THE RELATIONSHIP BETWEEN MATHEMATICS SUBJECT MATTER KNOWLEDGE AND INSTRUCTION: A CASE STUDY

A Thesis

Presented to the Faculty of

San Diego State University

In Partial Fulfillment

of the Requirements of the Degree

Master of Arts

in

Mathematics

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by

Barbara Ann Boyd

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CHAPTER I

SIGNIFICANCE AND PURPOSE

The Relationship Between Teachers' Subject Matter Knowledge and Their Instruction

Prior to 1980, educational researchers found little evidence to support any significant relationship between teachers' subject matter knowledge in mathematics and effective teaching or students' learning (Romberg, 1988; Stein, Baxter, & Leinhardt, 1990). Three factors may have contributed to these conclusions.

First, teachers' subject matter knowledge was typically measured by degrees received, courses taken, and grades earned (Romberg, 1988, p. 228). Second, students' learning, or effective teaching, was usually defined in terms of student achievement measured by standardized tests. Third, researchers were looking for a cause and effect relationship between their measure of teachers' subject matter knowledge and their measure of effective teaching (Fennema and Franke, 1992).

The assumption that teachers' subject matter knowledge could be measured by degrees received, courses taken, and grades earned is thought by current educational researchers to be unwarranted. Lee Shulman (1986) was one of the first to challenge this assumption, reviving interest in the relationship between teachers' subject matter knowledge and instruction.

Shulman contrasted teacher certification examinations given in 1875 with those given in the 1980's. He identified a dramatic change in these examinations, noting that earlier teacher examinations essentially ignored pedagogical questions and emphasized subject matter knowledge. In current examinations content knowledge is all but ignored while general pedagogy is stressed. He called this tendency to ignore subject matter the "missing paradigm" in research on teaching because of the failure to appropriately take subject matter into consideration.

Shulman (1986) identified various components of the knowledge base necessary for teaching. Specifically, he proposed a theoretical framework consisting of three categories of content knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular content knowledge.

Subject matter content knowledge was characterized as more than the information a teacher knows. Shulman indicated that teachers needed to know about both the content of their subject and the structure of their subject, that is, how the content fits together. They needed to know "what" is true, "why" it is true, and why it is worth knowing in the first place (Shulman, 1986).

Pedagogical content knowledge was "the particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9). In other words, "good teachers not only know their content but know things about their content that make effective instruction possible" (Grossman, Wilson, & Shulman, 1989, p. 25). This aspect of content knowledge included having a variety of representations available for specific topics, understanding the intrinsic difficulties that are a part of these topics, being aware of the variety of conceptions, preconceptions and misconceptions related to specific topics, and knowing and using strategies that would be helpful in overcoming these misconceptions.

Curricular content knowledge, Shulman's third category, had three parts. First was knowledge and understanding of alternative curriculum materials available for specific topics. Second was lateral curriculum knowledge, or familiarity with the curriculum concurrently studied in other subjects. Third was vertical curriculum knowledge, or familiarity with what the students have been taught in preceding years and what they will learn in the future about specific topics.

Shulman's work stimulated researchers' interest in the issue of teachers' subject matter knowledge. It also changed the focus of investigation. Rather than measuring teachers' subject matter knowledge, researchers are attempting to document or assess teachers' level of understanding and how it relates to their teaching. Recent studies have been interpretive, describing teachers in action, attempting to understand the actions of teachers and students from their points of view (Smith, 1987, p.176).

There is agreement among researchers on the significance of the role teachers' subject matter knowledge plays in instruction. The specifics about how much or exactly what teachers need to know is a different matter (Fennema & Franke, 1992).

Grossman, Wilson and Shulman (1989) noted that teachers' subject matter knowledge affects both the content, what one teaches, and the processes of instruction, how one teaches. Similarly, McDiarmid, Ball, and Anderson (1989) observed:

> Teachers' capacity to pose questions, select tasks, evaluate their pupils' understanding, and make curricular choices all depend on how they themselves understand the subject matter. [Furthermore], . . . to develop, select, and use appropriate representations, teachers must understand the

> content of what they are representing, the ways of thinking and knowing associated with this content, and the pupils they are teaching. Such flexibility in creating access to knowledge, in turn, demands a much deeper and more critical understanding of subject matter than that needed simply to tell pupils what they ought to know. (p. 198)

Bromme and Brophy (1986) made similar observations about the consequences on instruction of teachers' shallow mathematics subject matter knowledge.

> Teachers must be well versed in mathematics in order to teach the subject effectively. Without such breadth and depth of mathematical knowledge, teachers are likely to rely too heavily on the textbook, to present the content in a fragmented way without sufficient explanation of key concepts or problem-solving strategies, and to be ineffective at individualizing instruction, diagnosing error patterns, or responding to unanticipated difficulties or opportunities that arise during instruction. (p. 123)

Leinhardt, Zaslavsky, and Stein (1990) concurred with these observations. In a study related to the teaching and learning of functions and graphs, they concluded:

> The teacher's subject matter knowledge empowers the teacher with the confidence and capability to make interconnections, build analogies, create examples, take intellectual excursions, and point toward future use and interrelationships. . . . Limitations on subject matter knowledge, on the other hand, often reduce the flexibility and creativity of a teacher as well as create a kind of authoritarianism toward the subject and student that permits little or no exploration of ideas. (p. 46)

These educational researchers maintain that the actions and thoughts of mathematics teachers while teaching are significantly influenced by their mathematics subject matter knowledge. But the relationship, though significant, is subtle. That is, it is often difficult to directly observe or identify the consequences of having or not having extensive knowledge of the subject, and it is certainly difficult to know what teachers are thinking as they teach.

The Rationale for a Case Study

Stein, Baxter and Leinhardt (1990) justified the use of a case study in investigations of teachers' subject matter knowledge.

> Investigations of teachers' knowledge as it relates to their instruction are needed to illustrate and advance important theoretical analyses of the role of subject-matter knowledge in teaching. . . . A central assumption of current research is that, in order to build a solid understanding of how teacher knowledge relates to instructional practice, we need to develop and draw upon detailed, qualitative descriptions of how teachers know, understand, and communicate their subject matter. (p. 640)

A case study can provide a description of how one teacher knows, understands, and communicates certain mathematics subject matter. As an illustration of the relationship between subject matter knowledge and instruction, it can enrich our understanding of purely theoretical explanations for such a relationship.

The Purpose of the Study

The purpose of this study is to seek a better understanding of the role subject matter knowledge plays in teaching mathematics. As the context for my investigation, I will identify and analyze the relationship between subject matter knowledge in mathematics and instruction in one teacher teaching division concepts and a unit on fractions.

Motivation for the Investigation

As a community college mathematics instructor teaching mathematics courses for prospective elementary school teachers, I observed that many students who completed these courses possessed a fragmented knowledge of mathematics. For example:

- Students had been exposed to many procedures and could usually perform these procedures accurately. But many were not sure when to apply which procedure, especially when working in the context of fractions or in solving “word problems” that required the operation of division. They did not connect procedures with applications.
- Division, fractions, decimals, ratio and proportion, dimensional analysis, and problem solving were all isolated topics that were covered in various chapters and appeared not to be connected in most students’ thinking.

Fragmented knowledge of mathematics was not, in my opinion, a satisfactory outcome for students who were planning to become teachers themselves. However, I was not certain how fragmented knowledge of mathematics fit into students’ mathematics subject matter knowledge nor exactly how it might affect future teaching of mathematics. The more general investigation of the role teachers' subject matter knowledge plays in teaching mathematics could shed some light on the relationship between teachers' fragmented knowledge of mathematics and their teaching of mathematics.

Specific Issues to be Addressed

Before presenting the case study, I will summarize several current analyses of the nature of teachers' knowledge, specifically highlighting subject matter knowledge, how it fits into the broader knowledge base necessary for teaching, and how it might be observed. The case study will take the form of a story about the experiences of one teacher, a participant in the Quantitative Reasoning Project, teaching division concepts and a unit on fractions.

CHAPTER II

THE SUBSTANCE OF MATHEMATICS TEACHING

SUBJECT MATTER KNOWLEDGE

A Summary of Six Current Analyses of Teachers' Knowledge

Six current analyses are representative of the post-1980 view of the knowledge base necessary for teaching. To help clarify the nature of subject matter knowledge, one of the components of the broader knowledge base, I will summarize these six analyses, emphasizing their characterization of subject matter knowledge.

Shulman (1986) proposed a theoretical framework for teachers' knowledge. His analysis consisted of three categories of content knowledge that he designated as subject matter content knowledge, pedagogical content knowledge, and curricular content knowledge.

Subject matter content knowledge was characterized as more than the information a teacher knows. Shulman indicated that teachers needed to know about both the content of their subject and the structure of their subject, that is, how the content fits together. They needed to know "what" is true, "why" it is true, and why it is worth knowing in the first place (Shulman, 1986).

Peterson (1988) expanded Shulman's framework with an analysis that accounted for the cognitive processes of both students and teachers. Her analysis described the thinking processes of students and teachers and the interaction of these processes. She suggested that teachers needed to know how students learn in general, and specifically how students learn their particular subject.

Furthermore, Peterson emphasized the need for teachers to be aware of how they learn and think about their subject, and to reflect on their own thoughts and actions as they teach. She indicated that without this self awareness, the knowledge of content that a teacher has will not be functional in helping students learn.

Romberg (1988) presented an analysis of the "professed knowledge" of mathematics teachers that blends and extends Shulman's and Peterson's analyses. He included three related categories of teacher knowledge:

- General knowledge of mathematics and knowing how mathematical topics relate to other topics within and outside of mathematics.
- Pedagogical knowledge that includes "understanding how students process, store, retain and recall information" (p. 228), and having available a variety of examples, instructional techniques, and instructional materials for each mathematical idea.
- Knowledge of how to manage a complex classroom situation that includes "a large number of students, a variety of resources, space, and an increasingly complex instructional technology" (p. 228).

Romberg addressed mathematics subject matter content in his first category. He highlighted the need for teachers to know how mathematical topics relate to other topics within and outside of mathematics.

Fennema and Franke (1992) suggested an analysis of mathematics teachers' knowledge that consisted of three components and a separate factor of teachers' beliefs about mathematics. The description of the first component, designated as the content of mathematics, resembled Shulman's subject matter content knowledge.

> [The content of mathematics] includes teacher knowledge of the concepts, procedures, and problem-solving processes within the domain in which they

> teach, as well as in related content domains. It includes knowledge of the concepts underlying the procedures, the interrelatedness of these concepts, and how these concepts and procedures are used in various types of problem solving. Crucial also to teacher knowledge of content is the manner in which the knowledge is organized, indicating teacher knowledge of the relationships between mathematical ideas. (p. 162)

Within a given context, the content of mathematics component interacted with the other two components, pedagogical knowledge and knowledge of students' cognitions. These three components combined with teachers' beliefs to "create a unique set of knowledge which drives classroom behavior" (Fennema & Franke, 1992, p. 162). The inclusion of teachers' beliefs was a distinguishing feature of this analysis.

Fennema and Franke (1992) provided an explanation of teachers' mathematics knowledge which seemed to summarize the descriptions of teachers' subject matter knowledge given in the other analyses.

> Researching teacher knowledge means more then investigating the number of mathematics courses teachers have taken or the procedural knowledge of mathematics they possess. Knowledge of mathematics teaching includes knowledge of pedagogy as well as understanding the underlying processes of the mathematical concepts, knowing the relationship between different aspects of mathematical knowledge, being able to interpret that knowledge for teaching, knowing and understanding students' thinking, and being able to assess student knowledge to make instructional decisions. (p. 161)

In the analysis of mathematics subject matter knowledge developed by Leinhardt and her colleagues (Leinhardt and Smith, 1985; Leinhardt, Zaslavsky & Stein, 1990), teachers had two organized knowledge bases:

- General teaching skills and strategies that are used in lesson planning and presentation.
- "Domain-specific information necessary for the content presentation" (Leinhardt & Smith, 1985, p. 248), or what the teacher knows about the content.

Fennema and Franke (1992) explained how these two knowledge bases were related.

> The skill of teaching, according to Leinhardt and her colleagues, is determined by at least two fundamental, related systems of knowledge: subject matter (content knowledge) and lesson structure (practical knowledge). The structuring of a lesson takes priority and is both supported and constrained by the teacher's knowledge of the content to be taught." (p. 157)

Leinhardt & Smith (1985) used semantic nets to represent concepts and relationships in a given domain such as a lesson on reducing fractions. They used planning nets to show how knowledge is combined into actions, and flow charts to show the algorithmic aspects of procedures. All three of these cognitive science research techniques, semantic nets, planning nets, and flow charts, were combined to analyze "the many facets of knowledge that are involved in a mathematics lesson" (p. 249).

Leinhardt and Smith (1985) described the system of knowledge they called subject matter or content knowledge as including "the concepts, algorithmic operations, the connections among different algorithmic procedures, the subset of the number system being drawn upon, the understanding of classes of student error, and curriculum presentation" (p. 247). Their description is similar to those given in the other analyses.

Ball (1989) looked at teachers' knowledge by asking "what kind" of subject matter knowledge was necessary for teaching rather than "how much" a teacher should know about various aspects of mathematics. Her analysis included four dimensions of understanding: knowledge of the substance of mathematics, knowledge about the nature and discourse of mathematics, knowledge of mathematics in culture and society, and the capacity for pedagogical reasoning.

Ball gave examples to help explain what she meant by these four dimensions. Her first dimension, substantive knowledge of mathematics, included three categories.

- Connectedness. For example, was division of fractions connected to the meaning of division that students had for whole numbers?
- Meanings underlying concepts and procedures. For example, the statement that division by zero is impossible was not sufficient to convey to students why this is the case.
- Correctness. Ball stated that what a teacher knows about mathematics should certainly be correct, but then amplified the notion of correctness by asking whether or not it is correct for a first grader to claim that zero is the smallest number.

Ball explicitly linked teachers' subject matter knowledge to the improvement of teaching and learning mathematics. She used every-day examples of the ways teachers need to understand mathematics to clarify her theory that "how" a teacher knows mathematics is more important than "what" a teacher knows about mathematics.

The review of these analyses of what constitutes knowledge necessary for teaching reveals that characterizing teachers' subject matter knowledge in mathematics is not a simple matter, let alone trying to understand how it relates to effective instruction. The analyses point to the naivete´ of earlier attempts to equate subject matter knowledge to the number of mathematics courses taken by teachers and grades received in those courses.

A Synopsis of Mathematics Subject Matter Knowledge

The facts about mathematics that teachers know--"what" they know--is one part of mathematics subject matter knowledge. This information about mathematics is organized in some way in teachers' minds, and the organizational structure is also part of mathematics subject matter knowledge (Ball, 1989; Peterson, 1988; Shulman, 1986).

It is difficult to analyze a mental organizational structure. The six analyses of teachers' knowledge indicate five specific issues pertaining to teaching that might give insight into the organizational structure of teachers' mathematics subject matter knowledge. These specific issues were considered in the case study to gain insight into the organization of the mathematics subject matter knowledge of the observed teacher.

- Mathematical connections. How teachers connect mathematical topics to other topics within and outside of mathematics (Ball, 1989; Fennema & Franke, 1992; Leinhardt & Smith, 1985; Romberg, 1988).
- Procedures and concepts. What meanings teachers give to procedures and how they communicate the concepts underlying procedures. How they explain why a procedure is correct, relate it to other procedures that students already understand, and use the concepts and procedures in problem solving (Ball, 1989; Fennema & Franke, 1992; Shulman, 1986).
- Reflectiveness. How teachers reflect on their own thoughts and action as they teach (Peterson, 1988).
- Lesson structure. How teachers structure their lessons. What their objectives are for the lessons. What they choose to include, what they omit, and how the material is sequenced (Leinhardt & Smith, 1985).
- Understanding student errors. How teachers understand students' thinking and respond to student errors (Leinhardt & Smith, 1985; Peterson, 1988).

CHAPTER III

THE CONTEXT AND DESIGN OF THE CASE STUDY

The Quantitative Reasoning Project

The case study was carried out in the context of the Quantitative Reasoning Project (QRP), a four-year National Science Foundation project. The QRP investigated two issues. The first was how students in grades six through eight develop algebraic reasoning as they are taught from a curriculum that encourages them to reason quantitatively. The second was to investigate the transformations in teachers' beliefs and knowledge necessary for them to teach a curriculum that encourages students to conceptualize situations in terms of relationships and quantities (Thompson, A., & Thompson, P., 1992, April).

The QRP is grounded in situations. Students reason about situations, focusing on the relationships and quantities embedded in those situations. The conceptual aspects of the situations rather than their computational aspects are emphasized.

A major goal of instruction in the QRP is to engage students in thinking about situations in which quantities and quantitative relationships are embedded. This requires that the teacher steer students away from thinking exclusively in terms of numbers, numerical operations, and calculational procedures. In other words, the students have to approach mathematics differently than they typically do in their school mathematics experience. This makes the task of implementing the QRP curriculum all the more difficult for teachers.

Two teachers participated in the project. They met with the leaders of the QRP twice a week for a year, starting in August, 1990. Discussions were informal, but always centered around matters of the mathematics and pedagogy of the project. These discussions were typically grounded in conversations about events in the sixth grade QRP class (the target class in 1990-91) and conversations about the ideas learned by or intended for the QRP students.

In addition to basic ideas about reasoning quantitatively and the content of the QRP curriculum, the participating teachers and the project leaders discussed four pedagogical principles that were fundamental to the teaching envisioned by the project (Thompson, P., & Thompson, A. 1992, April). The first was the need for teachers and students to hold instructional conversations. Instructional conversations are whole-class discussions in which the students were expected to explain their reasoning, ask questions, make decisions about assumptions, alternatives, and agreements, and to challenge one another on these issues. Through these conversations, students have opportunities to develop the complex reasoning patterns that are required to reason quantitatively and they also have occasions to make explicit their understandings of situations so they could reflect on them. (Thompson, P., in press)

The second pedagogical principle was the need to keep students' attention on numbers as values of quantities, asking "This is a number of what?" If students had to explain what numbers stood for, they would become aware of the relationships that existed between the quantities whose values were represented by the numbers. This not only would help students reason their way through a situation, but also would help prepare them to reason about quantities whose values are unknown, as they would do in algebra.

The third principle was to help students keep the task in mind by having the teacher ask the question "What are you trying to find?" every time an operation was considered. Students needed to consider what quantity their choice of arithmetic operation was actually evaluating. They needed to reflect on their choice of operation and its result.

Finally, the fourth principle was to explicitly identify the quantity that had been evaluated by an arithmetic calculation. To this end, the teacher needed to ask "What did this calculation give you?" Students needed to be able to identify the result of their calculation as a quantity.

The thrust of the meetings held by the QRP leaders and the teachers was not to give the teachers prescriptions of how to teach. That is, the QRP leaders did not tell the teachers what to do. Rather, the intent was to help the teachers internalize the ideas of the project so that these ideas would act as a guide for their instructional actions as they taught the QRP curriculum.

The Unit on Division

One segment of the QRP curriculum was the topic of division. In preparation for this unit, the QRP leaders and the participating teachers specifically discussed the goals and details of instruction during their biweekly meetings.

The Aim and Intended Instructional Development of the Unit

The aim of the division unit was to help the students develop a conceptual understanding of division. This was to be implemented by guiding the students through a four part sequence of tasks. First, the students were given situations in which division as sharing was embedded to explore or “make sense of”. They were to discuss the situations with each other and to figure out how to accomplished the sharing that was required in these situations.

Second, the students would make explicit what they had done to accomplish the sharing. To this end they were to communicate to other students and to the teacher how they had done the sharing. This communication required that the students figure out how to record with paper and pencil the actual methods they had employed to share. The intended effect was that students would develop personal methods of sharing that were connected to their reasoning.

Third, issues of efficiency in sharing would be addressed. That is, the students would discuss ways to streamline their sharing methods to develop more efficient methods for sharing.

Fourth, the students would encounter division in other than sharing situations and would generalize what they had previously thought and done to include these division situations. The students could then use their thinking and their personal recording procedures to generalize from division in sharing situations to division in any setting.

Blocks Microworld

The students who had seen, albeit not mastered, procedures for “how to” divide in previous years, used Macintosh computers and the Blocks Microworld program (Thompson, P., 1992a) to explore division as sharing. The teacher and the class were already familiar with Blocks Microworld since it had been used earlier in the year for reviewing decimal numeration, and addition and subtraction concepts.

Using Blocks Microworld, the students saw blocks on their screens in the shapes of cubes, flats, longs and singles, similar to Dienes’ base ten blocks. They also saw the particular block that represented the unit and the numerical value represented by the blocks on the screen. Figure 1 illustrates two cubes, one flat, two longs and seven singles representing 2 thousands, 1 hundred, 2 tens, and 7 ones, or 2,127, when a single represented the unit.

For dealing with sharing situations, a number of containers could be chosen and displayed at the bottom of the screen. The students' task was to share the blocks shown evenly among the given containers.

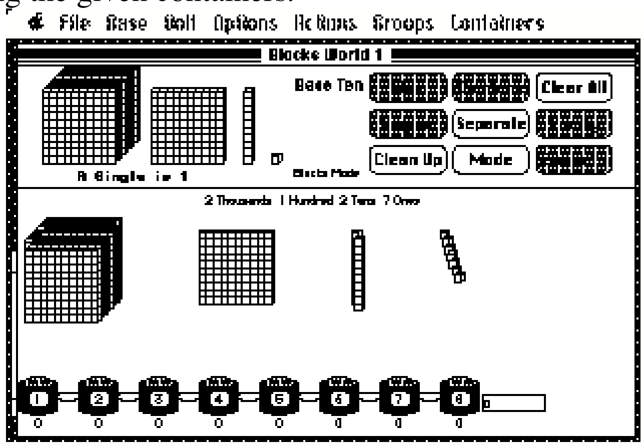

Figure 1

Students could evenly distribute blocks among the containers by selecting a number of similar blocks and dragging them down to the containers. If the selected blocks could be shared evenly among the containers, the value of the blocks in each container would appear beneath each container. The total value of the blocks in all the containers would appear to the right of the containers.

Students could also change the configuration of the blocks by using the "Unglue" command. For example, they could select and unglue the 1 flat, changing it into 10 longs. This would result in a configuration of 2 cubes, 0 flats, 12 longs and 7 singles as a representation for 2,127 as illustrated in Figure 2.

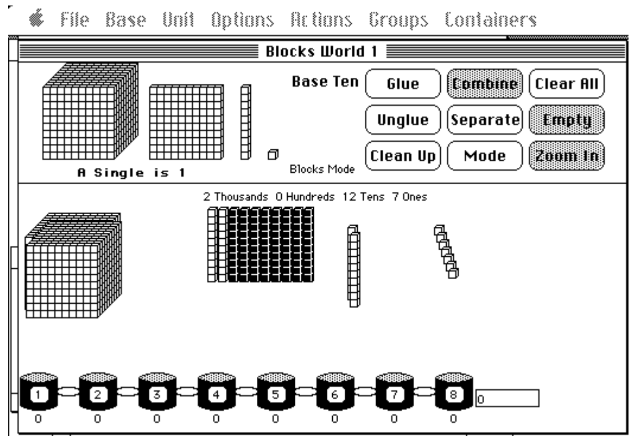

Figure 2

It would then be possible for students to share 8 of the 12 longs among the eight containers. This would put 1 long into each of the eight containers and would leave 4 longs left over. Below each container would be the number 10, showing that there was one ten in each container. To the right of the containers the number 80 appeared, showing that the value of all the blocks in all the containers is eighty. On the screen there would be 2 cubes, no flats, 4 longs and 7 singles left, indicating that they still had 2,047 left to share. Figure 3 shows the screen after this sharing has taken place.

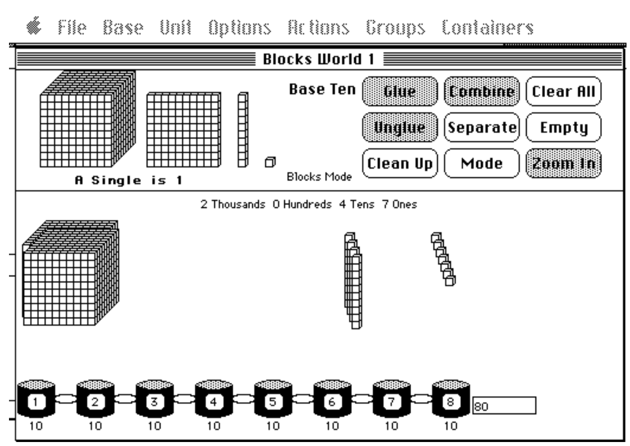

Figure 3

The Blocks Microworld program enabled the students to explore division as sharing in the context of reasoning about quantities or blocks. It also made explicit the connection between changes in the blocks representation of a quantity with changes in the numerical representation of that quantity. Thompson (1992b) explained this relationship between representations as follows.

> Blocks Microworld orients students to notational representations as things to be acted on in order to effect changes in blocks. The student intends to act on blocks, but that intention can be carried out only through actions on notation. It is hoped that, by this design, both blocks and numerals will be present in the student's experience at the moment of making a decision to act, and the student's decisions will be made according to systematic relationships between blocks, intended action on blocks, numerals, and actions on numerals. (p. 130)

Thompson stressed the importance that the relationship among blocks, numeral, and numerical value be supported by the teacher's instruction.

The Case Study

The case study took place in the context of the Quantitative Reasoning Project (QRP) during the units on division and fractions. It focused on the division segment of the QRP curriculum and addressed Mae, one of the participating teachers. It contributed to the second aim of the QRP, investigating the transformation in teachers' beliefs and knowledge necessary for them to teach a curriculum that encouraged students to conceptualize situations in terms of relationships and quantities (Thompson, A., & Thompson, P., 1992, April). It is against this backdrop that my analysis of Mae's teaching has been done.

Mae, the subject of the case study, was an experienced teacher who had been teaching sixth and seventh grade mathematics since 1988 at Local Middle School and who taught the sixth grade QRP course during the 1990-91 school year. Mae had twenty years of teaching experience. She had taught all subjects in the elementary grades as well as English, social studies, and mathematics as single subjects in middle school.

Mae’s Mathematical Preparation

Mae’s undergraduate degree was in elementary education with a social sciences major. She also had a master’s degree in education. Her master’s project involved studying student self-esteem in the classroom.

Mae’s mathematical content preparation consisted of an introductory high school algebra course, one semester of tenth grade geometry, and one undergraduate mathematics course for elementary school teachers required for the teaching credential. She dropped out of mathematics in high school after her first semester of geometry, and avoided mathematics during college except for the one mathematics course required for her education degree.

Mae pointed out that she had not originally intended to be a mathematics teacher, and said: “I’m definitely not what you [would] consider a math oriented person.” She explained that she had not studied to become a mathematics teacher in college, but in 1986 had obtained a special credential that allowed her to teach mathematics in grades one through eight. She earned this credential through an eighteen month program that “trained” (her word) credentialed teachers specifically for mathematics teaching. She had learned about this program from her supervisor while she was working as a part-time teacher. At that time there was a demand for full time mathematics teachers, and many of the participants in the program were attempting to obtain permanent full-time teaching positions.

The course-work for this special credential program consisted of courses in pre-algebra, algebra, geometry, trigonometry and one semester of calculus. Mae summarized her experiences with these courses in the following remarks: “It was like it [the course-work] was there and I did it. I probably did what the kids did, I learned it for the moment and that’s it. That tells you a lot.”

In Mae’s opinion the best instructor in the mathematics teacher training program was the one who emphasized hands-on experiences with manipulatives as well as projects to use in the classroom. Mae said that all the participants in the program knew they would be teaching mathematics in the near future and therefore were interested in practical methods of presentation that would help them communicate mathematics to their students.

Mae identified algebra as the subject she most enjoyed and as the course in the program that she found most useful, mainly due to its relevance for teaching pre-algebra in the middle grades. She also stated that she did not like theorems and had not used much of the material from the courses that emphasized theorems, like geometry, trigonometry, and calculus.

After receiving her supplemental credential for mathematics in 1986 and having taught mathematics exclusively for one year, Mae participated for three consecutive summers in a mathematics in-service program, the California Mathematics Project. She attributed certain changes in her teaching and in her attitude toward mathematics to this experience.

Mae described her mathematics teaching before the summer in-services as "traditional", and she added:

> Students entered the classroom and I began the class with a quick warm up of problems on the overhead. The emphasis was on the practice of the algorithm of multiplication, division, etc. Then I would correct homework and discuss questions the students might have. This was followed by a lecture on the day's topic and time for the students to work on their homework as practice.

She also described how her mathematics teaching had changed as a result of the three summers of in-service.

> My entire format and thinking of math changed completely. I liked teaching math originally because it was so exact and the agenda of the class was the same day in and day out. Now, every day is NEW and DIFFERENT. I find that students are excited because I am excited, and I try to make every lesson stimulating.

It appeared that the in-service programs had an affective impact on Mae that was reflected in her new excitement for the subject.

The Physical Setting

Mae's classroom conveyed her feeling of excitement. The walls were covered with mathematics posters, cartoons, and mottoes, and student projects were prominently displayed. A classroom set of calculators was easily accessible to the students. Around the ceiling was a very long string of Cheerios that one class had used to develop a feeling for large numbers. Mae had arranged the tables in groups rather than in rows to facilitate small group interaction. Often, manipulatives were found on the desks, ready for student use.

There was one Apple II computer in the classroom which Mae used to keep track of students' grades, and a Macintosh computer which she used as part of the QRP instruction. She used the Macintosh with an LCD panel and an overhead projector to project the Mac's screen onto a projection screen. Students could then watch what Mae did on her Macintosh as she spoke about what she was doing.

Around the perimeter of the classroom were sixteen more Macintosh computers that pairs of students used during parts of many class sessions. Prior to teaching the QRP class Mae reported having used school computers to demonstrate a geometry program and to provide some classes with incidental student use of both a logic program and a program designed to develop a feel for metric sizes.

An overhead projector was used daily in the presentation of lessons. QRP lessons often required the use of Mae's Macintosh computer and a computer screen image projector that would display her Macintosh screen on the overhead projector. Mae also used the overhead to encourage student participation, often calling on individual students to show their work on the overhead.

The Students

The QRP class for the 1990-91 school year was composed of average to below-average students according to the students' Comprehensive Test of Basic Skills scores. As a common practice above-average students at Local Middle School were placed in "enhanced" mathematics classes. Mae's sixth grade QRP class was an unenhanced class.

From the outset, it became apparent that most of the students in the QRP class were not accustomed to paying attention during the lessons either to the teacher or to other students. They often talked among themselves and were frequently observed off task. Most students appeared to be lacking interest in mathematics. Their motivation, in general, seemed exceptionally low.

Parental support was low at Local Middle School, but particularly so in the case of the target class. Attendance on parent night was sparse--9 out of 32 students were represented at the first parent meeting early in the school year. Lack of parental participation was noteworthy since this was a new school for the sixth grade students and this was the first opportunity for parents to meet the teachers. Furthermore, a special letter of invitation had been sent by the QRP staff to parents saying that the staff would explain the project and what the students would be doing. The main interaction Mae had with the parents of the QRP students was when she phoned them to report that their student was not doing his or her homework.

Mae's Teaching Style

In contrast to the parental lack of interest, Mae communicated to her students that she was concerned about them and wanted them to be successful in learning mathematics. Her teaching style was dynamic and lively. She spoke forcefully, used expressive gestures, and called students by name.

Mae demonstrated a willingness to participate in professionally related activities. In her school she had been a member of various committees that selected textbooks, chose standardized tests, and hired new teachers. She was involved in presenting in-service training to fifth grade teachers in her district on the use of manipulatives for teaching mathematics. She was active in the local mathematics council and was a member of a committee charged with implementing the NCTM Curriculum and Evaluation Standards (National Council of Teachers of Mathematics, 1989). At the time of this case study (1991-1992) she was field testing new curriculum materials in one of her sixth grade classes.

Mae's willingness to try new ideas and her receptiveness to mathematics reform as outlined in the NCTM Curriculum and Evaluation Standards (NCTM, 1989) and the California Framework (California State Department of Education, 1985) were part of the reason she was asked in the summer of 1990 to participate in the Quantitative Reasoning Project.

To summarize, Mae appeared to be an up-to-date mathematics teacher, committed to her students' learning, and enthusiastic about reform in mathematics teaching. She had the advantages of exposure to specific pedagogical and subject matter issues germane to the QRP and of regular interaction with the QRP leaders regarding both course content and the progress of her students. Her instruction on division and on fractions will help clarify our understanding of the sense Mae made of the curriculum, goals and orientation of the Quantitative Reasoning Project.

The Mechanics of the Case Study and Data Sources

Mae's QRP class was video taped daily during the spring of 1991. During the forty-nine taped lessons, she started and finished a unit on division and began a unit on fractions. These video tapes were analyzed and summarized.

In November, 1991, after Mae had completed the year with the QRP class, I asked her to view selected video tapes of her teaching the previous spring and to write answers to specific questions I provided. Mae was to furnish a reflective look at those lessons from her perspective. The questions were intended to probe her instructional objectives, her evaluation of the lessons and what she thought the students had learned, her own understanding of the QRP curriculum and its pedagogical basis, and how her own goals matched or did not match the goals of the QRP. I asked three types of questions:

- General questions: What did you hope the students would learn from this lesson? What were your intentions, goals, and objectives for this lesson?
- Specific questions: What did you mean here (in a specific incident on the tape)? What did you hope to have them learn from this (from a specific incident)?
- Evaluation questions: Do you feel that your goals for this lesson were met? If you had this lesson to do over, what might you do differently or in the same way?

I met with Mae on twelve occasions to discuss other questions I had formulated after reading her written responses to my initial questions on the video tapes. The interviews were intended to help clarify her instructional objectives, her understanding of the material she had taught, and her evaluation of both her communication of the material to her students and their understanding of that material. Examples of the written questions and the interview questions are in Appendices A and B.

During one of these interviews, Mae and I watched segments of several video tapes and tried to role-play the part of a student. That is, we tried to think like a typical student might be thinking as he or she was presented with Mae's teaching. This interview format was intended to help Mae see her teaching from the students' perspective and to help her reflect on her students' thinking. Since part of a teacher's subject matter knowledge is how that teacher understands his or her students' thinking (Peterson, 1988; Fennema and Franke, 1992), Mae's responses, which revealed how she understood her students' thinking both at the time the lessons were given and at the time of the interview, nine months later, provided insight into her subject matter knowledge.

In addition to my summaries of the video tapes, Mae's written answers to questions on the tapes, and my interviews with Mae which were recorded and transcribed, I also had access to the transcriptions of the biweekly meetings between the two participating teachers and the QRP leaders which provided information about Mae's interactions with the QRP leaders as they discussed matters of the curriculum and the pedagogical principles of the QRP.

Using the data from the video tapes and the transcripts of the meetings between Mae and the QRP leaders, I looked for incidents that illustrated similarities or differences between the intent of the lessons as described in the meetings and the actual lessons as Mae presented them. I used the information from the interviews and written questions to help clarify my observations and to understand how Mae perceived the incidents. Then I chose three specific incidents that I thought best illustrated the differences between what was intended by the QRP and what occurred in the classroom and analyzed them to uncover possible links between Mae's knowledge of the specific mathematics of these lessons and her instruction.

The mechanics of the study seemed reasonable in light of the question I was investigating. I was looking for insights into what changes in Mae's knowledge and beliefs might have been necessitated by her teaching of the QRP course. My observations of differences between the QRP's intent for the lessons and the actual lessons Mae presented highlighted areas where such changes in Mae's thinking would have been necessary, but had not occurred. Mae's responses to the written questions and interviews clarified my observations and indicated change or lack of change in Mae's knowledge and beliefs. My analysis will show that when changes had not occurred, Mae's subject matter knowledge may have acted as an obstacle. This suggests how a teacher's mathematical knowledge may have influenced her pedagogical actions, thus illuminating our understanding of the relationship between subject matter knowledge and instruction in mathematics.

CHAPTER IV

THE CONTRAST BETWEEN INTENT AND INSTRUCTION

The Intent of Instruction in the Division Unit

The Quantitative Reasoning Project (QRP) leaders' intent for the division unit was to help the students develop a conceptual understanding of division. This was to be implemented by guiding the students through a four part sequence of tasks.

First, the students were to explore division as sharing and to construct methods for sharing blocks having a given value evenly among a specified number of containers, within the constraints of the base-ten numeration system. To this end, the students were to use Blocks Microworld in pairs and discuss methods by which they might share blocks. The intended effect was that students would develop personal methods of sharing that were connected to their reasoning.

Second, the students were to make explicit what they had done to accomplish the sharing. To this end they were to communicate to other students and to the teacher how they had done the sharing. This communication required that the students figure out how to record with paper and pencil the actual methods they had employed to share. The intended effect was for the students to represent their personal methods in notation that would communicate their methods to others.

Third, issues of efficiency in sharing would be addressed. That is, the students would discuss ways to streamline their sharing methods to develop more efficient methods for sharing.

Fourth, the students would encounter division in other than sharing situations and would generalize their methods to apply them to these division situations. The students could then use their thinking and their personal recording procedures to generalize from division in sharing situations to division in any setting.

The reader should keep in mind that while guiding the students through this sequence of tasks, Mae was to treat situations as contexts for students to reason about and to focus the students' attention on the quantities and relationships among them that were embedded in these situations.

What Transpired in Mae's Classroom During the Division Unit

During the initial lessons of the division unit, students used Blocks Microworld (Thompson, P., 1992) in pairs to share a specified number of blocks among a specified number of containers. Mae and the QRP leaders circulated among the students, asking students questions about what they were doing and why they were doing it. Many students began the early lessons by ungluing all the blocks into singles, and then sharing them one single per container at a time.

To help students make explicit to themselves and others the methods of sharing they had used, Mae asked them in the fourth lesson of the division unit to write for another class a description in natural language explaining how to share a specified number of blocks evenly among a specified number of containers. No demands for a concise description were placed on the students, but they were encouraged to develop some notation that would make communication of their sharing methods easier and more efficient.

Notation, which Mae referred to as the recording process or recording, was further developed by the students in the next day's lesson when they were asked to figure out how to record with paper and pencil the actual methods they had employed to share blocks. Mae emphasized that alternative methods of solution and alternative methods of recording would be acceptable, and that there was not just one correct way to solve a problem and to record the solution.

During this lesson, as Mae helped pairs of students with their recording schemes, she observed that the students experienced difficulty as they attempted to make their methods for sharing explicit. They were not able to keep track of all the quantities involved in their solution process, that is, how many blocks they had shared, how many blocks were left to share and, in particular, the value of the blocks in each container.

That afternoon, in response to these difficulties, the QRP leaders suggested to Mae the use of a format to help the students record their solution methods. The next day Mae implemented this suggestion and gave the students the three column form shown in Figure 4 to assist them in keeping track of the quantities involved. This form provided a format for the students to organize and record their solution methods. The students used the three column form and continued to work on the development of a recording scheme for three more lessons.

Number of Containers	Number	Value of Each Container's Blocks

Figure 4
The Three Column Form

After eight class sessions devoted to sharing and recording, Mae tested the students. She asked them to use the three column form to write out two sharing tasks, and to explain in words what they had done on the first task. About half of the students were unable to finish the test during the class period and needed additional time to complete it. It should be noted that this was the only test given in the division unit.

Mae's Perspective on the Recording Process

To understand how Mae thought about the recording process, I devoted several interviews to discussing the first eight lessons that had focused on recording schemes. Mae indicated that she was at a disadvantage in helping the students with the development of procedures for recording their sharing method, stating: "I had no idea either what I was looking for [when helping the students individually], because I'm so ingrained that when you divide, you do it this way [using the standard algorithm]."

Mae's conception of division was clarified in an explanation she wrote in preparation for one of these interviews. She explained: "To do division means to follow the algorithm as we are used to seeing it. What does it mean to divide? It means we are sharing the number or whatever evenly among the groups specified."

Eight class periods had been devoted to the development of the individual recording schemes. I asked Mae to evaluate the use of time required for this development. She said the students had spent a lot of time producing methods that they would not use in the future. She was sure they would use the more efficient standard division algorithm. Mae concluded: "I'm not quite sure if it was worth having them try to come up with their own way of recording."

Mae seemed to envision what the students wrote as a procedure rather than as a communication of their reasoning about how they had shared. Her perspective on the recording process presented a stark contrast to the intent of the QRP for exploring division as sharing, constructing methods for sharing, and communicating these methods to others.

In a meeting with the QRP leaders, Mae had read a description from the NCTM Curriculum and Evaluation Standards (NCTM, 1989) that emphasized the need for such development of recording schemes.

> As they [the students] begin to understand the meaning of operations [in this case division] and develop a concrete basis for validating symbolic processes and situations, students should design their own algorithms and discuss, compare, and evaluate them with their peers and teacher. Students should analyze the way algorithms work and how they relate to the meaning of the operation [division as sharing] and to the numbers involved. (p. 95)

Mae's response to this excerpt from the NCTM Standards indicated that she was not convinced of the feasibility of this instructional development. She stated:

> I can't think of any teacher that follows that standard, that lets them discover a method and then have another student discuss with another student. . . . Mostly because it is so time consuming to do it that way. It's extremely time consuming.

Mae’s Explanation of Tasks with Remainders

The QRP leaders had suggested on January 9, 1991, that Mae give the students a sharing task with a remainder to see what sense they would make of it and what they might do to record it. They again raised the issue on January 21, 23, and 28. On January 28, Mae remarked that she was not ready to give problems with remainders: “. . . because I didn’t want to complicate matters for them.” On February 4, she presented a task with a remainder to the class, and demonstrated how the students should deal with it.

What follows is part of the transcript of Mae’s introduction of situations that entail a quotient and a remainder. It is offered as an illustration of three aspects of Mae’s instruction: her image of what she was teaching (and hence of what she wanted her students to learn), her understanding of a division algorithm as an accommodation to the constraints of a commitment to always represent numerical values within the decimal numeration system, and her understanding of Blocks Microworld’s relationship to decimal numeration and the constraints imposed by it.

The first excerpt is suggestive of how Mae’s students conceived of sharing tasks and Mae’s level of concern with their conceptions.

Excerpt 1: February 4, 1991, 10:46 a.m.

1. Mae: Some of you were saying to me as I was walking around, what do you do if you can’t share them all? And you said, “Oh, but you wouldn’t do that to us.” But, is that life?

2. Stu: Yes.

3. Mae: Do we always end up with nothing? Can you use everything up every single time?

4. Stu: No.

5. Mae: Ok, what are we going to do with that single that's left?

(Portion of transcript omitted.)

6. Mae: Do you have any idea of how we could share them [the blocks that are left]?

7. Sam: You could change the containers.

8. Mae: But if . . . Let me do a real problem. Ok? Jim, could you bring your chair here please. And Mary, could you come right here please? Thank you. Here are my four containers. And these are people that want part of what I have in here.[Mae is holding a bin of unifix cubes.] And I'm going to share these blocks with these people. Now, what I'm doing is, I'm going to demonstrate what Sam said.

Sam's remark [¶ 7] that "you could change the containers" seems to have meant "change the number of containers so that you can share everything that's left." Other students agreed with Sam's suggestion. Regardless of the mathematics of Sam's remark, it suggests that the students' understanding of Mae's sharing tasks was "get rid of all the blocks you start with" instead of "how many blocks will each container receive if they each receive the same number?"

Mae did not respond to the orientation reflected in Sam's remark; instead she tried to explain that this would change the problem. Mae enacted a problem of sharing ten unifix cubes among four students seated at a table directly in front of her. She gave each student two cubes, then gave the remaining two cubes to two of the students, asking if this was fair. Then she said that giving cubes to just two of the four students was like "changing the containers" and that this changed the problem. Many students still maintained that it was all right to change the number of containers.

Mae asked what they might do to share the remaining two cubes evenly. One student suggested cutting them in half, another student suggested giving each pair of students five cubes per pair. Mae acknowledged these responses as "one way" to share and continued asking for other ways--evidently with the hope that someone would think of them as blocks in Blocks Microworld. Finally, she raised the issue of Blocks Microworld herself, remarking that Blocks Microworld ("the program") could not split singles into smaller pieces. In the following excerpt Mae offers one solution to the inability to split a single: change it into a long and then split up the long.

Excerpt 2: February 4, 1991, 10:54 a.m.

1. Mae: Ok, in each container you have to put the same amount. What do you usually have left when you're all done and you have this remainder? What do you usually have that's left?

2. Stu: Singles.

3. Mae: Ok., they are singles usually because if they were longs, what would you do?

4. Stu: Break them up.

5. Mae: Break them up . . . You agree that you can break up a long? Well you're going to be able to go . . . when you're done with all the sharing, you can go into the computer program and you can change that single, and you can change it into a long. And then what can you do with the long?

6. Stu: Break it up.

7. Mae: Break it up! And then if you break up a long, what does it get broken up into?

8. Stu: Singles.

9. Mae: Ah! But we already have one, and it got split up, and what is it then?

10. Stu: (unintelligible)

11. Mae: All right. Here's the problem. I need to share what was a single. How much is a single worth? [Mae holds up one unifix cube.]

12. Stu: One.

13. Mae: So this [holding up one unifix cube] is one. I'm going to go into the computer program and I'm going to change it so that this [single unifix cube] is going to be represented now by a long [lowers the hand with the single unifix cube; raises the other hand with ten linked unifix cubes]. Just for an easy representation, because you can't split this [single unifix cube] up. You are right. Because the program has limitations. So now, what is this [ten linked unifix cubes] worth?

14. Stu: Ten? Eleven?

15. Mae: (Pause) This was a single before which is worth how much?

16. Stu: One.

17. Mae: I now traded and this is now a long which is worth what?

18. Stu: (Many students) Ten. Eleven.

(Some student) One.

19. Mae: David, one? All I'm saying is that you can't split this [single unifix cube] up, so I'm going to change what it's worth and it's going to be like this [ten attached unifix cubes]. And this is now one.

Mae succeeded in steering the discussion toward changing a single into a long and then breaking up the long. However, [¶ 14-18] identifying the values represented by the blocks after exchanging them was problematic for the students. Mae did not address the thinking of the many students [¶ 18] who had suggested that the long was worth values of ten or eleven. Once David [¶ 19] said that the long was worth one, she continued by demonstrating how she could share a single unifix cube among the four students at the table in front of her by replacing the single cube with the long formed from ten attached unifix cubes.

As Mae shared the ten cubes among the four students, one student identified that the value of the two blocks that remained after sharing was two tenths. Mae explained that these two singles could also be exchanged for longs and then shared as singles among the four students.

When asked about the value these singles would represent, students suggested four, four-fifths, and hundredths. The reasoning of the students who suggested four or four-fifths as values that might be represented by the singles was not discussed. Once hundredths were mentioned, Mae abruptly ended the discussion of unifix cubes and began to talk about Blocks Microworld.

In the next excerpt, Mae used her Macintosh computer, the LCD, and the overhead projector to demonstrate how to use Blocks Microworld repeatedly to change singles into longs and how to share the new longs among containers.

Excerpt 3: February 4, 1991, 11:00 a.m.

Mae has dragged one single into the blocks region to share among 8 containers. The display says "A Single is 1," but Mae does not mention this.

1. Mae: [Mae selects "Increase Unit", the single turns into a long, and Blocks Microworld changes its display to say "A Long is 1", but Mae does not mention this.]

   One. And you'd want to now unglue.

2. Stu: Could you increase it again?

3. Mae: Increase it again after. . . You can only do one at a time.

   Despite her statement, Mae nevertheless responds to this students question by selecting "Increase Unit" again, turning the long into a flat (The display also changed from "A Long is 1" to "A Flat is 1", but Mae did not mention this).

4. Stu: Whoa! What did you do?

5. Mae: Put it together. I increased it. But, if we had. . . [long pause; Mae erases the screen, getting rid of all the blocks]. Let me show you something. Let's bring down eight. How many blocks can I put into the containers?

In [¶ 3], Mae was hesitant to admit the possibility of selecting "Increase Unit" more than once at a time. It seems plausible that Mae's statement, "You can only do one at a time," stemmed from her having this procedure, or activity pattern, in mind: Get to a point where you have only singles to share, select "Increase Unit" (to turn singles into longs), unglue the longs (to turn longs into singles), and then share the singles.

This possibility seems further born out by Mae's evident disequilibrium when she found herself presented with a flat where before she had a long [¶ 5]. At this moment in the lesson, instead of demonstrating her intended activity pattern of increase-unglue-share, she would have had to demonstrate an activity pattern of increase-unglue-unglue-share (or some derivation of this pattern depending on how she herself segmented the entire situation from the beginning of the demonstration), which didn't fit her intentions--whence clearing the screen and starting with another situation involving only singles [¶ 5].

Mae had not mentioned Blocks Microworld's change of unit whenever she selected "Increase Unit". Students were confused about why a long was worth only one thousandth at the end of Mae's demonstration. Mae had not mentioned that the values of the blocks changed upon selecting "Increase Unit", only that singles are replaced by longs. In her two examples, she had selected "Increase Unit" a total of five times, so at the end of her demonstration Blocks Microworld's unit setting was "A Cube is 1/100." Students were confused about the current blocks' displayed value.

Excerpt 4: February 4, 1992, 11:02 a.m.

Mae continues demonstrating how to use the "Increase Unit" operation.

1. Mae: This, if I change . . . if I increase it a couple of times, and so instead of having it by one tenth it became what?

2. Stu: Eight. (unintelligible)

3. Mae: Thousandths place, but you can break it down. So don't keep going up and clicking "Increasing Unit" until you've done something with those, with that remainder you have. Don't keep going up and clicking and clicking because I clicked three times; I increased the unit and the long one wasn't broken down, was not a tenth, was not a hundredth, it became a thousandth. You don't want that to happen.

Mae's closing remarks [¶ 3], "don't keep going up and clicking . . . until you've done something with those, with that remainder you have" and "You don't want that to happen", again suggest that what she had in mind was an activity pattern of increase-unglue-share, where "increase" meant replace a single with a long. She appeared not to have in mind the activity of "increase" as systematically changing each block into a block ten times larger while simultaneously making the unit ten times larger, so that the collection's value remained the same.

Mae demonstrated a procedure that enabled students to represent fractional values with Blocks Microworld so that they could share a remainder evenly among the containers and, thus, share all the blocks on the screen. However, she did not make it clear that Blocks Microworld, in requiring the use of decimal fractions, was reflecting the constraints of base-ten numeration. Instead, the need to change the representation of the unit was presented as due to a shortcoming of the program, caused by the program's lack of ability to break up a single into common fractions like halves or quarters.

The entire class discussion on the change in representation of the unit from a single to a long reflected the difficulties the students faced in understanding the idea of letting a block other than a single have a value of one. It also reflected the difficulty Mae faced in presenting a coherent explanation. It was not always clear exactly what it was that she was representing with a long. Was it a single or was it the number one?

Further insight into this incident was found in the transcripts of a meeting between Mae, a QRP leader (L), and the second QRP teacher (T2) that occurred the afternoon after this lesson was presented. The reader should note indications of what Mae thought the students needed to learn and her evaluation of student understanding, and her understanding of Blocks Microworld's relationship to decimal numeration and the constraints imposed by it.

Excerpt 5: Meeting, February 4, 1991, p.m.

1. Mae: So today, we talked about what you call it when you have some left over, what do you do with this.[Mae describes how she shared the 10 unifix cubes among 4 children.]

(Portion of transcript omitted.)

2. Mae: [The discussion of today's lesson continues.] And I was trying to get across, "What would you do with the computer?" And they finally said, "The problem is that it's a single, and it's too small, you can't break it up. What could you do?" "You could change it." And some people said, "Well, you could go ahead and split it up." So we had a discussion about that and I showed them how you could do that.

3. L: How did that discussion go?

4. Mae: It went fairly well but I think they were still a little bit lost. I was hoping that the decimal would carry over because we had spent so much time on decimals, but I had a feeling they weren't quite sure what I was talking about.

5. L: What do you suppose was their . . . what is it, the hold-back?

6. Mae: I don't know, maybe if I had shown them a brief problem and had done the whole problem with them, maybe. . . but, I was trying. I was also thinking in the back of my mind that I really need to give these people enough time to finish their test.

Mae's goal [¶2] was to "get across, 'What would you do with the computer?'" She indicated that student understanding might have been improved if she "had shown them a brief problem and had done
the whole problem with them" [¶ 6]. Her image of what she was teaching seemed to be synonymous with the procedure on the computer which she had demonstrated to the class [¶ 2].

The meeting continued with a lengthy discussion of the choice of the problem, sharing ten cubes among four children, and the students' desire to express remainders as common fractions instead of as decimal fractions. Teacher 2 brought the discussion back to Blocks Microworld and what Mae had demonstrated to the class.

Excerpt 6: Meeting, February 4, 1991, p. m.

1. T2: Does the program that's on the computer have the thing he was talking about where you can go in and change a single to a long?

2. Mae: [Referring to the class response when she had changed the single to a long on the computer] They were like, "Ooh." I also showed them if you

> click it [the "Increase Unit operation] more than once, it keeps dividing that unit and making it smaller and smaller. Instead of breaking it up and getting tenths, two tenths, it ended up going to thousandths. I said, "What would that unit be?" I felt like we spent all that time with decimals. . . . [Mae continues to describe what she showed the class on the computer.]

(Portion of transcript omitted.)

3. T2: Does the program show. . . it can't show on the screen at the same time, a single that is one and a long that is one? You converted a single into a long so that you can divide it up. So you can have two different things representing one at the same time?

4. Mae: Well, what happens is that it's after you've already shared it all. You have singles and you can just click on actions and you can increase the unit; in other words, a single becomes a long, but its worth a ten. The others don't change at all, they stay. It's just that. . .

5. L. The idea that you're saying is that the single is still. . .

6. T2: One. Now the long is also one.

7. Mae: It is confusing.

Mae had been personally instructed in the use of Blocks Microworld by one of the QRP leaders and had used the program in instruction for three months prior to the division unit. She had also been given documentation for Blocks Microworld that included an explanation of the use of the "Increase Unit" operation in the "Actions" menu. Yet, in her eagerness to show the students "how to" share a remainder, she had not noticed that the block that represented the unit was always identified on the screen and changed when the "Increase Unit" operation was applied [¶ 6].

Furthermore, Mae said that [¶ 4] when you "have singles and you can just click on actions and you can increase the unit; in other words, a single becomes a long, but it is worth a ten. The others don't change at all, they stay." She thought that a single could change into a long without affecting anything else.

As the second QRP teacher pointed out [¶ 6], Mae seemed to think that both a long and a single could have a value of one at the same time. It appeared that she did not understand that the operation of “Increase Unit ”systematically changed each block into a block ten times larger while simultaneously making the unit ten times larger, so that the collection’s value remained the same.

What follows is a later portion of the same meeting, when the discussion returned to the topic of changing the representation of the unit.

Excerpt 7: February 4, 1991, p.m.

1. Mae: When you change it [the representation of the unit] on the computer does it show the long as one, on the screen itself? Do you recall? I don’t think it changes, does it?

2. L: I think it will show you that you have . . .

3. Mae: See, I don’t recall it showing anything. Because then, after we break that one down you want to change the unit again.

4. L: When you say to increase the unit and you only have one single there, what you’ll have on top is 1, right? It will say 0+0+0+1.

5. Mae: Right.

7. L: And I think when you go and you say increase the unit, right, I think then . . .

8. Mae: See, I’m thinking it still shows the same thing.

9. L: Well, we can try it. Do you want to try it on my computer?

10. Mae: Yes.

11. T2: It still shows the single as being one?

12. Mae: As being one, I think. Because for all the other problems it was one.

Mae seemed convinced that the single always represented the unit [¶ 1, 2, 9, 10, 12]. This may have been necessitated by her image of what she was doing: exchanging some singles for some longs, ungluing them, sharing them, and repeating the process with any singles that were left. The depth of her convincement

[¶ 8-12] may indicate the extent to which her strictly procedural understanding of sharing remainders affected her thinking.

After this meeting, there was no observable change in Mae's teaching. During the four subsequent lessons, which were devoted to recording tasks that involved quotients and remainders, Mae did not discuss with the class Blocks Microworld's identification of the block representing the unit.

It was the intent of the QRP that after the students had developed methods of recording that reflected their thinking, students would discuss ways to streamline their sharing methods to develop more efficient methods for sharing. In the fourteenth lesson, Mae told the students to share as many blocks as possible in each step of their processes and to write numerical values like "200" instead of writing "2 flats". There was no class discussion as to why these suggestions would result in efficiency nor why efficiency would be desirable. This was the extent of Mae's attempt to address the issue of efficiency.

Mae Shares Chairs

After the fourteenth lesson, the tasks given to the students changed from sharing a given number of blocks among a specified number of containers on the computer to word problems worked with paper and pencil. The change of task was intended by the QRP leaders to help the students generalize their understanding of division as sharing.

In the remaining three lessons of the division unit, Mae presented a total of seven word problems that she had selected. The pattern of most of the word problems was to share a given number of objects (like apples, chairs, or donut holes) among a specified number of recipients (like baskets, rooms or children).

In one of these three lessons Mae presented the students with the following situation: "We have 462 chairs in room B. We want to keep 30 there and put the others evenly into 13 rooms. How many chairs should we put in each room?"

The transcript of the ensuing class discussion which took place on February 14 beginning at 10:03 a.m., highlights Mae's implementation of the QRP objective to extend and generalize the students' concept of division using situations as contexts for reasoning. This episode also illustrates the effect Mae's understanding of Blocks Microworld and base-ten numeration had on her attempts to generalize division.

Excerpt 8: February 14, 1991, 10:10 a.m.

1. Mae: [Standing by the overhead projector, she has stated the task: share 432 chairs evenly among 13 rooms.] I'm up here now, tell me what to do. Tell me what we get.

2. Stu: (unintelligible)

3. Mae: 432 what?

4. Stu: Chairs.

5. Mae: All right. We are now going to find a spot for these chairs. How many rooms do I have?

6. Stu: 13.

A lot.

7. Mae: We're only using 13, right. Because some don't have the space for it at all, and so they just said, you know, "13 classrooms." Now what do we do? Come on guys, what do we do?

8. Stu: (unintelligible)

9. Mae: Split them up.

Mae began the discussion by asking the students [¶ 1]: "Tell me what to do. Tell me what we get." The situation of placing chairs in rooms appeared to be a context for doing something [¶ 1, 7], rather than a context for reasoning.

Mae asked for a student volunteer to show on the overhead projector what to do to solve the problem. Students discussed why they only needed to share 432 chairs and asked questions about the volunteer's work.

Excerpt 9: February 14, 1991, 10:11 a.m.

1. Mae: It's perfectly fine what he's doing. What's the 13?

2. Stu: 13 classrooms.

3. Mae: Thank you.

4. Stu: What's the 30?

   The 30 because you already took away 30.

5. Mae: Do we have to worry about that 30 then?

6. Stu: No.

   (unintelligible)

7. Mae: [A student works on the overhead showing how he would do the sharing. He begins by writing down 462 – 30 = 432. Mae interrupts him and holds up a classroom chair.] That's all right. 432. Now I want to tell you something guys. When we use computers we share blocks. These aren't blocks these are what?

8. Stu: Numbers.

   Chairs.

9. Mae: These are chairs. And I will tell you, Mrs. Hancock would be very unhappy with us if we took these chairs and decided to take them apart and unglue them. You can not unglue chairs!

In what respect does it make sense to think of "ungluing" chairs? Recall that in Blocks Microworld "Unglue" was the operation the students had used to change one block into ten of the next smaller kind. It appeared that Mae, and possibly the students, had made a connection between a chair and a cube, based, perhaps, on similarity of shape. Making a connection between cubes, which represented values in Blocks Microworld, and chairs was an indication that Mae thought of her previous activities as being about blocks as objects and not as being about blocks as representations of numerical value.

Looking back on this episode in an interview, Mae explained that her intention had been to remind the students that fractional parts of a chair would not be sensible. She said she had wanted the students to realize that a sensible answer would have been 33 chairs per room with 3 chairs left over. The remaining 3 chairs could not be shared using decimals. In the context of the video taped lesson, however, when Mae held up the chair, the students had just begun to think about how to share the set of chairs among 13 rooms and were not yet concerned about remainders.

Whatever Mae was thinking, her comments directed the students toward thinking in terms of chairs as blocks and not in the direction of generalizing the process of numeration-constrained sharing. This was the opposite of the QRP intent for extending and generalizing.

<u>Mae Presents the School-weeks Problem</u>

Another episode from the final three lessons on division that contrasts Mae's teaching with the QRP intent for extending and generalizing is found in the sixteenth division lesson. Mae presented the following situation to the class: "The school year for teachers consists of 184 school days. How many weeks is that?" This problem was selected by Mae and had not been discussed with the QRP leaders prior to its presentation.

The following transcript is offered as evidence of Mae's understanding of division, her use of situations in teaching, and her tendency to address calculations and to ignore conceptual issues.

At 10:49 a.m., Mae showed the problem on the overhead projector and asked a student to read it aloud. A lengthy discussion followed in which Mae directed the class to think of five-day weeks and to interpret the problem as asking "how many five-day weeks are in 184 school days". She begins trying to get the students to think about the problem in terms of objects to share and containers among which to share them.

Excerpt 10: February 19, 1991, 10:52 a.m.

1. Mae: So, the question is . . . how many did each receive? What's receiving the things this time?

2. Stu: Weeks.

3. Mae: How many?

4. Stu: Weeks.

   Days.

5. Mae: What's our container this time?

6. Stu: [The students look perplexed.]

   (unintelligible)

   Monday through Friday.

7. Mae: Monday through Friday? What's this right here? [Mae points to the number 184 on the overhead.]

   Stu: Days.

8. Mae: And we're going to put it into groups of what?

9. Stu: Five.

10. Mae: And we need to know how many will go into every (pause). How many days will go into every what?

11. Stu: Weeks.

12. Mae: Right. (Pause) So let's try doing it like this. Yes, you're writing this part down here.

Mae talked about sharing days among containers [¶ 5, 10], which she identified as representing the weeks in the problem [¶ 10-12]. She paused several times during the discussion [¶ 1, 10, 12], apparently troubled by the difficulty of establishing an analogy between the problem and the model of division as sharing that the students had been using. The reader should note that a more appropriate analogy would have been: "We are going to put five days into each container (week), but the problem is that we do not know how many containers we will need."

In the process of writing the solution to this problem on the overhead projector [¶ 12], Mae abandoned the three column form that the students had developed over the last twelve lessons. Instead, she used the standard algorithm for division, writing $5\overline{)184}$ and told the students to copy what she had written [¶ 12]. The students had not previously seen the standard algorithm in the QRP course, and Mae did not relate it to the three column form used up to that point. Interestingly, none of the students asked about the change.

It should be noted that in response to an earlier inquiry by Mae, the QRP leaders had suggested some ways Mae could relate the standard algorithm to the three column method if the students happened to ask about the standard algorithm. The QRP leaders had also emphasized that it would not be necessary or productive for Mae to present the algorithm.

Mae then begins to use the standard algorithm on the overhead projector, but continues to talk in terms of containers.

Excerpt 11: February 19, 1991, 10:53 a.m.

1. Mae: Ok. How many groups of . . . Remember our questions. How many did each [container] receive? How many did you use? How many are left? And how many total has each [container] received? Ok, we're going to keep going back to those questions. Ok,. I need five days to start off with. Do I have five days that I can share?

2. Stu: Yeah.

   No.

3. Mae: Can I share more than five days?

4. Stu: No.

Yeah.

5. Mae: How many do you think I can share?

Mae continues trying to establish a parallelism between what she is doing and the standard algorithm [¶ 1], but keeps having difficulty with the idea of sharing objects among an unspecified number of containers. The student responses [¶ 2, 4] and their facial expressions are indicative of a high level of confusion. In the next excerpt, Mae continues to ask the students questions about how much they can share.

Excerpt 12: February 19, 1991, 10:55 a.m.

1. Mae: How many can I share in (unintelligible) . . . Come on, kids, it's not that hard. Five.

2. Stu: You can take twenty-five.

3. Mae: Can I take twenty-five? Can I take twenty?

4. Stu: Yeah

5. Mae: Twenty. And, instead of putting it on the side I'm going to put it on top. [Mae writes 20 above the line drawn for the standard division algorithm.] And if I put twenty in every single one, how many would I have used up?

6. Stu: Four. Four school days.

7. Mae: These are days now. If I put twenty days in each one of these weeks, or in each one of these, uh . . .

8. Stu: Containers.

9. Mae: Containers. (unintelligible) We have so many. . .

(Portion of transcript omitted.)

10. Mae: If I put twenty school days and I group them together, and I put one into every single of these . . .

11. Stu: Weeks.

12. Mae: But it's not weeks though, is it? We want to put them in groups of five. We're looking for how many weeks right? Think about it a second.

In attempting to use both the standard algorithm and the model of division as sharing objects among containers, Mae found herself putting twenty days into each week [¶ 7], which did not make sense. Mae saw that [¶ 12]: “We want to put them in groups of five. We’re looking for how many weeks, right?”

At this point in the lesson, a student suggests using a calculator to do the division and claims that the answer is “Thirty-six point eight.” Mae asks the class what 36.8 means, and they reply: “Thirty six weeks and eight days left over.” The reader should note that if one were to share days among a specified number of containers, then the remainder should, indeed, be days. Thus, the difficulty of the students is a reasonable difficulty. Mae apparently interpreted the students’ difficulty as stemming from lack of understanding of decimals and used this occasion to review decimal notation.

In response to Mae’s questions about the meaning of “36.8”, and in particular the meaning of the numeral eight, the students suggest that it could represent eight, eight days, eight weeks, eight hours, or eight minutes. They appear to be playing a guessing game with little or no idea of what Mae is asking. In the next excerpt, Mae finds it necessary to ignore the problem in order to address issues of decimal notation.

Excerpt 13: February 19, 1991, 11:00 a.m.

1. Mae: Ignore the problem for a second.

2. Stu: Eight people.

3. Mae: Ignore the problem for just a moment. If I said to you, "Read this number. Just read this number. What would it mean?”

4. Stu: Thirty-six and eight-tenths.

5. Mae: Thirty-six and eight-tenths. That means . . . Let's put it back to this problem now. Thirty-six weeks. We're in weeks, right, that's the unit, and . . .

6. Stu: Eight-tenths days.

7. Mae: Is that our unit?

8. Stu: I don't know.

When taken out of the context of the problem situation, the number “means” thirty-six and eight tenths [¶ 3-5], but what is that? The discussion continues until Mae hears the students say that the calculator result means “thirty-six and eight tenths weeks”.

After the students agree that “36.8” means “36.8 weeks”, Mae uses the standard algorithm, with no references to containers this time, to get the same result.

Excerpt 14: February 19,1991, 11:01 a.m.

1. Mae: [Mae has completed using the standard algorithm to divide 184 by 5.] How many do I have?

2. Stu: Thirty-six.

3. Mae: Thirty-six what?

4. Stu: Thirty-six weeks and you have eight-tenths left.

Note that thirty-six and “eight-tenths left” [¶4] resembles what was previously accepted as the correct answer when the class had discussed the “meaning” of the calculator result. However, it is not the answer Mae wants.

Excerpt 15: February 19, 1991, 11:02 a.m.

1. Mae: I don't have eight-tenths left. What do I have left?

2. Stu: Four (unintelligible)

3. Mae: Four what?

4. Stu: You just have four.

Four-tenths.

(unintelligible)

5. Mae: What are those here? What are those [pointing to “184”]? Those are days. So I have four days.

6. Stu: Four days. (in unison with Mae)

7. Mae: Now, in my head I know now. Ok, I'm going to work thirty-six weeks and I'm going to work four days of another week. But if you give me that answer of thirty-six and eight-tenths weeks, I'm like, "Eight-tenths?" It doesn't really answer what I'm looking for. Right?

8. Stu: Right.

This time the remainder was four days. Mae did not address how the remainder of 4 days and the remainder of eight tenths of a week were related. The problem had been “find out how many weeks were in 184 school days.” The answer “36.8 weeks” actually answered this question, whereas “36 weeks and 4 days” did not.

Mae commented [¶ 7]: “Now, in my head I know now. Ok, I'm going to work thirty-six weeks and I'm going to work four days of another week. But if you give me that answer of thirty-six and eight-tenths weeks, I'm like, ‘Eight-tenths?’ It doesn't really answer what I'm looking for. Right?” This may be indicative of Mae’s mastery of decimals. It appeared that Mae much preferred to think in terms of whole numbers, 36 weeks and 4 days, rather than in terms of decimals, 36.8 weeks.

Situations were intended by the QRP leaders to be contexts for reasoning. In contrast, Mae seemed to use this situation as a context for calculating and for presenting the standard division algorithm. To use the standard algorithm, Mae had to abandon the three column form that the students had been using throughout the entire division unit. She made no attempts to relate the standard algorithm with the three column form, and was unsuccessful in her attempts to use the standard algorithm with the objects and containers of division as sharing.

Mae also abandoned the sense of the problem situation. She referred to containers and how many days each container received. However, in this case the students knew there were 5 days in each container, but needed to figure out the number of containers. Neither Mae nor the students identified that the sharing model for division, when one assumes beginning with a known number of containers, did not fit the situation given in the problem.

Later, after Mae had watched the video tape of this lesson, I asked her how to write about how this problem differed from other division problems she had previously done with the class. She wrote:

> This is a problem that seems more real to the students where they can use the information in their own lives. You cannot actually put days into a cup to share, nor can you really divide up the remaining days. This problem has some variables, which we did not take into account, which could have made the problem even harder. Example: There are holidays in some weeks, which would make for even more weeks.

We discussed this problem again in a later interview where Mae indicated that she had experienced difficulty with it while she was teaching. She explained:

> “I think that it was very difficult to think about the containers. I remember giving this problem and going, at the time kind of stuttering over it thinking, ‘oops’. . . it doesn’t feel right.”

Mae continued her explanation and drew a conclusion about the cause of her difficulty:

> “The problem was visualizing that there would be a problem at the end. What do you do with some that are left over?”

Mae’s lack of comprehension was evident in her interview responses. Her remarks were also revealing of her tendency not to reflect on her teaching, in particular, her tendency not to examine the source of critical instructional incidents such as the one described here. Her lack of reflection was manifested in various ways: First, when Mae wrote the problem, she did not consider how it fit in with the sharing model of division. Second, after she presented the problem to the class, she did not reflect on why it did not “feel right”. Third, when I asked her to write about the problem, her response was calculational in orientation and did not mention any conceptual difficulties she or the students had with it. Fourth, when Mae later discussed the problem in an interview, she still did not know why its presentation had been awkward.

The day after the presentation of the School-weeks Problem turned out to be the final day of the division unit. It was not planned as such by the QRP leaders. But once Mae had presented the standard algorithm, she seemed to feel that she was finished with division.

On the final day of the division unit, the class discussed one last sharing problem, sharing 60 donut holes among 23 students, which Mae solved using the standard algorithm. Mae said to a student who had suggested using half donut holes to share the remaining donuts : “Don’t tell me one-half if we’re doing decimals”. Once the problem was solved using decimals to express the remainder, Mae used the idea of remainders expressed as halves as a transition into the unit on fractions. This transition marked the rather abrupt end of the division unit. There was no summary discussion, no tying together of division concepts, nor any evaluation of the students’ understanding of division.

The remaining thirty-two video taped lessons spanning class meetings in February through April of 1991 document the beginning of the fraction unit, which lasted until the end of the year. Mae, not the QRP leadership, had the primary responsibility for the organization of the unit and the preparation of instructional materials. However, the biweekly meetings between Mae and the QRP leaders continued, allowing Mae the opportunity to ask questions and get feedback and advice on what she might do with the unit.

After summarizing the video tapes of the fraction unit, it seemed to me that the fraction unit was disjoint from the division unit. No connections were explicitly made between fractions and division even though these connections had been discussed in meetings with the QRP leadership.

Mae's approach to teaching fractions and operations with fractions was quite different from the pedagogical approach suggested by the QRP leaders for the division unit. Recall that the suggested development for the unit on division was to explore situations and construct a solution method, record and communicate the method, make the method efficient, and extend and generalize the method.

In contrast, notation and algorithms for fractions were presented to the students and illustrated through the use of manipulatives. Mae noted in the interviews that the students had difficulty making the transition between what they had done with objects to what they wrote with numbers. That is, their reasoning about objects was not connected to the notation they had been given.

CHAPTER V

THE RELATIONSHIP BETWEEN MAE'S MATHEMATICS SUBJECT MATTER KNOWLEDGE AND HER INSTRUCTION

In chapter four I highlighted the difference between the intent of the Quantitative Reasoning Project (QRP) leaders and Mae's teaching. In this chapter I propose a possible cause for the observed mismatch: Mae's subject matter knowledge of mathematics. First I will characterize Mae's mathematics subject matter knowledge. Then I will illustrate how her subject matter knowledge influenced her teaching, resulting in the observed disparity between the QRP intent and the content of the lessons.

The reader should recall that Mae was willing to try new things in teaching. Her lively and dynamic teaching style and her contemporary classroom arrangement indicated that she was an up-to-date mathematics teacher, enthusiastic about mathematics reform.

The Characterization of Mae's Subject Matter Knowledge

Mae's mathematics subject matter knowledge can be characterized in terms of objects and actions that she envisioned being applied to those objects. This was illustrated in the discussion of sharing chairs in which Mae appeared to make a connection between a chair per se and a block per se [Ch 4 Excerpts 8 and 9]. She stated: “When we use computers we share blocks. These aren’t blocks, these are what?” Students suggested “numbers” and “chairs”. Mae continued: “These are chairs” [Ch 4 Excerpt 9 ¶ 7-8]. Chairs were objects in the problem Mae had presented while blocks in Blocks Microworld were intended to always be portrayed as representing numerical value. Mae’s connection between the two indicated that she thought of her activities with Blocks Microworld as being about blocks as objects and not about blocks as representations of numerical value.

When Mae told the class: “You can not unglue chairs!” [Ch 4 Excerpt 9 ¶ 9], she indicated that the physical ungluing actions they had used with blocks would not be appropriate to use with chairs which could not be cut up into pieces. Mae did not mention that ungluing had been applicable to blocks because actions on blocks were actions, constrained by base-ten numeration, on numerical values. Sharing a quantity (chairs in this case) with a numerical value of 432 among 13 containers (rooms in this case) could be accomplished the same way as sharing 432 represented by blocks among 13 containers.

The word problems Mae chose to help students extend and generalize their understanding of division as sharing reflected her tendency to think of objects and patterns of action which employ objects. The problems she portrayed fit a common theme: share a given number of objects, like pieces of paper, among a specified number of recipients, like children, and were easy to visualize as physical sharing. As with sharing the chairs among the rooms, Mae talked concretely in terms of moving objects around into equal piles to accomplish the sharing. She did not address the abstract idea of division as mental partitioning, constrained by base-ten numeration, of some quantity that had a numerical value.

Mae's understanding of mathematics appeared to involve actions that were very precise and could be used only in specific situations. For example, the QRP leaders had repeatedly encouraged Mae to introduce a sharing task that resulted in a remainder as an open ended task in order to see what sense the students would make of it. Mae decided to present such a task in conjunction with a specific pattern of actions, "increase-unglue-share", that would produce the solution. The specific type of problem, division with a remainder, had a precise pattern of actions that accompanied it.

Mae's application of precise actions is further illustrated in her presentation of the school-weeks problem [Ch 4 Excerpts 10-15]. The situation described by the school-weeks problem did not fit the actions connected with the conception of division as sharing that Mae tried to use to solve the problem [Ch 4 Excerpts 10-12]. The students' puzzled expressions as Mae attempted to lead them through the solution of this problem reflected their confusion, thus attesting to the incoherence of her explanation. Moreover, her incoherent explanation could be taken as a reflection of how tightly her own understanding of division was tied to the particular actions she was trying to implement. She continued to apply the actions even when they resulted in 20 days being shared in every week which did not make sense [Ch 4 Excerpt 12 ¶ 7-10].

When Mae was unable to use the action-pattern related to sharing days among containers, which were weeks, she reverted to another pattern, the standard written division algorithm, to solve the problem. In preparation for an interview in which she would discuss the meanings of division, Mae wrote:

> To do division means to follow the algorithm as we are used to seeing it. What does it mean to divide? It means we are sharing the number or whatever evenly among the groups specified."

This is illustrative of Mae's tendency to think in terms of actions on objects, "sharing [an action] the number or whatever [objects] evenly among the groups specified" and her understanding of division in terms of following precise action-patterns like the standard algorithm.

A commitment to using a precise pattern of actions is also illustrated in the last lesson in the division unit. Mae said to a student who had suggested using half donut holes in a sharing situation: "Don't tell me one-half if we're doing decimals." It is possible that in Mae's thinking, the actions being taken to share the donut holes paralleled the actions taken to share blocks and thus required the use of decimals, even though in the situation of sharing donut holes, using halves was more sensible than using tenths.

Base-ten numeration and decimal numeration were problematic for Mae. She told the students that her procedure, "increase-unglue-share", used to share remainders in Blocks Microworld was necessary because "You can't split up [unglue] a single" [Ch 4 Excerpt 2 ¶ 13, 19 and Ch 4 Excerpt 4 ¶ 2]. When the students suggested using common fractions to share remainders, Mae acknowledged their suggestion as "one way" to share the remainder, but she did not pursue their suggestion. Instead, she said that there was a "problem with the program" that would not allow them to use common fractions [Ch 4 Excerpts 1,2,5]. There, in fact, was no flaw in the program. Rather, Mae had never noticed that Blocks Microworld always portrayed numeric value as a ratio relationship between blocks. The program "had a shortcoming" only insofar as one interpreted its depiction of blocks as a depiction of objects. Blocks Microworld was deliberately designed to reflect the constraints of base-ten numeration. Mae did not grasp the role numeration played in the program or in the division algorithm that she wanted to teach.

Another illustration of Mae's lack of comfort with decimal numeration was found at the end of the school-weeks problem [Ch 4 Excerpt 13 ¶ 7]. Mae said:

> "I'm going to work thirty-six weeks and I'm going to work four days of another week. But if you give me that answer of thirty-six and eight-tenths of a week, I'm like, 'Eight-tenths?' It really doesn't answer what I'm looking for."

Mae preferred to think in terms of whole numbers rather than in terms of decimals.

To summarize, Mae's mathematics subject matter knowledge can be characterized in terms of objects and actions she envisioned being applied to those objects. Furthermore, the actions on objects were very precise and could be used only in specific situations.

<u>The Effects of Mae's Subject Matter Knowledge on Her Instruction</u>

Mae's dynamic teaching style and use of up-to-date classroom techniques would make it easy for the casual observer of her class to miss the conceptual weaknesses in her instruction. When Mae attempted to teach ideas it became clear that her instructional actions were both guided and constrained by her fundamental images of what she intended students to learn: action patterns. The following examples illustrate the effects of this aspect of her mathematics subject matter knowledge on her unit organization, her lesson plans, group discussions, and interactions with individual students.

The overall organization of the division and fraction units reflected Mae's image of mathematics as consisting of a set of precise actions that applied to specific situations. Mae's organization of the division unit seemed to be based on procedural simplicity. First, she presented division problems with no remainder; next, division problems that result in a remainder; and finally, word problems. Each type of problem would require a different, though perhaps related, action-pattern for solution. This organization of the division unit took little account of the one suggested by the QRP leaders. They encouraged the development of division algorithms be organized as exploring, recording and communicating, and extending and generalizing. Similarly, Mae's organization of the fraction unit seemed to be centered around the type of manipulative being used, where she portrayed each manipulative as suggesting a distinct pattern of actions related to solution procedures.

An orientation to mathematics as action-patterns may also have contributed to the structure of individual lessons. Lessons that focused on solution methods, procedures, and correction of student errors rather than on situations, reasoning, and expressions of reasoning. Situations were used as contexts for doing and computing rather than contexts for reasoning or giving meaning to the doing and computing. It seems plausible that Mae taught sequences of actions because that is what she had in mind for students to learn.

That Mae had difficulty initiating and sustaining the instructional conversations which were one of the basic pedagogical principles of the QRP was reflected in the incoherence of many of her explanations as documented in the excerpts in chapter four. What Mae knew about mathematics was closely tied to objects and actions on objects. She could discuss how to do specific procedures and how to avoid common errors in carrying them out, but she lacked the language required to discuss ideas and concepts.

Stein, Baxter, and Leinhardt (1990) connected subject matter knowledge with meaningful student discourse.

> In general, teachers with more explicit and better organized knowledge tend to provide instruction that features conceptual connections, appropriate and varied representation, and active and meaningful student discourse. On the other hand, teachers with limited knowledge have been found to portray the subject as a collection of static facts; to provide impoverished or inappropriate examples, analogies, and/or representations; and to emphasize seatwork assignments and/or routinized student input as opposed to meaningful dialogue. (p 641)

When Mae attempted to help individual students develop notational methods for recording the results of their concrete sharing methods, she emphasized the recollection and application of previously learned procedures for recording. Recording was intended to reflect the students' reasoning and the numerical effects of their actions as they had shared the blocks on the computer. It was not intended to be an extension of previously developed notation nor a polished algorithm that could be efficiently used to solve a particular type of problem. Mae said in response to an interview question about the goals she had for students as they developed their personal notational methods for recording their sharing:

> I had no idea either what I was looking for [when helping the students with their individual recording procedures], because I'm so ingrained that when you divide, you do it this way [using the standard algorithm].

This suggests that Mae's ability to help individual students develop methods for recording their sharing processes was constrained by her thinking of an activity pattern, the standard algorithm, as giving meaning to division.

I have characterized Mae's mathematics subject matter knowledge in terms of objects and actions on those objects. This aspect of her knowledge affected her ability to carry out the intent of the QRP leaders when extending and generalizing division. When Mae attempted to do this, she only guided the students to situations with objects other than blocks. It was intended that Mae help them come to understand notational methods as reflecting operations on numbers which were constrained by the numeration system used to represent them. That she did not provide this orientation, despite the assistance and coaching provided by the QRP leaders is strongly suggestive that she, herself, did not have this understanding.

Mae had difficulty responding to student comments or situations that did not fit her "object and actions" thinking. For example, when students suggested ways of sharing remainders that were not related to Mae's image of "increase-unglue-share", she was unable to address their conceptual difficulties [Ch.4, Excerpt 1, ¶ 5].

Bromme and Brophy (1986) acknowledged that it is difficult for teachers to gain an awareness of student thinking. They related this difficulty to teacher subject matter knowledge.

> Even teachers who do not have serious classroom management problems and who appear to be generally effective instructors, however, do not seem able consistently to respond effectively when events dictate a shift in plans or when pupils present them with 'teachable moments' that could be exploited. The reasons for this are not yet clear, but several investigators have suggested that the teachers they studied lacked knowledge of the subject matter (both the content itself and how to teach it) that was specific and detailed enough to enable them to diagnose learning problems on the spot and respond appropriately with prescriptive instruction. (p. 117)

Mae's instruction appeared to be filtered through her mathematical knowledge. That is, she seemed to decide what to teach and how to teach it based on her own knowledge of mathematics rather than on information she received from the QRP leadership. Thompson said:

> Much of what a teacher makes of a particular experience depends on the conceptual schemas available to the teacher into which the experiences are assimilated or on the accommodation of the schemas that takes place. (Thompson, A, 1991, pg 2)

Mathematics subject matter knowledge appeared to act like a conceptual schema to which Mae assimilated the basic principles of the QRP. An unusual lack of reflectiveness on Mae's part, which could be a personality characteristic rather than something specifically related to her knowledge of mathematics, added to the difficulty of her assimilation of the basic principles of the QRP. The instructional decisions she made were based on this same conceptual schema, mathematics subject matter knowledge, which, due to her lack of reflectiveness, remained untouched by the influence of the QRP leaders.

Mae’s orientation toward actions and objects is not uncommon, and is to some extent understandable. What is perplexing is its robust persistence, despite intensive and sustained efforts to reorient her thinking and her instruction. This should be taken as a warning to designers of in-service programs or methods courses intended to bring about change in mathematics instruction. Mathematics subject matter knowledge has an effect on instruction that cannot be easily addressed.

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