held the significance of numerical operations, and that ten was an arithmetic unit, or number, for her. We have yet to examine the relationship that Rho had established between ten and one, and her significance for an increment by ten when counting by ten.

In Excerpt 4.4.6 (page 189), Rho made seventy out of MAB longs, and then traded a long for ten unit cubes. This in itself would not be sufficient to conclude that Rho equated one ten with ten ones, for "trading" could be a portion of an action schema for "take away" problems. We recall that Rho <u>conserved</u> 70 by way of the operational reversibility of integrating and separating, however, and if we take note of her remark ". . . seventy, <u>including these</u> . . ." (15), it seems fair to conclude that, in that task, Rho considered seven tens to be equivalent to six tens and ten ones, and hence that one ten and ten ones are equivalent numbers.

Another episode shows that Rho could keep ten and one, <u>qua</u> abstract unit items, conceptually distinct. The episode in the following excerpt followed those in Excerpt 4.4.10.

I: Here is one last problem, but instead of writing it down I'm going to tell it to you. Are you ready? How many is it from thirty-six up to fifty?
R: (Pauses.) Two . . . twenty . . . twenty-six (counts on fingers).
I: How did you get that?
R: Thirty. Forty, fifty (putting up one finger, then another on left hand). Fifty-one, fifty-two, . . ., fifty-six (sequentially putting up other 3 fingers of left hand and 3 fingers of right hand). That's six (indicating last six fingers that she put up) and that's twenty (indicating first 2 fingers she put up).

Excerpt 4.4.14

Let us first assume that, for whatever reason, Rho understood the problem to be "How many from 30 to 56?" It seems clear that Rho could not have been counting fingers as such, for she put up eight fingers--not 26. Given that she was not counting fingers, we have to ask what she was counting; the answer is apparently extensions by ten and by one--abstract unit items, of which her fingers served merely as records. That Rho kept the two conceptually distinct is apparent from the way she categorized her fingers (9-10), which, as fingers go, were otherwise indistinguishable. Recall that Kappa attempted to use his fingers to record both tens and ones (page 155). but could not conceptually distinguish between ten and one qua abstract unit items. The mechanism by which Rho distinguished between ten and one as abstract unit items was likely by labelling as "ten" those constructed in the context of incrementing by ten.

The significance that Rho gave to an increment by ten appears to have been ten increments by one. This inference can be drawn indirectly from her appropriate coordination of counting by ten and counting by one in addition and subtraction problems, and directly from the following episode.

I: In this little bag (places bag filled with unit cubes onto the table) there are . . .
R: A hundred.
I: Wrong.
R: Two hundred.
I: Two hundred and thirty-five little blocks.
R: Did you count them all?
I: I did. I counted them all out before I put them in.
R: God, I bet that took a long time.
I: Not very long.
R: Did you count by tens?
I: Uh-uh (no).
R: Twos.
I: No, I counted by ones.

Excerpt 4.4.15

In (9-14), we see that Rho understood that 235 cubes could be counted by tens, and that you would produce the same number sequence (235), only faster. Similarly for counting by two. Implicit in this is the understanding that a sequence of 235 units can be organized into subsequences each of length ten: "(one, two, . . ., nine) ten, (eleven, . . ., nineteen) twenty, . . ., two hundred thirty-five." That is, an increment by ten had the significance for Rho of ten increments by one.

Concept of one hundred. Excerpts 4.4.2 and 4.4.3 show that Rho could sequence by one hundred, but not from an initial number-name. The linguistic transformation of ((WORD)(HUNDRED)) to (((ONE)HUNDRED)(WORD)) was still problematic for Rho. However, Rho's concept of one hundred was quite elaborate. We shall see that she understood one hundred as an arithmetic unit, and one that is composed of ten units of ten.

The following episode shows that Rho had constructed one hundred as an arithmetic unit, and suggests that it was also a unit of numeration for her.

I: In this little bag (places bag filled with unit cubes onto the table) there are . . .
R: A hundred.
I: Wrong.
R: Two hundred.
I: Two hundred and thirty-five little blocks.
R: Did you count them all?
I: I did. I counted them all out before I put them in.
R: God, I bet that took a long time.
I: Not very long.
R: Did you count by tens?
I: Uh-uh (no).
R: Twos.
I: No, I counted by ones. If we wanted to make piles so that there were exactly a hundred in each pile, with just that many, how many piles could we make?
R: Two.
I: How do you know that?
R: Because a hundred plus a hundred is two hundred.
I: Have we got enough to make more? More than two piles with exactly one hundred?
R: You could only make two piles because thirty-five isn't a hundred.

Excerpt 4.4.16

The crucial portion of this episode is in lines (19-23), especially (22-23). Rho apparently attributed intensive meaning to "one hundred" at a symbolic level, and understood that the number of hundreds in 235 was to be found by constructing 235, as much as possible, in increments of one hundred. That is, there are <u>two</u> hundreds in 235 because you can increment by one hundred twice ("a hundred plus a hundred is two hundred" (19)), but 35 doesn't leave enough to increment again by one hundred. Thus one hundred was both one number and a number of ones for Rho, and as one number, it could be used as a unit of construction for others. The base-ten structure of Rho's number-names allowed her to carry out these constructions at the level of language, with her linguistic transformations carrying the significance of numerical operations.

The next excerpt will be given in three parts. Each part illustrates in a way slightly different from the others the relationship that Rho had established between ten and one hundred as units.

I: (Hands R some MAB longs.) I want you to count these until you get to one hundred sixty.
R: Thirty, sixty, ninety, one hundred (putting out 3 longs at a time, then 1 long). A hundred and what?
I: One hundred sixty.
R: (Recounts the longs.) A hundred. One hundred ten (continues to count out longs) one hundred sixty (recounts the 6 longs).
I: There are one hundred sixty little tiny blocks, if we could saw these up (points to longs). How many hundreds are there?
R: One.
I: How do you know that?
R: Cause this is ten tens.
I: What's ten tens?
R: (Slides 11 longs to her left one at a time.) That's ten tens (placing her hand over the 11 longs). That's a hundred. See. Ten, twenty, . . ., one hundred (pointing to first 10 of the 11 longs). Wait (places 1 of the 11 with the other 5). That's ten tens. One . . . ten, twenty, . . ., one hundred (pointing to each of the 10 longs). And that's sixty (picking up the 6 longs).

Excerpt 4.4.17a

It is interesting to note that Rho's reason for saying that there is one 100 in 160 was because "there is ten tens" (12), and yet she never _explicitly_ counted ten longs as "one, two, . . . ten." This suggests quite strongly that Rho's understanding of 100 as ten tens was a conceptual relationship, and not one based literally on, say, the composition of MAB flat as a collection of longs. She appeared to be appealing to the _necessity_ of having ten tens, since she counted by tens to "one hundred."

I: Keep counting. I'm going to put some more out.
R: One hundred sixty, one hundred seventy, one hundred eighty, one hundred ninety, two hundred (as the interviewer hands her 4 more longs).
I: (Places a cover over the 20 longs). Keep counting. No, don't count. I'm going to put this many down (places 4 longs on top of cover). How many tens underneath and on top are there altogether now?
R: (Pauses.) Three hundred.
I: Three hundred tens?
R: No. That's two hundred seventy right there. (Places 1 long from top of cover to side of cover; pauses.) Eighty, two hundred ninety, three hundred (pointing to each long on top of cover).
I: Let's see now. How many were under here (lifts cover).
R: Two hundred. Wait. Let me see (lines up longs, without counting them, so that there are 11 in one row and 9 in another; the left ends of the rows are aligned). It's not two hundred.
I: Do you want to check? I thought it was.
R: (Counts the row of 9 longs; begins counting the row of 11 longs.) Now it is (places 1 long from the row of 11 into the row of 9 making two rows of 10). That was ninety down there.
I: Was it two hundred before?
R: Yeah. Wait (counts each of the 20 longs). Yeah that's two hundred.

Excerpt 4.4.17b

Rho's error in (9-14) appears to have been superficial. If we assume that the word "tens" didn't register with her, and that she confused this with the previous episode (where she counted to 160), it would follow quite naturally that she recalled having ended at "two hundred sixty," and that she understood the interviewer to be asking _how many_ altogether. The interesting aspect of the episode is when she decided to check that there were, in fact, two hundred under the cover (16-18). Her "check" was to make two groups of ten; when the groups didn't align properly Rho recognized this as violating a necessary condition of having two hundred given that at least one group had ten tens. Again we see that Rho understood one hundred as a number to be necessarily composed of ten numbers of ten.

I: (Replaces cover; places 4 longs on top of cover.)
R: Two hundred ten, two hundred twenty, two hundred thirty, two hundred forty (pointing to each long on top of the cover). That's two hundred forty.
I: How many tens altogether?
R: (Pauses.) Twenty-four.
I: How do you know that?
R: There's twenty tens down there, and there's four more (pointing to longs on top of cover).
I: (Places another long on top of the cover.)
R: Twenty-five.
I: How many little blocks like this are there (holds up unit cube)?
R: A hun . . . two hundred and . . . (points to each of the longs on top of the cover; looks at the long in the interviewer's hand). Two hundred and sixty.
I: How many hundreds are there?
R: Twenty-five.
I: Twenty-five piles of one hundred blocks?
R: Not twenty-five--yeah, twenty-five.

Excerpt 4.4.17c

At the end of Excerpt 4.4.17b we saw that Rho recounted the longs. It was not apparent from viewing the tape whether she counted them by one or by ten. Whether she did or not may have an implication for how she knew that there were 24 tens after the interviewer placed four more on top of the cover (1-9). Regardless, Rho certainly displayed flexibility in moving from the context of counting a number by ten t o the number of tens counted. The origin of Rho's mistake in (16-19) is not clear. Perhaps, since all but one of the questions up to that point had been about either "altogether" or "tens," Rho expected any question dealing with units other than one to be about ten, and assimilated the question accordingly. Rho's method of answering the interviewer's question in (12) is also understandable. To linguistically compute 25 tens, she would have had to apply an equivalent of the distributive property of multiplication over addition, attribute to "twenty" the meaning of two tens, and make the substitution of "hundred" for ten tens, such as: (((TWEN)TY)(FIVE))TENS) → ((((TWEN)TY)TENS)((FIVE)TENS)) → (((TWO)((TEN)TENS))((FIVE)TENS)) → (((TWO)HUNDRED)((FIVE)TENS)) → (((TWO)HUNDRED)((FIF)TY)). More likely, she recalled the episode of equating the 20 tens under the cover with 200, and then counted-on with the visible longs.

Concept of place value. We have already established that Rho's concepts of ten and one hundred were fairly complete, and that she had formed the relation "ten of" between one and ten, and ten and one hundred. Only one episode sheds light on whether or not Rho could compose these relationships so that one hundred was ten of (ten of one).

I: (Places MAB flat on the table; holds up MAB long in hand.) How many of these (indicates long) could you make out of this piece of wood (indicates flat)?
R: What are you saying?
I: If we could get a saw and cut up this piece of wood (indicates flat) into pieces like this (indicates long), how many pieces could we make?
R: A hundred.
I: How do you know that?
R: Cause that's a hundred (points to the flat), and if you had a saw, you could cut across those lines (vertically) and there would still be ten . . . a hundred.
I: One hundred of these (holds up long)?
R: (Pauses.) You use that thing?
I: That's what you told me. One hundred.
R: It's going to be one hundred . . . Oh, one hundred and ten.

Excerpt 4.4.18

Rho's behavior in the above episode might be taken as counter-indicative of the conclusion drawn earlier that she knew one hundred as composed of ten tens. The interpretation given here is that her behavior is irrelevant to that conclusion. If we first suppose that she tacitly named the flat "hundred" and thought of it initially strictly as a number, then it is quite reasonable that she maintained that it remains one hundred even if "you cut across these lines" (10-12). If Rho's aim was to maintain the structure of 100 as a number of ones, then to assimilate the question as the interviewer intended, she would have had to reconstruct 100 as ten of (ten of one)--composing the two relations she apparently had. The following episode lends credence to this interpretation.

I: (Places MAB flat on table; holds unit cube in hand.) Let's try this one. How many of these little blocks (indicates unit cube) make up this piece of wood (indicates flat)?
R: A hundred.
I: But you told me there were a hundred of these (holds up long) that we could make out of that (points to flat).
R: No, there are ten of those. See, one, two, . . ., ten (pointing to each unit along lower edge of flat; rotates flat 90 degrees). One, two, . . ., ten (pointing to each unit along lower edge of flat). There's ten both ways. Ten this way (horizontal) and ten this way (vertical).

Excerpt 4.4.19

When the interviewer confronted Rho with her conflicting answers (5 - 6), she resolved the conflict by making the pairwise relationships between 100 and one and between 100 and ten. She did not have to resort to composing the relations between 100, ten, and one.

The counting-board tasks presented no difficulty whatsoever for Rho. She counted each piece as it was uncovered by appropriately incrementing the name of the number uncovered up to that point. Several times, almost playfully, she recounted the blocks by first counting the flats, then the longs, and last the unit cubes. All of this is consistent with the analyses of her behavior on the other tasks. In terms of constructing number-names as referring to numbers, she would do so beginning with the largest possible numeration unit and proceed with successively smaller ones.

Case Study 4.5: Gamma

Gamma was a second-grader (age 7 years at the beginning of the 1977 school year). In November of 1977 she correctly solved Problem 1 by counting "six, seven, eight" while looking at the visible squares, and solved Problem 3 by putting out five and four fingers and then counting-all. She could not solve Problem 2. When asked to "count by tens," Gamma sequenced by five. She continued the interviewer's example of "ten, twenty" by sequencing to "one hundred," but when asked for the next one, said "two hundred." Gamma could not sequence by ten from "two," but knew that 12 is ten more than two. When asked for ten more again, she answered "thirteen . . . fourteen." Gamma responded, without counting, that a bundle of ten and four single sticks make 14 altogether, and that there are three tens in 32. Her reason for the latter answer was "cause there's a three in front and two is in last." However, when given two bundles of ten and five single sticks, she responded that there were 15 in all--acknowledging that there were "two tens" and "five ones" on the table. When asked to find the total number of sticks in ten bundles of ten, Gamma first counted each bundle as one (getting "ten"), and then, after the interviewer had reiterated the objective, counted the individual sticks in the bundles--ending with "eighty-two." This was all after acknowledging that there were ten sticks in each bundle. The final interviews were given to Gamma on May 5, 10, and 18 of 1977.

Gamma's case study is an interesting one. She had apparently constructed numbers and could use extending and declending reasonably well. However, she had created (errorful) empirical routines from school instruction on standard paper-and-pencil addition and subtraction algorithms which competed with her numerical operations for activation. In several instances this produced quite odd behavior.

Writing and reading numerals. Gamma had no difficulty writing numerals, and only occasionally misread them. She read "201" as "twenty hundred and one," "31" as "thirteen," "46" as "sixty-four," "52" as "twenty-five," and "61" as "sixteen." The common ingredient of these instances is that Gamma was in the context of solving a problem when reading the numerals. Quite possibly, she was concentrating on the problem and gave only fleeting glances to the numerals.

Sequencing. Though there were no tasks aimed directly at assessing Gamma's ability to sequence by one, she chose to count by one in several tasks. On one occasion Gamma counted by one from 20 to 89, and on another from one to 64--both times making counting errors, but not sequencing errors. More will be said about Gamma's counting by one in "Numerical operations."

Gamma's routine for sequencing by ten, like Kappa's, seems to have been based around the sequence "ten, twenty, . . ., ninety," as seen in the following episode.

I: Start at ninety-seven and count-back by tens. (Pause.) What's ten back?
G: Eighty-seven.
I: Okay, keep going.
G: Seventy-seven, sixty-seven, forty . . . thir--, fifty-seven, forty-seven, thirty-seven, twenty-seven . . . seventeen . . . te . . . seven.

Excerpt 4.5.1

Two incidents in Gamma's production of her backward sequence suggest that she was sequencing "eighty, seventy, . . ., ten" and appending "seven" to each term. The first is in (5), where she said "sixty-seven . . . forty . . . thir--fifty-seven . . . ." She appears to have been checking for a reciprocal "next" relationship between successive "ty" words, which suggests that it was them that she was operating upon. Second, in (6), Gamma caught herself saying "seventeen . . . ten." Perhaps in going from "seventeen" to "ten," Gamma anticipated arriving at the beginning of the forward sequence through which she was "backtracking"--which would put her at "ten" if the significant sequence was "eighty, seventy, . . ., ten." The fact that she caught herself and said "seven" could be explained by the supposition above that she intended to append "seven" to each term--and she had already said "seventeen."

The hypothesis that Gamma's routine for sequencing by ten was centered around her sequence "ten, twenty, . . ., ninety" also fits well with the fact that she had tremendous difficulty making transitions between centuries. The next episode illustrates her problem. The context of the episode is that the interviewer had asked Gamma to count as he placed MAB longs on the table; we join the interview after the longs had been placed.

I: Let's keep counting now. We have one hundred and sixty . . . one hundred and sixty.
G: Hundred and seventy, hundred and eighty, hundred and ninety, one hundred and . . . twenty (as the interviewer places 4 more longs). Wait.
I: What comes after one hundred ninety?
G: A hundred and . . . twenty. No.
I: One hundred ninety, two hundred.
G: Two hundred.

Excerpt 4.5.2

If we assume that Gamma's method of sequencing was to construct the next "ty" name and then precede it by "one hundred," then her dilemma in (4) becomes more apparent. When she reached "ninety," she knew the next name should be "one hundred"; but she was already <u>beyond</u> "one hundred," so she took it as the completion of her second episode of counting to one hundred. This, along with being in the context of sequencing by ten, gave her the conceptual basis for linguistically computing "hundred (two)(ten) $\rightarrow$ hundred and twenty."

The above interpretation of Gamma's difficulties gains support from another episode.

I: Start at three hundred forty and count-on by tens.
G: Three hundred sixt . . . four hundred and . . . forty.
I: Okay. Three hundred forty—three hundred fifty, three hundred sixty . . .
G: Three hundred and seventy, three hundred and eighty, three hundred and ninety, three hundred and . . . three hundred and . . . .
I: Three hundred ninety.
G: Three hundred and ninety . . . three hundred and . . . ninety-one, three hundred and ninety-two, three hundred and ninety-three . . . .
I: (Interrupting.) Are you counting by tens?
G: No.
I: What's ten more than three hundred ninety?
G: Three hundred and . . . fifty. Three hundred and sixty, three hundred and seventy, three hundred and eighty . . . wait.
I: We got off track. We got up to three hundred ninety. We want ten more than three hundred ninety.
G: (Long pause; mumbles to herself.)
I: What are you doing?
G: Three hundred eighty.
I: We could do it this way. Three hundred ninety and ten more. Three hundred ninety, three hundred ninety-one, . . ., three hundred ninety-nine . . . .
G: One hundred.
I: Four hundred.

Excerpt 4.5.3

Gamma's initial confusion may have stemmed from an indecisiveness about which part of the number-name to operate upon. The significant aspects of the episode are in lines (8-11) and (22-25), where Gamma changed to sequencing by one when she could not construct the successor to "three hundred and ninety" and where she completed the interviewer's sequence by one with "one hundred." The latter shows quite strongly that it was the sequence "ninety-one, . . ., ninety-nine" that Gamma took as significant, and that "three hundred" was merely an appendage to be "tacked on." This in turn suggests that it was "fifty, sixty, . . ., ninety" that she took as significant in the former, and that she could not continue because she had no *episodic* recollection of repeatedly sequencing by ten to one hundred.

Gamma seemed also to have an underlying method of sequencing by ten: emphasize the first-said part of the number-name and use the "next" relation for sequencing by one.

I: Can you start at eight and count-on by tens?
G: Eighteen, twenty-eight, . . ., ninety-eight . . . ninety-eight, one hundred and eight, two hundred and eight, three hundred and eight, four hundred and eight, five hundred and eight, six hundred and eight . . . .
I: Okay, that'll be fine.

Excerpt 4.5.4

Apparently, Gamma sequenced "eight, eighteen, *twenty*-eight, *thirty*-eight, *forty*-eight, *fifty*-eight, . . ., *ninety*-eight, *one* hundred and eight, *two* hundred and eight, . . ."--the added emphasis meaning that at some point she fell into the pattern of incrementing the first-said part of the number-name and continued applying it to the currently-held one. This routine likely genetically underlay Gamma's "automatic" routine for sequencing "ten, twenty, . . ., ninety."

Gamma had no routine for sequencing by one hundred. In tasks where she was asked to begin at 30 and 73, she could not begin, nor continue after the interviewer gave the second term of the sequences. She continued only after being given the third terms, and then stopped at "nine hundred . . .," maintaining that there were no more to be said.

The question of the operationality of Gamma's system for generating number-names and sequences of number-names can easily be answered: it was not. Gamma apparently had the structure depicted in Chapter 3 (Figure 3.5, page 58), but only in action. She assimilated the tasks of placing the numeral cards in ascending order to her routines for sequencing rather than making pair-wise comparisons among cards. She would first select the "teen" cards and then the smallest among them, then the "twenty" cards and then the smallest among them, and so on. Her attempts at seriating the cards in descending order best show Gamma's lack of operationality.

Cards: 30 47 48 49 52 61 67 76 (shuffled)

I: This time I want you to place these cards on the board starting with the biggest number in the first place (indicates) and the smallest number in the last place (indicates).
G: (Spreads cards on table.) (76) Seventy-six. (76-67) Sixty-seven . . . sixty-seven . . . . (76-67-48) Forty-eight. (76-67-48-47) Forty-seven. (76-67-48-47-61) Sixty-one . . . wait. (76-67-48-47) (Removes all cards.) (76) Seventy-six.
I: That's what you had before.
G: (76-67) Sixty-seven. (76-67-49) Forty-nine. (76-67-49-48) Forty-eight. (76-67-49-48-30) Thirty. (76-67-49-48-30-52) Twenty-five. (76-67-49-48) (76-67-49-48-47) Forty-seven. (76-67-49-48-47-30) Thirty. (76-67-49-48-47-30-52) Twenty-five. (76-67-49-48-47-30-52-61) Sixteen.
I: Satisfied?
G: Yeah.

Excerpt 4.5.5

In (6 - 8), we see that Gamma was aware that "61" should not follow "47" in backward order, but that she could not decide where it should have been placed among those that were already positioned. If we assume that Gamma relied on her sequencing structure ("teen" < "twenty," "twenty" < "thirty," . . ., "eighty" < "ninety"; "<" means "precedes"), as she did when seriating in ascending order, then her failure to place "61" suggests that the sequence she constructed was rigid. In order to place "61" Gamma had to destroy the sequence up to the point of placement of "61" and begin anew. That is, Gamma's relationships between number-names were not operationally transitive. In (13) we see that Gamma ended up placing "61" as if it were "16." (Whether this was an accommodation to her conflict or a subsequent misreading of the numeral is not clear.) In the task following that in Excerpt 4.5.5, Gamma displayed similar behavior--destroying the sequence up to the proper point of placement of a newly-found card.

Numerical operations. It is not clear what type of counter Gamma was. She was at least a counter with motoric unit items, and possibly a counter with abstract unit items. The difficulty in deciding between the two may be seen in the next excerpt.

I: You can see we have twenty-four little blocks under this screen (lifts screen to show blocks; covers all the blocks; places MAB long adjacent to screen.) Can you write down how many we have altogether in pieces of wood?
G: (Writes "34.")
I: (Advances screen so that all blocks are covered; places three unit cubes next to screen.) How many little blocks are there altogether now?
G: (Pause; holds fingers of her right hand in her left; writes "37.")
I: (Advances screen so that all blocks are covered; places two MAB longs.) How many now?
G: (Pause; counts 20 times by one on her fingers; writes "67.")
I: (Advances screen so that all blocks are covered; places 1 MAB long and 1 unit cube next to screen.)
G: (Pause; counts 10 times by one on her fingers while looking at the blocks; writes "58.")
I: What's that number (points to "58")?
G: Fifty-eight.
I: What's that number (points to "67")?
G: Seven . . . oh (erases "58"; softly says 67; counts 10 times by one on her fingers; looks at unit cube; writes "78").
I: (Advances screen so that all blocks are covered; places two unit cubes and two MAB longs.) Write down how many little blocks I have altogether now.
G: Seventy-nine . . . (Counts 10 times by one on her fingers; looks at blocks; writes "90.")
I: How did you get that? What did you do?
G: Counted by ones.
I: (Advances screen so that all blocks are covered.) How many little pieces of wood are under here now?
G: Ninety.
I: Would you like to write ninety there (points to box at bottom of G's sheet of paper)?
G: (Writes "90" in the box.)

Excerpt 4.5.6

It is clear from lines (13), (21-22), and (26) that Gamma was at least a counter with motoric unit items. She not only made the functional substitution of fingers for blocks in her counting, but, and more significantly, counted the motoric activity of putting up a finger, as opposed to counting fingers per se. In (13) she counted each finger twice, yet considered them as different counts.

The reason that we cannot attribute counting with abstract unit items to Gamma in this task is that she apparently referred to the blocks for her criteria for stopping. It could well have been that Gamma's substitution of fingers for blocks was literally just that, and hence that she had not abstracted the numerical criterion of counting, say, 20 more (13). She could have instead equated both hands open with a long (by calling both "ten"), counted both hands for one long, counted both hands again for the second, and stopped because there were no more blocks to count. Her attention to the blocks in forming a criterion for stopping is most apparent in (21-22) and (26), where she counted ten fingers and then finished by counting unit cubes.

Gamma almost certainly constructed the task as (at least) successive extensions of a lot by another, and integrated the extensions--providing meaning for the number-name which she used as the starting place for the next step in the task. The question remains as to whether she reflected upon the lots she constructed, forming arithmetic lots. The answer to this is apparently yes, as will be seen in the discussion of the sequel to the above episode.

I: Now I'm going to take some wood from behind here. I want you to tell me how much I've got left (removes 2 longs). How much have we got left behind the screen? (Pause.) How much did we start with?
G: Ninety. (Pause.) Seventy.
I: Would you like to write it there (points to bottom of page)?
G: (Writes "70.")
I: (Removes 1 unit cube and 1 long.) Now how much do I have left behind now?
G: Sixty-one (writes "61.")
I: (Removes 1 long.) How much left now?
G: (Pause.) Fifty-one (writes "51").
I: (Removes 6 unit cubes.) Now how much do I have left?
G: (Pause; points to 2 unit cubes.)
I: What are you doing?
G: (Pause.) Forty-eight . . . forty-seven . . . forty-six . . . forty-five (while pointing to the last 4 unit cubes with her pencil). Forty-five (writes "45").
I: (Removes 1 unit cube and 2 longs.) How much is left?
G: (Pause; waves her pencil over the wood.)
I: How much was left before I took this out (picks up unit cube and 2 longs)?
G: (Points to "45.") Forty-five.
I: (Replaces the 2 longs and unit cube behind screen.) Now I'm taking out this much (takes out unit cube and 2 longs).
G: Forty-three . . . no, forty-four, forty-three (pointing to the unit cube), forty-two, forty-one, forty, . . ., twenty-four (while pointing to each unit of the 2 longs). Twenty-four (writes "24").

Excerpt 4.5.7

If Gamma had not constructed arithmetic lots, then she could not have separated her final construction (a lot named "ninety") into a known (2 longs) and unknown amount (1 - 5)--assuming she was actually thinking of a quantity of blocks under the cover. It is possible that Gamma had made a strong association between the action of things being "taken away" and declending, and did not construct a separation at all. We cannot decide between the two on the basis of this task alone. The following episode suggests that she could, in fact, construct separations, and that the operands were at least arithmetic lots.

I: (Places card with "__ + 9 = 79" onto the table.)
G: Blank plus nine equals twen . . . seventy-nine (pause; gets 1 long from the box.) Nine . . . teen.
I: (Interrupting.) Can you do it without the wood?
G: Huh-uh (no). Nineteen, twenty-nine, . . ., seventy-nine (while getting 6 more longs from the box). Ten, twenty, . . ., seventy (pointing to each long).
I: So what's the number?
G: Seventy.

Excerpt 4.5.8

Gamma clearly conceived of 79 as composed of nine and something else, and apparently used the MAB longs as a convenience for recording the amount she needed to extend nine by in constructing 79. Her placement of the longs was apparently a means to construct a *nameable* lot in place of the unnamed one that she had constructed in her conceptualization of the problem.

Gamma's behavior in Excerpt 4.5.8 also suggests that she could construct numbers. She had nothing in her experiential field from which she could derive boundaries for the lots she constructed in conception, so the boundaries must have been conceptually introduced--forming numbers. However, the fact that she needed a record of extending for later quantification suggests that Gamma did not extend with abstract unit items. That is, extending for Gamma was a means to an end, but was not an object of reflection--she did not count her counts.

The following episode suggests that declending was at the same level of operationality as extending for Gamma.

I: (Places card with "74 - ___ = 70" onto the table.)
G: Forty take away blank equals seventy.
I: What's this number (pointing to "74")?
G: Seventy-four take away blank equals seventy. (Pause.) Go backwards.
I: Okay.
G: Seventy-three, seventy-two, seventy-one, seventy. (Looks at her hands under the table). Four.

Excerpt 4.5.9

It seems clear that declending was an operation through which Gamma constructed a nameable lot that, when separated from one named "seventy-four," left one named "seventy." Whether or not Gamma meant "seventy" as a name of a lot *when she said it*, or whether saying "seventy" was her functional criterion for stopping--empirically abstracted from school experiences--is not clear. In either case, the main point remains substantiated: Gamma did not count her counts. Rather, she constructed a record of them and then quantified after counting. In no task did she do otherwise.

Though Gamma could integrate and separate arithmetic lots and numbers, it is not clear what relationship she had established between the two. Only two situations arose where it appeared that she was about to behave in a way that would be suggestive of an answer to the question (both were in the context of subtraction with blocks involving a trade), and both times the interviewer interrupted her, asking her if she wanted to trade a long for ten unit cubes. In the first of the two (70 - 31 = __), Gamma appeared to be ready to put a unit cube from the box alongside three longs that she had removed from seven. The interviewer was too quick with his question, though, for us to see if she would have done so without compensating the difference. It seems likely that she would not have compensated, for after counting the difference between 70 and 31 Gamma wrote "39," asking, "Are you sure?" The "trade" was an imposition by the interviewer, not a necessity for Gamma.

It is also not possible to tell what relationship Gamma had established between extending and declending. The set of tasks aimed directly at this relationship is largely irrelevant to the question; Gamma solved them figurally.

I: (Places card with "70 → 92" onto the table.) How many is it from seventy up to ninety-two?
G: (Pause.) Twenty.
I: Twenty?
G: No, thirty.
I: Thirty?
G: No, twenty.

I: Make up your mind now.
G: Twenty.
I: (Places card with "92 → 70" onto the table.) How many is it from ninety-two down to seventy?
G: Twenty.
I: Why?
G: Cause seventy to nine is twenty and twenty to ninety is seventy . . . twenty. It's the answer.
I: Why is that?
G: Cause they're both the same, but they're mixed around different.

Excerpt 4.5.10

Let us ignore, for the moment, the incorrectness of Gamma's answer and focus instead on her way of answering the second question (10-17). In (14-15) we see that Gamma apparently arrived at her original answer by comparing "7" of "70" and "9" of "90," and took into account their position in the numeral. Her criterion for saying that 92 → 70 has the same answer as 70 → 92 was figural, as opposed to operational, in that "92," "70," and "→" had changed positions but nevertheless were all still there. Gamma's way of relating the two "problems" was, in principle, the same as saying that if 92 - 70 = 20, then 70 - 92 = 20 because nothing has changed—they've merely been "mixed around different."

Gamma had developed an interesting figural routine for dealing with subtraction sentences containing two two-digit numerals.

I: (Places card with "60 - 20 = __" onto the table.)
G: Sixty take away twenty. (Pause.) Forty.
I: How did you get that?
G: Six take away two is four, and zero plus zero is . . . zero take away zero is zero.

Excerpt 4.5.11

It is quite possible that Gamma, like Rho, had developed this routine as a "shortcut" for more meaningful numerical operations. That this is not the case may be seen in the next excerpt.

I: (Places card with "70 - 31 = __" onto the table.)
G: Seventy take away thirteen equals blank. (Pause.)
I: How would you do that one?
G: Forty-one.
I: How did you get that?
G: One take away zero is one, and three take away seven is four.

Excerpt 4.5.12

If pressed, Gamma would most likely have admitted that three takeaway seven is not four, but within the constraints of operating upon <u>numerals</u>, that was the best she could do.

Gamma also applied her figural routine to "91 - 29 = __," "47 - 21 = __," and "84 - 30 = __." She also appeared to apply a variant of it to "__ + 20 = 25." That and her subsequent behavior prove to be quite interesting.

I: (Places card with "__ + 20 = 25" onto the table.)
G: Blank plus twenty equals twenty-five. (Pause.) Three . . . I mean, thirty. Wait (counts to herself).
I: What are you doing? Counting to yourself? I can't hear you.
G: Fifty . . . twenty-one, twenty-one . . . twenty-five. Five . . . (pause; mumbles something). Thirty.
I: Thirty plus twenty is twenty-five?
G: No. Twenty . . . no . . . .
I: What are you doing down there (referring to G's hands under the table)? Tell us what you're counting.
G: I'm not counting.
I: You can put your fingers up. (Pause.) What do we want to know here (points to card)?
G: (Pause.) Forty-five.. . . firty . . . five.
I: Would that be the number here (points to blank)?
G: I think so.
I: Are you sure you're not looking at the forty under there (referring to card directly underneath "__ + 20 = 25"; removes all cards but "__ + 20 = 25").
G: I didn't look at that one.
I: So you think forty-five plus twenty is equal to twenty-five. Is that it?
G: Huh-uh (no). It's too high. That'd be . . . fifty-five. (Pause.) Nineteen, eighteen, . . ., five, four, three, two, one.
I: What are you doing?
G: Counting backwards.
I: Counting back from where?
G: From twenty-five.
I: Okay.
G: Twenty-four, twenty-three, . . ., twenty. (Pause.) Nineteen . . .
I: What did you just do? You just counted what? Twenty. What did you just count back? Say it again.
G: Twenty-five—twenty-five, twenty-four, . . ., sixteen . . . .
I: Would it help you to use some of these (hands G box of MAB blocks). Tell me out loud what you're doing.
G: (Removes 2 longs from the box.) I'm going ten, twenty. Twenty-one, twenty-two, . . ., twenty-five (while removing a unit cube from the box with each count); places them beside the longs).
I: All right, twenty-five. How can that help you to do this one (points to the card)? What number goes in the blank?
G: Twenty here (slides the 2 longs away from her; removes 2 more longs from the box; places them next to the 5 unit cubes; points

to the original 2 longs; picks up the 2 longs just removed from the box). Wait.
I: Tell me what you're thinking.
G: I'm thinking that if I put these two there (indicates the 2 longs in her hand) then it would be forty . . . forty-five. (Places the 2 longs back on the table.) Ten, twenty (pointing to each of the original longs), thirty, forty (pointing at the 2 longs just placed on the table), forty-one, . . ., forty-five (pointing to each unit cube).
I: How many would you have altogether (places finger above "25") when you know the number here (points to the blank)? How many do you have altogether? (Pause.) What's that number (places finger above the "5" of "25")?
G: Five.
I: This number here (traces a circle around "25" with his finger).
G: Twenty-five.
I: That's how many you have altogether when you know this number (points to blank), isn't it?
G: Yeah. (Pause.)
I: So the most you would have is twenty-five. Is that right?
G: (Places both pairs of longs together.) Forty, thirty, forty (removes a long from the box; combines it with the 4 already out; combines the five longs with the 5 unit cubes; places 3 longs back into the box.)
I: Well, the most you would have is twenty-five. Read what this says again (points to card).
G: Blank plus twenty equals twenty-five.
I: Can you make what that would say with these (points to the 2 longs and 5 unit cubes)?
G: Yep. (Slides the 2 longs to her right.)
I: We know one of the numbers here is twenty (points to "20" on the card).
G: (Reaches for 2 more longs from the box; drops them back into the box.)
I: We know also that the most we want is twenty-five. We've got twenty-five here (places hand on the 2 longs and 5 unit cubes). One of the numbers is twenty (slides the 2 longs to the right). So what else have we got?
G: (Stares at wood.) Nothing. Two tens and . . . .
I: Two tens. That's twenty.
G: And five ones.
I: Five ones. What's five ones plus twenty?
G: Five ones plus twenty. Twenty-five. (Pause.) It's five!

Excerpt 4.5.13

The excerpt above can be subdivided into six fairly distinct episodes. In (2-3), Gamma apparently understood the problem as a difference between 20 and 25 (the amount necessary to extend 20 by), and called upon her figural routine to compute it. She had to modify it, however, because there was no difference between "2" of "20" and "2" of "25," and there had to be a difference--thus leaving "5" of "25" as the remaining candidate. "Three" became "thirty," since "2" of "20" was in the tens place. We see a confounding of routines in (5) as Gamma attempted to reconstruct her answer--"fifty" being a holdover of treating "5" of "25" as a tens-digit, "twenty-one, twenty-two" being a switch to extending, and "thirty" being her answer after changing back to her initial method. The second episode occurs in lines (7-23). It seems that, for whatever reason, Gamma understood the interviewer to be saying that her answer, and hence method, was wrong. So she tried something else--adding 20 and 25. If the interviewer had not asked her to think about her answer (21-22), Gamma would likely have ended there. But upon reflection, she realized 45 could not be correct because it was "too high" (23). There we see the beginning of the third episode in which Gamma gave declending a try (24-25), but failed because of not having kept a record of her count. In the fourth episode, with the blocks (37-53), we again see Gamma trying to add 20 and 25. It appears that by this time she was more concerned with getting an answer that would satisfy the interviewer than with thinking about the problem. In the fifth episode (54-85) Gamma had apparently given up, and answered each of the interviewer's questions in isolation from what had gone on. The sixth episode (86-87) is where Gamma "discovers" the answer. If the interviewer had asked "What is five ones plus twenty?" early on, Gamma probably would have answered then just as she did at the end.

The flexibility with which Gamma could call upon extending and declending makes it fairly safe to assume that she had fully routinized and labelled them. Her tendency to count by one in extending or declending (e.g., Excerpts 4.5.6 and 4.5.7) suggests also that Gamma had a strong sense of intensive meaning for number-names. However, Gamma's number-names were not symbolic of counting, for they did not comprise an operational base-ten structure.

To summarize this section, Gamma could construct arithmetic lots and numbers, which themselves could be operands of integrating and separating, but she did not count with abstract unit items. Rather, she constructed arithmetic lots or numbers through reflecting upon either her representation or record of having counted. Gamma had not fully established extending and declending at an operational level, for she could use them only to construct lots that she would later quantify. She had, however, routinized and labelled them, for they could be called as means to an end. It is not clear what relationships Gamma had established between integrating and separating and between extending and declending, though there was one indication that she had not operationally related integrating and separating. Finally, Gamma had developed a figural routine for subtraction that at times interfered with operations based upon an initial numerical understanding of a problem.

<u>Concept of ten</u>. Gamma's figural routine for subtracting was of such a nature that we might think that her concept of ten was fairly extensive, albeit misapplied in that context. If we analyze her routine for what she needed to know to apply it, though, we find that linguistic and perceptual competencies and memorized subtraction facts between numbers from 0 to 9 would be sufficient to account for her behavior. As a basis for a discussion of this, Excerpt 4.5.12 is repeated below with a new number.

1 I: (Places card with "70 - 31 = __" onto the table.)
2 G: Seventy take away thirteen equals blank. (Pause.)
3 I: How would you do that one?
4 G: Forty-one.

I: How did you get that?
G: One take away zero is one, and three take away seven is four.

Excerpt 4.5.14

The following procedure accounts for Gamma's behavior in the above episode.

1. Separate the numerals into left-most and right-most digits.

2. Subtract left-most from left-most; homonymically translate name of difference to a "ty" name.

3. Subtract right-most from right-most.

4. Append name of last-found difference to derived name of first-found difference.

5. Say name.

The above procedure requires no conceptual understanding of ten as a unit of numeration. It requires only that one abstract a correspondence between the way one reads numerals and writes numerals. As we shall later see, Gamma did not understand ten as a unit of numeration.

Gamma had structured her number-names, as may be inferred from the above procedure that we have imputed to her for subtracting. (A structure for her individual number-names should not be confused with a structure for her linguistic system for producing number-names. Gamma had the former, but not the latter.) This inference is directly substantiated by the episodes below.

I: How many tens are there in thirteen?
G: One.

- - - - - - - - - - - - - - -

I: How many tens are there in sixty-seven?
G: Six.

- - - - - - - - - - - - - - -

I: (Places pile of sticks on table; covers pile with hand.) Let's imagine that this pile has seventy-two sticks in it. If you took all the tens out of seventy-two, how many sticks would be left?
G: Two.

Excerpt 4.5.15

Since there were no numerals involved in the above tasks, Gamma evidently used linguistic transformations of her number-names themselves to arrive at her answers. Given her facility with numerals, however, it would not be surprising if she supported her linguistic computations upon figural representations of numerals.

If Gamma had based her answers in Excerpt 4.5.15 upon conceptual understandings of numeration, then we would expect to see similarly structured behavior in situations where she could apply her knowledge. As it turns out, she does not.

In Excerpts 4.5.6 and 7 (pages 212, 214) we saw that Gamma only incremented or decremented by ten in the initial stages of the tasks. Thereafter, she incremented or decremented only by one--even when counting in correspondence to MAB longs. The following episode shows that Gamma's understanding of ten at a conceptual level is of a cycle of incrementing ten times by one.

I: What is two tens and ninety more?
G: (Long pause; counts rapidly to herself on her fingers; pauses slightly after each time all fingers are extended.) Eighty-nine.
I: Eighty-nine? How did you get that? What did you do?
G: I counted.
I: Counted how? Did you start at two tens or did you start at ninety?
G: Two tens.
I: And what did you do?
G: Counted on.
I: Counted on? By what? On your fingers by what?
G: Tens.
I: Did you count on your fingers?
G: Yeah.
I: Did you count by ones?
G: No.
I: How did you do it?
G: Tens.
I: Show me how you did it. Two tens . . . .
G: Two tens.
I: Then what did you do?
G: Then it would make thirty, forty, . . ., ninety (moving a finger with each number-name) ninety . . . ninety . . . .
I: Ninety. How many tens did . . . .
G: One hummm (long pause).
I: Let's start again. Two tens. Okay.
G: Okay.
I: How many more do we want to count?
G: Ninety.
I: So we want to count nine tens.

G: Thirty, forty, . . ., ninety (while the interviewer sequentially puts up seven fingers; G does so also).
I: How many have we counted so far?
G: Seven.
I: Ninety. What's next?
G: A hundred . . . one hundred one . . . two hundred (as the interviewer puts up two more fingers).
I: What have we got? Two hundred? Is that nine tens?
G: Yeah.
I: So two tens and ninety more is how many?
G: Eighty-seven.

Excerpt 4.5.16

Apparently, Gamma attempted to extend two tens by 90 in cycles of ten (2 - 3), but could not keep track of the number of cycles that she had constructed. That Gamma was counting in cycles of ten is substantiated again by her later insistence that she was counting by ten (6 - 12). When Gamma later counted by ten to show how she had originally done it (19-25), we see that Gamma has difficulty making the transition from 90 to 100. This suggests that an increment by ten held the significance of a cycle of ten for Gamma only within her established sequencing routine ("ten, . . ., ninety"). In (26-41) we see that the interviewer's comment that she had to count nine tens had, for Gamma, no relationship to her original problem. After extending nine times by ten (arriving at "two hundred"), Gamma gave her recollection of her original answer to "What is two tens and ninety more?" That is, it appears that when Gamma originally set out to extend two tens by ninety in cycles of ten, she did not know in advance how many cycles she would create.

Another episode suggests even more strongly Gamma's lack of conceptual relationship between ten and one as units of numeration. We join the episode after Gamma had counted (with some difficulty) 20 longs by ten to 200.

I: (Screens (20) MAB longs from view; places 4 longs next to cover.) How many are under the cover?
G: Two hundred.
I: How many tens are there altogether now?
G: (Subvocally utters "210-220-230-240" while pointing to each long on the cover.) Two hundred and forty.
I: Two hundred and forty tens?
G: Yeah.
I: How many tens are there (points to 4 longs on top of the cover)?
G: Four.
I: How many tens are under the cover?
G: Two hundred and twenty . . . two hundred.
I: Two hundred tens?
G: Yeah.
I: Two hundred tens under there, and how many on top?
G: Four.
I: How many tens altogether?
G: Forty (Pause.) Two hundred and forty. Wait. Two hundred . . . two hundred and one . . . and ten (pointing to 1st of the longs on the cover) two hundred and twenty, two hundred and thirty, two hundred and forty (pointing to each of the remaining 3 longs on the cover).
I: Two hundred forty?
G: Two hundred and forty, yeah.
I: How many of these little blocks are there altogether (holds up a unit cube)? If we could saw all these up into little blocks this size?
G: Two hundred and twenty . . . two hundred and forty.
I: Two hundred forty like this (holds up a unit cube)? How many tens are there underneath and on top?
G: (Long pause.) I think . . . about nine.

Excerpt 4.5.17

We have to ask ourselves how Gamma could have said that there were both 240 and 240 tens on the table, and have felt no conflict between her answers. If we recall that Gamma could construct numbers, and if we assume that she reflectively abstracted a number from either her actions of counting or perceptions of the blocks, then it seems quite reasonable to conclude that Gamma answered that there were 240 tens with the meaning that the 240 was composed figurally of tens. The number was 240, its figural composition was of things called "ten." "Two hundred forty" referred to a number; "ten" referred to a figural representation of a block or an increment by ten.

Concept of one hundred. Gamma had essentially no concept of one hundred other than "hundred" is the name of a number and an MAB flat. She could not sequence by one hundred (except possibly "one hundred, two hundred, . . ."), nor did she understand one hundred as a unit of numeration. The relationship that she had established between ten and one hundred was that there are "ten tens in one hundred, but it was only at an empirical level that she knew this (she knew there were ten longs in a flat). The following episode shows the tenuousnous of her relationship between one hundred and ten.

I: Count for me (begins placing MAB longs on the table).
G: Ten, twenty, . . ., one hundred, two hundred (as the interviewer places 11 longs on the table).
I: (Takes last long away from others; points to previously lain long.) A hundred. A hundred and (placing a long on the table) . . . .
G: (Softly) Ten.
I: A hundred and--(still keeping hand on 11th long) . . . .
G: Twenty.
I: A hundred and ten more?
G: Hundred and ten. Hundred and twenty, hundred and thirty, . . . , hundred and sixty (as the interviewer places 5 more longs onto the table).
I: How many little blocks are there here? Same as this (points to unit cube)?
G: Two hundred and forty-five . . . wait.
I: You just counted them. What did you get?
G: (Points to each of the 16 longs.) Hundred and sixty.
I: How many hundreds are there? How many piles of exactly one hundred can we make of these? If we could saw them up in little blocks just like this (holds up unit cube)?
G: (Points to each of the 16 longs.) Sixteen.
I: Sixteen piles with a hundred in each pile?
G: No. (Slides 10 longs one at a time towards herself; points to each of the remaining six longs; pause.) I think about nine.

Excerpt 4.5.18

The portion of interest in the above excerpt is contained in lines (18-25). There it seems that Gamma attempted to accommodate her understanding of the situation (having counted 160) so that she could measure 160 by a unit other than one. The closest that she could come was by reconstructing the lot she had named "one hundred sixty" so that it was composed of units created from the longs. That is, Gamma could not conceptually reconstruct 160 as a composition of abstract unit items, which themselves were compositions of units, so she did what she could--a figural reconstruction.

The following episode substantiates the claim at the beginning of this section that one hundred was not a unit of numeration for Gamma.

I: In this little bag I have lots of little blocks (holds up bag with 235 unit cubes). How many do you think there are?
G: (Pause; picks up bag.)
I: A hundred?
G: Yeah. A hundred.
I: You know how many are in there? There are two hundred thirty-five little tiny blocks. Let's say I want to take them out of there and make piles with a hundred in each pile—exactly a hundred in each pile. How many piles could I make?
G: (Long pause.) Ten hundred.
I: How many piles?
G: Eight.
I: How did you get that?
G: Ummm . . . I don't know.
I: How many were in here again (points to bag)? Do you remember what I told you?
G: Huh-uh (no).
I: Two hundred and thirty-five. And we want to make piles with exactly one hundred in each pile.
G: (Long pause.)
I: Eight piles?
G: Uh-huh (yes).

Excerpt 4.5.19

Whatever the basis of Gamma's answers (she could have thought of the digits-names and added), it seems clear that she did not understand "two hundred thirty-five" as referring to a number that is constructed by counting two units of one hundred (then three units of ten, and then five units of one), where each unit had the significance of being composed of other units.

<u>Concept of place value</u>. The discussions following Excerpts 4.5.18 and 19 could also be given in support of the claim that Gamma did not have a conceptual understanding of place value. Hence they will not be repeated here. One episode that we have not seen so far is particularly interesting, in that it suggests that Gamma could attribute both conceptual and figural meaning to "one hundred" that, from an observer's point of view, was contradictory but with which she was quite comfortable.

I: (Places MAB flat on the table; holds MAB long in his hand.) How many of these pieces (indicates long) could you make from one of these (indicates flat)?
G: A hundred.
I: If we got a saw and we could saw up this piece of wood (indicates flat) into pieces this size (indicates long)?
G: Ten.
I: How do you know that?
G: Cause ten, twenty, . . ., one hundred makes a hundred.

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I: (Holds unit cube in hand.) How many of these little blocks (indicates unit cube) make up this piece of wood (indicates flat)?
G: (Pause; counts the unit cubes around the outer edge of the flat; begins to count the cubes which lie just inside the outer layer.) Sixty-four.
I: How did you work that out? By counting all those?
G: Yeah.
I: How many of these (holds up unit cube) could you make out of these pieces (holds up long)?
G: Ten.
I: And we can make ten of these (holds up long) out of one of these pieces (points to flat). Is that right?
G: Uh-huh (yes).
I: So we have ten of these (points to unit cube) makes one of these (points to long) and ten of these (points to long) make one of these (points to flat)?
G: Yeah.
I: And how many of these (holds up unit cube) make one of these (points to flat)?
G: Sixty-four.

Excerpt 4.5.20

The episode speaks for itself. "A hundred" can refer to a flat, but the flat can also be named "sixty-four" when its units are counted. Moreover, Gamma's behavior suggests quite strongly that she was not capable of constructing a necessary (i.e., reciprocal) relationship between one hundred as a number and one hundred as composed of ten of (ten of one).

Though Gamma did not have a conceptual understanding of place value, she did have one at the level of action. In the counting board tasks, she would construct a name for successive collections of MAB blocks by first counting the flats, then the longs, and then the unit cubes. This action schema probably provided the basis for her abstraction of the structure of individual number-names and numerals.

Case Study 4.6: Sigma

Sigma was a first-grader (age 7 years at the beginning of the 1977 school year). In November of 1977 Sigma solved Problem 1 of Figure 4.1 (page 105) by counting the visible squares and then pointing at the four corners of the cover (getting "seven"). He correctly solved Problems 2 (and 3) by putting seven (five) "in his head" and counting to ten (nine). Sigma could sequence "ten, twenty, . . ., one hundred," but could not sequence by ten from "two," instead sequencing by two. He said that there were 100 tens in 32. He counted five bundles of ten by ten to make 54 (and thought he had counted three tens), but did not put out any single sticks; and named two bundles of ten and five single sticks "thirty" without counting. When "using ten" to find how many sticks (33), Sigma counted individual sticks while sequencing "ten, twenty, . . ., fifty," whereupon the interviewer stopped him. To find out how many sticks in ten bundles of ten, Sigma counted the individual sticks by one, acknowledging that there were ten in each bundle. The final interviews were given to Sigma on May 5, 10, and 16 of 1977.

Sigma's case study will show him to have been unique among the children. He had (empirically) abstracted a number of linguistic routines that he applied quite automatically, and which at times caused him difficulty because of their lack of flexibility. Also, Sigma, like Delta and Gamma, frequently reasoned heuristically with the aim of merely giving a "right" answer (i.e., an answer which the interviewer would accept, but which held little more necessity for Sigma than any other).

Writing numerals. Sigma appeared to have little difficulty writing numerals, except on one occasion in the numeral-writing task. There he began to write the numeral for "two hundred nine" by writing "200." He corrected himself, but subsequently went on to write "219" as "2019," "267" as "2067," and "934" as "9034." It should be noted that prior to writing "209," Sigma had correctly, and without hesitation, written five numerals for number-names beginning with "one hundred." Perhaps Sigma's errors subsequent to "209" were a result of his having corrected himself (when writing "200," then "209") with a remark something like "there should be one zero, not two," and subsequently continuing to apply his reminder.

Reading numerals. Reading numerals was more problematic for Sigma than was writing them. Though he would usually end up saying the correct number-name, he would quite frequently make several false starts (e.g., for "73" Sigma said "thir . . . thir . . . seventy-three"). This was especially true when Sigma read numerals beyond "100," and the difficulty seemed to be a lack of a standardized way of partitioning the numeral. He read "110" as "ten hundred," "120" as "twenty hundred," though he later corrected himself. Also, in reading "201," "311," and "594" he would pause after saying "hundred," as if sorting out the way to say the remainder of the name. One could explain Sigma's behavior by supposing that his routine for reading numerals was not standardized in terms of perceptually partitioning the numeral but that he had a fairly well-established grammar for number-names into which the products of his routine had to fit--hence, the false starts and eventually correct reading.

<u>Sequencing</u>. Sequencing by ten posed no difficulty for Sigma. He correctly sequenced by ten from eight to 158, 340 to 600, and 97 to seven. Sigma, however, had essentially no routine for sequencing by one hundred, other than possibly "one hundred, two hundred, . . .," as is shown in the following excerpt.

I: Start with thirty and count-on by hundreds.
S: (Pause.)
I: Can you count by hundreds? Just count by hundreds. (Pause.) One hundred . . . .
S: One hundred and ten . . . no, one hundred and twenty.
I: No, just say one hundred—and then a hundred more would be . . .?
S: Two hundred, . . ., nine hundred, ten hundred, eleven hundred . . .
I: Okay. Start with thirty and count-on by hundreds.
S: Forty, fifty, sixty . . . .
I: No. That's by tens. Count by hundreds. Start at thirty and a hundred more.
S: Thirty, four hundred . . . .
I: Start at thirty and count-on by hundreds.
S: I said three hundred, four hundred, five hundred, . . ., nine hundred, ten hundred.

Excerpt 4.6.1

Sigma's behavior in (5) will later have significance when we discuss his concept of one hundred. For the rest of the episode, we may classify the majority of his behavior as heuristical. He had no routine for sequencing by one hundred, so he attempted to do the best he could with what he had--sequencing by one and by ten. In (7) Sigma apparently abstracted the criterion of incrementing the "hundred" digit-name as when sequencing by one. In (9), he apparently chose sequencing by ten from "thirty" as the more appropriate of sequencing by one and by ten--perhaps because "ten" is closer to "one hundred." In (12), after having perceived the interviewer as "eliminating" sequencing by ten as a relevant routine, Sigma applied his newly constructed routine--forcing "thirty" into the required form of an input condition ("three") for its implementation (Sigma even thought he <u>said</u> "three hundred").

Sigma's behavior on the seriating tasks suggests that if Sigma's linguistic system for constructing number-names and sequences of number-names was not operational, then it was at least reasonably close. Sigma had no difficulty putting numerals into ascending order, and committed only two errors of placement--later correcting them--when putting numerals in descending order. However, a mistake that he made on an ascending task and another on a descending task suggest that he had made the problems simpler than one might imagine.

Cards: 20 30 60 70 90 100 110 120 (shuffled)

I: Here are some more cards. Can you put these cards in order on the board?
S: (Spreads cards on table.) (20) Twenty. (20-30) Thirty. (20-30-60) Sixty. (20-30-60-70) Seventy. (20-30-60-70-90) Ninety. (20-30-60-70-90-100) One hundred. (20-30-60-70-90-100-110) Ten hundred. (20-30-60-70-80-90-100-110-120) Twenty hundred.
I: What's this number ("120")?
S: Twenty . . . one hundred and twenty.
I: What's this number ("110")?
S: One hundred and ten.

Excerpt 4.6.2

Sigma's naming of "110" as "ten hundred" and "120" as "twenty hundred" (6-7) tells us something. He was translating each of "twenty, thirty, . . ." into "two, three, . . ." and putting them in sequence accordingly. Thus when he came to "110" and "120," he assimilated them as "ten" and "twenty"--to continue <u>his</u> sequence. That is, he was comparing, for the most part, digit-names and ordering the numerals accordingly. With this in mind, we must take his success on each of the ascending tasks with a grain of salt.

The descending tasks give us a better idea of Sigma's level of operationality. On one he had constructed the sequence "76 67 61 52 48 47" and, when coming upon "49," inserted it appropriately. However, the nature of "49 48 47" may have been such, for Sigma, that he knew nothing else <u>could</u> go to the right of "49" except "48" and "47," and hence that there was no need to remove them. The next excerpt suggests this to have been the case.

Cards: 97 103 107 113 117 124 134 143 (shuffled)

I: I want you to do this just like the last one. Put the cards across the board from the biggest to the smallest.

S: (Spreads cards on table.) (143) One hundred and forty-three. (143-134) One hundred and thirty-four. (Pauses; 143-134-124) One hundred and twenty-four. (Pauses; 143-134-124-107) One hundred and seven. (143-134-124-107-103) One hundred and three. (143-134-124) (143-134-124-117) One hundred and seventeen. (143-134-124-117-113) One hundred and thirteen. (143-134-124-117-113-107) One hundred and seven. (143-134-124-117-113-107-103) One hundred and three. (143-134-124-117-113-107-103-97) Ninety-seven.

Excerpt 4.6.3

We may infer from Sigma's removal of "107" and "103" from the board after having found "117," and from his reconstructing the sequence anew (7), that his criterion "place largest of those remaining" was really "place largest of those remaining (<u>that I have looked at</u>)." The nonadjacency (in sequencing) of "117," "107," and "103" seems to have meant to Sigma that there was the possibility of other numerals falling within the sequence, and hence that "107" and "103" needed to be removed to prepare for that possibility. That is, "before" in Sigma's linguistic system was not operationally transitive, and "before" and "after" were not operationally reciprocal to one another. Instead, Sigma apparently relied on a figurative base of sequencing forward or backward as his criterion for searching for the next numeral to be placed (as did Gamma). However, Sigma's success in adapting his linguistic system to the task suggests that it had been very firmly established.

<u>Numerical operations</u>. Sigma was a counter with abstract unit items, for in several episodes he counted his counts. It also appears that Sigma could construct at least arithmetical lots, and possibly numbers, and could integrate and separate them. These will not be elaborated upon, for there is another aspect of Sigma's numerical operations that is worthy of exploring--the connections that he had established between his numerical operations and his linguistic system. The following episodes illustrate this very nicely.

I: I have some number problems on these cards. I want you to think out loud while you do these so that I can hear what you're doing. Are you ready? Here's the first one.
I: (Places card with "10 + 7 = __" onto the table.) Read the problem out loud.
S: Seventeen.
I: Read it out loud. How did you get that?
S: Because seventy plus seven is seventy-seven and ten plus seven is seventeen.

I: (Places card with "10 + __ = 13" onto the table.)
S: Ten . . . (pauses). Three.
I: How did you get that?
S: Cinchy. The same way as the first one.
I: How's that?
S: Thirty plus three is thirty-three.
I: So, ten plus three is . . . thirteen.
S: Thirteen.

I: (Places card with "40 + __ = 46" onto the table.) Read it out loud.
S: Forty plus . . . forty plus six.
I: Six?
S: Yep.
I: Same way, huh?

Excerpt 4.6.4

Given that Sigma did understand the sentences as referring to an integration of numbers, we can conclude that he had abstracted a routine for naming the integration when both addends are named and the names may be concatenated to form a number-name. In (8 - 9), and again in (15) Sigma seems to have been saying, "Look, there's a pattern that these things follow." The episode below suggests, however, that the pattern was fixed--in that if the sentence didn't begin with the decadal numeral, Sigma would not see the relevance of concatenating.

I: (Places card with "___ + 20 = 25" onto the table.)
S: Blank plus twenty equals twenty-five. Ten.
I: Ten? If we wrote ten there (in the blank), would it be right?
S: (Pause.) Five.
I: Five? Why not ten?
S: (Points to the blank.) Five, ten, fifteen, twenty, twenty-five.
I: So which is right? Five or ten?
S: (Pause.) Five.
I: How did you get five?
S: Like five, ten, fifteen, twenty, twenty-five.

Excerpt 4.6.5

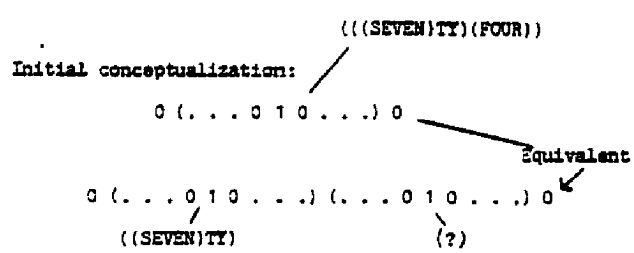
It is not at all clear how Sigma arrived at his initial answer ("ten"). By whatever method, it is clear that he did not think of decomposing "twenty-five," or of searching for a digit-name to concatenate with "twenty" to produce "twenty-five." However, it appears that Sigma did search for a (linguistic!) routine to connect "twenty" with "twenty-five." In (6-10), Sigma seems to have tried to say something like, "Look, when you count by five you get to 'twenty' and then 'twenty-five,' so twenty plus five is twenty-five."

Another episode suggests that Sigma had abstracted linguistic correspondent to separating.

I: (Places card with "74 - ___ = 70" onto the table.)
S: Four.
I: Right away! You know it's four. How?
S: Seventy plus four is seventy. I mean seventy-four take away four is seventy. It's like seventy plus four is seventy-four. Anyway, I knew it from before.

Excerpt 4.6.6

Apparently, Sigma understood the problem as a separation of 74 into 70 and something else--decomposing "seventy-four" into "seventy" and "four," and then searched for a name to concatenate with "seventy" to produce "seventy-four." This would explain why he first explained himself in terms of "plus" (4). One might conclude from his behavior that Sigma had operationally related integrating and separating as inverses of one another--that he solved a subtraction problem by transforming it into an equivalent addition problem ("70 + __ = 74"). However, because this problem can be solved linguistically, we must exercise caution in making a conclusion of operational reversibility. Even though Sigma (assumedly) conceptualized the problem as a separation, further processing might well have amounted to heuristically juggling the number-names till they were in a position which fit the conditions of the problem. For instance, Sigma might have done something like the processing depicted in Figure 4.6.1. There, integrating enters the picture not because it is the inverse of separating, but because concatenating "seventy" and "four" held the significance, for Sigma, of integrating. To <u>say</u> what he had done, he would have had to use "plus."



Processing:	(((SEVEN)TY) (FOUR))	((SEVEN)TY)	74, 70
	((SEVEN)TY) (FOUR)	((SEVEN)TY)	70, 4, 70
	((SEVEN)TY)	((FOUR) ((SEVEN)TY))	70, (4, 70)
	((SEVEN)TY)	(((SEVEN)TY) (FOUR))	70, (70 + 4)[a]

"Seventy plus four is seventy."

[a]This is written (70 + 4), as opposed to (70, 4), because concatenating number-names, where the result is a number-name, had the significance of "plus," or integrating, for Sigma.

Figure 4.6. Sigma's linguistic computation of "70 + __ = 74."

That Sigma had not reversibly related integrating and separating is quite clear from the following episode.

I: (Places card with "91 - 29 = __" onto the table.)
S: That one is too high to do in my head. (Counts out 9 MAB longs and 1 unit cube.)
I: Now you're going to take away twenty-nine?
S: (Subvocally utters "10-20" while sliding 2 longs to his right; picks up the unit; pauses.)
I: What's the problem?
S: (Subvocally utters "10-20-30- . . . -70" while sliding the remaining longs to his left.) Seventy.
I: Now wait a minute. How many did you take away?
S: Twenty-one.
I: But you have to take away twenty-nine.
S: (Counts 8 more unit cubes out of the box; subvocally utters "22-23- . . . -29" and combines them with the 2 MAB longs and 1 unit cube; subvocally utters "10-20- . . . -70" while counting the 7 MAB longs.) There's still seventy.
I: How many do you have altogether? How many are you starting with? You have seventy there (pointing to the 7 longs). How many do you have altogether?
S: Ninety . . . ninety-nine. Altogether ninety-nine.
I: How many did you start with?
S: Ninety-one.

Excerpt 4.6.7

Sigma's first answer of "seventy" may have resulted from his having interpreted "What's the problem?" (7) as "What's the answer?" That is, Sigma may not have finished at that time. However, in (10-16), Sigma went on to satisfy the conditions of his problem (take away 29) by adding eight cubes to his "take away" pile--without seeing any implication for the minuend. If Sigma had operationally related separating and integrating, then he would have realized that integrating the result of separating necessarily produces the original number. Put another way, adding blocks to the subtrahend changes the problem, since you have also increased the minuend. Sigma seemed not to be bothered by the lack of correspondence between the total number of blocks and the original problem (20-22).

Extending and declending were well-established routines for Sigma. His method for carrying them out was also unique among the children. In extending or declending by a number he would create the number while counting by tens and ones.

I: (Places card with "70 → 92" onto the table.) How far is it from seventy to ninety-two?
S: (Pause.) Twenty-two.
I: How did you get that?
S: Don't ask so many questions.
I: I won't know anything if I don't ask questions.
S: (Long pause.) Seventy plus ten is eighty, seventy plus twenty is ninety, and one . . . two more is ninety-two.

Excerpt 4.6.8

In (7 - 8) we see Sigma carrying on two counts—one in which his aim was to count from 70 to 92, the other in which his aim was to count how many he had counted in counting from 70 to 92. That is, Sigma created the extension in cycles of ten in order to keep track of his progress toward 92. He did much the same when declending.

I: (Places card with "84 - 30 = __" onto the table.)
S: (Pauses; subvocally utters "84 take away 30 . . . 80 take away 10 is 74 . . . 84 take away 20 is 64 . . . 84 take away 30 is fifty-four.
I: How did you get it?
S: Same way as the other one. Except higher.

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I: (Places card with "47 - 21 = __" onto the table.)
S: (Pauses.) Twenty-six.
I: Do it again, but out loud. Okay?
S: Forty-seven take away ten is thirty-seven, take away twenty is twenty-seven, take away one more is . . . you know.

Excerpt 4.6.9

In both problems, Sigma apparently formed a separation, and as a means of naming the unknown number, called upon declending. In declending, he would construct the subtrahend (by counting forward in increments of ten and one) while maintaining a backward count of the minuend.

Sigma's behavior in Excerpt 4.6.9 at least suggests that he may have operationally related extending and declending as inverse operations. His coordination of two counts in opposite directions of each other would seem to imply that he was compensating one by the other. However, his linguistic behavior is also very much like what he might do while removing MAB longs and units from a collection. This is not to say that he might have been thinking specifically in terms of longs and unit cubes, but that he may have abstracted his method from those actions. In that case, he would be coordinating counts, but not compensating one by the other.

At this point we cannot choose between the two interpretations above. Each is a viable explanation of his behaviors in Excerpt 4.6.9, so we must look elsewhere for additional information that might make one more viable than the other. The following episode is at least somewhat helpful in that regard. It is the continuation of the episode in Excerpt 4.6.7.

I: How many do you have altogether? How many are you starting with? You have seventy there (pointing to the 7 longs). How many do you have altogether?
S: Ninety . . . ninety-nine. Altogether ninety-nine.
I: How many did you start with?
S: Ninety-one.
I: We're not starting with ninety-nine. So we have to put these back (removes the 8 unit cubes). Let me give you a hint. We'll change it a little bit. We have ninety-one. Let's take away ten (slides 1 long to S's left).
S: Twenty.
I: Take away twenty (slides 1 long to S's left). Let's take away thirty (slides 1 more long to S's left). Did we take away too many?
S: Yes.
I: How many too many?
S: Three too many.
I: We want to take away twenty-nine, right? And I took out . . .
S: Two too many . . . one too many!
I: We took out one too many, right? How many are left?
S: Seventy-one.
I: But we took out one too many . . . are there seventy-one? Count them.
S: (Counts MAB longs.) Sixty.
I: Sixty . . . .
S: One.
I: So we took out one too many and that left sixty-one. How many should there be if we hadn't taken out one too many?
S: Should be two taken out.
I: No. We took out thirty. That's one too many. That leaves how many?
S: Sixty-one.
I: But we took out one too many. How many should there be?
S: (Pauses.) Seventy.
I: Let's try it. You say I took out one too many. So I'll give you one (hands S 1 unit cube). So now is there the right number there?
S: (Pause.) It has to be seventy.

Excerpt 4.6.10

The reason that the above episode is only "somewhat" helpful is that we may argue that Sigma never understood the problem as involving a compensation of + 1 and - 1. Rather, he may have understood the problem in terms of tens. In (17), "three" apparently referred to three tens having been removed ("too many" being superfluous) as is suggested by Sigma's remark that it "should be two (tens--and nine ones) taken out" given that the interviewer removed one ten too many. Thus Sigma appears to have been rigid in his method of constructing the declension, and hence unable to operationally coordinate extending and declending. (This argument is not <u>sufficient</u> to establish the preoperationality of extending and declending, it is only suggestive of it.)

<u>Concept of ten</u>. In Excerpts 4.6.8 and 4.6.9 of the previous sections we saw Sigma extending and declending in increments of ten and one. The question addressed here is: what significance did Sigma give to an increment by ten when counting? In the discussions following those excerpts, it was allowed that each increment had at least the significance of a cycle of ten. What remains to be established is whether or not Sigma had constructed ten as a numeration unit. We will begin this investigation with the following episodes.

I: I'm going to give you some more cards, and you tell me what goes in the blank (places card with "50 - 20 = __" onto the table).
S: Sixty take away twenty. (Pause.) Forty.
I: How did you get forty?
S: It isn't right?
I: I'm not saying it's wrong--I'm just wondering how you got forty.
S: (Pause.) Sixty take away one is fifty; take away two more is forty.
I: Two more tens?
S: Yeah.
I: So sixty take away two tens is forty?
S: Is forty.

I: (Places card with "70 - 31 = __" onto the table.)
S: (Long pause.)
I: Do you want to use these things (pointing to MAB blocks)?
Can you do it in your head?
S: Yeah. (Pauses; speaks softly to himself; "Seventy take away
one is sixty, take away two is fifty, take away three is forty.")
Thirty-nine.
I: You got that fast. How'd you do that?
S: Seventy take away ten is sixty. Seventy take away thirty is
forty. Seventy take away one more is . . . thirty-nine.

Excerpt 4.6.11

It is clear that Sigma enumerated his decrements by ten. What is not clear is the criterion for stopping that he operated by. If he elaborated the number-names of the declension into a number of tens and a number of ones and used them as criteria, then he used ten as a numeration unit. However, he could have held the number-name of the declension as a criterion and counted, say, "Seventy take away one (ten = 10) is sixty, take away (one more ten is) two (tens = ten more = 20) is fifty, take away (one more ten is) three (tens = ten more = 30) is forty," and so on. If this was his method, then he merely used an increment by one to count cycles of ten while constructing declensions, and did not use ten as a numeration unit.

Several episodes suggest that Sigma had not, in fact, constructed ten as a numeration unit. The first of these occur in tasks aimed at investigating whether he had developed a base-ten structure for individual number-names.

I: How many tens are there in sixty-seven?
S: Seven. (Pauses.) Is that right?
I: How many tens are there in sixty-seven?
S: Six.
I: How do you know that?
S: Is it seven?
I: Is it seven or is it six? I'll say the name again—sixty-seven.
S: (Pauses.) Seven.

- - - - - - - - - - - - - - -

I: (Places pile of sticks on table; covers pile with hand.)
Let's imagine that this pile has seventy-two sticks in it.
If you took all the tens out of seventy-two, how many sticks
would be left?
S: All the tens out of seventy-two?
I: How many sticks would be left? If we took out bundles of ten
sticks?
S: Seven . . . two . . . two.

I: Which is it?
S: Two.
I: Why did you say that?
S: Seven.
I: You're guessing. (Pause.) What do you want me to write down? Seven or two?
S: Two.

Excerpt 4.6.12

Apparently, Sigma knew that he could get the number of tens in a number by focusing upon a part of the number-name, but he didn't know what part. Thus, the discussion following Excerpt 4.6.11 can be concluded by supposing that Sigma did not elaborate a number of tens and ones as a criterion for terminating an extension or declension, but instead constructed declensions and extensions while holding a number-name as a criterion for stopping.

Several other episodes suggest that Sigma had not operationally related ten and one by "ten of." Rather, he constructed numbers or arithmetical lots whose units had a value that was made implicit by the context (e.g., sequencing by ten, counting MAB longs, etc.).

The following excerpt is from a task in which Sigma counted MAB longs by ten to 200 and was asked the number of tens he had counted. Sigma did not know, and eventually counted the longs by one. We join the episode at that point.

I: (Screens (20) MAB longs from view; places 4 longs next to screen.) How many tens are there altogether now?
S: Twenty (pointing to screen). How many tens?
I: Right. All together.
S: Two hundred . . . (subvocally utters "210-220-230-240" while sliding the 4 MAB longs on the screen). Forty.
I: Forty? How many tens under the cover?
S: (Pause.) Twenty under the cover.
I: How many tens on top of the cover?
S: (Pause.) Forty. There's forty tens on top of the cover (picks up the 4 longs).
I: How many tens up here? Count them.
S: Ten, twenty (while placing 2 longs on the cover) . . . .
I: No. How many tens is this (pointing to 1 of the longs)?
S: (Picks up the long.) How many tens? Forty.
I: Is it forty ones--forty of these kind (holds up a unit cube) or forty of these kind (holds up long)?
S: Forty of these kind (picks up 1 long).

Excerpt 4.6.13

It appears that Sigma's confusion stemmed from two sources. The first is that referring to the number of covered tens with "twenty" took him outside of his concept of ten; he had a number of abstract units--ones, if you will. He then took "altogether" as referring to all (as ones), and to find out how many, he continued sequencing by ten, where each figural unit item made from a long (named "ten") served as an impetus for a linguistic increment by ten. Sigma's insistence that there were forty tens on top of the cover, even when indicating the "unit" (a long) of which there were forty, leads to the inference that Sigma was implicitly saying "there are forty [abstract units made up] of these [figural items called 'ten']." Hence the second source of difficulty: Sigma did not coordinate the units of ten and one as abstracted entities. He did not "label" his units. When in the context of his concept of ten he had no need to, for the label was implicit. When outside of the concept, however, units arising from different conceptual contexts needed to be distinguished, and at the level of abstract units, this required some sort of label.[1]

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[1]While the author was pondering Sigma's behavior on this task, it occurred to him that it would follow from this assumption that Sigma was operating by some sort of "homogeneity" criterion. This is that otherwise unlabelled units that are grouped together must be considered as being of the same type, and, contrapositively, that units that are not considered of the same type cannot be put together. To test this notion, he asked his unsuspecting (and oft bewildered) wife "What is twenty apples plus four dogs?" Her response was, "That doesn't make any sense!"

Sigma went on for some time in the above task insisting that there were "twenty down there (under the cover) and forty up there (on top)." The following excerpt shows what happened once Sigma ceased to take a long as an impetus to count by ten.

I: How many tens is that (pointing to 1 MAB long on the cover)? Count.
S: One ten (pointing to first long), twenty tens (pointing to second long) . . . .
I: No. That's not twenty tens. One ten (pointing to first long). How many tens is that (pointing to second long)?
S: One . . . there's twenty. (Pause.) You're getting me mixed up.
*C: How many tens in twenty?
S: Two.
I: How many tens in forty?
S: Four.
I: How many tens do you have up here then?
S: Four.
I: How many tens down here? You know that.
S: You know it too.
I: It's twenty, right?
S: Yeah.
I: How many altogether?
S: Twenty-four.
I: You did get mixed up, didn't you?
S: Yeah!

*"C" stands for the camera operator.

Excerpt 4.6.14

In responding to the camera operator's questions ("How many tens in twenty? forty?"), Sigma reentered the context of his concept of ten, and the unit, implicitly, became ten.

Sigma's difficulty did not end, however. Even after having resolved the "conflict" (from our point of view, not his) of mismatched unit-types, Sigma still failed to coordinate his numeration units, as seen in the continuation of the above excerpt.

I: (Places one more MAB long on the cover.)
S: Five.
I: Five tens, right?
S: Um-hmmm (yes).
I: So, there's twenty-five tens. How many of these kind are there (holds up 1 unit cube)?
S: (Pause.) Five . . . twenty-four.
I: What?
S: Five.
I: Five what?
S: Five tens.
I: And how many under the cover?
S: Twenty te . . . two hundred.
I: How many altogether?
S: Two hundred and four. (Pause.) Two hundred and twenty-four.
I: How did you get that? There's two hundred here. How many up here?
S: Oh . . . two hundred and five.

Excerpt 4.6.15

At this point Sigma's reasoning was becoming almost entirely heuristical. Sigma's answer in (7) can be explained by supposing that "twenty-four" as he had constructed it at the end of Excerpt 4.6.14 referred to a number of ones (abstract unit items constructed from his perception of the longs) as did "five"--referring to the number of visible longs. Thus, the unit cube held by the interviewer (6) could have been taken by Sigma as similar to either the units in 24 or 5. Finally, in (13-18) S i gma appears to be combining names in any reasonable way in search of an answer that might satisfy the interviewer--"two hundred," "four," "twenty," and "five" all at one time or another referred to the bunch of blocks on the table, so perhaps somewhere in their combination lay a "right" answer.

Out of all of this one thing stands out: Sigma had not constructed ten as a unit by which other numbers could be measured, nor had he operationally related ten and one as units.

Concept of one hundred. Sigma's concept of one hundred was far from well formed. His major achievements toward forming it were: (1) one hundred as a number (of the same status as, say, forty-seven); (2) "one hundred" as referring to particular figurations (mainly an MAB flat); and (3) sequencing by hundred (but only 100, 200, . . .).

Another peculiarity of Sigma's concept of one hundred was that he considered "one hundred" as signifying two increments by ten. So, in a sense, Sigma had also given a significant (an extension by two tens) to an increment when sequencing by hundred. Making an increment by one hundred, however, was a problem--one that he solved by extending twice by ten. The following excerpt illustrates this.

I: Start with thirty and count-on by hundreds.
S: (Pause.)
I: Can you count by hundreds? Just count by hundreds. (Pause.) One hundred . . . .
S: One hundred and ten . . . no, one hundred and twenty.

Excerpt 4.6.16

One feature of Sigma's behavior, that the reader has most likely already noticed, is that he would try plausible alternatives to "incorrect" answers. That is, Sigma made extensive use of a sort of "means-end analysis" heuristic, in the sense that if one set of operations didn't work (to satisfy the interviewer), then try a related set. This aspect of Sigma's "problem" solving showed up again and again in the context of questions aimed at tapping his concept of one hundred. The following excerpt illustrates how Sigma's use of his version of means-end analysis in combination with his limited concept of one hundred produced some quite interesting behavior.

I: Hands S a bag of MAB unit cubes.) How many do you think are in there? Can you guess?
S: (Long pause.) Thousand.
I: A thousand?

.
(Portion of dialogue omitted.)
.

S: One hundred.
I: One hundred? That's pretty good. You know what? I counted those (cubes) and there's two hundred thirty-five in there.
S: Include one more (picks up 1 unit cube) and that'd be two hundred thirty-six.
I: That would be two hundred thirty-six. There's two hundred thirty-five in there. If I let you take those out and make piles of hundreds, what would be the most number of piles you could make? Remember how many are in there?
S: Two hundred thirty-five.
I: You made piles of one hundred. The most number of piles of one hundred, how many could you make?
S: I don't know.
I: How many hundreds do you think would be in two hundred and thirty-five?
S: (Pause.) Thirty-five.
I: Thirty-five hundreds?
S: Thirty-five.
I: Thirty-five?
S: There's thirty-five tens.

Excerpt 4.6.17

When Sigma was thinking in terms of the number of actual piles he would make, he found no connection between that and a set of operations that he might apply (17). When the interviewer modified the task ("How many hundreds in two hundred thirty-five?"), Sigma knew that he could answer questions of that sort by taking part of the number-name--thus "thirty-five" (20). When that answer didn't seem to satisfy the interviewer, Sigma seemed to have thought something like "Well, if they're not hundreds, maybe they're tens" (21-24).

Sigma did know, in a sense, that there are ten tens in one hundred. This item of knowledge, however, was largely isolated from others pertaining to one hundred. When asked how many longs are in a flat, Sigma said "ten." But when asked to extend twenty by one hundred ("What number is one hundred more than twenty?"), he extended by two tens rather than ten. Sigma's meaning for one hundred was not entirely linguistic--he imposed his operational meaning for hundred upon objects as well. In the following excerpt the interviewer had placed a board in front of Sigma that had a number of MAB flats and longs in a row and under a cover. Sigma was asked to count the blocks as they were uncovered.

Board: 100 10 10 100 10 10 10 100 100 (covered)

I: (Uncovers MAB flat.)
S: One hundred.
I: (Uncovers MAB long.) One hundred and . . . .
S: (Pause.) Ten.
I: (Uncovers MAB long.)
S: Twenty. One hundred and twenty.
I: (Uncovers MAB flat.)
S: One hundred and forty.
I: How much?
S: One hundred and forty.
I: One hundred and forty? Where do you see the forty?
S: (Pause.) One hundred (pointing to the first MAB flat). There's one hundred and ten (pointing to the first MAB long). One hundred and twenty (pointing to the second

long). And plus one hundred is forty. Cause one hundred counts on two more instead of one more.
I: Oh. You're adding two more onto one hundred twenty to get one hundred forty. I see . . . .

Excerpt 4.6.18

Sigma went on to count each of the remaining longs by an increment of ten and each of the remaining flats by two increments of ten.

The last excerpt that will be discussed shows how Sigma's shortcoming in both his concepts of ten and one hundred sometimes came together to produce somewhat "bizarre" behavior.

Board: 10 10 4 100 2 10 (covered)

I: (Uncovers MAB long.)
S: Ten.
I: (Uncovers MAB long.)
S: Twenty.
I: (Uncovers 4 unit cubes.)
S: (Long pause; points to each unit cube.) Twenty-five.
I: Twenty how many?
S: Twenty-five.
I: Twenty-five? Look at that again.
S: One, two, three, four (pointing to each unit cube). One hundred and twenty-four.
I: How much?
S: One hundred and twenty-four.
I: Look down and tell me how many again.
S: Two, four (pointing to pairs of unit cubes). Ten, twenty (pointing to each MAB long).
I: Ten, twenty . . . .
S: (Long pause.) Oh, one hundred and . . . .
I: Where do you see one hundred?
S: (Pause.) I mean twenty-six.
I: Where do you see twenty-six? Show me twenty-six.
S: Twenty! Plus four. Four plus two is six.
I: Two what?
S: You're getting me mixed up.

Excerpt 4.6.19

A tenuous hypothesis is that Sigma first miscounted the four unit cubes, getting "twenty-five," and then inferred that he was supposed to say something "hundred" (6-11). A stronger inference can be made about the source of "twenty-six" (20). Sigma counted the two longs as "ten, twenty" making abstract (and unlabelled) units of his increments by ten. After he had extended twenty by a subitized four, getting twenty-four, he still had two uncounted unit items of the same type as the four--the increments associated with the longs. So the longs were "counted" twice--first as tens, and then as ones (22).

The task proceeded from this point as follows.

I: Let's start over (covers the board; uncovers MAB long).
S: Ten.
I: (Uncovers MAB long.)
S: Twenty.
I: (Uncovers 4 unit cubes.)
S: (Very long pause.) Twenty-four!
I: (Uncovers MAB flat.)
S: (Long pause.) Forty-four.
I: Forty-four? Where do you see forty-four? (Long pause.) This is one hundred and (pointing to the flat), right?
S: Yeah. One hundred and . . . forty-four.

Excerpt 4.6.20

In exclaiming "twenty-four!" Sigma seems to have had the feeling that he finally "straightened things out." However, when subsequently encountering a flat, he found himself faced again with a problem of satisfying some unknown (from his perspective) wish of the interviewer. Sigma knew there were forty-four with the addition of the flat, and if the interviewer wanted him to say "one hundred," then he would say it--but "forty-four" had to be in there somewhere (7-11).

<u>Concept of place value</u>. Sigma clearly had little concept of place value beyond being able to read numerals correctly. What he had instead were empirical routines, mainly of a linguistic nature, that he had abstracted from counting. Sigma constructed numbers as abstract units, but a unit was a unit was a unit unless there was some way to figurally disassociate them (e.g., longs and unit cubes). Even when Sigma's units arose from figurally dissimilar objects, he would not always coordinate them as having different values. In short, Sigma's linguistic abstractions, albeit essential in his overall development of numeration concepts, made him appear to know more than he actually did.

Case Study 4.7: Alpha

Alpha was a first-grader (age 6 years at the beginning of the 1977 school year). In November of 1977 Alpha solved Problem 1 of Figure 4.1 (page 105) by pointing to the cover, saying "five," and then counting "six, seven, eight" while pointing to the visible squares. He solved Problem 2 by counting "eight, nine, ten" to himself, and then explained his answer by saying "cause seven plus three is ten." Similarly, Alpha counted "eight, nine, ten, eleven, twelve--it's five" when the interviewer asked him to suppose that there were 12 in all. Alpha solved Problem 3 by putting up five and four fingers and then counting them all against his lip. Alpha sequenced by ten only after the interviewer said "ten, twenty, thirty," and even then continued "forty, fifty, sixty, seventy, eighty, twenty." When asked to start at "two," Alpha sequenced "twenty, thirty, . . ., ninety, twenty." To find the total number of a bundle of ten and four single sticks, Alpha counted from ten, and similarly counted from 20 for two bundles and five. Alpha said that there were two tens in 32, and made 54 by putting out five bundles of ten, but did not put out any single sticks. He grouped 29 sticks (not knowing beforehand that there were 29) into two bundles of ten and nine single sticks, but had to recount them to find how many there were. Finally, Alpha counted ten bundles of ten as "ten, twenty, . . ., ninety, twenty," saying there were 20 sticks in all. The final interviews were given to Alpha on May 8, 11, and 15 of 1977.

Alpha's case study provides an interesting contrast to those already presented. He committed many of the "low-level" errors common to Delta, Lambda, Kappa, and Gamma (e.g. misreading and miswriting numerals), and yet he had very well-developed concepts of numeration.

Writing numerals. The only type of error Alpha made in writing numerals was to include an extra zero following the hundreds digit. He was quite consistent in his commitment and noncommitment of this error: he correctly wrote numerals such as that for "two hundred nine"; he incorrectly wrote numerals such as that for "two hundred nineteen." Apparently, Alpha felt compelled to write "0" as he said "hundred" to himself in elaborating the number-name.

Alpha's method of writing "teen" numerals suggests also that he based his writing of a numeral on an elaboration of its number-name. For, say, "eighteen," he first wrote "8," paused with his pencil held to the right of "8," then placed his pencil to the left of "8" and wrote "1." For "two hundred nineteen" he wrote "209" and then inserted "1" between the "0" and "9"--producing "2019."

Reading numerals. Alpha correctly read each of the numerals presented to him in the task designed to directly assess his ability. However, misreading numerals was by far his most frequent error throughout the interviews. It is not clear why Alpha could correctly read numerals when he understood that to be the aim of the task but not necessarily otherwise. Perhaps his routine for reading numerals was not so well established that he could successfully implement it without devoting his full attention--elaborating each step. In the context of a problem, his attention was necessarily divided between understanding the problem and constructing a solution.

Sequencing. Alpha's routines for sequencing by ten and one hundred were firmly established. He correctly sequenced by ten from eight to 228, 97 to seven, and from 340 to 510. He also sequenced by hundred from 30 to "ten hundred and thirty" and from 73 to 973. He had learned that "thousand" comes after "hundred," but had not routinized sequencing by one hundred beyond 1,000.

I: Start at thirty and count-on by hundreds.
A: One hundred and thirty, two hundred and thirty, . . ., nine hundred and thirty . . . ten hundred and thirty.
I: Any more? Have you ever counted that far before?
A: I know what ten hundreds is.
I: What is it?
A: A thousand.
I: What comes after ten hundred and thirty?
A: Three thousand, four thousand, . . ., nine thousand . . . ten thousand.

Excerpt 4.7.1

The error committed by Alpha in sequencing "three thousand, four thousand, . . ., ten thousand" when asked to continue is, in principle, exactly that of some of the other children's when sequencing "ninety, one hundred, two hundred, ..." More will be said about this in the next chapter in a discussion of the development of concepts of numeration.

Alpha's linguistic system for constructing number-names and sequences of number-names was operational (as far as it had been established). The following episode suggests that the relation "after" within his system was operationally transitive.

Cards: 11 21 31 51 81 91 101 111 (shuffled)

I: Let's try doing the same thing with these cards.
A: (Spreads cards on the table.) Eleven (11). (Pauses; shuffles cards.) Thirteen (11-31). Twelve, I mean (11-21). Thirteen (11-21-31). Eighteen (11-21-31-81). Nineteen (11-21-31-81-91). (Picks up "51"; moves "81" and "91" each one space to the right;

11-21-31-51-81-91). A hundred one (11-21-31-51-91-101). A hundred eleven (11-21-31-51-91-101-111).

I: Now, I want you to read them starting here, please (points to left end of board).

A: Eleven, twelve, thirteen, fifteen, eighty-one . . . eighteen, nineteen, hundred and one, hundred and eleven.

I: Are you sure they're in the right order from smallest to biggest?

A: (Nods head affirmatively.)

Excerpt 4.7.2

The insertion of "51" into the sequence that Alpha had constructed (5 - 6) provides grounds for inferring operational transitivity of "after" and its reciprocal "before." Alpha had only to find the two numerals for which "51" was between--then (by transitivity of "before") all those to the left of "51" preceded it and (by transitivity of "after") all those to the right of "51" succeeded it. It may be objected that this was not a difficult enough task for Alpha to infer operational transitivity. After all, Alpha misread the numerals so that, for the most part, he was working with number-names in the "teens," and hence could have assimilated the task to sequencing by one. The next episode suggests that even if this was the case, Alpha still made pairwise comparisons between cards and placed them according to the criterion "smallest (largest) of all."

Cards: 97 103 107 113 117 124 134 143 (shuffled)

I: I want you to do this just like the last one. Put the cards across the board from the biggest to the smallest. I'm going to trick you this time, so look at the cards carefully!

A: (Spreads cards on the table; pauses; points at several cards; stares at "143" and "107.") Hundred and seventy (107; pauses; looks at each card). Hundred and forty-three (107-143). Hundred and thirty-four (107-143-134). Hundred and thirty (107-143-134-103). Hundred and twenty-four (107-143-134-103-124). Hundred and seventeen (107-143-134-103-124-117). Hundred and thirteen (107-143-134-103-124-117-113). Ninety-seven (107-143-134-103-124-117-113-97).

I: Which one is the smallest one?

A: Ninety-seven.

I: And the biggest one?

A: Hundred and seventy.

I: You want to check them one last time to make sure I haven't tricked you. Say them out loud, and check them very closely.

A: Hundred and seventy, hundred and forty-three, hundred and thirty-four, hundred and thirty, hundred and twenty-four, hundred and seventeen, hundred and thirteen--haven't tricked me!

I: Which number is that (pointing to "103")?
A: Hundred and thirty.
I: Is that how you say it? (Pause.) Can you say it that way? . . . Okay. Which number is this (pointing to "107")?
A: Hundred and seventy.

Excerpt 4.7.3

Alpha's behavior in (4-5) suggests that he was using something akin to a sorting routine. At each step he placed his finger upon or near the largest numeral he had found up to that point in time and compared each numeral yet to be evaluated only with that one. If he found a larger numeral, he moved his finger to it. Given that Alpha's aim was to put the numerals in descending order, his behavior strongly suggests not only the operational transitivity of "before" and "after" within his linguistic system of number-names, but operational reciprocity. If the number-name *x* precedes the number-name *y*, then *y* is necessarily after *x*, and vice versa. That it was the number-names, and not the numerals, that Alpha was relating is apparent when we consider that he correctly ordered the numerals *according to the way he read them*, but not according to a lexigraphic order of the numerals themselves. This was true for each of the numeral seriating tasks in which he misread a numeral.

Numerical operations. Alpha could create numbers and could integrate and separate numbers. One episode in particular substantiates all three of these claims.

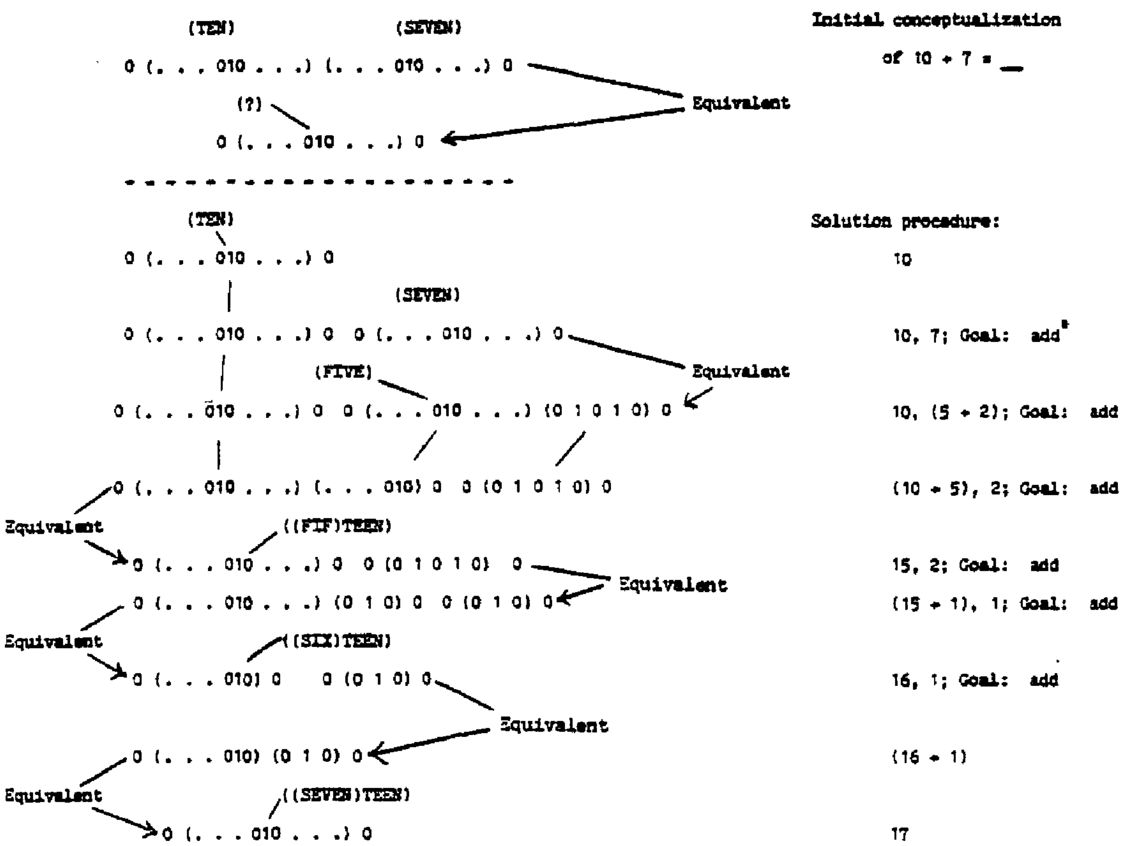
I: I have some number problems on these cards. I want you to think out loud while you do these so that I can hear what you're doing. Are you ready? Here's the first one. (Places card with "10 + 7 = __" onto the table.)
A: (Pause; looks at card; begins to extend a finger of his left hand; shakes hand over the card; touches two fingers of his right hand.) Seventeen.
I: Okay, how did you know that?
A: Seventeen.
I: How did you get that?
A: Ten in my head and I counted seven. Eleven, twelve, . . ., seventeen (putting up a finger with each utterance).

I: Did you do it that way, or did you just know?
A: Both. I did it that way and knew it. But I forgot it--that's why I did it that way.

Excerpt 4.7.4

Alpha's behavior in (5 - 6) provides the grounds on which the above claims are made. It appears that he created two numbers, named "ten" and "seven," and formed the goal of naming their integration. In implementing a solution Alpha intended to extend ten by seven, but made the association between seven and the integration of five and two--the association likely being based upon the figural composition of his "seven" finger pattern. He then separated five from two, integrated ten and five, named that number "fifteen," and then extended fifteen by two. This procedure is depicted in Figure 4.7.1.

Several further points about Alpha's behavior in Excerpt 4.7.4 are worthy of mention. First, his comment that he "did it that way and knew it," but that he forgot it (14-15), suggests that adding <u>meant</u> extending one number by another for Alpha--whether he actually extended or not. Second, Alpha employed a linguistic transformation of ((TEN)(FIVE)), obviating his need to implement his operation of extending, as a partial solution to his problem. It is not clear why he didn't employ the same transformation on ((TEN)(SEVEN)). Nevertheless, the fact that he did transform ((TEN)(FIVE)) into ((FIF)TEEN) in the context of extending suggests that concatenating number-names carried the significance of extending. This point is further substantiated in the following episodes.



*"Add" means to integrate the two numbers producing a named integration.

Figure 4.7.1

I: (Places card with "10 + __ = 13" onto the table.) Read it out loud for me.
A: Ten plus blank equals thirteen.
I: Well . . . .
A: (Long pause.) Ten plus three equals thirteen.
I: How come? Did you just know that one?
A: Same thing.

- - - - - - - - - - - - - - -

I: (Places card with "40 + __ = 46" onto the table.)
A: Forty plus blank equals forty-six. (Pause.) Six.
I: How did you know that?
A: Same way.

- - - - - - - - - - - - - - -

I: (Places card with "__ + 20 = 25" onto the table.)
A: Blank plus twenty equals twenty-five. (Pause.) Five.
I: How did you know that? Same thing?
A: Uh-huh (yes).
I: Are you sure now?
A: Yes, I'm sure.
I: You said five in your head?
A: Huh-uh (no). Twenty, and I put five . . . equals twenty-five.

Excerpt 4.7.5

We do not see until (19) that by "the same way" Alpha meant that he held one number-name in mind and searched for another to concatenate with it. That is, his solution strategies were based literally upon his understandings of the problems and the meaning that he attributed to concatenation: "What do I need to extend twenty by to make twenty-five?" Even though in each problem his criterion must have been a linguistic match with the name of the sum, Alpha appears not to have analyzed the name of the sum for the "missing" name. This suggests two things: first, that separating a number-name did not have the significance of separating numbers; and second, that extending and declending were not operationally related. While Alpha's behavior in Excerpt 4.7.5 cannot be taken as substantiating these points (we would have to see him make some critical error), they would at least explain why he did not decompose the names of the sums for his answers. Also, these two points will help us make sense of some episodes seen later.

Alpha appears to have operationally related integrating and separating. The following episode is one of the few which substantiates this (however, no episode suggests that he hadn't).

I: What number is three tens more than that number (places card with "50" written on it onto the table)?
A: (Pause.) Two.
I: How did you get that?
A: Cause I know that three plus five . . . three plus two is five.

Excerpt 4.7.6

Let us first assume that Alpha understood the question as "What number is fifty three tens more than?" and that he converted ((FIF)TY) to ((FIVE)(TENS)). We may then conclude that he understood that a separation of five into three and some number is equivalent to the integration of three and some number into five. More concisely, he understood that a problem requiring separating can be solved by solving an equivalent problem requiring integrating. The "equivalence" between problems, of course, was provided by Alpha's having related integrating and separating as inverse operations.

It was mentioned earlier that Alpha might not have related extending and declending as inverse operations. There the idea was offered as a plausible argument as to why Alpha solved missing addend problems as he did (working from the addends as opposed to working from the sum). The next episode gives more direct support for this conclusion.

I: (Places card with "70 - 31 = __" onto the table.)
A: Seventy take away thirteen. (Long pause.)
I: Do you know that one?
A: Seventy take away thirteen. (Pauses.) This number (picks up a pencil; draws "63" in the air).
I: Tell me what it is.
A: Sixty-three.
I: How did you get that?
A: I just counted back ten and put that three in. Back ten and put that three in.

Excerpt 4.7.7

It is clear that Alpha was solving "70 - 13 = __" rather than "70 - 31 = __," and that his approach was to solve it as 70 - (10 + 3). The aspect of his behavior that suggests his lack of having operationally related extending and declending is the way he treated + 3 of 13. If Alpha had operationally related the two, then he would have made the transformation 70 - (10 + 3) = 70 - 10 - 3. That is, he would have <u>reversed</u> the direction he had associated with 3. As it was, his initial direction for 3 dominated, and he made the transformation 70 - (10 + 3) = 70 - 10 + 3. (It is worth mentioning that if Alpha had counted <u>back</u> three from 70 we could not infer that he had operationally related extending and declending. Successful behavior could be accounted for by supposing that he initially assigned a backward direction to 3 and did not reverse himself. Only if we concluded that he conceived of 3 as an extension of 10 and then reversed himself could we infer operational reversibility.)

Numbers were very real for Alpha. They were objects of thought as much as anything else. He talked about operating on numbers in the same way he might have talked about playing with marbles. The following episode is an example of this.

I: What is two tens and ninety more?
A: Two tens and ninety more tens?
I: Two tens and ninety more.
A: It would be about . . . twelve . . . tens.
I: Let's see, two tens and ninety more.
A: A hundred and twelve.
I: How did you get that?
A: I meant to say a hundred and twelve, but I just said twelve. I just put all the tens together (sweeping his hands together) and got a hundred and twelve.
I: How did you do that in your head?
A: Well, I was thinking about ninety, and I was thinking about two more tens. And . . . and . . . I count from ninety to a hundred to a hundred and twelve.
I: Do it for me. I want to hear you do it.
A: Ninety . . . (touches right middle finger with right thumb) a hundred . . . (touches right index finger with right thumb) a hundred and twelve.
I: In other words, it goes ninety, a hundred . . .
A: And.

21 I: Ten more.
22 A: Ten more and then (puts up two fingers) two.
23 I: Why did you do two more?
24 A: I just did it.
25 I: You understand what I'm asking? Two tens and ninety more.
26 A: A hundred and twelve.

Excerpt 4.7.8

Before discussing how Alpha might have arrived at his answer, let us first note his manner of expressing what he did (9-10). When trying to explain that he "just put all the tens together," he supported this explanation by bringing his hands together--as if he was pushing together a bunch of marbles. This may be interpreted as the figural correspondent of his mental operation of integrating--capturing an iteration of units between conceptual boundaries.

Turning to Alpha's errors, we must ask ourselves why he insisted that the answer was necessarily 112. An indication of his reasoning comes from his insistence in (16-22) that one must count 90, 100 (+ 10) 112 (+ 10 + 2)--that it was necessary to go "ten more and then two." A reasonable hypothesis is that he included in his extension the two units he had created as a <u>criterion</u> for terminating extending by tens. With this explanation we can in return form a hypothesis about the source of his initial answer of "twelve . . . tens" (4). His answer would be accounted for by supposing something like the following chain of reasoning: 90 is nine tens; two tens more; ten tens (+ 1); then ten plus two--twelve . . . tens. To explain why he subsequently said "I meant to say a hundred . . .," we would have to suppose that he knew that 10 tens is 100. It will be shown later that he knew this very well.

Concept of ten. We have seen in Excerpts 4.7.6 and 4.7.8 that Alpha could quite flexibly move back and forth between thinking of ten as a number and of ten as a numeration unit by which other numbers could be constructed. The question remains, however, as to whether or not Alpha had realized that ten as a numeration unit is itself composed of units. The following episodes allow us to answer in the affirmative.

I: How many tens are there in thirteen?
A: (Pause.) One.
I: One ten. How do you know that?
A: Cause there isn't another ten after ten. Like one, two, . . ., nine, ten. There's only three more. There couldn't be two tens, three tens, or more.

- - - - - - - - - - - - - - -

I: How many tens are there in sixty-seven?
A: Six.
I: How do you know that?
A: Same thing.
I: How did you do it, now? Sixty-seven, now.
A: I know that if there are six tens, it's got to be sixty. And after that six tens, you can't make another ten, cause you only have seven more.
I: And that's not enough, huh?
A: (Shakes head negatively.)

- - - - - - - - - - - - - - -

I: (Places pile of sticks on table; covers pile with hand.) Let's pretend that this pile has seventy-two sticks in it. If you took all the tens out of seventy-two, how many would be left?
A: Two.
I: How did you know that?
A: I just did.
I: If we're pretending, now. If I let you take all the tens, how do you know there would be two sticks left?
A: Cause there's seven tens and only two ones.

Excerpt 4.7.9

It seems clear that the criterion that Alpha operated by was one of constructability: if (one were to construct 67--first in units of ten, then of one) six tens could be made, then (if one were to count them by ten) it has to be 60. And (if one were to try) you can't make another ten--you only have seven more (12-14). From this we may infer that, for Alpha, incrementing once by ten when counting by ten had the significance of incrementing ten times by one when counting by one, and that ten was, in fact, a numeration unit.

Though Alpha had a strong sense of the base-ten structure of numbers and number-names, he did not use it to the extent we might think he was capable. The arithmetic problems which he attempted to compute mentally and those for which he insisted he needed base-ten blocks are listed below.

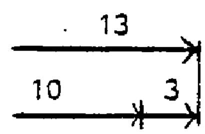
Computed mentally		Insisted on using blocks	
1. 10 + 7 = __	(OK)	8. 47 - 21 = __	(WRONG)
2. 10 + __ = 13	(OK)	9. 74 - __ = 70	(OK)
3. 40 + __ = 46	(OK)	10. 91 - 29 = __	(OK)
4. __ + 20 = 25	(OK)		
5. 60 - 20 = __	(6 - 4; OK)		
6. 70 - 31 = __	(70 - 13 = __; WRONG)		
7. 84 - 30 = __	(8 - 3; OK)		

Figure 4.7.2. Problems that Alpha missed compared to those for which he was successful.

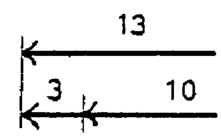
In asking ourselves why Alpha insisted on using the blocks for what would seem to be relatively easy problems given his notion of ten as a unit of numeration (especially problems 8 and 9), several hypotheses come to mind. The first is that he could not create a conceptual understanding of the problems as a separation of one number into two. The second is that he could conceptualize the problems, but had not developed the strategic knowledge of either declending by tens and ones or declending by tens (from tens) and by ones (from ones). The first hypothesis seems untenable--in terms of conceptualization, "47 - 21 = __" is no more difficult than "84 - 30 = __." The second is viable and would account for his use of the blocks in problems 8 and 10, but not 9.

A third hypothesis, and the one preferred by the author, is that Alpha created his own difficulties by assigning <u>directions</u> to each of the quantities involved. Let us return to the discussion of Excerpt 4.7.7 ("70 - 31 = __"). There it was argued that Alpha assigned a direction to 3--as extending 10, or + 3, and that his error occurred as a result of not having operationally related extending and declending. Figure 4.7.3 shows what is meant by the use of operational reversibility in solving 70 - (10 + 3).

Extending by 13 as extending by 10 and by 3:

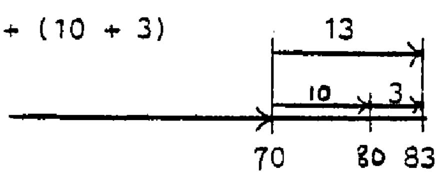


Declending by 13 as declending by 10 and by 3:



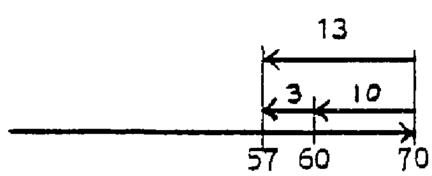
Extending 70 by 13 (as 10 + 3):

70 + 13 = 70 + (10 + 3)



Declending 70 by 13 (as 10 + 3):

70 - 13 = 70 - (10 + 3) = 70 - 10 - 3



10 + 3 (as an extension) <u>gets turned around</u>--it becomes a declension of 10 and 3.

Figure 4.7.3. Formal compensation when declending by an extension.

When Alpha considered 3 as extending 10, and then came to apply it in the context of declending 13, he did not realize that the direction of 3 was relative to the direction of 13. Hence, it retained its initial direction. If we examine again Alpha's explanations of the compositions of 13 and 67 (Excerpt 4.7.9), we see that he conceived of the quantity of ones as extending the quantity comprised by the tens. Thus, in problems 8, 9, and 10 of Figure 4.7.3 Alpha's difficulty stemmed from his assignment of directions to (at least) the quantity of ones--and the multiplicity of directions was too much for him to deal with. If Alpha had not assigned directions to numbers, but instead understood the problems in terms of quantities to be "removed" or "taken away," he likely would not have had difficulty in mentally computing at least problems 8 and 9.

Concept of one hundred. We have already established that Alpha possessed most of the major components and relationships of the concept of one hundred: his linguistic system for generating number-names and sequences of number-names was operational into the "hundreds," and he could give both intensive and extensive meaning to number-names. The two remaining questions are whether he had constructed one hundred as a numeration unit and whether he had operationally related one hundred to his concept of ten. The following episodes suggest that Alpha had constructed one hundred as a unit of numeration.

I: (Places bag with 235 unit cubes in it onto table.) Here are two hundred thirty-five little blocks. How many piles of one hundred could you make from them?
A: (Long pause.) What's the question?
I: There are two hundred thirty-five little blocks in there (the bag). If you were to use those little blocks, how many piles of one hundred could you make?
A: (Pause; holds up two fingers.)
I: Two? How did you know that? Did you count them?
A: I didn't count those (points to bag). Because I know how much makes two hundred.

. . .

I: Let's have you count for me (begins placing MAB longs on the table).
A: Ten, twenty, . . ., hundred and sixty (as the interviewer places 16 MAB longs one at a time onto the table).
I: There are one hundred sixty little blocks there. How many hundreds are there?
A: One.
I: How do you know?
A: I just know.
I: Okay, keep counting.
A: Hundred and seventy, . . ., two hundred (as the interviewer places 4 more MAB longs).
I: All right.
A: Two.
I: (Screens (20) MAB longs from view.)
A: Two hundred under here (places hand on cover).

Excerpt 4.7.10

In the first episode (1 - 11), Alpha appears to have appealed to a criterion of constructability--make as many one hundreds as you can while not exceeding 235 (5 - 11), and to the structure of the number-name "two hundred thirty-five." He again appears to have relied on the structure of the number-names in the second episode: "one hundred sixty, two hundred." Also, in (25-27) Alpha spontaneously moved from considering the blocks as two units of 100 to considering them as 200 units.

The question of whether Alpha had related one hundred to his concept of ten was addressed in the sequel to the second of the above episodes.

I: (Places 4 longs next to screen.) How many tens are there altogether now?
A: Two hundred and forty (as the interviewer asked above question).
I: How many tens are there altogether?
A: (Pause.) Two hundred would be twelve tens . . . sixteen. (Pause.) No! Thirteen . . . (looking at the 4 longs).
I: (Interrupting.) How many tens?
A: Sixteen.
I: How did you get that?
A: Counted.
I: How many tens under here (places hand on screen)?
A: Twelve. Thirteen, . . ., sixteen (pointing to the 4 MAB longs on top of screen).
I: How do you know there are twelve tens under there?
A: I know how many tens is two hundred.
I: (Places 1 more MAB long on table; uncovers other 20.)
A: Seventeen . . . ten . . . one, two, . . ., eight (while pointing to MAB longs; continues subvocally counting). Twenty-five.
I: How many hundreds are there?
A: Two.
I: There are twelve tens in two hundred. Right? Is that what you said?

A: Yeah, but I'm wrong.
I: Oh, why are you wrong? How do you know you're wrong?
A: I counted them.
I: How many tens are there? Do you want to change your mind?
A: I forgot how much I counted. (Counts 13 longs; separates them from the others.) There are thirteen tens in this pile and thirteen tens in this pile. One, two, . . ., twelve (as he points to each MAB long in 2nd pile). I always get 'em mixed up.
I: You told me there were twelve tens in two hundred. But you think you're wrong. Exactly how many tens are there in two hundred? Is it thirteen tens instead?
A: Uh-uh (no; looks in the air; pauses). Twenty.
I: Twenty? Why do you think twenty?
A: I counted two hundred. One, two, . . ., eleven . . . wait. (Removes 2 longs from one pile, 3 from the other; sets them aside.) These don't even need to be in here. One, two, . . ., twenty (pointing to each of the remaining longs).

Excerpt 4.7.11

Alpha's error in saying that there are 12 tens in 200 can be easily explained. Before we do this, however, let us take note of his behavior in (17-39). Alpha never explicitly counted (just) 20 longs--he counted 25. Yet in (34), we see him answer that there were 20 tens in 200--apparently on the basis of a mental computation. Second, when he proved to the interviewer that there were 20 tens in 200 (36-39), he imposed his knowledge that to have 200 in tens, he needed two groups of ten longs each--and then counted to 20. Thus, Alpha had operationally related one hundred with his concept of ten. Finally, to explain Alpha's initial error, he apparently reasoned something like "ten tens in one hundred; two hundreds; ten plus two is twelve."

Concept of place value. Alpha's behavior on the counting-board tasks was exemplary. He looked back only once in the five tasks given to him, and that appeared to have been because of a shift from counting by one hundred to counting by one. He would pause, however, with each change in unit value of the uncovered blocks (e.g. uncovering a flat and then uncovering two longs). The pauses were more than likely due to his changing from one sequencing routine to another.

The criterion used throughout the case studies for determining the operationality of a child's concept of place value has been that we must be able to infer that one hundred may be conceived of as ten of (ten of one). In this regard, we will let Alpha speak for himself.

I: (Places MAB flat on table; holds up MAB long.) How many of these pieces of wood (long) are there in this piece (flat)?
A: (Pause.) All of these (sliding hand over MAB flat) aren't right. Just that one is right (indicating one column within the flat).
I: Is that how many of these (holding up MAB long) there is in that? Just one?
A: Huh-uh (no).
I: Can we saw off just one of these (long) from that piece of wood (flat)?
A: No, we can saw off that one, and that one, . . ., and that one (pointing to each column of the flat).
I: How many would that be?
A: Ten . . . tens.
I: How many of these would there be (holding up MAB long)?
A: Ten.

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I: (Holds unit cube hand.) How many of these little blocks could I saw out of that piece of wood?
A: A hundred.
I: How do you know that?
A: Cause ten tens is . . . all little ones. And ten tens is a hundred. So those little ones are tens and they're a hundred.

Excerpt 4.7.12

Case Study 4.8: Mu

Mu was a first-grader (age 6 years at the beginning of the 1977 school year). In November of 1977 Mu correctly solved each of Problems 1, 2, and 3 of Figure 4.1 (page 105) by counting on his fingers. He counted-all for Problems 1 and 3, and counted "eight, nine, ten" while folding three fingers for Problem 2. Mu sequenced by ten to 130, but could not sequence by ten starting from "two." Mu counted from ten (to 14) to find the total of a bundle of ten and four single sticks, and similarly counted from 20 (to 25) for two bundles and five. He said that there were three tens in 32, but in making 54 with sticks counted five bundles by ten and thought there were four tens in 54; he then went on to place four single sticks. After grouping a pile of sticks into two bundles of ten and nine single sticks, the interviewer had to remind him that there were two tens and nine before Mu could give the total number. Mu counted ten bundles of ten as "ten, twenty, . . ., one hundred" to find the total number. The final interviews were given to Mu on May 11, 12, and 16 of 1977.

Mu's case study will show him to have been by far the most advanced of the eight children. Though he committed a number of low-level errors, such as misreading numerals (just as did Alpha), he had very well-developed concepts of numeration.

Writing numerals. Mu committed only one error in writing a numeral. It was to write "1014" for "one hundred fourteen." He realized his error while writing the next numeral ("121"), corrected it, and then returned to correctly write "121."

Reading numerals. Reading numerals was more problematic for Mu than was writing them. Though he correctly read each in the numeral-reading task, he expressed uncertainty about "201" and "311," though he didn't explain why he was uncertain. Also, his most frequent error during the interviews was to read a two-digit numeral as if the digits were reversed (e.g., "twenty-four" for "42").

Sequencing. Mu's linguistic system for constructing number-names and sequences of number-names was very well developed. He correctly sequenced by ten from eight to 208, from 97 to seven, and from 340 to 550. He also correctly sequenced by one hundred from 30 to 2,130. He made one error that suggests the nature of his routine for sequencing by one hundred.

I: Start at thirty and count-on by hundreds.
M: One hundred and thirty, two hundred and thirty, . . ., nine hundred and thirty, one thousand and thirty . . . two . . . oops . . . one thousand and . . . and one hundred and thirty, one thousand and two hundred and thirty, one thousand and three hundred and . . . thirty, one thousand and four hundred and thirty, . . ., one thousand and nine hundred and thirty . . . two thousand and thirty. Two thous . . . could we stop now?
I: What comes next?
M: Hmmm . . . two thousand and one hundred and thirty.

Excerpt 4.8.1

It seems that Mu's routine was to identify the current digit-name preceding "hundred," increment it, and then say the new current digit-name followed by "hundred and thirty." It also seems that Mu fell into the pattern of incrementing the first-said part of the number-name, as is suggested by his having begun to say "two thousand and thirty" as the next name after "one thousand and thirty" (10). How did Mu catch his error? Perhaps by his violation of a necessary condition that he say "hundred" immediately after the digit-name produced by incrementing.

Another aspect of Mu's routine is that he must have had a special-case rule for incrementing when there was no current "hundred" digit-name. Such a rule might take the form of

Next-hundred of (WORD) is (((ONE)HUNDRED)(WORD)),

while his more general rule might have taken the form of

Next-hundred of (((WORD1)HUNDRED)(WORD2)) is
(((Next of (WORD1))HUNDRED)(WORD2)).

Thus, in (4), Mu's pause is likely due to moving from his misapplied rule (increment the first-said part of the name) to his general rule (but not finding a "hundred" digit-name) to his special-case rule.

Mu's insertion of "and" between each of the digit-name/label-name combinations ("one thousand <u>and</u> two hundred <u>and</u> thirty") is interesting. Perhaps this was his way of keeping separate the components of a number-name while at the same time holding it as one name.

Mu's behavior in the numeral seriating tasks leaves no doubt of the operationality of his linguistic system for constructing number-names and sequences of number-names. In fact, we see an added feature not seen in the previous case studies: Mu's meanings for number-names were so well established that he ordered the numerals as a <u>number</u> sequence, as opposed to a number-name sequence.

Cards: 11 21 31 51 81 91 101 111 (shuffled)

I: Let's try doing the same thing with these cards.
M: (Spreads cards out.) Oh, boy. Not very much to start with. (Begins to place a card in left-most space; replaces it in pile and places "11"; places "21" next to "11.") Missing a lot (points back and forth between "11" and "21"). Jumps to this (11-21-31). Jumps to this (11-21-31-51). Jumps to this (11-21-31-51-81). Eleven, twenty-one, thirty-one, fifty-one, eighty-one. (Places "101"; picks up "111.") One hundred

eleven . . . I get mixed up with the one and one . . . the one, one . . . eleven and twelve. (11-21-31-51-81-101-111; picks up "91.")
I: What's that one you've got?
M: Oops. (Slides "111" and "101" each one space to the right; 11-21-31-51-81-91-101-111.)
I: Satisfied?
M: Uh-huh (yes).

Excerpt 4.8.2

The basis for inferring that Mu considered the task as creating a number sequence is his remarks in (4 - 5) that he was "missing a lot" (between "11" and "21") and that the sequence was "jumping" from one numeral to another--as if he were skipping over objects that were in the sequence, but not named.

We may also infer that Mu's relations of "before" and "after" were operationally transitive. In (13-14), we see that he spontaneously moved "101" and "111" each one space to the right in order to make room for "91." Mu inserted a card in this manner in three of the six seriating tasks.

Mu produced correct sequences in each of the three backward seriating tasks. This in itself would suggest, given his other behavior, that he had operationally related "before" and "after" as reciprocal relationships within his linguistic system. Mu's behavior on the third of these tasks, however, leaves no room for doubt.

Cards: 97 103 107 113 117 124 134 143 (shuffled)

I: I want you to do this just like the last one. Put the cards across the board from the biggest to the smallest.
M: (Spreads cards in front of him.) Oh my lord. This is the hardest one. One hundred and thirty (pointing to "103"). One hundred and .... oops. Now I got it. (Looks at all the cards; picks one up.) I better put them in order first. (Looks at all the cards; begins picking them up one at a time in ascending order; holds them in one hand.)
I: What are you doing?
M: I'm putting them in order (picks up remainder of cards one at a time in ascending order; largest is on top, smallest on bottom; places each card in appropriate space--largest first in left-most space and smallest last in right-most.)
I: Satisfied? Let's read them now.
M: One hundred and forty-three, one hundred and thirty-four, one hundred and twenty-four, one hundred and seventeen, one hundred and thirteen, one hundred and seven, one hundred and three, one hundred and . . . oops. Ninety-seven.

Excerpt 4.8.3

The operational reciprocity of "before" and "after" for Mu is clear. To put the numerals in descending order, he picked them up in ascending order--realizing that in putting them down they would be in descending order.

Mu's behavior in Excerpt 4.8.3 actually suggests much more than the operationality of his linguistic system. He held the goal of creating a sequence in descending order, and equated that anticipated descending sequence with the inverse of an ascending sequence (i.e., he knew he would reverse his ascending sequence to create a descending one). This sounds as if he had also related extending and declending as inverse operations. We will see that this was, in fact, the case.

Numerical operations. There are no episodes in the records of Mu's interviews which help us to specify the nature of his operations of integrating and separating, nor are there any which help us to characterize the relationship he had established between the two. There are two reasons for this. First, Mu made very few errors, and those he did make are irrelevant to the issue. Second, whenever Mu explained himself in situations where we might think integrating and separating were involved, he did so in ways best accounted for by supposing that he conceived of the problems in terms of extending and declending. That is, Mu appears to have been beyond thinking of numbers strictly as absolute quantities. Instead, he appears to have reconceptualized numbers as signed magnitudes.

This is a tenuous hypothesis, and it is not expressed very well. One might think that the system of integers is being attributed to

Mu, but this is not what the author means. To impute the system of integers to Mu we would have to have evidence that his signed magnitudes existed within a relational system having the structure of a commutative ring, or at least a group (if we eliminate multiplication). This certainly is not the case. What is meant is that Mu conceived of numbers as extensions or declensions <u>to be created</u>. Extending and declending, then were operators upon signed numbers. For Mu, extending by, say, three was an operation that could be thought of being performed independently of where he started--it was an object of his thinking.

It will be difficult to fully substantiate the above picture painted of Mu. The problems were, for the most part, too easy for him. It is only by his manner of explaining himself that we get a hint of the ways in which he understood the problems--and all we get are hints. Thus, rather than try to substantiate the position taken above, the author will try to make it at least plausible.

Mu, correctly, mentally solved each of the problems in Figure 4.8.1 by declending by ten or by ten and then one. His solution to "91 - 29 = __," however, did not quite fit the pattern. He at

1. 60 - 20 = __	4. 84 - 30 = __
2. 70 - 31 = __	5. 91 - 29 = __
3. 84 - 30 = __	

Figure 4.8.1. List of problems that Mu solved through declending by ten or by ten and one

first intended to use his fingers (most likely to keep track while declending nine), but the interviewer asked him not to.

I: (Places card with "91 - 29 = __" onto the table.)
M: Ninety-one take away twenty-nine. Just a minute. That's hard. I have to use my fingers.
I: Try it without your fingers first.
M: Ohhh. I know I'm going to make a big mistake.
I: That's okay. Everyone makes mistakes.
M: Ninety-one . . . eighty-one . . . seventy-one. Now the hard part. I need to use my fingers.
I: Okay.
M: Seventy-one (places two pencils on right, extends 5 fingers of right hand and 1 finger of left). Seventy-one . . . Oh, that's hard!
I: Seventy-one. What do you need to take away from seventy-one?
M: Nine.
I: Okay, you need to take nine away from seventy-one.
M: Seventy . . . seventy . . . sixty-nine, oops . . . .
I: How many is that we've taken away?
M: Two. (Pause.) Sixty-eight--sixty-seven, sixty-five, sixty-four, sixty-three, sixty-two, sixty-one. How much has that been?
I: I don't know. Have you been counting?
M: (Brings hands from under table with 5 fingers extended on right and 4 on left; giggles.)
I: Let's do it. Seventy-one. Let's do it.
M: I did it.
I: What did you end up with?
M: I forgot.
I: Seventy-one take-away nine was?
M: Didn't you remember?
I: No. You didn't ask me to. Was I supposed to?
M: Yeah!
I: Let's do it this way. What's seventy-one take away ten?
M: That's a cinch. Sixty . . . .
I: Sixty, is it? Seventy-one take away ten?
M: Oh. Sixty-one.
I: But we really want to take away nine.
M: Oh, I know it! I know it! I know it! Sixty-two!

Excerpt 4.8.4

Mu's insistence upon using his fingers to form nine as a declension from 71 is understandable. Given his aim to construct it, he needed some way to keep track of his progress. However, when in (31-35) the interviewer set the stage for an alternate strategy, Mu immediately saw a relationship--and used it. In exclaiming "I know it! I know it!" Mu seems to have equated a declension by nine with a declension by ten followed by an extension by one, or that $-9 = -(+10 + -1) = -10 + +1$. His formal change of direction of -1 also suggests that Mu had operationally related extending and declending as inverses.

An indication that Mu thought of extending as an operator on numbers is his expressed sense that even when extending by ten he was "sort of," but not really, counting.

I: What is two tens and ninety more?
M: Two tens?
I: And ninety more.
M: (Pause.)
I: Tell me what you're thinking as you work it out.
M: Two tens and ninety more. Hmmm . . . one hundred and ten?
I: How did you get that?
M: Well . . . I have to think. Well. No, I can't explain it.
I: You can't? You just guessed it?
M: Well, I can't explain it. And I didn't just guess.
I: You didn't guess . . . did you count? Did you use counting to help you? Or did you just know it because you had done a problem like that before?
M: I think I counted.
I: Uh-huh.
M: Well, I said, if ninety and ten more make a hundred, then it must be one hundred and ten.

Excerpt 4.8.5

Clearly, Mu went through something like the following chain of reasoning: $2 \text{ tens} + 90 = 90 + (10 + 10) = (90 + 10) + 10 = 100 + 10 = 110$. The significance of this episode is that Mu was correct in that he did not count, yet he felt somehow that he had. This could be indicative of his having recently elevated an increment by ten in language to a symbol for the operation of extending once by ten--which itself had the significance of extending ten times by one.

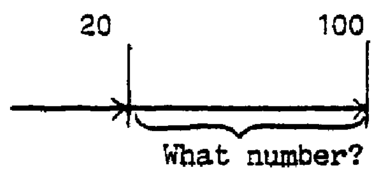
Another indication of Mu's having constructed signed numbers comes from an episode wherein he completely misunderstood the question, and understood it in a way that we might think requires signed numbers.

I: (Places card with "20" written on it onto table.) What number is one hundred more than this number?
M: One hundred more? I don't understand. One hundred more?
I: What number is ten more than that number? What is that number?
M: Twenty.
I: What's ten more than that number? (Pause.)
M: It's less . . . so it's ten less.

Excerpt 4.8.6

Mu's remark in (7) that "it (ten) is less . . . so it's ten less," suggests his difficulty with the original statement. The understanding that he apparently constructed was of comparing 20 and 100 with the aim of specifying the operation upon 20 necessary to transform it into 100. This is shown in Figure 4.8.2. This interpretation is worded as it is to

What number is 100 more than 20?



What number is 10 more than 20?

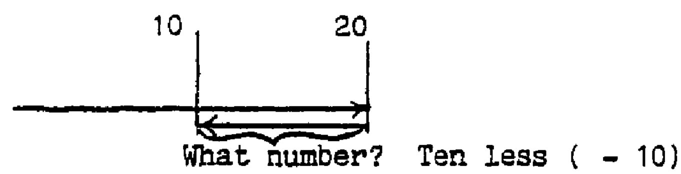


Figure 4.8.2. Mu's understanding of "What number is twenty more than one hundred?"

account for his behavior in comparing 10 and 20. There, he apparently compared the two with the aim of determining an operation to be performed upon 20 in order to transform it into 10. The operation was "ten less," or - 10--a declension by ten.

The only instances in which Mu might have remained at the level of separating and integrating numbers were in problems involving physical materials. The following episode suggests that, even then, he could move to the level of conceiving of a problem as extending or declending. We join the episode after the interviewer has successively placed and then covered up two longs and four unit cubes, one long, three unit cubes, two longs, one long and one unit cube, and two longs and two unit cubes. Mu wrote the numeral of the number of blocks at each step, producing the list "24 34 37 57 68 90."

I: (Advances screen so that all blocks are covered.) How many pieces of wood, little blocks, have I got altogether under here now?
M: Little blocks? Little teensy-weensy ones?
I: Yeah.
M: Ninety-eight . . . . Let me count (looks at numerals which he wrote below the screen; pauses). Seven . . .
I: What are you doing?
M: (Pauses; continues to look at the numerals he'd written below the screen; starts at left end and moves right.) Ten.
I: How many little . . . what's the number . . . .
M: Those little ones, those little ones. There's ten of them.
I: Oh, ten of those little ones.
M: Uh-huh (yes). Now, those long ones?

Excerpt 4.8.7

Mu apparently took the interviewer literally. He computed the number of unit cubes that had been placed. There are two ways that he could have accomplished this, supposing that he based his calculations on the numerals (which seems a safe assumption). He could have compared successive pairs of numerals in his list with the aim of determining the extension necessary to transform the first into the second (e.g., 24, 34 → + 10; 34, 37 → + 3), or he could have done the same, but focusing on the "ones" digit (e.g., 4, 4 → + 0; 4, 7 → + 3)—accumulating the changes in the number of ones. Whichever way he performed his calculations, it seems clear that he was conceiving of pairs of numbers as being connected by way of transformations of the first into the second, where the transformations were extensions.

It was noted in the discussion following Excerpt 4.8.5 that Mu's linguistic increment of a number-name by ten perhaps was a symbol for the numerical operation of extending a number by ten. Mu's solutions to the problems "10 + 7 = __, 10 + __ = 13, 40 + __ = 46, __ + 20 = 25, and __ + 9 = 79" are consistent with that point. He responded immediately to each, at times remarking how easy they were. In explaining why 10 + 7 = 17, Mu's rationale was that "if you don't have a ten you, you don't have a seventeen." At other times he said "I just know it" as his reason for his answer, and that when he didn't know it he would count. The inference drawn from all of this is that Mu had constructed linguistic transformations for this type of problem where concatenating (say, "forty" and "six" to make "forty-six") was symbolic of extending (40 by 6), and decomposing ("forty-six" into "forty" and "six") was symbolic of declending (46 by 6). Likewise, in the sequel to Excerpt 4.8.6 ("100 more than 20"), Mu eventually changed his understanding of what the interviewer was asking and said "One hundred and twenty." The concatenation of "one hundred" and "twenty" (by "and") served to symbolize, for Mu, the numerical operation of extending 100 by 20.

Concepts of numeration. The sections on Mu's concepts of numeration, rather than culminating the case studies, are anti-climatic in light of the discussions of his linguistic system and his numerical operations. Needless to say, Mu's concepts of ten, one hundred, and place value were all very well-developed. He responded that there is one ten in 13, six in 67, seven in 72, and 24 in 240; and that there is one one hundred in 160, two in 235, two in 250, and that 470 is 100 less than 570. As we have seen, Mu could move flexibly from one concept to another--changing from thinking of a number as a unit of ones to thinking of it as a unit composed of units of ten and one, or of units of one hundred, ten, and one.

Comment. A remark given in the opening discussion of Mu's numerical operations deserves to be expanded here. The author does not intend to attribute a concept of integer to Mu. Rather, he wishes to attribute to him the _germ_ of the concept. Even then, the tasks used in the interviews were not sufficient to establish boundaries for Mu's understandings. None was sufficient to induce Mu to construct a problem that he couldn't fully understand or solve. As a result, the author may have over-interpreted Mu's behavior, and attributed more to him than might be the case had more difficult tasks been given.

Chapter 5

VARIETIES OF UNDERSTANDING

WHOLE NUMBER NUMERATION

This chapter will summarize the case studies and present schematic representations of the respective children's numeration concepts. It will end with a cross-sectional analysis of the case studies so that the development of numeration concepts may be discussed.

Summaries of the Case Studies

The summaries will have the same organization as did the case studies themselves. The form will differ, however, in that the dialectical style of the case studies will be abandoned and only the end result (conclusions) will be presented. The reader should note that conclusions presented in the summaries are not intended as statements of fact. Rather, they are conclusions drawn from examining the children's behavior from the perspective of the theoretical framework presented in Chapter 3.

Summary 5.1: Delta

Delta began the 1977 school year with little conceptual and procedural knowledge of arithmetic. He could add only after he had created perceptual collections and then he would count-all to find the total. Delta could sequence by ten only as "ten, twenty, . . ., ninety," and otherwise had no concept of ten.

Writing and reading numerals. While writing a numeral Delta would elaborate the number-name, writing a digit for each major part. When writing a numeral beyond 100 Delta would say "hundred" and write a zero as he said it--regardless of the remainder of the number-name. Delta had no difficulty reading numerals preceding 100, but for those beyond 100 he was at times indecisive about how to partition a numeral for subsequent naming.

Sequencing. Delta knew the sequence "ten, twenty, . . ., one hundred" quite well. He had abstracted the structure of compound names preceding "one hundred," but in order to sequence by ten from a digit-name he had to explicitly separate the "ty" digit-name from the currently-held name in order to use sequencing by one to produce an increment. That is to say, Delta's routine for sequencing by ten was actually an augmented routine for sequencing by one as opposed to a routine in its own right that was operationally related to sequencing by one. As a result, Delta had difficulty transcending centuries when sequencing by ten.

Delta did not have a routine for sequencing by hundred. In one instance he continued the interviewer's example of "thirty, one hundred thirty, two hundred thirty" by empirically abstracting the criterion for constructing a next term--sequencing by one.

His behavior on the numeral seriating tasks showed that Delta had not developed an operational structure in his linguistic system for constructing number-names and sequences of number-names. His method for seriating numerals was to assimilate them, when possible, to his routine for sequencing by one. That is, he would form the number-name which would normally follow that of the last placed numeral, and then look for the corresponding numeral. He did not order the numerals by making pair-wise comparisons of them. Moreover, when Delta did compare numerals in terms of relative order he based his judgments on the relative order of the names of fortuitously chosen digits from the respective numerals.

Numerical operations. Integrating and separating were empirical routines for Delta. In terms of understanding, Delta could not conceptualize addition and subtraction problems. Rather, he had to assimilate the situations to his action schemas for putting together (integrating) and taking apart (separating) perceptual collections. There were some instances where Delta did not require a full elaboration of the operands, but these were situations where he could apply his heuristic of putting "the biggest number in his head." Moreover, Delta could not conceptualize missing addend problems even when he had the operands elaborated as collections. This suggests that Delta's operations of integrating and separating were even less than intuitive, for he could not coordinate them as complementary routines.

Extending and declending were also empirical routines for Delta. With the exception of situations in which he "put a number in his head," Delta used them only as routines for constructing number-names for perceptual collections.

Concept of ten. Delta's concept of ten was figural and action-based. He knew that to construct a collection corresponding to a number-name, he could do so most efficiently by first counting MAB longs by ten up to the "ty" part of the number-name, and then count unit cubes by one up to the digit name. However, ten was not a numeration unit for Delta. He could not conceptualize a number, let alone a number that is constructible in terms of units of ten and one.

Delta had empirically abstracted a criterion for answering questions like "How many tens in . . . ." It was to homonymically translate the first-said part of the number-name. But this was an isolated bit of knowledge for Delta; it had no quantitative significance, for he did not employ it in any situation where it might have been used appropriately.

Figure 5.1 is a schematic representation of Delta's concept of ten. Although sequencing by one was not directly assessed in the interviews, he is attributed it at an intuitive level because of his use of it in coordination with sequencing by ten. Similarly, Delta is attributed figural patterns at an intuitive level, at least in connection with sequencing, since he could subitize a pattern of number-names (e.g., "fifty, sixty, seventy") when continuing a sequence. Otherwise, the components in Delta's concept of ten were preoperational, having strict contextual requirements for their implementation. Relationships among the components were largely empirical, in that it was only after one component's modification of the context that another might come into play. There was little coordination among components in Delta's concept of ten.

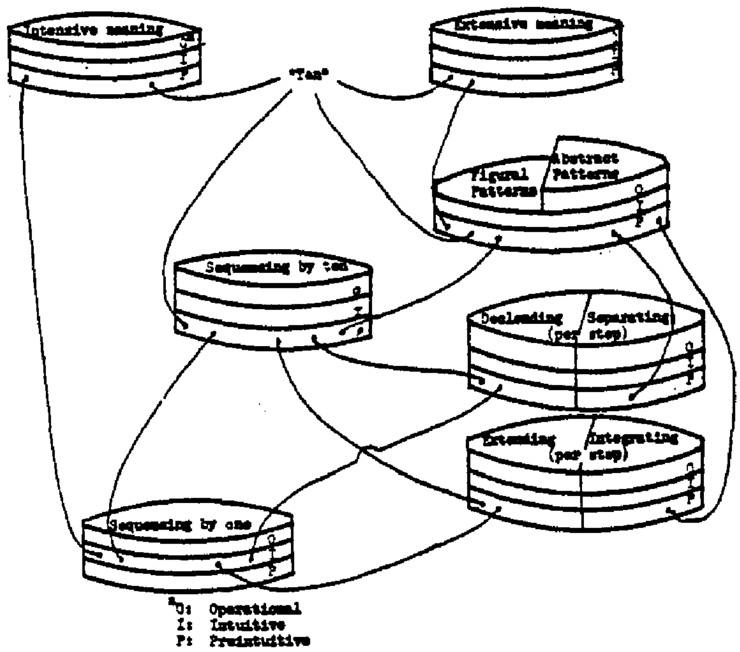


Figure 5.1. Delta's concept of ten.

Concepts of one hundred and place value. Delta's concept of one hundred comprised "hundred" as the name of an MAB flat and as a part of a number-name that appeared if one sequenced long enough. He had no concept of place value.

Summary 5.2: Lambda

Lambda began the 1977 school year as a counter with motoric unit items who could solve relatively simple addition problems, but not missing addend problems. She had essentially no concept of ten beyond sequencing "ten, twenty, . . ., one hundred." She could not sequence by ten in any other manner. Also, she failed to differentiate between perceptual items labelled "ten" and "one" when counting--counting everything as one item.

<u>Writing and reading numerals</u>. Lambda elaborated a number-name to write the digits of its numeral. Her procedure for doing so was unreliable, in the sense that at times she elaborated the name by syllables (e.g. "702" for "seven-ty-two") and at times into digit-names, suppressing the digit-labels (e.g., "42" for "<u>for</u>ty-<u>two</u>"). The conditions under which she chose one over the other were not clear. Also, Lambda would occasionally reverse the digits while writing a two-digit numeral.

Lambda had little difficulty reading numerals less than one hundred. The only mistake she made was an occasional reversal of the digit-names. When Lambda applied her routine for reading numerals beyond 100, however, the result was not always predictable. She had not established a convention for partitioning three-digit numerals as # ##, instead at times partitioning them as ## #. When the left-most pair of digits did not constitute a "ty" name, she would drop the name of the right-most of the two and then append the name of the very right-most digit (e.g., "fifty-nine" for "549").

<u>Sequencing</u>. Lambda could sequence by one and by ten with little difficulty--as long as she remained within a century. Crossing centuries was extremely problematic for Lambda. When sequencing by one, she went from "hundred ninety-nine" to "hundred and two." When sequencing by ten she went from "hundred and ninety" to "hundred and thirty." That is, within centuries Lambda sequenced by one or ten and appended "__ hundred" as a prefix.

Her performance on the numeral seriating tasks showed that Lambda's linguistic system for constructing number-names and sequences of number-names had yet to become operational. She lacked operational transitivity of both the relations "after" and "before" within her system and had not related the two as reciprocal to one another, as she could not construct descending sequences of numerals. Her routines for sequencing were established well enough, however, so that she could construct ascending sequences quite well. But once constructed, a sequence was rigid--Lambda could not insert a numeral amidst it.

Numerical operations. Lambda was a counter with motoric unit items. She did not require perceptual collections prior to counting. However, Lambda could not quantify a count originating beyond "one." That is, she could not double count.

Integrating and separating, and extending and declending, were intuitively related for Lambda. She could complement one by the other, say, in solving missing addend problems. Integrating and separating were not operations, however, since she could not construct numerical operands; likewise, extending and declending were not operations. Rather, they were yet empirical routines to construct named collections or lots.

Lambda attributed intensive meaning to number-names at the level of signs. A number-name could stand for the sequence she would produce were she to count up to it. A number-name was not a symbol, though, in that the meaning of it was still a figural representation of the activity of counting up to it. Lambda did not have a structure of relationships that would be maintained regardless of the names actually related. Similarly, extensive meaning for number-names was at most intuitive, in that Lambda could create at most lots as meanings for number-names.

Concept of ten. The significance of an increment when counting by ten for Lambda was a cycle of increments by one. That is, an increment by ten had meaning, but only intuitive meaning. Lambda could not create ten as a number--a unit of units, and hence could not reconstitute a number as a number of tens together with a number of ones.

Lambda did not conceptually distinguish ten and one as units. Two hundred as 20 tens and four more tens was "twenty--thirty (one long), forty (another long), fifty (another long), sixty (final long)."

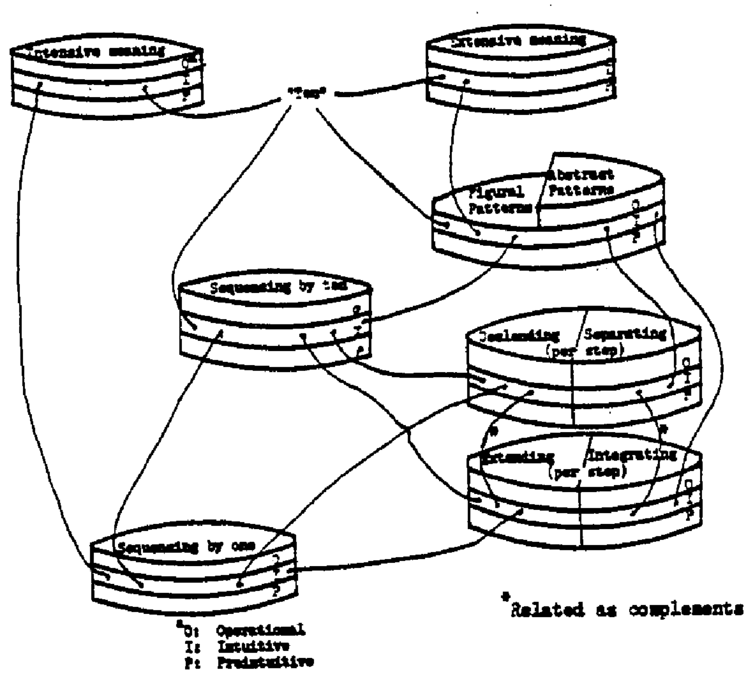


Figure 5.2. Lambda's concept of ten.

Figure 5.2 gives a schematic representation of Lambda's concept of ten. Her concept was intuitive, as each of the relations among components was one of functional coordination or complementarity.

Concepts of one hundred and place value. "One hundred" as a number-name could be attributed intensive and extensive meaning by Lambda (as a sign for counting and as a sign for a lot), just as could any other number-name. Beyond this, the only special significance that she gave it was as a name for an MAB flat. Nor had Lambda related 100 and ten--not even figurally as a flat being literally composed of ten longs.

Lambda had no concept of place value. She could not coordinate her concepts of one hundred, ten, and one. When she counted MAB blocks, she would begin with the block first presented her, but would change her counting routine only with great difficulty. If at all possible she would remain within the context of the concept first coming into play. If it was not possible to remain within her concept, say, of ten, she would move to her concept of one and stay with it.

Summary 5.3: Kappa

At the beginning of the 1977 school year Kappa's conceptions in arithmetic were limited to counting perceptual collections. Kappa had no concept of ten. He could not sequence by ten, count by ten, or mentally compute the sum of ten and a single-digit number.

Writing and reading numerals. Kappa made many errors in both writing and reading numerals. When writing two-digit numerals Kappa occasionally reversed the digits. When writing three-digit numerals he would often write a zero as he said "hundred" whether it was warranted or not.

Kappa's most consistent error when reading numerals was to reverse the digit-name for a two-digit numeral. This was apparently a result of his check for a "1" in the numeral in anticipation of a "teen" name.

Sequencing. To sequence by ten from a digit name Kappa required that he have both the initial name and "ten" currently in his thinking. He did not have the special-case rules for sequencing from a digit-name to the teens. Once beyond that point Kappa had little difficulty. He based sequencing by ten around his sequence "ten, twenty, . . ., one hundred," but when sequencing beyond "one hundred" he would continue incrementing the first-said part of the number-name (e.g. "ninety-eight, one hundred eight, two hundred eight," and so on).

Though Kappa errorfully sequenced by ten across centuries, he could correctly count by ten across centuries. Apparently this was because he would place a constraint of distance between number-names related by "ten more" upon the terms in his sequence. When he saw a collection named, say, "ninety" and then placed two more longs, he would not say "one hundred, two hundred," for he did not see enough of a difference between the collection named "one hundred" and the next one to warrant calling it "two hundred."

Similarly, Kappa could not sequence by hundred, yet counting by hundred was a solvable problem for him. The reason again was that sequencing by hundred as such did not signify a change in a referenced quantity, and hence Kappa had no basis for choosing to increment one part of the name instead of another.

His performance on the seriating tasks showed that Kappa had a sense of a series, but could not implement it within his linguistic system for constructing number-names and sequences of number-names. He had a limited sense of transitivity of "after," but could not relate a numeral to its entire sequence of predecessors--only the two or three immediately preceding it.

Numerical operations. Kappa could construct arithmetic lots, and could integrate and separate arithmetic lots. He was not, however, a counter with abstract unit items. Also, Kappa was restricted in his application of integrating and separating arithmetic lots to situations where each lot contained at most three units. For example, 82 - 20 (as 82 under cover and two longs removed) was solvable for Kappa, as 82 was an arithmetic lot of three units (80 and 2 ones), and 20 was a lot of two units, each unit being abstracted from a long. But 47 - 21 was not understandable for Kappa at that level--he could not create 47 as an attentional pattern.

Similarly, extending and declending by ten, by one, or by ten and one was understandable at the level of arithmetic lots for Kappa only when he could subitize a criterion for ending his count. Otherwise he had to create collections.

Kappa appeared to have little intensive meaning for number-names. He would count only in situations where he already understood a problem in terms of subitized attentional patterns. When he did not have such an understanding Kappa would not count except to name collections or lots.

Concept of ten. Ten as a numeration unit existed essentially as an empirical routine for constructing collections for a number-name by counting by ten to the "ty" part of the name, and then counting by one to complete the name. "Ten" did not refer to an arithmetical unit composed of units, but rather simply to a unit or lot. However, Kappa had empirically abstracted a correspondence between the products of his empirical routine and the structure of his number-names. He could answer questions of "How many tens in . . ." for number-names preceding "one hundred." However, the tens that Kappa spoke of in answering these questions were not units of units. He did not equate, say, 67 as six tens and seven ones with a number of ones.

Figure 5.3 gives a schematic representation of Kappa's concept of ten. What it does not show well is the dual nature of his relationships between integrating and separating, extending and declending, and each with sequencing. He would operate abstractly when he could construct a problem in terms of arithmetic lots having no more than three units each, but would operate quite perceptually otherwise. Also, Kappa's number-names appeared to be much closer to being indices of counting than signs of counting, even though he would use sequencing to name arithmetical lots. Finally, Kappa's routine for sequencing by ten is listed as less than intuitive, since it was only "ten, twenty, . . ., one hundred"

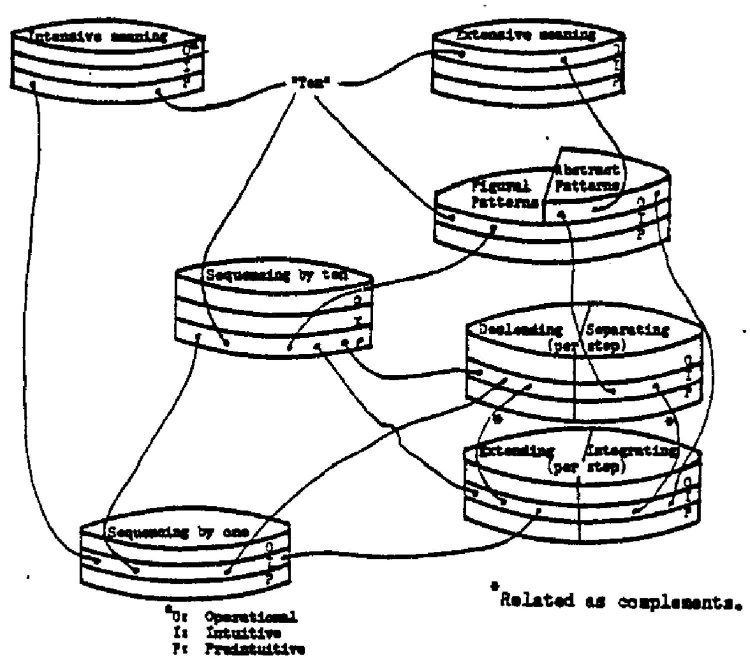


Figure 5.3. Kappa's concept of ten.

that was well established. He used this sequence as a basis for accommodating to situations requiring more complex sequencing (e.g., going backward by ten from "ninety-seven").

Concept of one hundred. Kappa did not have a routine for sequencing by one hundred, and counting by one hundred was problematic for him. "Hundred" was a name for an MAB flat, as well as a label that could be applied to an arithmetical unit or associated with an arithmetical lot. His concept of one hundred was only figurally related to his concept of ten, in that MAB flats had ten "bundles" of ten (i.e., ten longs).

Concept of place value. Kappa's concept of place value consisted of an empirical routine that he could intuitively apply for constructing named collections: count the hundreds, then the tens, then the ones. He had also empirically abstracted a correspondence between parts of a number-name and his "place-value" routine for constructing names. However, the value of a digit-name was determined by its label, and not by its position within a hierarchy of implied units. "Two hundred thirty-five" referred to two hundreds because "two" was labelled by "hundred," not because of an operational relationship among numeration units.

Summary 5.4: Rho

At the beginning of the 1977 school year Rho solved simple addition problems (5 + 3 = __, 7 + __ = 10, 5 + 4 = __) by use of figural patterns and subitizing. Rho's concept of ten was limited to sequencing "ten, twenty, . . ., ninety."

Writing and reading numerals. Rho had no difficulty reading and writing numerals. She only occasionally missed a numeral, and normally caught her error. Rho made no errors in writing numerals.

Sequencing. Sequencing by ten posed no difficulty, but sequencing by one hundred was problematic for Rho when she had to begin from a digit-name, "ty"-name, or combination of the two. Once beyond the first increment, however, Rho would continue the sequence by using her routine for sequencing by one. Rho could form an increment by one hundred from a "ty" number-name, but only by reconstituting the problem as one of addition. She would then switch the order of the addends and count-on from 100.

Rho's linguistic system of number-names was close to being operational. She had established the transitivity of "after," but not for "before." Nor had Rho operationally related before and after as being reciprocal relations within her linguistic system.

Numerical operations. Integrating and separating had been established by Rho as numerical operations that were related as inverse to one another. Extending and declending were yet at an intuitive level, for she used them primarily to construct named amalgams. Rho apparently did not use them in conceptualizing an arithmetical problem.

Concept of ten. Except for her linguistic system and the intuitive nature of extending and declending, Rho's concept of ten was fairly complete.

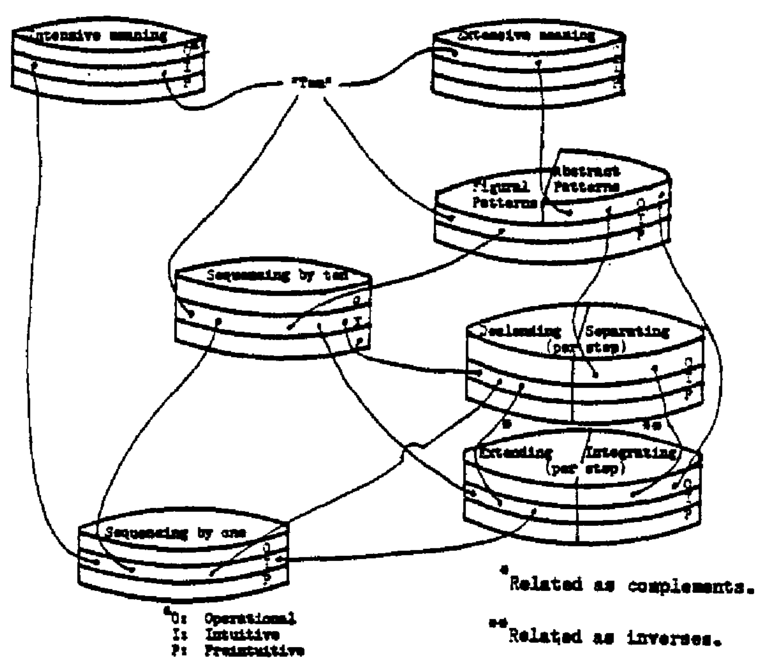


Figure 5.4. Rho's concept of ten.

An increment by ten had the significance for Rho of ten increments by one. She could also construct numbers, so "ten" had abstract extensive meaning for an increment of ten.

Ten constituted a numeration unit for Rho. She could reconstitute numbers as constructed by a number of tens and a number of ones, where each unit of ten could be composed of ones.

Concept of one hundred. Rho's routine for sequencing by one hundred was incomplete, but her concept of one hundred was quite elaborate. She understood one hundred as a number, and one that could be composed of a number of tens. She also understood one hundred as a numeration unit. This is shown in Figure 5.5.

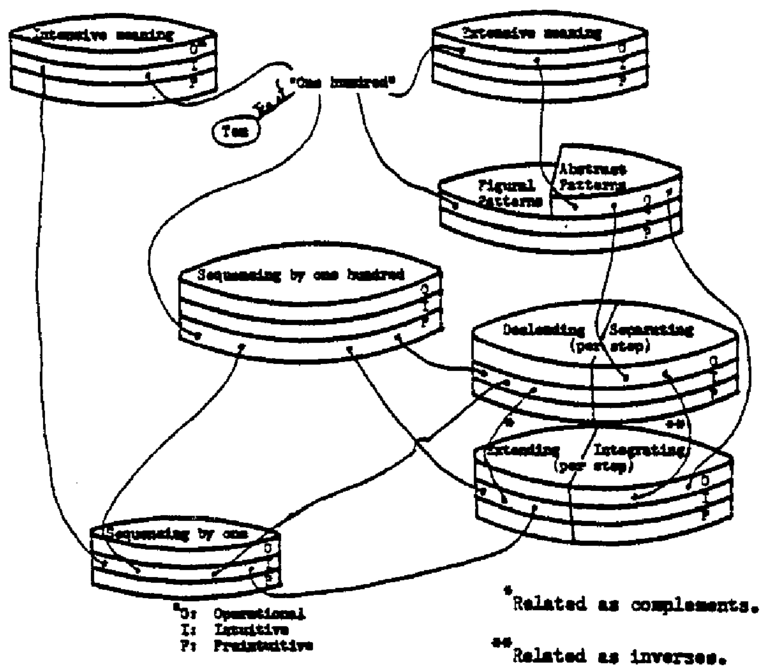


Figure 5.5. Rho's concept of one hundred.

Rho's routine for sequencing by one hundred is placed at the pre-intuitive level because of her heuristic use of sequencing by one to produce sequencing by one hundred. Also, no connection is inferred between sequencing by one hundred and subitizing largely because none of the problems that she solved demanded it.

Concept of place value. Rho had related 100 and ten, and ten and one, by the relation "ten of," but could not compose the relations so that 100 was ten of (ten of one). She also showed that she had an empirical routine (at the level of intuition) for place value in that she could construct a named collection by first counting the hundreds in it, then the tens, and then the ones. However, she did not need to count them in that order; she could flexibly move from one counting routine to another to accommodate to a change in unit.

Summary 5.5: Gamma

At the beginning of the 1977 school year Gamma could solve simple addition problems by counting-on or counting-all (5 + 3 = __, 5 + 4 = __), but could not solve missing addend problems (7 + __ = 10). She had no routine for sequencing by ten. Gamma could answer questions of "How many tens in . . ." but could not use tens to construct a named collection.

Reading and writing numerals. The only errors Gamma made were occasional misreadings of numerals, sometimes reversing the digit names and sometimes mispartitioning the numeral. She had no difficulty writing numerals.

Sequencing. Gamma could sequence by one, but her routine for sequencing by ten was based upon her sequence "ten, twenty, . . ., ninety." When sequencing by ten from a name with a "ones" digit-name, she would increment or decrement her basic decadal sequence and then append the "ones" digit-name. Because of this, Gamma found it very difficult to transcend centuries. Also, she found it difficult to transcend centuries when sequencing by one (except from the first to second century). Thus Gamma's sequencing routines were less than intuitive, since she was bound to "one, two, . . ., ninety-nine, one hundred" when sequencing by one and to "ten, twenty, . . ., one hundred" when sequencing by ten. Gamma did not have a routine for sequencing by one hundred.

When seriating numerals, Gamma assimilated the tasks to her routine for sequencing by one. When constructing an ascending sequence she placed each of the numerals within a decade, and then went on to the next decade. She similarly constructed descending sequences, but she had not reciprocally related "before" and "after," nor were these relations transitive for her. Thus Gamma's linguistic system was somewhat intuitive, but only for number-names preceding "one hundred."

Numerical operations. Gamma could integrate and separate arithmetical lots and numbers, but she had not related integrating and separating as inverse numerical operations. That is, integrating and separating were numerical operations for Gamma, but their relationship was only intuitive. They were complementary operations.

Extending and declending were yet empirical routines for constructing named amalgams. Gamma could use them as a means to an end, but apparently did not conceptualize problems in terms of them. Although it was not determined what relationship Gamma had established between extending and declending, it will be inferred that they were intuitively related, since each was a firmly established routine that she could flexibly call upon.

Concept of ten. Even though Gamma could correctly answer "How many tens in . . .," she did not understand ten as a numeration unit. Rather, she had abstracted a linguistic criterion for answering such questions.

The significance for Gamma of an increment when sequencing by ten was that of a cycle of ten increments by one. However, Gamma did not anticipate the number of cycles she would create when counting by ten to a number. Figure 5.6 gives a schematic representation of Gamma's concept of ten.

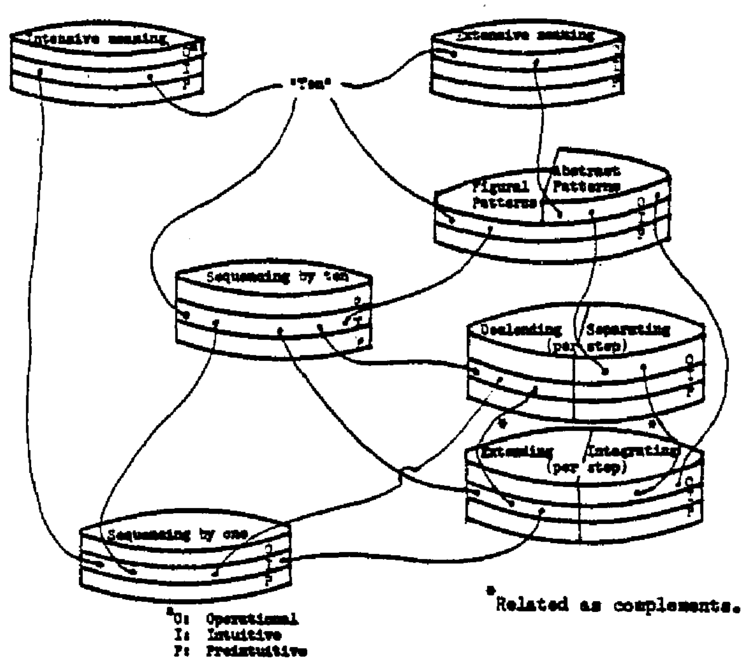


Figure 5.6. Gamma's concept of ten.

Concepts of one hundred and place value. Gamma did not have a concept of one hundred beyond "hundred" as the name of an MAB flat and as a label that occurred in number-names. In one instance she said that there were "one hundred" little cubes in a flat, but after later counting them said that there were "sixty-four"--reiterating that there were ten longs in a flat, ten cubes in a long, and 64 cubes in a flat.

Summary 5.6: Sigma

In November of the 1977 school year Sigma could solve (not always correctly) simple addition problems by counting figural patterns and by subitizing a count. His concept of ten was limited to sequencing "ten, twenty, . . ., one hundred."

Writing and reading numerals. Writing numerals posed little difficulty for Sigma. Reading numerals was more problematic for him. He would make false starts when reading three-digit numerals, but eventually say the correct name.

Sequencing. Sequencing by ten was a well established routine for Sigma. He had no difficulty transcending centuries, nor with beginning the sequence with a name other than "ten." Sigma did not have a routine for sequencing by one hundred except as an empirical abstraction of "one hundred, two hundred," and so on.

Sigma's linguistic system for constructing number-names and sequences of number-names was highly intuitive. He adapted it with remarkable success in the numeral seriating tasks. But it was not operational. He had not operationally related "before" and "after" as operationally reciprocal to one another, nor was either operationally transitive.

Numerical operations. Sigma was a counter with abstract unit items, and could create numbers. He had established integrating and separating as numerical operations, but had yet to operationally relate them as inverse to one another. They were yet intuitively related as complementary operations.

Extending and declending were empirical routines for constructing named amalgams. Sigma had not reconstituted them as numerical operations--he did not conceptualize problems in terms of them. Rather, he used extending and declending as means to ends.

Sigma had related the linguistic concatenation of number-names as the linguistic correspondent of integrating. Concatenating "forty" and "six" meant combining two numbers--one named "forty" and the other named "six."

Concept of ten. Sigma gave an increment when sequencing by ten the significance of a cycle of ten increments by one. However, he had not abstracted the base-ten structure of number-names so that parts of the name referred to numbers of numeration units. Sigma knew that one could answer "How many tens in . . ." by focussing on a part of the number-name, but he did not know what part.

Ten was not a numeration unit for Sigma. Rather, it was a unit which was conceptually indistinguishable from any other. He did not label his units when attempting to coordinate ten and one. Figure 5.7 gives a schematic representation of Sigma's concept of ten.

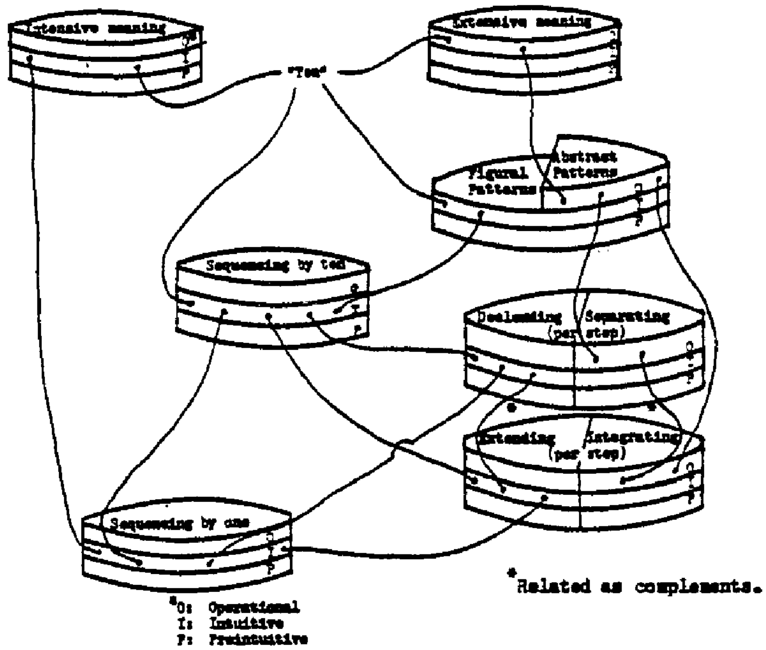


Figure 5.7. Sigma's concept of ten.

<u>Concepts of one hundred and place value</u>. Sigma's concept of one hundred amounted to "hundred" as a number and as a name for an MAB flat, and (possibly) sequencing by one hundred as "<u>one</u> hundred, <u>two</u> hundred," and so on. Sigma's significance of an increment when sequencing by one hundred was that of <u>two</u> increments by ten.

Sigma had no concept of place value. He did not possess an empirical routine for constructing named amalgams (e.g., counting hundreds, then tens, then ones), and had not related one hundred and ten nor ten and one by "ten of" beyond the literal composition of base-ten blocks.

<u>Summary 5.7: Alpha</u>

At the beginning of the 1977 school year Alpha could solve simple addition problems by counting-all and counting-on (5 + 3 = __, 7 + ___ = 10, 7 + ___ = 12, 5 + 4 = ___). Alpha sequenced by ten by saying "ten, twenty, . . ., eighty, twenty," and counted ten bundles of ten by "ten, . . . ninety, twenty." He could partition collections into tens and ones, but would recount the collections by one to determine the total.

Writing and reading numerals. Both writing and reading numerals were problematic for Alpha. When writing three-digit numerals he would systematically write "0" as he said "hundred." When writing a numeral for a "teen" name he would first write the numeral for the digit-name and then decide which side of the digit he would place the "1."

Reversing the digit-names was Alpha's most frequent reading error. This occurred only when he was in the context of solving a problem.

Sequencing. Alpha could sequence by ten and one hundred with no difficulty. He knew that "thousand" was the next label to appear after "hundred," but had not routinized sequencing by one hundred in the thousands.

His performance on the numeral seriating tasks showed that Alpha's linguistic system for constructing number-names and sequences of number-names was operational (as far as it had been established). Sigma's relations of "before" and "after" were each operationally transitive, and he had related the two as operationally reciprocal to one another.

Numerical operations. Extending, declending, integrating, and separating had each been constituted as numerical operations. Alpha could conceptualize problems in terms of extending and declending, but he had not operationally related them as inverse to one another. They were yet only intuitively related as complements. Alpha had, however, operationally related integrating and separating as inverses.

Concept of ten. Figure 5.8 gives a schematic representation of Alpha's concept of ten. Aside from his intuitive relationship between extending and declending, Alpha's concept of ten was complete.

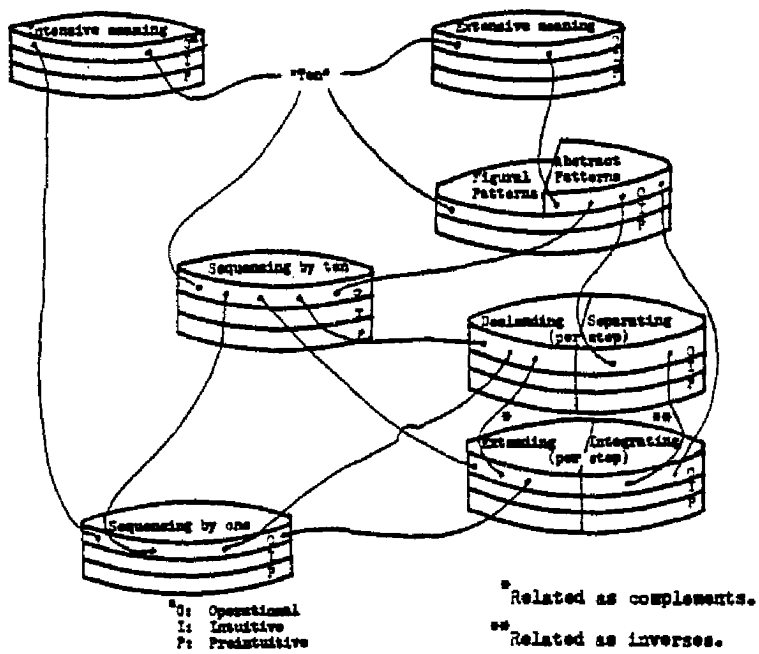


Figure 5.8. Alpha's concept of ten.

Alpha had constructed ten as a numeration unit, and had abstracted the linguistic base-ten structure of number-names. Moreover, he could symbolically operate on numbers via his linguistic system for constructing number-names and sequences of number-names.

Concept of one hundred. Alpha had constructed one hundred as a numeration unit, and had operationally related it to his concept of ten. One hundred was ten tens, so two hundred was 20 tens, and so on. Figure 5.9 gives a schematic representation of Alpha's concept of one hundred.

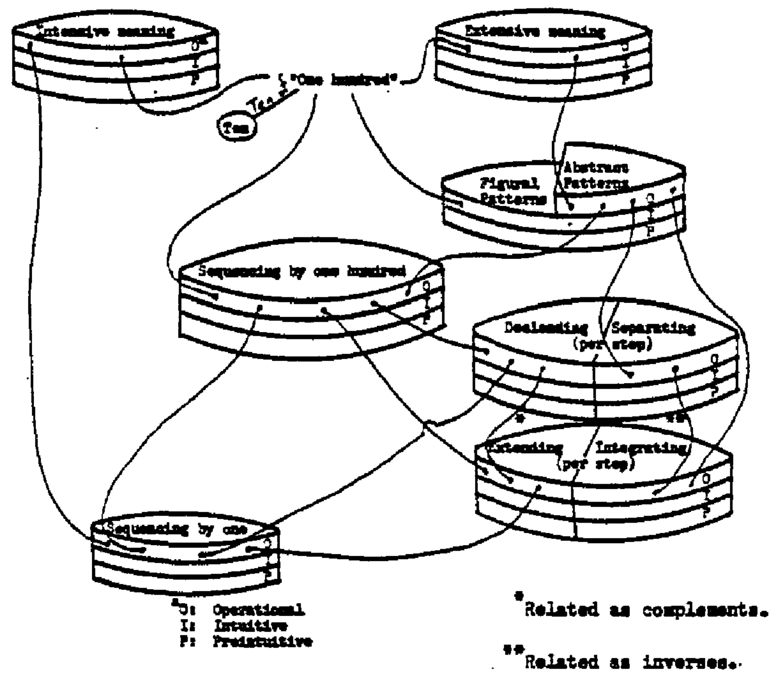


Figure 5.9. Alpha's concept of one hundred.

<u>Concept of place value</u>. Alpha had not only related one hundred and ten, and ten and one, as numeration units by "ten of," he could <u>compose</u> the relations. One hundred was ten of (ten of one). He could conceive of numeration units as themselves composed of a hierarchy of numeration units.

<u>Summary 5.8: Mu</u>

At the beginning of the 1977 school year Mu could solve simple addition problems (5 + 3 = __, 7 + __ = 10, 5 + 4 = __) by counting-all and counting-on using his fingers. Mu had an intuitive aspect of concept of ten even then. He said there were three tens in 32, counted-on from 20 (two bundles of ten) to find a total of 25 sticks, and counted ten bundles by ten to find a total of 100 sticks. However, he made 54 with five bundles and four single sticks, but thought that there were four tens in 54.

Writing and reading numerals. Reading numerals was more problematic for Mu than was writing them. He would frequently hesitate when reading two- and three-digit numerals, and would occasionally reverse the digit-names for a two-digit numeral.

Sequencing. Mu's sequencing routines were all well-developed. He could sequence by ten, without limit, and could sequence by one hundred well into the thousands.

Mu's linguistic system for constructing number-names and sequences of number-names was fully operational. His relations of "before" and "after" were each operationally transitive and were operationally related as reciprocal to one another. Moreover, when Mu put numerals into order he created number sequences, at times taking note of how many numerals were missing between terms of a sequence.

Numerical operations. Extending, declending, integrating, and separating had each been established by Mu as numerical operations. Moreover, he had operationally related extending and declending, and integrating and separating, as inverse of one another.

Concept of ten. Mu's concept of ten was complete, as shown in Figure 5.10. Ten was a numeration unit for him. He could conceptualize a number as simultaneously being composed as a number of ones and a number of tens combined with a number of ones.

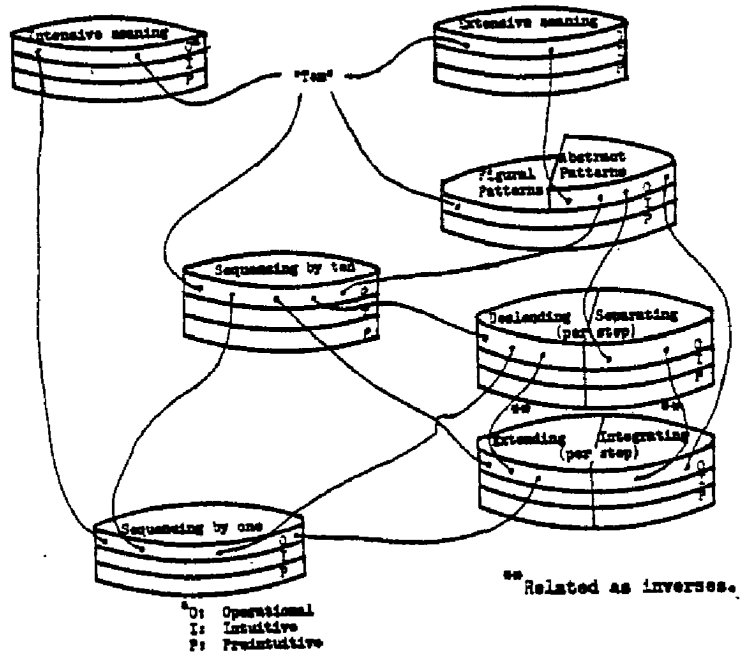


Figure 5.10. Mu's concept of ten.

<u>Concept of one hundred</u>. Mu's concept of one hundred was also complete, as shown in Figure 5.11. One hundred was a numeration unit for him. Mu could conceptualize a number as both being a number of ones and as a number formed from a number of hundreds. He had also operationally related one hundred and ten by "ten of." One hundred is ten tens, so two hundred forty is 20 tens plus 4 tens, or 24 tens.

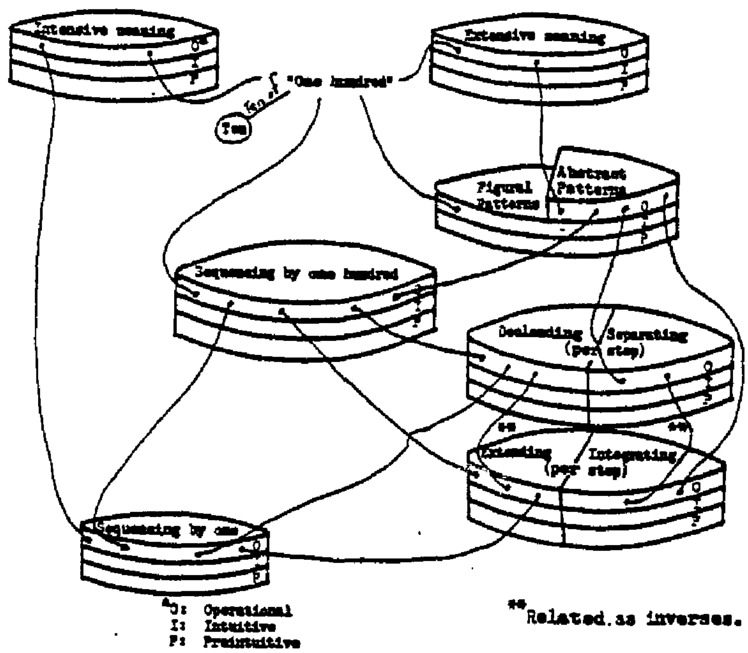


Figure 5.11. Mu's concept of one hundred.

Concept of place value. One hundred and ten, and ten and one, had each been related by Mu by "ten of." Moreover, Mu could compose the relations so that one hundred is ten of (ten of one).

Cross-Sectional Analysis

The summaries of the previous sections now provide a basis for looking across the case studies with an eye toward the development of numeration concepts. The remarks in this section will be divided into two parts: general observations about the development of relationships among components of the respective numeration concepts, and statements about the development of the components themselves.

General Observations

One thing stands out among the concepts of ten and one hundred attributed to these eight children: it is that the development of operations involving counting lags behind operations on amalgams. Only in the case of Mu did we see operational reversibility between both extending and declending and integrating and separating. Alpha had operationally related integrating and separating, and had established extending and declending as numerical operations but had not operationally related them. Rho had operationally related integrating and separating, but had not established extending and declending at the level of numerical operations. Gamma had established integrating and separating as numerical operations, but had not operationally related them, while extending and declending were still empirical routines, and similarly with Sigma. Kappa could create arithmetic lots, but could operate on them only when they were within his subitizing range, while extending and declending were employed almost exclusively to name collections. No child was further advanced in extending and declending than he or she was in integrating and separating. Similarly, no child was more advanced in intensive meaning for number-names than in extensive meaning, and they were frequently less advanced.

Why might this be so? Possibly for two reasons. First, children naturally focus upon the objects of perception and less on their actions upon them. This may be so because of the necessity of actively and purposely refocussing thought upon transformations (which cannot be perceived), brought about by actions.

Also, counting is a much more complex scheme than is the construction of collections. So reflectively abstracting the numerical structure of a collection involves less, and less "purposive," reprocessing than does reflectively abstracting the numerical structure of a counting episode. Second, it may be that intensive meaning for number-names at the operational level becomes a variant of extensive meaning--in the sense that the child may construct a number, but associate a direction to the construction. The direction may be that of left-right, right-left, forward-backward, etc. (again at an operational level).

Another observation that may be made is that a child's ability to correctly read and write numerals, and to answer questions such as "How many tens in . . ." had little relationship to his or her knowledge of numeration units. The children who had advanced concepts in numeration could answer "How many tens in . . .," but so could most of the others. The relationship was necessary, but not sufficient. The relationship between reading numerals and concepts of numeration was neither necessary nor sufficient. One could not make a prediction of the state of a child's numeration concepts by his or her ability to read numerals, or vice versa.

Finally, we did not see true flexibility in the children's thinking with regard to numeration till the case studies of Alpha and Mu, wherein we also, for the first time, saw operational linguistic systems. Moreover, Rho's system was close to being operational, and she showed signs of flexibility in her thinking. This supports the contention expressed in Chapter 3 that it is not until the child establishes his or her linguistic system for constructing number-names and sequences of number-names at the level of mental operations that he or she may operate symbolically upon numbers. It involves much less processing to manipulate symbols than it does to manipulate their meanings. (However, the meanings must exist, for otherwise there are no constraints on the manipulations other than empirical ones.) A child's operations upon numbers may then be reflected in operations within his or her linguistic system, with a subsequent gain in flexibility.

Development of Components

This section will focus upon the development of sequencing, intensive and extensive meaning, and numerical operations.

Sequencing. The qualities of the errors that the children made in sequencing by ten are suggestive of a pattern of development. Delta, Lambda, Kappa, and Gamma each made errors such as ". . ., eighty-eight, ninety-eight, one hundred eight, two hundred eight, ..." That is, they would focus upon the first-said part of a number-name as the significant part in sequencing. Similarly, when sequencing by one hundred, Alpha sequenced ". . ., nine hundred thirty, ten hundred thirty," and when asked to continue, went "three thousand, four thousand, . . ., ten thousand." It seems that in extending their sequencing routines, they must first routinize the "left-most side" of the newly added routine and then assimilate their existing routines to it. Thus, in sequencing by one they routinize "ten, twenty, . . ., one hundred," and then coordinate it with "one, two, . . ., nine." To continue into the hundreds they routinize "one hundred, two hundred, . . ., nine hundred" and then coordinate it with their existing routine (which itself is a coordination) for "one, two, . . ., ninety-nine." The assimilation process perhaps is one of forming relationships out of the acts of coordinating what at first are separate routines. A relationship likely comes by differentiating an intuitive coordination of routines (by empirical abstraction) and then extending the routine by the addition of a rule. For example, a child might learn names in order up to "twenty," then "thirty," and then possibly "forty" before noticing that there is a pattern in the way one makes the next name out of "(blank)ty-nine," namely "(next of blank)ty." The differentiation is of the pattern in the first-said part of the name from that of the second. The assimilation of "one, . . ., nine" to "ten, . . ., ninety" is then of the form of adding a rule for transcending decades. This process likely reoccurs in extending the routine to sequencing in the hundreds.

The construction of a routine for sequencing by ten requires that the child abstract similarities and differences of names connected by "ten more" and "ten less," and these are quantitative criteria. So in constructing a routine for sequencing by ten we see that the child has to go beyond sequencing _per se_ and relate sequencing to quantity--the child has to count. The child's empirical abstraction of the _rules_ for sequencing by ten is based on the number-names _per se_. But to first get two number-names with the proper relationship so that a comparison may take place, the child must count ten from a number-name. Thus we see a necessary link with numerical operations.

The construction of a routine for sequencing by one hundred likely follows the pattern of the construction of sequencing by ten. However, it is possible that the child reflectively abstracts the operation of acquiring a next name so that the detailed empirical abstractions are obviated. For example, the child may realize that "it's just like 'ten, twenty, thirty'—only it's 'one hundred, two hundred, three hundred,'" and relate the "next" operation for hundred to his or her already established routines.

Intensive and extensive meaning. This issue was indirectly addressed in the general observations. The development of intensive meaning for number-names appears to lag behind that of extensive meaning. More specific statements about their development can, however, be made.

Von Glasersfeld's model remains a viable explanation of the development of extensive meaning. Extensive meaning apparently develops as a result of the child's being human--he or she reflectively abstracts numerosity from interactions with his or her physical environment. Intensive meaning appears to be more influenced by instruction (or the lack of it). If a child does not count, or is prohibited from counting, or does not reflect on his or her counting, then it is more likely than not that counting will remain at the level of an empirical routine for naming collections or lots. Given the diffuse nature of the activity of counting *per se*, it requires prolonged reflection on the part of the child to internalize it as a numerical meaning for number-names. What the motor might be for bringing about the child's reflection upon counting is not clear.

The development of intensive meaning appears to parallel that of extensive meaning, except that the <u>context</u> of a figural representation of counting must differ from that of a collection of perceptual items, since the "object" being represented was never really "there." Perhaps it is a representation of the rhythmic actions involved in counting. Once the child is able to form representations of counting as such, it is possible that he or she then reflectively abstracts the unit structure of the items, and so on. If this is the case, then operational extensive and intensive meaning for number-names would differ only in their genesis, but would be structurally identical. This is not to say that they would be <u>semantically</u> identical, for the child would associate them with different episodic representations.

<u>Numerical operations</u>. The pattern of development of numerical operations seems to be "elevate, relate; elevate, relate." As the meanings of number-names (operands of the operations) develop, the child must elevate the operations and then relate them. The clearest examples of this came in the case studies of Gamma, Sigma, and Alpha. Gamma and Sigma had each established integrating and separating as numerical operations, but the operations were yet only intuitively related as complements to one another. Alpha had established extending and declending as numerical operations, but had also maintained only an intuitive relationship.

The examples of Gamma, Sigma, and Alpha raise an interesting question. We have postulated a mechanism for the establishment of extending, declending, integrating and separating at the level of numerical operations, namely reflective abstraction. But how may one establish an operational relationship? An answer to this is critical for taking the theoretical framework used in this study as a basis for instruction on numeration. Without an answer we have no rationale for choosing between one instructional approach and another when the objective is that the children relate numerical operations. Perhaps operational relationships are established only when the operations become intuitive at the next level of thought. For instance, it could be that extending and declending are operationally related when they can intuitively be related as formal operations. This would mean that instruction aimed at establishing relationships among extending and declending would amount to an intuitive treatment of addition and subtraction of integers. Instruction aimed at establishing integrating and separating as inverse numerical operations would amount to an intuitive treatment of multiplication and division of whole numbers. Instruction aimed at the establishment of operational relationships among numeration units would amount to an intuitive treatment of composition of functions. That is to say, the child would need not only to routinize the operations at his current level of thinking, he would have to be challenged with problems that would take him beyond it.

Chapter 6

CONCLUSION

Four issues will be addressed in this chapter. They are: the viability of the theoretical framework, the shortcomings of the study, the "next step" alluded to in Chapter 2, and pedagogical implications of the study.

Viability

The viability of a theoretical framework may be assessed in two different, but complementary ways (Thompson, in press). One way is to ask if it sufficiently accounts for observed behavior, where "sufficiency" is relative to the level of detail at which the framework is specified. Another is to examine the constructs of the framework in relation to those of others.

Sufficiency

The theoretical framework of this study (Chapter 3) can be judged sufficient on two grounds. First, it was able to explain discrepancies, some of them quite dramatic, within individual children's behavior across a variety of tasks. The most outstanding instance of this can be seen in Kappa's case study (pages 155-183). Kappa could count by ten, yet had difficulty sequencing by ten; he could coordinate his sequencing routines when counting base-ten blocks of varying size, yet he apparently had little concept of place value; he could create abstract unit items labelled "ten," yet could not operate numerically on them beyond his subitizing range. Each of these instances was seemingly paradoxical, but the framework was able to account for them in terms of his meanings for number-names and in terms of the nature of his routines and the quality of the relationships that he had established among them.

Second, the framework was able to account for differences among a group of children who varied widely in the quality of their behaviors. The children ranged from Delta, a counter with perceptual unit items whose routines were entirely empirical and data driven, to Mu, who was largely beyond counting, having created the beginning of the additive group of integers. The framework, in each case, was able to specify a task environment which would manifest itself in behavior similar in kind to that which the child actually displayed.

Related Frameworks

The theoretical framework presented in this study is viable on two additional grounds. First, it provides a natural link with future frameworks for multiplication and division of whole numbers and integers. The conceptual foundation of multiplication of whole numbers is quite like the Biblical "multitudes"--creating many from one. That is to say, multiplication of whole numbers is the systematic creation of units of units. At an intuitive level this is repeated addition; at an operational level it could be either a transformation brought about by composing, extending with itself a number of times (e.g., 5 x +7 = the transformation brought about by 5 applications of extending 7), or, equivalently, a hierarchy of integrations (e.g., 5 x 7 = 5-units each of 7 units). Similarly, division of whole numbers may be conceptualized intuitively as repeated subtraction or successive partitioning. Operationally, it could be either a transformation brought about by repeatedly composing declending with itself a number of times, or equivalently, as a hierarchy of separations--decomposing a number into a number of subunits, each of which is a number. When multiplication and division, and addition and subtraction, have been reflected in the child's thinking as formal operations, then he or she has the essential ingredients of the ring of integers (cf. the discussion of extending and declending as germs of the additive group of integers in Chapter 3). The child's task then is to establish formal relationships among them.

Second, the framework presented in this study fits well with studies by Stake (1980) and Lawler (1981). Stake (1980) clinically investigated several third-grader's understandings of arithmetic from the perspective of a schema theory of cognition (Rumelhart & Ortony, 1977; not cited by Stake). Although the focus of Stake's investigation was not whole number numeration, it addressed several aspects in common with this study--place value, skip counting, and counting. The results of her investigations of skip counting are especially significant for the framework of this investigation. Stake found that her children were quite satisfied with counting a collection of 25 discs by one, saying there were 25, and counting the same collection by five, saying that there were 125. She characterized this as arising from the existence of two separate schemas only loosely related: one-to-one correspondence in counting by one, two, and five, and quantity. One-to-one correspondence in counting, according to

Stake, is the coordination of an object of a collection with one and only one term in a sequence of number-names. The quantity schema is the association of the last number-name of a counting sequence with the collection of objects so counted.

In the language of the framework of this investigation, Stake's children merely sequenced by five, where each linguistic increment had the significance of being associated with a perceptual unit item. The children did not attribute intensive meaning to an increment either as a cycle or as an operational extension of a number.

The similarity between this and Stake's study with respect to place value is difficult to assess. Stake characterized the essence of a place value schema as being the "recognition of the 1s, 10s, 100s, and 1000s in numerals" (p. 130), whereas place value in this study was characterized as a structure emanating from the recursive composition of the relation "ten of" with itself.

Lawler (1981) carried out a six-month study of his daughter (Miriam) as she came to grips with place value in addition. He characterized Miriam's advancements in terms of the development of "microworlds," or cognitive structures that emerge from experiences similar enough in kind to warrant focused mental activity. The most primitive of Lawler's microworlds was the Count structure; emerging epigenetically from Count were the Money and Decadal microworlds, wherein Miriam began to structure her counting activities and to draw upon similarities of the decadal number-names and "one, two, ... , nine" in doing mental arithmetic. Money and Decadal were integrated through the emergence of Serial, whose primary function was to mediate between the two as a control structure.

Miriam's microworld for processing written addition, named Paper Sums, operated independently of Money and Decadal. It became integrated with them when she created a microworld, named Conformal, whose primary function was to relate Serial, and hence Money and Decadal, with Paper Sums.

Two aspects of Lawler's study are in strong agreement with the framework presented here. The first is that a child's concepts of numeration comprise a structure of interdependent domains of knowledge. In both Lawler's and this study, the linguistic domain appears to be crucial for numeration. (However, the two differ in the significance attributed to numerical operations--Lawler does not address them.) Second, the motor of development in Lawler's characterization is the "elevation of control." That is, the emergence of microworlds whose primary function is to serve as control mechanisms, mediating among two or more ancestral microworlds.

The similarity between "elevation" and "reflection" should be apparent. The reflection of a current state of affairs of an ancestral microworld in its control structure amounts to a representation of it (MacKay, 1954/1969). Likewise, a change in the state of a control structure can be characterized as an internalized action--a mental operation. Thus Serial's mediation as a control structure appears to be analogous to reflectively abstracting integrating to the level of numerical operations.

Shortcomings

Many interpretations in the case studies were based upon fortuitous episodes occurring in the context of tasks aimed at uncovering a different aspect of the child's thinking. This was especially true of interpretations of the relationships that the children had established between the aspects of their concepts of numeration. It is quite possible, for instance, that Mu (Case Study 4.8, pages 268-279) had not completely established extending and declending as inverses of each other, as he only compensated +1 and -1 (Excerpt 4.8.4, page 274). Another shortcoming was that none of the tasks were explicitly designed to assess whether or not a child had actually established extending and declending at the level of numerical operations. Questions that might have proved helpful in these regards are given below.

Relationship between integrating and separating:

1.a. Put out 70 in MAB longs; have the child count them. Ask "What is left when we take thirty-one away?"

b. After the child has completed part (a), ask "How many are here ('take away' pile)? How many are here ('difference' pile)? How many all together?"

c. Place two unit cubes next to the "take away" pile; cover the "difference pile." Ask "How many are we taking away now? How many are there all together? How many are left?"

2. Put out a pile of 40 sticks or unit cubes. Say "There are forty sticks here. I'm going to take these away (separate into piles; make the spatial separation quite large). Now I'm going to put these sticks with this pile (place 6 sticks next to "take away" pile). How many sticks are there <u>all</u> together now?"

3.a. Put out two piles of MAB blocks (10 and 20). Say "There are ten here (indicate) and twenty here (indicate)." Push the two together. Ask "How many in all?"

b. Place 3 unit cubes alongside "sum" pile. Say "I'm going to put these ten back where they were (do so). How many are left?"

The aim of each of these problems is to assess whether or not the child sees a necessary implication for the initial number when a change is made in one of the lots in its separation, and for the numbers in an initial separation when a change is made in their integration. Of course, it is assumed that it has already been established that the child can create at least arithmetical lots.

Relationships between extending and declending:

4. Place 2 longs and 2 unit cubes under a cover (without letting the child see them). Place 6 longs and 8 unit cubes alongside the cover. Say, "There are some blocks under here (indicate) and these (indicate). There are ninety in all (indicate). I'm going to count backward from ninety to find how many are under here." Do so, but quickly, and make a mistake so that you end with 20. Say, "Let's see if I'm right (lift cover). Oh! I made a mistake! There are twenty-two! What did I do wrong?"

5. Say, "Suzy went to the store to buy some candy. She took the candy bar to the check stand; it cost twenty cents. She gave the cashier a quarter, and got her change. Outside the store she counted her money, and she had five cents more than she started with. How can that be? How much money did the cashier give her in change?"[1]

[1]This task is similar to several devised by Vergnaud (1981) in his investigations of relational calculi.

The aim of these tasks is to see if the child can operationally relate extending and declending. To correctly answer the question in Task 4, the child will have to reverse a surplus of two (blocks) so that it means a deficit of two (counts). To appropriately answer the question in Task 5, the child will have to realize that the deficit Suzy should have had was over-compensated by the change the cashier gave.

Another shortcoming of the study was that it did not directly assess the children's ability to determine pairwise order of number-names, nor were there any questions directed at the level at which number-names represented their meanings. Questions 6 and 7 might have been helpful in these regards.

6.a. Say, "I'm going to read you some numbers. You tell me which one comes first. (Read: 37, 62; 74, 58; 66, 57; 47, 41; 83, 88; 112, 86; 375, 509; 254, 234.)

b. Same as the numeral seriating tasks used in this study, except after the child has (successfully) put the numerals in order the interviewer gives him or her two cards to insert. Say, "Oh, I forgot to give you these two. Can you put them in their proper place?"

7. Say, "Can you say the letters of the alphabet?" Let child say them as far as he or she can. Say, "Instead of counting with 'one, two, three' we're going to count 'a, b, c.'" Can you count these using letters instead of numbers (place card with 9 squares on it)?" After the child has counted, place board with one square showing, and several others covered. Say, "There are some squares under this cover, and this one. I counted this one last and it was 'h.' How many are under the cover?"

Task 6a focusses on the child's ability to make pairwise comparisons between number-names, while 6b explicitly focusses on the child's capability of reasoning transitively. Task 7 focusses on whether or not the child has divorced number-names _per se_ from the operations and relationships in which they occur, and has realized that any word may be used in place of a number-name as long as the network of relationships remains intact.

Finally, the tasks used in this study as entries into examining a child's relationships among numeration units may have been too familiar (in type) to force a child into reasoning at an operational level. Task 8 below might have been useful as a supplement to those.

8. Say, "How many tens are there in this (MAB flat)?" After child says "ten," place an MAB cube on the table. Say, "There are ten of these (indicate flat) in this (indicate cube). How many tens (hold up MAB long) are there in this (indicate cube)?"

Question 8 aims directly at the child's ability to compose "ten of" with itself. The questions used in this study were weak in that the child could say 100 is ten of (ten of one) without actually composing the relations--"one hundred" is the name of a flat. Question 8 forces the child into a situation where he can only answer correctly by composing relations--so that ten of (ten of _x_) is 100 _x_, and 100 _x_ is _not_ the name of the physical object being used.

Finally, another shortcoming of the study is that no mention has been made of how children might construct either a general understanding of whole number numeration or of what a general understanding might "look like." The numeration units of ten and 100 each appear to be constructed anew. But surely at some point many children establish an operational routine for creating numeration units and all the associated routines, so that the need for an elaborate construction of each succeeding numeration unit is obviated.

The Next Step

The author actually sees three "next" steps following from this study. These are teaching experiments, computer implementations, and further interviewing of children. Each will be discussed below.

Teaching Experiments

In order to get a better understanding of children's construction of numeration concepts they must be observed as they grapple with the subject matter of instruction. The term "observe," however, is loaded. The author means observations in the sense described by Steffe and Richards (1981), Steffe and Cobb (1981), and Thompson (1979). Each of these authors has in mind systematic intervention, and the recording of both the child's actions and the investigators' instruction in terms of the theory or framework that is guiding the instruction. The methodology of teaching experiments of this kind is well described in Steffe and Richards (1981).

The children of this study did participate in a teaching experiment. In fact the aim at the outset of writing this dissertation was to trace the genesis of the children's numeration concepts. It quickly became apparent, though, that that was an impossible task until the author first made clear to himself what it was that was developing. That is to say, it very early became apparent that a theoretical framework was necessary before one could capture a collage of models attributed to a real child within a single, functioning ideal child. As the development of the framework progressed it also became apparent that going back to trace the genesis of the children's concepts was beyond the scope of a dissertation.

Computer Implementations

One test of the viability of a framework is whether or not it can sufficiently account for children's behavior in the content area for which it was built, where the level of specificity of the explanations corresponds with that at which the framework is expressed. Another test is whether it can survive scrutiny upon a finer analysis of its composition. The presentation in Chapter 3 was a qualitative expression of the author's thinking, sometimes moving to a formal level of description, such as in the discussion of sequencing and unit items. But there is really no way to check for internal inconsistencies in the framework when it is as complex as the one presented here and when it is expressed qualitatively. A much more rigorous test is to express the components of the framework at a level of formal description that may in turn be implemented by an impartial observer. Nothing is more impartial than a computer.

Further Interviewing

The children in this study had a great deal of experience with counting--likely much more than they would have had in a "normal" instructional setting. However, the framework presented here was not predicated upon a particular teaching method. Thus, for it to remain viable one must be able to give sufficient explanations of any child's behavior in problem-solving situations involving numeration. A critical test of the framework would be the interviewing of children whose counting had been suppressed (to the extent that it "normally" is) and whose instructional experiences were focussed more toward manipulating numerals. The author's expectation is that one would have to draw heavily upon the aspects of the framework falling under heuristic reasoning and empirical routines, and that one would find few operational relationships among components, but that overall the framework would largely remain viable.

Pedagogical Implications

The author sees two major implications of this study for the pedagogy of early school arithmetic. These are tailoring instruction to individual children and distinctions that can be made among qualities of instruction.

Tailoring Instruction

There are two issues a teacher must address when individualizing instruction for a child: where the teacher wants the child to go and where the child is with respect to that goal.

The first issue may be addressed by specifying specific behaviors that the teacher wants the student to display, or by specifying cognitive structures that would enable the child to display those and related behaviors (Greeno, 1976b). The latter approach is the more powerful. By specifying instructional objectives in terms of cognitive structures the teacher has the flexibility of "seeing" the achievement of the objectives in a host of behaviors, instead of being constrained to requiring that the child solve particular tasks in particular ways (Petrie, 1977). The theoretical framework of this study provides a vehicle for specifying cognitive objectives for instruction in the concepts of whole number numeration. The routines and structures given in Chapter 3 (numerical operations, sequencing, ten, and one hundred) could each specify a goal state that the teacher wishes the child to achieve. The teacher's assessment of the child's achievements would then be judgments of the child's level of operationality within domains of knowledge and the nature of the relationships the child had established within and among domains.

The second issue, characterizing where the child is with respect to a set of cognitive objectives in whole number numeration, was directly addressed by the framework. In a sense, this is its reason for existing. Moreover, since the framework is developmental, it provides an entry into the area of <u>prescribing</u> instruction.

Let us take Sigma as a case in point with respect to instruction aimed at helping him to develop an operational concept of ten. His strengths were that he had a highly intuitive linguistic system, that he could create numbers, and that he could integrate and separate numbers. His weaknesses were that he had not established operational intensive meaning of number-names, neither extending nor declending had been established as numerical operations, integrating and separating were only intuitively related, and he only differentiated among types of units at a figural level (cf. Figure 5.7, page 301 and Case Study 4.6, pages 229-249). Instruction for Sigma would best be focussed around problems and situations where he is forced to reflect upon his <u>activities</u> of counting, as opposed to asking him merely to do more counting. These activities should also focus upon distinctions between units created through counting (e.g., "Can you count by tens to find out how many twenties there are in one hundred?"). Similarly, his instruction should include activities which force him to reflect upon his relationship between integrating and separating (e.g., Problems 1, 2, and 3, pages 320-321). These sorts of activities would be appropriate for Sigma, since his thinking in numeration is at a level (intuition) where reflection is a reasonable aim of instruction. They would not be appropriate for, say, Delta, whose thinking was pre-intuitive.

Qualities of Instruction

Much has been made in this study of the difference between figural and operational thought. It should be clear that the author's bias is that operational thought is "good" while figural thought should only be transitory. Yet one finds the majority of elementary school instruction aimed at the establishment of figural thought. One finds elementary textbooks whose emphasis on place value amounts to the establishment of figural associations between digits and pictures. For example, a common approach to teaching the value of the tens-digit is to have the children fill out pages of worksheets where there are collections of "ten-bars" and "one-cubes" and the child is expected to match the tens-digit with the picture of the ten-bars and the ones-digit with the picture of the one-cubes. The children learn to do this quickly, and they even learn to say such as "there are four tens in 42 because the four is in front and the two is in last" (cf. the introduction to Gamma's Case Study, page 206). However, the significance of activities such as this needs be no more in principle than if the ten-bars were apples and the one-cubes pears. "There are four apples in 42 because the four is in front and the two is in last."

To establish good, sound elementary school instruction, the teacher must put figural thought in its proper perspective. It is a necessary level that the child must pass through, but it is only a stepping stone to where the teacher _really_ wants the child to arrive.

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APPENDIX I

THE INTERVIEW TASKS

Appendix I

The Interview Tasks

Post-Interview #1

<u>Ordering numerals</u>:

Warm-up:	Numeral cards: 16, 17, 18, 19, 20, 21, 22, 23; shuffled. Order least to greatest.
Task A:	Numeral cards: 20, 30, 60, 70, 90, 100, 110, 120; shuffled. Order least to greatest.
Task B:	Numeral cards: 11, 21, 31, 51, 81, 91, 101, 111; shuffled. Order least to greatest.
Task C:	Numeral cards: 8, 12, 13, 17, 19, 21, 31, 102; shuffled. Order least to greatest.
Task D:	Numeral cards: 30, 47, 48, 49, 52, 61, 67, 76; shuffled. Order greatest to least.
Task E:	Numeral cards: 97, 103, 107, 113, 117, 124, 134, 143; shuffled. Order greatest to least.

<u>Concept of ten</u>:

Task A:	HOW MANY TENS ARE THERE IN THIRTEEN? (No materials.)
Task B:	HOW MANY TENS ARE THERE IN SIXTY-SEVEN? (No materials.)
Task C:	THIS PILE OF STICKS HAS SEVENTY-TWO STICKS IN IT. IF WE WERE TO TAKE ALL THE TENS OUT OF SEVENTY-TWO, HOW MANY STICKS WOULD BE LEFT? (With materials.)

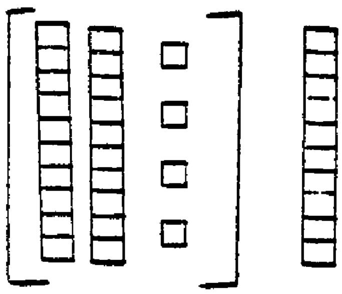
<u>Sequencing by Ten</u>

Task A:	START AT EIGHT AND COUNT-ON BY TENS. (Help S start if necessary.) GO AS FAR AS YOU CAN. (Stop S at 148.)
Task B:	START AT NINETY-SEVEN AND COUNT-BACK BY TENS. (Help if necessary.)
Task C:	WHAT IS TWO TENS AND NINETY MORE? HOW DID YOU KNOW THAT?
Task D:	(Place "50" numeral card on the table.) WHAT NUMBER IS THREE TENS MORE THAN THIS NUMBER? HOW DID YOU GET THAT?

Counting-on and Counting-back by Tens and Ones:

Task Summary (Counting-on):

Start with 24 blocks under screen. Let S verify that there are 24. Place tens and ones as indicated in figure below. Have S compute and record total number of blocks up to that point. Slide screen to cover all blocks. Proceed as indicated.



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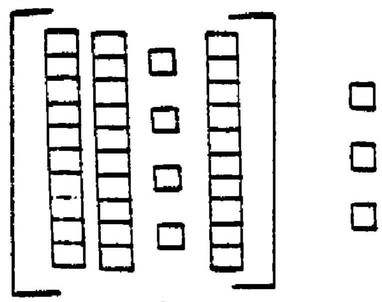

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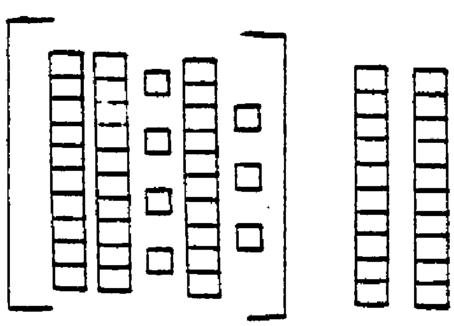

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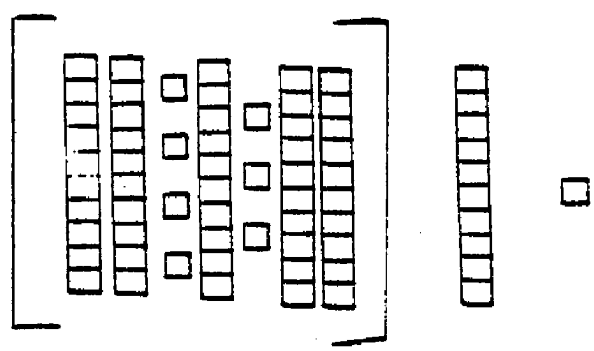


4.

Task Summary (Counting-back):

Remove blocks as indicated below. Have S compute and record the number of blocks left under the screen after each set of blocks has been removed.

(1) 2 tens (2) 1 ten & 1 unit (3) 1 ten (4) 6 units

(5) 2 tens & 1 unit

Finding Remainders:

Display numeral cards "56" and "66".

* HOW FAR IS IT FROM FIFTY-SIX TO SIXTY-SIX?

* HOW MANY MORE IS SIXTY-SIX THAN FIFTY-SIX?

If S "comfortably" answers previous question, ask:

HOW MANY IS IT FROM FIFTY-SIX TO SEVENTY?

HOW MANY IS IT FROM SIXTY-SIX TO FIFTY-FIVE?

Transformation of units:

Place twenty (20) MAB longs; have S count by tens as they are placed.

* HOW MANY HUNDREDS ARE HERE?

Screen the 20 longs from child's view; place 4 longs adjacent to the screen.

* NOW, HOW MANY TENS ARE THERE ALL TOGETHER?

Post-Interview #2

Addition and Subtraction Sentences -- tens and ones

Place sentence card in front of S; allow S to use materials if necessary.

(1) 10 + 7 = ___	(2) 10 + 3 = ___	(3) 40 + __ = 46
(4) __ + 20 = 25	(5) __ + 9 = 79	

(6) FIND TWO NUMBERS THAT ADD UP TO FIFTEEN.

If S has difficulty, say: LIKE THIS--14 + 1.

NOW, FIND AS MANY MORE NUMBERS AS YOU CAN THAT ADD UP TO FIFTEEN.

(7) 60 - 20 = ___	(8) 70 - 31 = ___	(9) 84 - 30 = ___
(10) 47 - 21 = ___	(11) 74 - __ = 70	(12) 91 - 29 = ___

Reversibility problems (presented with numeral cards)

HOW FAR IS IT FROM SEVENTY TO NINETY-TWO?

HOW FAR IS IT FROM NINETY-TWO TO SEVENTY?

HOW FAR IS IT FROM SIXTY-FOUR DOWN TO FORTY?

HOW FAR IS IT FROM FORTY UP TO SIXTY-FOUR?

No numeral cards.

HOW FAR IS IT FROM THIRTY-SIX UP TO FIFTY?

HOW FAR IS IT FROM FIFTY DOWN TO THIRTY-SIX?

Post-Interview #3

Concept of One-Hundred

1. Place "20" numeral card.
   WHAT NUMBER IS ONE-HUNDRED MORE THAN THIS NUMBER?

2. Place 235 unit cubes in front of S.
   HERE ARE TWO-HUNDRED THIRTY-FIVE LITTLE BLOCKS. USING THESE LITTLE BLOCKS, HOW MANY PILES OF ONE-HUNDRED COULD YOU MAKE?

3. Place 16 MAB longs in front of S.
   HOW MANY LITTLE BLOCKS ARE HERE? HOW MANY HUNDREDS ARE THERE? HOW DO YOU KNOW THAT?

4. Place 20 MAB longs in front of S; have S count by tens as they are placed. Screen the 20 longs; place 4 longs adjacent to the screen.
   HOW MANY TENS ARE THERE ALL TOGETHER? WHY?

5. Place 25 MAB longs in front of S.
   COUNT THESE AND TELL ME HOW MANY LITTLE BLOCKS ARE HERE. HOW MANY HUNDREDS ARE THERE?

6. WHAT NUMBER IS ONE-HUNDRED LESS THAN FIVE-HUNDRED SEVENTY?

<u>Oral Counting tasks</u>

1. START AT THREE-HUNDRED FORTY AND COUNT BY TENS.

2. START AT THIRTY AND COUNT BY HUNDREDS.

3. BEGIN AT SEVENTY-THREE AND COUNT BY HUNDREDS.

<u>Counting tasks with materials</u>

1. Place MAB flat in front of S. Hold MAB long in hand.
   HOW MANY OF THESE (indicate long) ARE IN THIS PIECE (indicate flat)?

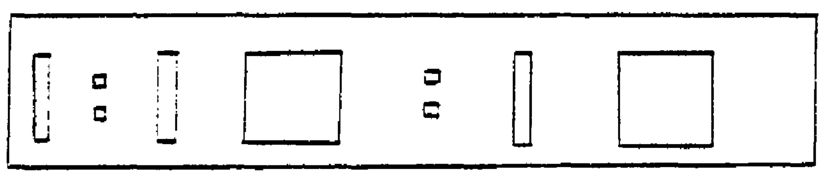
Counting tasks with materials (cont.)

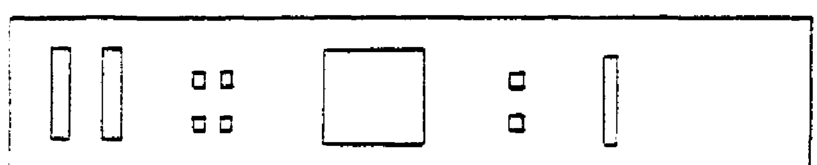
2. Place MAB flat in front of S. Hold MAB unit cube in hand. HOW MANY OF THESE (indicate unit cube) ARE IN THIS PIECE (indicate flat)?

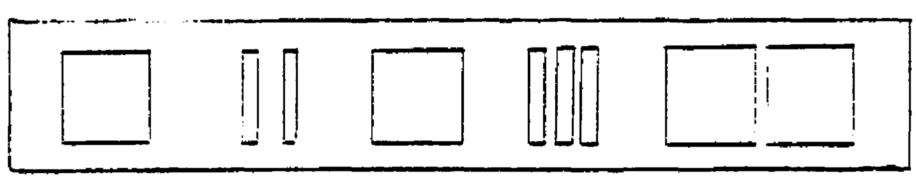
Counting by Hundreds, Tens, and Ones

Task Summary:

Place indicated display of MAB base ten flats, longs, and units in front of S, screened from S's view. Slide screen to uncover successive groups of blocks and have S record the total number of blocks uncovered up to that point.







APPENDIX II

TRANSCRIPT OF ALPHA'S INTERVIEWS

Appendix II

Alpha's Interview Transcript

Post-Investigation Interview - A1

Part I

I: I'm going to tell you some numbers. As I say them I want you to write them down. Go across the page. You can leave a small space between them as you go. Are you ready?

Thirteen.	"13"
Forty-one.	"41"
Eighty-four.	"84"
Eighteen.	"18"
One hundred one.	"101"
One hundred seven.	"107"
One hundred fourteen.	"1014"
One hundred twenty-one.	"1021"
One hundred seventy-eight.	"1078"
Two hundred nine.	"209"
Two hundred nineteen.	"209"; "2019"
Two hundred sixty-seven.	"2067"
Nine hundred thirty-four.	"9034"

- - - - - - - - - - - - - - - -

I: On these cards I've written some numbers. Can you say them for me?

"18"	Eighteen.
"26"	Twenty-six.
"73"	Seventy-three.
"120"	One hundred and twenty.
"174"	One hundred and seventy-four.
"210"	Two hundred and ten.
"311"	Three hundred and eleven.
"410"	Four hundred and ten.

"594" Five hundred and ninety-four.

"936" Nine hundred and thirty-six.

- - - - - - - - - - - - - - - -

Cards: 16 17 18 19 20 21 22 23.

I: I've got eight cards with numbers on them and a board with eight places for the cards. Can you put the cards in order on the board so that each place has a card on it?

Al: (Spreads cards on the table.) Sixteen (16). Seventeen (16-17). Eighteen (16-17-18). Nineteen (16-17-18-19). Twenty (16-17-18-19-20). Twenty-one (16-17-18-19-20-21). Twenty-two (16-17-18-19-20-21-22). And twenty-three (16-17-18-19-20-21-22-23).

- - - - - - - - - - - - - - - -

Cards: 20 30 60 70 90 100 110 120 (shuffled).

I: Here are some more cards. Can you put these cards in order on the board?

Al: (Spreads cards on the table.) Twenty (20). Thirty (20-30). Sixty (20-30-60). Seventy (20-30-60-70). Ninety (20-30-60-70-90). Hundred (20-30-60-70-90-100). Hundred and ten (20-30-60-70-90-100-110). Hundred and twenty (20-30-60-70-90-100-110-120).

- - - - - - - - - - - - - - - -

Cards: 11 21 31 51 81 91 101 111 (shuffled).

I: Let's try doing the same thing with these cards.

Al: (Spreads cards on the table.) Eleven (11). (Pauses; shuffles cards.) Thirteen (11-13). Twelve, I mean (11-21). Thirteen (11-21-31). Eighteen (11-21-31-81). Nineteen (11-21-31-81-91). (Picks up "51"; moves "81" and "91" each one space to the right; 11-21-31-51-81-91). A hundred one (11-21-31-51-91-101). A hundred eleven (11-21-31-51-91-101-111).

I: Now, I want you to read them starting here, please (points to left end of board).

Al: Eleven, twelve, thirteen, fifteen, eighty-one ... eighteen, nineteen, hundred and one, hundred and eleven.

I: Are you sure they're in the right order from smallest to biggest?

Al: (Nods head affirmatively.)

- - - - - - - - - - - - - - - -

Cards: 8 12 13 17 19 21 31 102 (shuffled).

I: Here are some more cards. Can you put these in order?

Al: (Spreads cards on the table.) Eight (8). Twelve (8-12). Thirteen (8-12-13). Seventeen (8-12-13-17). Nineteen (8-12-13-17-19). Twenty-one (8-12-13-17-19-21). Thirty-one (8-12-13-17-19-21-31). Hundred twenty (8-12-13-17-19-21-31-102).

I: Would you read them to me now?

Al: Eight, twelve, thirteen, seventeen, nineteen ... twenty-one, thirty-one, hundred and twenty.

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Cards: 30 47 48 49 52 61 67 76 (shuffled).

I: This time I want you to place these cards on the board starting with the biggest number in the first place (indicates) and the smallest number in the last place (indicates).

Al: (Spreads cards on the table.) Seventy-six (76). Sixty-seven (76-67). Sixty-one (76-67-61). Fifty-two (76-67-61-52). Forty-nine (76-67-61-52-49). Forty-eight (76-67-61-52-49-48). Forty-seven (76-67-61-52-49-48-47). And thirty (76-67-61-52-49-48-47-30).

I: Now, they go from biggest to smallest?

Al: Right.

- - - - - - - - - - - - - - - -

Cards: 97 103 107 113 117 124 134 143 (shuffled).

I: I want you to do this just like the last one. Put the cards across the board from the biggest to the smallest. I'm goint to trick you this time, so look at the cards carefully!

Al: (Spreads cards on the table; pauses; points at several cards; stares at "143" and "107".) Hundred and seventy (107; pauses; looks at each card). Hundred and forty-three (107-143). Hundred and thirty-four (107-143-134). Hundred and thirty (107-143-134-103). Hundred and twenty-four (107-143-134-103-124). Hundred and seventeen (107-143-134-103-124-117). Hundred and thirteen (107-143-134-103-124-117-113). Ninety-seven (107-143-134-103-124-117-113-97).

I: Which one is the smallest one?

Al: Ninety-seven.

I: And the biggest one?

Al: Hundred and seventy.

I: You want to check them one last time to make sure I haven't tricked you. Say them out loud, and check them very closely.

Al: Hundred and seventy, hundred and forty-three, hundred and thirty-four, hundred and thirty, hundred and twenty-four, hundred and seventeen, hundred and thirteen--haven't tricked me!

I: Which number is that (pointing to "103")?

Al: Hundred and thirty.

I: Is that how you say it? (Pause.) Can you say it that way? ... Okay. Which number is this (pointing to "107")?

Al: Hundred and seventy.

- - - - - - - - - - - - - - -

I: How many tens are there in thirteen?

Al: (Pause.) One.

I: One ten. How do you know that?

Al: Cause there isn't another ten after ten. Like one, two, ..., nine, ten. There's only three more. There couldn't be two tens, three tens, or more.

- - - - - - - - - - - - - - -

I: How many tens are there in sixty-seven?

Al: Six.

I: How do you know that?

Al: Same thing.

I: How did you do it, now? Sixty-seven, now.

Al: I know that if there are six tens, it's got to be sixty. And after that six tens, you can't make another ten, cause you only have seven more.

I: And that's not enough, huh?

Al: (Shakes head negatively.)

- - - - - - - - - - - - - - -

I: (Places pile of sticks on table; covers pile with hand.)
Let's pretend that this pile has seventy-two sticks in it.
If you took all the tens out of seventy-two, how many would be left?

Al: Two.

I: How did you know that?

Al: I just did.

I: If we're pretending, now. If I let you take all the tens, how do you know there would be two sticks left?

Al: Cause there's seven tens and only two ones.

- - - - - - - - - - - - - - - -

I: Start at eight and count-on by tens.

Al: Eighteen, twenty-eight, thirty-eight, forty-eight, sixty-eight, seventy-eight, eighty ... eighty-eight, ninety-eight, a hundred and eight, ..., a hundred and ninety-eight ... a hundred and ninety-eight ... two hundred and eight ... two hundred and eighteen, two hundred and twenty-eight ...

I: That's good.

- - - - - - - - - - - - - - - -

I: Start at ninety-seven and count-backward by tens.

Al: Ninety-seven, eighty-seven, sixty-seven ... sixty-seven, fifty-seven, forty-seven, thirty-seven, eighty ... thirty-seven! Twenty-seven! Seventeen, seven.

I: And that's the last, huh?

Al: Huh-uh (no). Zero.

- - - - - - - - - - - - - - - -

I: What is two tens and ninety more?

Al: Two tens and ninety more tens?

I: Two tens and ninety more.

Al: It would be about ... twelve ... tens.

I: Lets see, two tens and ninety more.

Al: A hundred and twelve.

I: How did you get that?

Al: I meant to say a hundred and twelve, but I just said twelve. I just put all the tens together (sweeping his hands together) and got a hundred and twelve.

I: How did you do that in your head?

Al: Well, I was thinking about ninety, and I was thinking about two more tens. And ... and ... I count from ninety to a hundred to a hundred and twelve.

I: Do it for me. I want to hear you do it.

Al: Ninety ... (touches right middle finger with right thumb) a hundred ... (touches right index finger with right thumb) a hundred and twelve.

I: In other words, it goes ninety, a hundred ...

Al: And.

I: Ten more.

Al: Ten more and then (puts up two fingers) two.

I: Why did you do two more?

Al: I just did it.

I: You understand what I'm asking? Two tens and ninety more.

Al: A hundred and twelve.

- - - - - - - - - - - - - - -

I: What number is three tens more than that number (places card with "50" written on it onto the table)?

Al: (Pause.) Two.

I: How did you get that?

Al: Cause I know that three plus five ... three plus two is five.

I: Now listen. What number is that?

Al: Fifty.

I: What number is three tens more than this number?

Al: (Pause.) Eighty.

I: How did you get that?

Al: Counted up by tens.

I: Do it for me.

Al: Sixty, seventy, eighty.

- - - - - - - - - - - - - - - -

I: You can see we have twenty-four little blocks under this screen (lifts screen to show blocks; replaces screen so that all blocks are covered; places MAB long adjacent to screen.) How many little blocks are there altogether now?

Al: (Writes "34".)

I: How did you get that?

Al: Put in another ten.

I: I want you to do it out loud for me, now.

Al: I went from twenty-four, then thirty-four.

I: (Advances screen so that all blocks are covered; places 3 MAB units.) Now watch ... Don't write it down ...

Al: (Writes "44".)

I: Did you wait until I did more? I don't think you did.

Al: (Erases "44".)

I: Now I want you to tell me how many altogether.

Al: Thirty-four, thirty-five, thirty-six, thirty-seven.

I: All right, write it down then.

Al: (Writes "37" below unit cubes.)

I: (Advances screen so that all blocks are covered; places two MAB longs.) How many now?

Al: (Writes "57".) Fifty-seven.

I: How did you get it?

Al: Count by tens again. Forty-seven, fifty-seven.

I: (Advances screen so that all blocks are covered; places one MAB unit and one MAB long.) Now you can do it.

Al: (Writes "68".) Fifty-seven, sixty-seven, sixty-eight.

I: (Advances screen so that all blocks are covered; places two unit cubes and two MAB longs.) Write down how many little blocks I have altogether now.

Al: Seventy ... eight, eighty-eight, eighty-nine, ninety (writes "90").

I: How much have I gotten hidden under the screen?

Al: Ninety.

I: Would you put that down in that box, right there (lower part of Al's record sheet)? Ninety.

Al: (Writes "90" in box.)

- - - - - - - - - - - - - - -

I: I'm going to take some wood out now. Each time I do it, I want you to write down what is left (removes 2 MAB longs). Write down how much is left.

Al: It would be ... ninety, eighty ... seventy (writes "70").

I: (Removes 1 MAB long and 1 unit cube.)

Al: (Looks at "70", then at blocks; pause; looks at "70", then at blodks; pause; writes "59".) Fifty-nine.

I: (Removes 1 MAB long.) Now how much is left?

Al: (Writes "6".) Not sixty-nine (erases "6" and writes "49"), but forty-nine.

I: (Removes 6 unit cubes.) Now I'm going to take those out. How much is left?

Al: (Points to each unit cube.) Six (places hands under table; pauses; writes "42").

I: What did you get?

Al: Forty-two.

I: How did you get that?

Al: Counted back.

I: Do it out loud.

Al: I went back forty-eight, forty-seven, forty-six, forty-five, forty-three (while putting up each finger of right hand), forty-two (while putting up thumb of left hand).

I: (Removes 2 MAB longs and 1 MAB unit.)

Al: (Writes "22".)

I: How did you get that?

Al: Just did.

I: Well, do it for me.

Al: Oh, I didn't see that one (erases 2nd "2" of "22"; looks at blocks; writes "23").

I: What are you going to get?

Al: Twenty-three.

I: Tell me how you did that.

Al: Just like I did the others.

I: Well, do it for me. Where are you at? You started at ...

Al: Twenty-two.

I: Forty-two, and then what did you say.

Al: (Looks at the 2 longs and the unit cube.) Thirty-two, twenty-two. Then I saw one and I made it twenty-three.

I: Is that right?

Al: Yeah.

I: How much is left under here? I don't know. (Pause.) You're not sure, huh?

Al: (Shakes head negatively.) I forgot what we started off with.

I: What did it say when we started off?

Al: Oh, yeah. Ninety.

I: And we took out this wood. So how much should be left under here?

Al: Ten.

I: Ten under here? Do you want to check it (raises cover)?

Al: Twenty-two.

I: Not ten, huh.

Al: (Shakes head no.)

Part II

I: I have some number problems on these cards. I want you to think out loud while you do these so that I can hear what you're doing. Are you ready? Here's the first one. (Places card with "10 + 7 = __" onto the table.)

Al: (Pause; looks at card; begins to extend a finger of his left hand; shakes hand over the card; touches two fingers of his right hand.) Seventeen.

I: Okay, how did you know that?

Al: Seventeen.

I: How did you get that?

Al: Ten in my head and I counted seven. Eleven, twelve, ..., seventeen (putting up a finger with each utterance).

I: Did you do it that way, or did you just know?

Al: Both. I did it that way and knew it. But I forgot it--that's why I did it that way.

- - - - - - - - - - - - - - - -

I: (Places card with "10 + __ = 13" onto the table.) Read it out loud for me.

Al: Ten plus blank equals thirteen.

I: Well ...

Al: (Long pause.) Ten plus three equals thirteen.

I: How come? Did you just know that one?

Al: Same thing.

- - - - - - - - - - - - - - - -

I: (Places card with "40 + __ = 46" onto the table.)

Al: Forty plus blank equals forty-six. (Pause.) Six.

I: How did you know that?

Al: Same way.

- - - - - - - - - - - - - - - -

I: (Places card with "__ + 20 = 25" onto the table.)

Al: Blank plus twenty equals twenty-five. (Pause.) Five.

I: How did you know that? Same thing?

Al: Uh-huh (yes).

I: Are you sure now?

Al: Yes, I'm sure.

I: You said five in your head?

Al: Huh-uh (no). Twenty, and I put five ... equals twenty-five.

\- - - - - - - - - - - - - - -

I: (Places card with "__ + 9 = 79" onto the table.)

Al: Blank plus nine equals seventy-nine. Seven.

I: Seven. How did you get that?

Al: Same way.

I: Tell me how you did it now. I forget. Did you put seventy-nine in your head?

Al: No. I knew that seven tens would be seventy, and nine ones had to be seventy-nine.

I: So what goes there (points to blank)?

Al: So, I put seven tens there (points to blank).

I: Oh, so seven tens ...

Al: And nine ones.

\- - - - - - - - - - - - - - -

I: On this piece of paper I want you to write down as many numbers as you can that add up to be fifteen. Like fourteen plus one. Do you know what fourteen plus one is?

Al: Yes. Can I do some take away's too?

I: Well, do some plusses first. (Pause.) Read it out as you do it.

Al: (Writes "14 + 1 =".) Fourteen plus one equals (writes "15").

I: You don't need to write the fifteen. Okay, do you know another one?

Al: Thirteen ("13") plus ("+": "13 +") nine equals ("2 =": "13 + 2 =").

I: Don't try to trick me.

Al: (Giggles.)

I: Do you know some more?

Al: (Writes "21 + 3 =".) Twelve plus three.

I: Keep going. (Pause.) Is that all there are?

Al: (Writes "11 + 4 =".)

I: The next one. The one after that.

Al: (Pause.) Ten plus five.

I: The one after that?

Al: (Pause.) Nine plus six.

I: The one after that?

Al: Seven ... eight plus ... what was it? Eight plus seven? Six plus eight ...

I: Do you know a take away that's equal to fifteen?

Al: Huh-uh (no).

- - - - - - - - - - - - - - -

I: (Places card with "60 - 20 = __" onto the table.)

Al: Sixty take away twenty ... should be ... forty. Forty ... right?

I: Don't you know?

Al: Forty.

I: Are you asking me or telling me?

Al: Ahh ... telling.

I: How can you be sure it's forty?

Al: By counting back.

I: How?

Al: Sixty, fifty (putting up two fingers of left hand), then you end up with forty. Like I had like sixty lollipops. I ate up twenty. I'd have forty left.

- - - - - - - - - - - - - - -

I: (Places card with "70 - 31 = __" onto the table.)

Al: Seventy take away thirteen. (Long pause.)

I: Do you know that one?

Al: Seventy take away thirteen. (Pause.) This number (picks up a pencil; draws "63" in the air).

I: Tell me what it is.

Al: Sixty-three.

I: How did you get that?

Al: I just counted back ten and put that three in. Back ten and put that three in.

- - - - - - - - - - - - - - -

I: (Places card with "84 - 30 = __" onto the table.)

Al: Eighty-four take away thirty. (Long pause; holds hand--held in a fist--in front of him.) Thirty-four.

I: Are you sure? Tell me how you would do it.

Al: Same way I did that one (searchs for, and then points to, "60 - 20 = __").

I: Let's do this 'eighty-four" one. What does it say to do?

Al: Eighty-four take away thirty.

I: Okay, how would you do that?

Al: I got eighty-four lollipops and I take away five.

I: Why do you want to take away five?

Al: Ummmm.

I: That just makes it a bit easier, does it?

Al: (Nods head affirmatively.)

I: What do you get when you take away five?

Al: (Pause.) Thirty ... thirty-four.

I: What? If you have 84 lollipops and take away five, then you have thirty-four? All right, lets do it with this (hand Al a box of MAB longs and units). Tell me what you're doing.

Al: I'll show you. (Counts out 8 MAB longs). I'm getting out eighty. (Counts out 4 MAB units.) Now I got eighty-four. Now I need to take away five. One, two, three, four, five (as he picks up 5 of the longs).

I: What did you take away?

Al: Five tens.

I: Why did you do that?

Al: I just did. It makes it easier. And I got thirty-four.

I: Read this again (points to the open sentence). What does it say.

Al: Eighty-four take away thirty.

I: Why did you take away five tens?

Al: (Drops the 5 MAB longs back onto the table; picks up 3 MAB longs.)

I: What are you taking away now?

Al: Thirty.

I: So what have you got?

Al: (Counts the MAB longs; counts the MAB unit cubes.) Forty-four.

I: Sure now?

Al: I'm sure.

I: Okay, lets go to the next one.

Al: Wait ... fifty-four (after recounting the MAB longs).

I: (Removes blocks.) Tell me how you would do that without any wood.

I: I don't know how I'd do it without any ...

I: Think about what you did when you did have the wood. (Pause.) You've got eighty-four and you want to take away thirty. What can you do? (Very long pause.) Let's leave that one and go on.

I: (Places card with "47 - 21 = __" onto the table.)

Al: Forty-seven take away twelve!

I: Read it again. I didn't hear you.

Al: Forty-seven take away twelve.

I: How would you do that one?

Al: Forty-seven take away twelve (long pause).

I: Do you need to do it with the wood?

Al: Yeah (takes box of MAB blocks; removes 5 MAB longs; removes 4 MAB unit cubes). Forty-seven take away twelve (returns 1 MAB long and 2 unit cubes to box).

I: What have you got?

Al: Thirty-three.

I: (Places card with "74 - __ = 70" onto the table.)

Al: Forty-seven take away blank equals seventy. (Reaches for box of blocks; looks at card; removes 7 MAB longs and 4 MAB unit cubes from the box.)

I: What have you got there?

Al: Seventy-four ... take away ... (looks at card) blank equals seventy (groups the 7 MAB longs together; returns the 4 MAB unit cubes to the box). Four.

I: So what goes here (pointing to the blank).

Al: Four.

I: (Places card with "91 - 29 = __" onto the table.)

Al: Is that nineteen?

I: Is it?

Al: (Picks up card.) Ninety-one take away ninety-two. (Pause.) Can't do it.

I: What's this number?

Al: Twenty-nine.

I: Is it twenty-nine or ninety-two?

Al: Ninety-two.

I: Hmmm?

Al: Twenty-nine! (Removes 7 MAB longs and 1 unit from the box.)
One, two, three, ..., seven (while pointing to each long;
removes two more longs from box). Eight, nine ... take away
... (returns 1 MAB long to the box).

I: What are you doing?

Al: Trading. (Removes 9 MAB units from the box.)

I: How many do you want?

Al: Ten (lines the 9 unit cubes next to 1 MAB long; gets 1 more
unit cube from box).

I: Did you get ten?

Al: Yeah.

I: You got ten?

Al: No (moves 1 unit cube aside the longs; removes 1 more unit
cube from box and places next to the 9 cubes; unit cube
originally taken out with the longs has been pushed to Al's
extreme right; he no longer sees it).

I: Have you got ten now? Count them all.

Al: Yeah, I got ten. One, two, ..., ten ... eleven (pointing to
each unit cube; returns one unit cube to the box).

I: Are you sure now? Let me count them. One, ..., nine
(pointing to each unit cube except one).

Al: (Removes 1 unit cube from box.)

I: How about this way? Ten, twenty (pointing to 2 of the MAB
longs).

Al: Thirty, forty, ..., eighty (as the interviewer points to
each long; lines up 10 unit cubes against a long). Now I
have to take away twenty (removes 2 MAB longs) and nine
(separates 1 unit cube from those lined up against a long;
returns other 9 to the box). Sixty-two.

I: Are you sure now?

Al: Yeah, I'm sure. Three and three is sixty (grouping the MAB
longs together; holds up the 2 unit cubes), and two ones.

- - - - - - - - - - - - - - - -

I: (Places card with "70 --> 92" onto the table.) How far is it
from this number (points to "70") up to this number (points
to "92")?

Al: (Pause; looks at fingers.) Twenty-two.

I: From seventy up to ninety-two? Are you sure?

Al: Yeah, I'm sure. I'm positive.

I: (Places card with "92 --> 70" onto the table.) How far is it from ninety-two down to seventy?

Al: (Immediately.) Twenty-two.

I: How did you know that?

Al: Same question. Just it was turned around.

- - - - - - - - - - - - - - - -

I: (Places card with "64 --> 40" on it onto the table.) How many is it from sixty-four down to forty?

Al: (Pause.) Twenty.

I: How did you get that?

Al: I just counted back in my head.

I: How many is it from forty up to 64 (places card with "40 --> 64" written on it onto the table)?

Al: (Immediately.) Twenty.

I: Sure about that? How many is it from forty up to sixty-four?

Al: Twenty.

I: How many is it from forty up to sixty?

Al: (Long pause; looks back at "64 --> 40".) Forty ... fifty.

I: What are you looking at that one for? Do you want to change your mind?

Al: Sixty-four down to forty. Twenty-two. And twenty-two (indicating "40 --> 64").

- - - - - - - - - - - - - - - -

I: Here is one last problem, but instead of writing it down I'm going to tell it to you. Are you ready? How many is it from thirty-six up to fifty?

Al: (Pause.) Twenty-six.

I: Sure about that? Thirty-six up to fifty?

Al: Twenty-six.

I: How did you do that?

Al: Counted up.

I: Go on. Tell me how you did it.

Al: I just told you. I counted up.

I: Show me. Just do it this time for me, okay?

Al: (Shakes head negatively.)

I: Thirty-six. I'll tell you how you did it. Thirty-six ... forty-six. Thirty-six ... I carried on ten. That's forty-six.

Al: Fifty-six ... and that makes twenty-six.

I: Forty-six and ten more is fifty-six. I counted up twenty. Fifty-six and I count up another six ... fifty-seven, ..., sixty-two. That's twenty-six.

Al: Twenty-two!

I: I didn't get fifty. I got sixty-two. And I counted up twenty-six.

Al: Thirty-two.

I: Tell me, thirty-six to fifty ... how many is it?

Al: Twenty-six ... No! it's sixteen.

I: Sixteen? Sure now?

Al: Yeah.

I: How many is from fifty down to thirty-six?

Al: Sixteen.

I: How did you know that?

Al: Just turned it around.

Part III

I: (Places card with "20" written on it onto table.) What number is one hundred more than this number?

Al: One hundred more? (Pause.)

I: What is that number?

A1: Twenty.

I: What number is one hundred more than that number?

A1: (Pause.) One hundred and twenty.

- - - - - - - - - - - - - - - -

I: (Places bag with 235 unit cubes in it onto table.) Here are two hundred thirty-five little blocks. How many piles of one hundred could you make from them?

A1: (Long pause.) What's the question?

I: There are two hundred thirty-five little blocks in there (the bag). If you were to use those little blocks, how many piles of one hundred could you make?

A1: (Pauses; holds up two fingers.)

I: Two? How did you know that? Did you count them?

A1: I didn't count those (points to bag). Because I know how much makes two hundred.

- - - - - - - - - - - - - - - -

I: Let's have you count for me (begins placing MAB longs on the table).

A1: Ten, twenty, ..., hundred and sixty (as the interviewer places 16 MAB longs one at a time onto the table).

I: There are one hundred sixty little blocks there. How many hundreds are there?

A1: One.

I: How do you know?

A1: I just know.

I: Okay, keep counting.

A1: Hundred and seventy, ..., two hundred (as the interviewer places 4 more MAB longs).

I: All right.

A1: Two.

I: (Screens (20) MAB longs from view.)

Al: Two hundred under here (places hand on cover).

I: (Places 4 longs next to screen.) How many tens are there altogether now?

Al: Two hundred and forty (as the interviewer asked above question).

I: How many tens are there altogether?

Al: (Pause.) Two hundred would be twelve tens ... sixteen. (Pause.) No! Thirteen ... (looking at the 4 longs).

I: (Interrupting.) How many tens?

Al: Sixteen.

I: How did you get that?

Al: Counted.

I: How many tens under here (places hand on screen)?

Al: Twelve. Thirteen, ..., sixteen (pointing to the 4 MAB longs on top of screen).

I: How do you know there are twelve tens under there?

Al: I know how many tens is two hundred.

I: (Places 1 more MAB long on table; uncovers other 20.).

Al: Seventeen ... ten ... one, two, ..., eight (while pointing to MAB longs; continues subvocally counting). Twenty-five.

I: How many hundreds are there?

Al: Two.

I: There are twelve tens in two hundred. Right? Is that what you said?

Al: Yeah, but I'm wrong.

I: Oh, why are you wrong? How do you know you're wrong?

Al: I counted them.

I: How many tens are there? Do you want to change your mind?

Al: I forgot how much I counted. (Counts 13 longs; separates them from the others.) There are thirteen tens in this pile and thirteen tens in this pile. One, two, ..., twelve (as he points to each MAB long in 2nd pile). I always get em mixed up.

I: You told me there were twelve tens in two hundred. But you
think you're wrong. Exactly how many tens are there in two
hundred? Is it thirteen tens instead?

Al: Uh-uh (no; looks in the air; pauses). Twenty.

I: Twenty? Why do you think twenty?

Al: I counted two hundred. One, two, ..., eleven ... wait.
(Removes 2 longs from one pile, 3 from the other; sets them
aside.) These don't even need to be in here. One, two, ...,
twenty (pointing to each of the remaining longs).

- - - - - - - - - - - - - - -

I: What number is one hundred less than five hundred seventy?

Al: (Pause.) Four hundred and seventy.

- - - - - - - - - - - - - - -

I: Start at three hundred forty and count-on by tens.

Al: What?

I: (Repeats question.)

Al: Three hundred and fifty, ..., three hundred and ninety ...
four hundred, four hundred and ten, ..., four hundred and
ninety ... five hundred, five hundred and ten ...

I: Okay. You can stop there.

- - - - - - - - - - - - - - -

I: Start at thirty and count-on by hundreds.

Al: One hundred and thirty, two hundred and thirty, ..., nine
hundred and thirty ... ten hundred and thirty.

I: Any more? Have you ever counted that far before?

Al: I know what ten hundreds is.

I: What is it?

Al: A thousand.

I: What comes after ten hundred and thirty?

Al: Three thousand, four thousand, ..., nine thousand ... ten
thousand.

- - - - - - - - - - - - - - -

I: Begin at seventy-three and count-on by hundreds.

Al: Seventy-three? One hundred and seventy-three, two hundred and seventy-three, ..., six hundred and seventy-three, eight hundred and ... seven hundred and seventy-three, eight hundred and seventy-three, nine hundred and seventy-three .... ten hundred and ... nine hundred and seventy-three, ten hundred and seventy-three ... seventy-three thousand.

- - - - - - - - - - - - - - - -

I: (Places MAB flat on table; holds up MAB long.) How many of these pieces of wood (long) are there in this piece (flat)?

Al: (Pause.) All of these (sliding hand over MAB flat) aren't right. Just that one is right (indicating one column within the flat).

I: Is that how many of these (holding up MAB long) there is in that? Just one?

Al: Huh-uh (no).

I: Can we saw off just one of these (long) from that piece of wood (flat)?

Al: No, we can saw off that one, and that one, ..., and that one (pointing to each column of the flat).

I: How many would that be?

Al: Ten ... tens.

I: How many of these would there be (holding up MAB long)?

Al: Ten.

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I: (Holds unit cube hand.) How many of these little blocks could I saw out of that piece of wood?

Al: A hundred.

I: How do you know that?

Al: Cause ten tens is ... all little ones. And ten tens is a hundred. So those little ones are tens and they're a hundred. See, one, two, ..., eleven (counting the "individual" cubes of the flat) ...

I: (Interrupting.) You don't have to count them all, do you?

Al: Yeah.

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Board: 10 2 10 100 2 10 100

I: I want you to count these pieces of wood as I show them (uncovers MAB long).

Al: Ten.

I: (Uncovers 2 unit cubes.)

Al: (Pause.) Twelve.

I: (Uncovers MAB long.)

Al: (Pause.) Twenty-two.

I: (Uncovers MAB flat.)

Al: (Pause.) A hundred and twenty-two.

I: (Uncovers 2 unit cubes.)

Al: A hundred and twenty-four.

I: (Uncovers MAB long.)

Al: (Pause.) A hundred and thirty-four.

I: (Uncovers MAB flat.)

Al: (Pause.) Two hundred and thirty-four.

\- - - - - - - - - - - - - - -

(Board: 10 10 4 100 2 10)

I: (Uncovers MAB long.)

Al: Ten.

I: (Uncovers MAB long.)

Al: Twenty.

I: (Uncovers 4 unit cubes.)

Al: (Long pause.) Twenty-four.

I: (Uncovers MAB flat.)

Al: (Pause.) A hundred and twenty-four.

I: (Uncovers 2 unit cubes.)

Al: (Pause.) A hundred and twenty-six.

I: (Uncovers MAB long.)

Al: A hundred and thirty-six.

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Board: 100 10 10 100 10 10 10 100 100.

I: (Uncovers MAB flat.)

Al: A hundred.

I: (Uncovers MAB long.)

Al: A hundred and ten.

I: (Uncovers MAB long.)

Al: A hundred and twenty.

I: (Uncovers MAB flat.)

Al: A hu ... two hundred and twenty.

I: (Uncovers MAB long.)

Al: Two hundred and thirty.

I: (Uncovers MAB long.)

Al: Two hundred and forty.

I: (Uncovers MAB long.)

Al: Two hundred and fifty.

I: (Uncovers MAB flat.)

Al: Three hundred and fifty.

I: (Uncovers MAB flat.)

Al: Four hundred and fifty.

\- - - - - - - - - - - - - - -

Board: 100 3 10 10 100 1 10 100.

I: (Uncovers MAB flat.)

Al: A hundred.

I: (Uncovers 3 unit cubes.)

Al: A hundred and three.

I: (Uncovers MAB long.)

Al: (Pause.) A hundred and thirteen.

I: (Uncovers MAB long.)

Al: (Pause.) A hundred and twenty-three.

I: (Uncovers MAB flat.)

Al: Two hundred and twenty-three.

I: (Uncovers 1 unit cube.)

Al: (Pause.) Two hundred and twenty-four.

I: (Uncovers MAB long.)

Al: (Pause.) Two hundred and thirty-four.

I: (Uncovers MAB flat.)

Al: Four hundred and thirty-four.

\- - - - - - - - - - - - - - -

Board: 100 10 10 4 100 100 100 10 1 10.

I: (Uncovers MAB flat.)

Al: A hundred.

I: (Uncovers MAB long.)

Al: A hundred and ten.

I: (Uncovers MAB long.)

Al: A hundred and twenty.

I: (Uncovers 4 unit cubes.)

Al: (Pause.) A hundred and twenty-four.

I: (Uncovers MAB flat.)

Al: Two hundred and twenty-four.

I: (Uncovers MAB flat.)

Al: Three hundred and twenty-four.

I: (Uncovers MAB flat.)

Al: Four hundred and twenty-four.

I: (Uncovers MAB long.)

Al: Four hundred ... thirty-four.

I: (Uncovers 1 unit cube.)

Al: Four hundred ... thirty-five.

I: (Uncovers MAB long.)

Al: Four ... hundred ... forty-five.

END

APPENDIX III

PRELIMINARY ANALYSIS OF ALPHA'S

INTERVIEWS, AND WORKSHEET

Appendix III
Analysis of Protocol
Post-Interview — Alpha

(ON: Observational note, TN: Theoretical note.)

Writing Numerals

ON1: Okay, except for numerals in the hundreds with a non-zero second digit—e.g., "2067" for "267".

Reading Numerals

ON1: Okay. Concatenates hundred word with remainder by "and".

Ordering Numerals

ON1: In first three tasks, Alpha appeared to search for the "next card according to a sequence criterion—e.g., regurlar counting sequence in <56-68>, counting by ten in <69-80>, and regular counting in <81-103>.

TN1: Alpha seemed to make his perceptions conform with his expectations in <81-102>. He expected to produce a sequence in increments of one, so he saw "18" rather than "81". The conflict is made clear in <96>, where Alpha read the numeral cards "eighty-one ... eighteen."

TN2: Alpha's misreading was not a casual slip of the tongue. When he misread a numeral, he acted according to what he said it was, and not what it was—and acted correctly with respect to what he said it was. For example, "107-143-134-103-124-117-113-97" was "170-143-134-130-124-117-113-97" when he read them.

How many tens in thirteen?

ON1: "One — cause there isn't another ten after ten."

TN1: Alpha's justification suggests that there is an order involved in the construction of thirteen--"isn't another ten after ten. Like one, two, ..., ten." That is, one makes tens first, and then tacks on the remaining ones.

TN2: Alpha's justification also suggests that thirteen as a number of tens and a number of ones is the result of an operation, viz. extending.

How many tens in sixty-seven?

ON1: Same sort of justification as for thirteen—"six tens and not enough for another ten."

Seventy-two minus all the tens in seventy-two.

ON1: "Two—cause there's seven tens and only two ones."

Start at ninety-seven and count backwards by ten.

ON1: Paused at 67 —> 57, 37 —> 27 (started "80 ...").
17— > 7 —> 0.

TN1: In transition 37 —> 27, Alpha started "eighty ...". Perhaps he focused on the "seven" of "thirty-seven" and began to increment.

Wouldn't this imply that Alpha generated feedback as part of a n necessary-condition check while sequencing backward? Otherwise how could he have caught himself?

What is two tens and ninety more?

ON1: "About ... twelve ... tens."

ON2: "A hundred and twelve [he meant to say] ... I just put all the tens together.

ON3: Brought his hands together as he said that he put all the tens together.

ON4: Interviewer said "Niney, a hundred, and ten more." Alpha said, "Ten more and then two."

TN1: Alternative interpretations:

a) 9 + 2 = 12; Alpha saw "consternation" in the interviewer's face, so he reflected on his answer, realized his answer had to be in the hundreds, so he made it 112 rather than 12—relating it to his original procedure of converting 90 into tens. Said "ten more and then two" because he need two more to get from 110 (the ten more) to his answer of 112.

b) He really was thinking 112, but said only the last construction (12). [If so, then why did he say "tens"?] Got 112 by counting the two tens (100, 110), but still had the abstract pattern for "two" actively in mind, so extended by it as well.

TN2: The action of bringing his hands together seems to be a figural content of the conceptual operation of integrating.

What number is three tens more than that number (50)?

ON1: "Two—cause I know that three plus five ... three plus two is five."

TN1: Understood the question as "What number is this number three tens more than? Then dropped "tens" as a unit label?

Operational reversibility between integrating and separating?

ON2: After interviewer asked "What's this number?", Alpha counted-on "sixty, seventy, eighty."

TN2: Reconceptualized the problem as an extension of 50 by a number of tens. Choose sequencing by ten as a relevant routine to name the result of the transformation of 50 by

Extensions by tens and ones (blocks under cover).

ON1: Correct performance. Counted by ten for longs; one for unit cubes.

Declensions by tens and ones (blocks under cover).

ON1: State: 59. Removed 1 long. Alpha started to write "69," said "No, not sixty-nine." and wrote "49."

TN1: Direction must be part of the extending and declending. How else does one decide upon the direction? Must be a conceptual condit- tion (make less?). Also, to catch an error, one must anticipate (expect) certain necessary characteristics of the result, which if not present causes reperformance of the operation (e.g., "new name should precede old name when declending.").

ON2: State: 42. Removed 2 longs 1 unit cube. Result: 22. Alpha: "Oh, I didn't see that one (unit cube)." Made it "23."

ON3: Reperformed operation of counting back 21 from 42 as "32, 22, then I saw that one and made it 23."

TN2: Why did he change direction? Perhaps because he thought something like "Oh, and one more!"

TN3: His "reperformance" seems to have been a historical account rather than a true reperformance. He seems to have held his original answer constant and looked for a way of arriving at his answer once more.

Arithmetical Sentences

10 + 7 = __

ON1: Shook one hand over the card, then counted two fingers.

TN1: 10 + 7 = 10 + (5 + 2) = (10 + 5) + 2. Utilization of a known fact.

TN2: Alpha started to count-on, but stopped. Perhaps he was thinking of the figural criterion for counting on seven (one hand and two fingers), and took the hand as a criterion for five, produced 15 as (10 + 5) from memory, and then counted-on what was left of the criterion.

10 + __ = 13

ON1: "Three--same thing."

TN1: The long pause may have been because of

a) Alpha's conceptualizing the problem in terms of a "missing" segment.

b) counting-on.

c) both (a) and (b).

__ + 5 = 25

ON1: "Five--same thing." In answer to the interviewer's question of w whether he put five in his head Alpha said "No. Twenty, and I put five ... equals twenty-five."

TN1: In each of 10 + 7 = __, 10 + __ = 13, and __ + 5 = 25 Alpha treated them as addition problems and not as problems of "find the missing digit."

TN2: If one conceptualizes "__ + 20 = 25" as

$$\frac{?\quad\vdots\quad 20}{25}$$

then how does one get commutativity? One answer might be reversibility, in the sense of "it's the same 'reading' it right to left as from left to right. That is, the operation of extending takes eighter of the two numbers first, and whichever it takes first determines the structure of a new conceptualization of the problem. But wouldn't this just be a functional knowledge of commutativity, as opposed to operational knowledge?

TN3: The comment in TN2 ("new" conceptualizations) implies that extending would have to have an underlying goal transformation procedure that inputs and outputs goal structures.

__ + 9 = 79

ON1: "Seven--I know that seven tens would be seventy, and nine ones had to be seventy-nine."

TN1: Did he begin with seventy and make seven tens of it? The way he explains it the seven tens came about because he had seventy in mind. But if that was the case, why did he not say "seventy" to begin with? Perhaps, this time, Alpha began by focusing on the "7" of "79", interpreting it as 7 tens, or 70.

60 - 40 = __

ON1: "Sixty, fifty (holding up two fingers), then you end up with forty."

70 - 31 = __

ON1: Read it as "70 - 13"; "Sixty-three--just counted back and put that three in."

TN1: Why does Alpha change directions (sometimes) when taking away the ones?

TN2: Alpha thought of this problem in terms of ordinal segments that were to be created through extending and declending.

84 - 30 = __

ON1: Read the problem as "84 - 30", but operated according to "eighty- four take away five tens."

ON2: The interviewer asked if taking away 5 made the problem easier, and Alpha concurred. Then Alpha remarked again, when asked why he took away 5 longs, that taking away 5 made the problem easier.

TN1: Alpha appears to have meant that taking away 5 tens was easier than taking away 50 ones--not that taking away 5 made the problem easier.

ON3: Alpha noticed, after rereading the sentence, that he shouldn't have taken away five tens--but three. When asked how he might do it without the MAB blocks, Alpha answered "I don't know how I'd do it."

TN2: What did Alpha mean by "I don't know how I'd do it?" Perhaps in all the confusion (between taking away 5 or 3 tens), he "forgot" about counting back. Or, perhaps he meant that he didn't know how he'd take away all the tens at once to find the result without the blocks. Or, perhaps he meant that he didn't know how he would do what he did with the blocks without them.

74 - ___ = 70

ON1: Made 74 out of 7 longs and 4 unit cubes, then separated the total into 70 and the rest -- getting 4.

TN1: This sort of strategy requires the same sort of conceptualization as extending. Seventy-four had to be viewed, before separating the blocks, as seventy and something else. That is, both seventy and the remainder must be viewed as being included in seventy-four, and as comprising seventy-four.

91 - 29 = ___

ON1: Solved it with blocks -- made 91 (9 longs, 1 unit), traded 1 long for 10 units; took away 2 longs and 9 units. "Sixty-two (very quickly)."

ON2: "Sixty-two -- three and three is sixty and 2 ones."

TN1: Remarks given for "74 - ___ = 70" apply here.

TN2: Why did the instructors introduce trading? Was it in anticipation of the standard algorithm? Why not instead have them reason by compensation with a long

$(^{-}9 = {}^{-}(10 - 1) = {}^{-}10 + {}^{+}1)$, or at least count-back the required number of units in a long.

TN3: Alpha used subitizing and facts (subitized 3 & 3; fact: 3 & 3 is 6) and converstions(six tens is sixty) to get his answer.

70 --> 92 How far is it from seventy up to ninety-two?

ON1: Answered "22" to 70 --> 92; no apparent process. Answered (quickly) "22" to 92 --> 70, saying, "Same question. Just turned around."

TN1: Did he mean that it was the same operation, except going in the opposite direction, or that it was the same set of numerals, only written in reverse order? It appears that the latter was the case.

64 —> 40; 40 —> 64

ON1: "Twenty—I just counted back in my head."

TN1: Counted-back from what to waht? Sixty to forty? Sixty-four to forty-four?

ON2: Answered immediately "20" to 40 —> 64.

TN2: Answered on the basis of his first answer, apparently with no further processing.

ON3: The interviewer asked "How far is it from 40 up to 60?" Alpha a answered, "Forty ... fifty ... ", and then began looking at the first card (64 —> 40). He changed his answer to "22" for <u>both</u> problems.

TN3: Perhaps Alpha began looking at 64 —> 40 after he realized that he was going to have to count <u>beyond</u> 20 to get from 40 to 64 (since he was counting 20 to get from 40 to 60). But where did "two" come from?

36 —> 50; 50—36 (verbal—no card)

ON1: Alpha answered "twenty-six," but wouldn't explicate. The inter- viewer tried showing that 26 would be too much by counting "Thirty- six, forty-six. Thirty-six—I carried on ten. That's forty-six." Alpha continued "Fifty-six, and that makes twenty-six." [Then changed his answer to 16.]

TN1: Perhaps Alpha's extension went something like "Thirty-six, forty- six (ten)—fifty-six (twenty), and six too much, so twenty-six."

ON2: Responded immediately that it is 16 from 50 to 36. "Just turned it around."

<u>What number is one hundred more than this number (20)?</u>

ON1: Only after Alpha had read the numeral card and <u>said</u> "twenty" did he answer "one hundred <u>and</u> twenty."

TN1: Perhaps he need the <u>symbols</u> (sound images) in order to carry out the operations of reversing and concatenating.

TN2: Perhaps this type of symbol manipulation assists in the construction of operational commutativity.

<u>How many hundreds in two hundred thirty-five (bag)?</u>

ON1: "Two — I didn't count those. ; Because I know how much makes two hundred."

TN1: "I didn't count..." Perhaps Alpha was attempting to distinguish between unit iteration (which is involved in counting) from linguistic transformations (which do not necessarily involve counting)—noting that it was by operating upon the symbol structure (((TWO)(HUNDRED)) (AND) ((THIRTY)(FIVE))) that he knew that it was "two", and knew it quantitatively by associating (TWO) with ((01010)(label.HUNDRED)).

<u>Ones —> Hundreds —> Tens</u>

ON1: 160 blocks counted by tens. How many hundreds? "One — I just know."

TN1: "I just know." — Signifies <u>symbolic</u> transformation.

ON2: Not only knew that there are two hundreds in two hundred, but "knew" that this was the question to be answered—without it being asked!

ON3: 20 longs covered and 4 on top—named it 240 (by counting by ten). How many tens in 240? Sixteen — 12 hunder the cover and 4 on top. How did he know that 12 were under the cover? "Because I know how many tens in 200."

TN2: How did he get 12 tens for the 200 under the cover? Perhaps as follows: 10 tens in one hundred, 2 hundreds, so 12 (10 + 2) tens.

ON4: Alpha counted 25 longs (each long being a unit item) and said "twenty-five." When asked how many hundreds, he said "two."

TN3: Because there are two tens in twenty-five?

ON5: To show the interviewer that there are 20 tens in 200, Alpha made (and counted) two groups of longs — one with 12 and one with 13. He started to count the total, then stopped and removed 2 from one group and 3 from the other. Then he counted the 20 longs.

TN4: Alpha must have remembered that he had counted 12 and 13, wanted two groups of ten tens, and conceptualized 12 as <u>10 + 2</u>, 13 as <u>10 + 3</u>. So to get 2 tens (of tens), he removed 2 and 3 respectively.

<u>What number is one hundred less than five hundred seventy?</u>

ON1: "Four hundred and seventy." (No indication of process.)

<u>Start at three hundred forty and count-on by tens</u>.

ON1: Okay. Paused at transition 390 —> 400 and 490 —> 500.

<u>Start at thirty and count-on by hundreds.</u>

ON1: Initiated sequence on his own. Sequenced: 130,230, ..., 930, ten hundred thirty.

TN1: Was he able to make the transition 30 —> 130 because he was able to <u>reverse</u> the two—100 and 30 more for 30 and 100 more?

TN2: Once the sequence is started a <u>pattern</u> takes over—increment the "hundred" word.

ON2: Alpha knew that ten hundreds is a thousand, yet continued (at the interviewer's request) his sequence: "ten hundred thirty ... three thousand, four thousand, ..., ten thousand."

TN3: Perhaps, in continuing the sequence, he applied the routine for incrementing as he did for counting-on by hundreds in the hundreds, namely by incrementing the first-said part of the name.

<u>Begin at seventy-three and count-on by hundreds.</u>

ON1: Began to go from 673 to 873, but caught himself.

ON2: Had difficulty going to next-of-973, so repeated it--973, ten hundred and seventy-three ... 973, ten hundred and seventy-three.

TN1: Again, some sort of check against expectations must have been present.

TN2: If an operation takes you into unfamiliar ground, start over but keep an eye out for previously missed branches.

<u>How many longs in a flat?</u>

ON1: "All these (sliding his hand over the flat) aren't right. Only this one is right (indicating one column of the flat).

TN1: Perhaps this was maore a remark to himself than anything else. Namely, that he must <u>construct</u> a column to correspond to a flat, whereas "free" squares (unconceptualized into columns) don't qualify according to the criterion of being a column.

ON2: Ten longs in a flat (ten tens in a hundred).

<u>How many unit cubes in a flat?</u>

ON1: One hundred--cause ten tens (made into "all little ones") is a h hundred.

<u>Counting by hundreds, tens, and ones (boards).</u>

ON1: Correct performance. Was <u>sure</u> to pause when unit cubes were uncovered. Also, when going from unit cubes to longs.

TN1: Symbolic versus "concrete" operations? Perhaps Alpha operated with ten and one hundred at a <u>symbolic</u> level, wherein numbers were representationally manipulated by operating upon number-names, whereas an increment by <u>one</u> was done as if he were counting.

Worksheet for Alpha's Case Study

Writing Numerals

Standard.

Reading Numerals

Digit reversals—mostly in the context of problems, especially ordering numerals.

Sequencing

By one

By ten

<223-231> 8 —> 228
<234-242> 97 —> 0
<1058-1061> 340 —> 510

By hundred

<1071-1087> 30 —>
<1090-1097> 73 —>

Numerical Operations

Integrating

<469-490> 10 + 7 = __.
<493-529> 10 + __ = 13;
40 + __ = 46;
__ + 20 = 25.
<245-297> 2 tens + 90.

Separating

<290-298> 3 tens more than 50.

Relationship

<290-298>

Extending

<469-490> 10 + 7 = __.
<826-844> 70 —> 92.

Declending

<601-622> 60 - 20 = __.
<625-641> 70 - 31 = __.
<724-744> 47 - 21 = __.
<845-871> 64 —> 40.

Relationship

<625-641>

Concept of Ten

<176-220> How many tens in ... ?
<245-287> 2 tens + 90.
<532-552> __ + 9 = 79.

Concept of One Hundred

<946-960> 235 blocks.
<963-985> 16, 20 longs; how many hundreds?
<987-1051> Tens in 240; 200.
<1054-1055> One hundred less than 570.

Concept of Place Value

<1100-1123> Longs in a flat.
<1126-1140> Unit cubes in a flat.