FUTURE TEACHERS' IMAGERY WHILE REASONING QUANTITATIVELY

A Thesis

Presented to the

Faculty of

San Diego State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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by

Donna B. Troy

Fall 1993

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ACKNOWLEDGEMENTS

I wish to give my sincere thanks to Dr. Patrick Thompson who was not only my adviser and professor but also my mentor and friend. His suggestions and patience were invaluable. I also wish to thank Dr. Randy Philipp for his encouragement and advice. I wish to extend my gratitude to Dr. Larry Sowder for his assistance in refining the rough edges of this research report. This study was supported by the National Science Foundation Grants No. MDR 89-50311 and 90-96275.

TABLE OF CONTENTS

CHAPTER I

COMMUNICATION IN THE MATHEMATICS CLASSROOM

The *Curriculum and Evaluation Standards for School Mathematics* developed by the National Council of Teachers of Mathematics (NCTM, 1989) stressed the significance of communication of ideas in the mathematics classroom. One standard addressed the need for students to be able to talk about symbols and operations while having a deep understanding of the underlying concepts. They state that,

> It is essential that mathematical concepts be firmly attached to the symbols that represent them; the need for symbolic representation arises out of the exploration of these concepts. In the process of discussing mathematical concepts and symbols, students become aware of the connections between them. Unless students frequently and explicitly discuss relationships between concepts and symbols, they are likely to view symbols as disparate objects to be memorized. (p. 78)

Teachers should foster communication in the mathematics classroom. To do so, they might alter their instructional patterns by engaging students in more problem solving activities, organize small groups of students to work together, and organize whole-class discussions.

Over recent years there has been an increase in the volume of mathematical education research in the area of communication in the classroom. These researchers have focused on how the social norms, teacher and student roles, and the structure of classroom events affect the knowledge students gain (Cobb, Yackel, Wood, 1989; Lampert, 1990; Mehan, 1982). Bauersfeld (1980) and Thompson and Thompson (in press b) focused on the language teachers use when they converse with students. They studied the meanings, intentions, and interpretations associated with teachers' use of language when conversing with students.

Thompson, Philipp, Thompson, and Boyd (in press a) claim that a teacher's orientation to teaching mathematics plays a key role in forming instructional conversations between teachers and students. They identified two orientations to teaching mathematics that result in different levels of discourse among teachers and students. A teacher who has a calculational orientation to teaching mathematics engages in conversations with students about numbers, symbols, and numerical operations. A teacher who has a conceptual approach to teaching mathematics engages in conversations with students about how the students think about a situation, what a particular number is a number of, what the students are trying to find, what a calculation yields, etc. (Thompson et al., in press a).

Thompson et al. (in press a) state that a conceptual orientation is undoubtedly the most effective for teachers in regard to the quality of students' learning. However, this type of communication is rare among many mathematics teachers today, who feel most comfortable when they focus their discourse on calculational aspects of a problem (Thompson and Thompson, in press b). Von Glasersfeld would support this claim by stating that, “[Teachers] continue to act as though it were reasonable to believe that the verbal reiteration of facts and principles must generate the desired understanding on part of the students” (1989, p. 135). He claims that the best teachers realize that “telling” students is not a sufficient means for eliciting understanding. In recent years, there has been a call for teachers to use a language that engages students in thinking about situations. Teachers need to appreciate the power of mathematics language and symbolism (MAA, 1991; MSEB, 1990).

There are two communicative skills teachers must have to conduct conceptually grounded conversations with students. First, teachers must be able to express their ideas within the context of a given occasion, topic, idea, problem, or, in general, a domain of discourse. This means that teachers must have a deep conceptualization of the situation being discussed (Thompson et al., in press a). Also, teachers must use a language that students will be able to understand. Their language must orient students toward thinking conceptually about the situation. Teachers with a calculational approach to teaching mathematics have “a tendency to speak exclusively in the language of numbers and numerical operations” (Thompson et al., in press a). This language does not suffice when trying to discuss what is going on conceptually in a situation. Teachers must develop a language and an orientation that helps students develop an understanding of what is going on in situations.

Skemp (1978) made a distinction between two ways of understanding mathematics: relational and instrumental understanding. The kind of understanding students gain depends on the kind of understanding teachers want students to develop. Teachers who want students to gain a relational understanding of mathematics expect students to know both what to do and why when the teacher presents them with a mathematical problem. This orientation to teaching mathematics is similar to a conceptual orientation identified by Thompson et al. The objective of these teachers is to have students understand the situation before they perform the appropriate skills to solve the problem. Skemp describes students with an instrumental understanding of mathematics as students who know rules and how to use them. Teachers who cultivate this level of understanding believe that the students truly understand mathematics when they become proficient at performing many skills. This orientation to teaching mathematics is similar to a calculational orientation identified by Thompson et al. These teachers focus their instruction on skills, numbers, and numerical operations instead of on understanding situations.

Thompson and Thompson (in press b) discuss how the language used by teachers with a calculational orientation to teaching mathematics may lead to dysfunctional instructional conversations. That is, it may lead to students and teachers speaking past one another or it may lead to teachers working at cross-purposes with their initial goals and objectives. They analyzed a discussion of rates between Bill, a sixth-grade mathematics teacher, and Ann, a sixth-grade student. Bill had a deep conceptualization of the situations he discussed with Ann, and he had a strong reasoning ability in regard to speed in particular and proportional relationships in general. However, he did not have conceptually-oriented conversations with Ann. Bill's difficulties in communicating with Ann appeared to be rooted in the language he used. He spoke in terms of numbers, symbols, and calculations. For Bill, these words had a great deal of meaning embedded in them. For Ann, they did not have any embedded meaning except that affiliated with knowledge of procedures. Thompson and Thompson explain, "When what one knows is bound tightly to numbers, operations, and symbols, then one lacks a language to talk about the ideas bound up in them" (Thompson and Thompson, in press b). Thus, his language was not sufficient for his student because the student struggled to try to make sense of what he was saying without possessing a compatible meaning of the words he used. Von Glasersfeld (1989) states, "The analysis of linguistic communication shows that knowledge cannot simply be transferred by means of words. Verbally explaining a problem does not lead to understanding, unless the concepts the listener has associated with the linguistic components of the explanation are compatible with those the explainer has in mind" (p.136).

Thompson suggests that "mathematical reasoning at all levels is firmly grounded in imagery" (Thompson, in press b), where images are reconstructions of past experiences. For example, consider the following conversation recorded by Thompson and Thompson (in press b) between two mathematics educators about why some people develop images of proportionality and others do not:

Pat: You may have reflected when the person next to you didn't. One of your parents may have said something to you that you thought about. Clearly, one key is reflection. My sense is that you have an intuition about these situations of which we speak that is more than, "Oh, that's where I use this rule." Well, to get that requires reflection. So, for whatever reasons I suspect you reflected on the situations themselves, trying to make sense of them.

John: As you were speaking there I was just kind of envisioning, too, all the times that I was growing up I was in a ranch or farm kind of environment, very active, and did a lot of craft work and outside work where I had to use that kind of reasoning. Cutting wood to make something, for example, I was doing that kind of thing all the time.

At some point before Thompson and Thompson's study was done, Bill had developed a strong ability to reason about situations. His reasoning was developed by constructing and organizing images of past experiences. Thompson and Thompson (in press b) claim that teachers who are strong reasoners are often able to use the language of arithmetic as a personal representational system. Bill used numbers, symbols, and numerical operations to represent his reasoning. Perhaps if teachers such as Bill became aware of their images they would be able to expand their language and hence express themselves more clearly. That is, teachers may be able to express their ideas in a way that would give students the opportunity to develop images of situations.

Teachers also should be able to hear what their students say. There is a difference between listening and hearing. When teachers listen to students, they are not necessarily searching for insight into the students' understanding of a situation. As a result, teachers are less likely to respond appropriately. When teachers intend to hear what students are saying, they seek an understanding of the students' understanding. As a result, teachers have a better chance of responding in a way that may enable students' to gain a stronger understanding.

Thompson and Thompson (in press b) state that in order to hear students, teachers need to understand the subject matter in a way that will enable them to teach conceptually. They add, "We see teachers gaining this knowledge through sustained and reflective work with children and mathematical ideas, analyzing their experiences in light of the children's thinking and learning, and reflecting on their findings" (Thompson and Thompson, in press b). Thompson et al. (in press a) stress the importance of teachers being familiar with students' thinking.

> Furthermore, to be able to orient students' thinking in productive ways, it is extremely helpful to have an image of students' thinking as they develop these ideas. Any teacher can begin building this image

> by encouraging students to reason and express themselves accordingly, by listening to their reasoning, respecting it, and asking students to do likewise.

The Purpose of the Study

The study will focus on two aspects of teachers' imagery. One is what imagery they bring to the solution of elementary quantitative problems. The other is to characterize their elaboration of imagery and operations as they work in an environment which constrains them from thinking calculationally. The broad purpose of this study is to begin to provide groundwork for developing a theory of the reflexive relationships among teachers’ orientations, imagery, and language. This first step in developing this theory is to gain insight into teachers' imagery and orientation while reasoning mathematically.

CHAPTER II

CONSTRUCTS OF THE STUDY

This chapter discusses various constructs the researcher used to analyze the interviews. First, the researcher discusses general psychological constructs: the notions of actions, mental operations, schemes, imagery, and reflective abstraction. Second, the researcher discusses theoretical constructs: the meaning of quantity and quantitative operations. Third, the researcher includes a discussion of two orientations to teaching mathematics.

General Psychological Constructs

Actions

We constantly interact with objects in the world. Each interaction is a new experience. Each experience is composed of a series of actions. These actions are either mental actions or physical actions. These actions are goal-oriented actions (Piaget, 1950). Consider the mental and physical actions one takes while driving and coming upon a stop sign. An example of a mental action in this context is thinking of the shape and color of the sign. We employ this mental action to achieve the goal of determining the type of road sign that is ahead in order to engage the appropriate physical actions. An example of a physical action in this context is to apply pressure to a brake pedal. This action's goal is to stop the car at the stop sign.

Mental Operations

We often need to perform more than one action to reach a goal. All the actions one requires to obtain a goal are part of a schema of actions. The "stop sign schema" consists of recognizing the stop sign, raising your foot, applying pressure to the brake pedal, etc. We carry out all of these actions to achieve the goal of stopping the car at the stop sign.

A mental operation is a special network of schemas. Each action is capable of being carried out in thought. In addition, we constitute the actions in such a way that any one action can be annulled by some combination of others within the schema. For example, thinking of building a model car is a mental operation. We can think of the pieces of a model car and how to connect them to make the car. Here we have an image of the initial state, the individual pieces, while thinking of the final state, the model car. We can also think of a model car, then think of how you joined the pieces to make the car and then think of each part individually. Here we have an image of the initial state, the model car, while thinking of the final state, the individual pieces. At the same time, we have the assembled model car, the final state, and we can imagine it in relation to the pieces which compose it, the initial state. We can imagine each state in relation to the other.

Schemes

A scheme is a coordinated system of mental operations. Von Glasersfeld (1989) lists three qualities of schemes identified by Piaget:

1) recognition of a certain situation

2) association of a specific activity with that kind of item

3) expectation of a certain result

The “model car scheme” has each of these qualities. First, one recognizes the plastic parts as being parts of a model car. Second, one thinks of each piece as being joined with one or more of the other pieces. Third, one expects that gluing the appropriate pieces together will make a model car.

Von Glasersfeld (1991) discusses Piaget’s constructs, assimilation and accommodation, and how they relate to schemes. He states that one assimilates new experiences to a scheme. The actions one takes that form the experience may or may not yield the expected result. When one’s actions do not produce the expected result, the scheme accommodates to the new action. He explains these occurrences:

> When a novel item (“novel” in the *observer’s* judgement) is assimilated to the initiating element of a scheme, it triggers the associated activity. If the activity leads to the expected result, the acting subject in no way differentiates the item from those that functioned like it in the past. But if, for one reason or another, the activity does *not* lead to the expected result, this generates a perturbation, which could be described as either disappointment or pleasant surprise. In either case, the perturbation may focus the subject’s attention on the configuration that triggered the activity this time, and it may then discriminated from those past experiences where the activity functioned in the expected manner (cf. Piaget,

> 1974b, p. 264). If the failure of the scheme and the ensuing discrimination of the novel item or situation leads to the tightening of the criteria of assimilation that determine what can and what cannot be taken as a trigger for the particular scheme, this would constitute an *accommodation* of the initiating conceptual structure. Similarly, if the outcome is a pleasant surprise, this, too, may lead to an accommodation, in the sense that a new scheme will subsequently be triggered by the newly isolated experience. (p. 56)

Imagery

We do not engage in actions randomly. Instead, we act according to what we have learned from past experiences. Our mental reconstructions of past experiences form our images that guide our actions. We form an image by remembering having acted, schemed, or operated.

Thompson (in press e) gives these attributes of an image:

> By 'image' I mean much more than a mental picture. Rather I mean 'image' as the kind of knowledge that enables one to walk into a room full of friends and expect to know how events will unfold. An image is constituted by coordinated fragments of experience from kinesthesia, proprioception, smell, touch, taste, vision, or hearing. It

> seems essential also to include the possibility that images entail fragments of past affective experiences, such as fearing, enjoying, or puzzling, and fragments of past cognitive experiences, such as judging, deciding, inferring, or imagining. Images are less well delineated than are schemes of actions or operations (Cobb & von Glasersfeld, 1983). They are more akin to figural knowledge (Johnson, 1987; Thompson, 1985) and metaphor (Goldenberg, 1988). A person's images can be drawn from many sources, and hence they tend to be highly idiosyncratic.

For example, consider the image a student may form when asked to find the derivative of $3x^2$. Let's say the student forms an image of swinging down and to the left while at the same time subtracting and multiplying. The student's thinking of a swinging motion is like a bodily movement. The student formed this image by activating their "finding the derivative scheme." This scheme consists of a coordinated system of schemas of actions. First, the student thinks of taking whatever is "up there," in this case the 2, and bringing it down to multiply it by the number "in front," in this case the 3. Next, the student thinks of recording the product and placing the letter, in this case x, next to it. Finally, the student decreases the number "up there" by 1 and writes the result "up there" next to the x in the new expression.

Reflective Abstraction

Von Glasersfeld (1991) explains that Piaget identified the act of thinking on a higher level of thought as reflective abstraction. Thompson (in press e) refers to this higher level of thought as the level of mental operations. Von Glasersfeld (1991) interprets some of Piaget's discussion of reflective abstraction:

> Reflective abstraction always involves two inseparable features: a "reflechissement" in the sense of the projection of something borrowed from a preceding level onto a higher one, and a "reflexion" in the sense of a (more or less conscious) cognitive reconstruction or reorganization of what has been transferred. (Piaget, 1975, p.41) (p. 58)

Consider the example of finding the derivative of a function again. Let's say students are asked to find the derivative of the expression 3. If they apply the same scheme to find this derivative as they do to find the derivative of $3x^2$ then they will most likely produce an incorrect answer or stop in a state of confusion. Let's suppose they take the following steps to find the derivative of 3: first, they think of 3^1, multiply the 3 and 1 to get 3 and record it; second, because there is no letter they simply do not place one next to the newly recorded 3; third, they decrease the number "up there" by 1 to get 0 and put it "up there" next to the newly recorded 3; next, they simplify 3^0 to get 1; they determine the derivative of 3 is 1. They are told this answer is incorrect. They check their computational procedure again and still come up with the same answer. There are several possible ways the student's thinking can progress: One way is to create a new procedure that handles expressions without variables; a second is to think about the situation on a higher level of thought that is void of procedures.

Let's suppose students choose the latter (i.e., they engage in reflective abstraction). Let's suppose students employ reflective abstraction in the following way: They project the elements of the situation as they have constructed it onto their "derivative is slope scheme" (i.e. they assimilate the new experience, taking the derivative of 3, to their “derivative is slope scheme”). They think of the slope of 3. They form an image of a line parallel to the x-axis and intersecting the y-axis at three. Next, they think of the slope of this line and determine it is 0. Engaging this scheme enables the student to think about the slope of the line y=3 as also being the derivative of 3. There was a perturbation in their "finding the derivative scheme". Now they must reconstruct their "finding the derivative scheme" so that it accommodates their reflections.

Theoretical Constructs

In this section, the researcher discusses the following theoretical constructs developed by Thompson (1990): quantity, quantitative operations, ratio, and rate. He identifies eight quantitative operations in which a student can engage in while solving algebra word problems. The researcher will provide a brief explication of each quantitative operation.

Quantity

The perceptions one has of objects or phenomenon they come across depends on one's past experiences. No one has precisely the same experiences, therefore the qualities one attributes to each object may vary from person to person. One quality a person might perceive is an object’s measurability. Thompson (1990) defines a quantity as "a quality of something that one has conceived as admitting some measurement process" (p.3). It is not necessary for one to know the value of the quantity to conceive of the quantity. For example, consider walking on a misty night. When one conceives of “mistiness” as a measurable quality of water vapor, one has conceived of mistiness as a quantity.

Thompson's notion of quantity differs from the one Schwartz (1988) explicates. Schwartz identifies a quantity as something formed by counting, measuring or performing a numerical operation. A quantity in Schwartz’s terminology is a quantity’s value in Thompson’s terminology. Schwartz’s system does not have a counterpart to Thompson’s notion of quantity.

Quantitative Operations

Quantitative operations are mental operations. Thus, they are composed of schemas of goal-directed actions. These mental operations differ from others in that the schemas have the quality of reversibility. That is, there are actions in the schema that enable one to undo the effects of the actions actually engaged. One performs a quantitative operation to achieve the goal of conceiving of a new quantity made of two already-conceived quantities.

Schwartz’ idea of quantitative operations differs from the one Thompson discusses. A quantitative operation in Schwartz’s terminology is a numerical operation in Thompson’s terminology. Schwartz’s system does not contain a construct similar to Thompson’s idea of a quantitative operation.

Combine quantities additively. When one employs this quantitative operation, one thinks of putting parts together to make a whole. One does this action to obtain the goal of thinking of two parts as a whole. One conceives of each part as a quantity before thinking about uniting them. When one thinks of this union, one conceives of a new quantity, the whole. Reversibility is a quality of this operation because one can think of the new quantity, the whole, in relation to the parts composing it.

This quantitative operation is not simply addition. Addition is a numerical operation that adds together the values assigned to two quantities. Combining quantities additively is thinking of two different entities and combining them into one entity. Understanding a situation as an additive combination of quantities does not require one to know values for the quantities.

<u>Compare quantities additively.</u> People typically employ this operation in order to think of how much one quantity exceeds another. One can also think of how much one quantity falls short of another.

For example, consider comparing the lengths of two pencils additively. The following actions are part of this schema: comparing by holding each pencil side-by-side, thinking of one pencil B as matched with part of pencil A, thinking of pencil A as being split into two parts— the matched part and the excess part— and thinking of pencil A as being made by combining matched and excess parts. The excess part is the newly created quantity. One reaches the goal, to think of by how much the length of one pencil exceeds another, once one conceives of the excess in relation to the pencils in their comparative states.

To continue the above example, one can also think of undoing the effects of these actions: comparing by holding each pencil side-by-side; thinking of part of pencil A being matched with the part that pencil B falls short of pencil A; thinking of pencil B as being split into the pencil part and the "fall short" part, thinking of pencil B as being made by combining the pencil part and the “fall short” part. The “fall short” part is the newly created quantity. One reaches the goal, to think of by how much the length of one pencil falls short of another, once one conceives of the “fall short” part in relation to the pencils in their comparative states.

Combine quantities multiplicatively. One employs this operation to achieve the goal of thinking of combining two quantities, such as length and width, to form the new quantity, an area. Another example is thinking of force (measured in Newtons) acting at a distance (measured in meters). When one combines these quantities multiplicatively, they constitue the quantity torque (measured in Newton-meters).

This quantitative operation is not the same as multiplication of numbers. One can solve a multiplication problem by simply referring to multiplication tables. To employ this quantitative operation, one needs to conceive of the quantities in a situation and think of combining them multiplicatively to form the desired third quantity.

Compare quantities multiplicatively. The goal of the schema of actions for this quantitative operation is to think of how many times as large is one quantity than another. For example, consider comparing multiplicatively the amount of marbles in two bags. Let’s say bag A has more marbles than bag B. The schema of actions is the following: thinking of the two sets of marbles as side-by-side, thinking of bag B as being matched with the same amount of marbles in bag A, thinking of subdividing the marbles in bag A into divisions equal to the matched portion, and thinking of how many subdivisions are made. One reaches the goal, to think of how many times as large the amount of marbles in one bag is than in the other, when one conceives of the two bags in either of two comparisons: the marbles in bag A measured in units of bag B, or each marble in bag B being matched with some number of marbles in bag A.

Ratio

In the system of quantities employed in this study, there are four categories of quantities: a number of something, a difference, a ratio, and a rate. For this study, I explicate only the ideas of ratio and rate because they are central to the situations addressed by subjects in this study.

A ratio is a multiplicative comparison of two quantities made in their "independent, static states" (Thompson, in press c). For example, consider a fruit basket with six bananas and three kiwis. One conceives of two quantities, how many bananas and how many kiwis are in the basket. One conceives of the multiplicative comparison, how many times as many bananas there are as kiwis. It is important to note that one does not conceive of the number of bananas and kiwis changing. One only compares the number of bananas and kiwis in the basket. The ratio is six bananas to three kiwis or vice versa.

Thompson's definition of ratio differs from other definitions given such as the one given by Vergnaud (1988) who describes ratios as composed of a multiplicative comparison of two quantities with the same unit. Vergnaud would not consider the six bananas versus three kiwis as a ratio if the units were banana versus kiwi, but would consider it to be a ratio if the unit were expressed as fruit versus fruit. Thompson's definition includes all multiplicative comparisons of quantities regardless of their units. Thompson's definition hinges upon the way one conceives of the situation.

Rate

One who conceives of a rate thinks of one quantity accumulating with respect to another quantity in a constant ratio. Also, one who conceives of a rate thinks of an accumulation of a quantity that goes on and on with no end. The values of the quantities vary indefinitely.

For example, consider riding a bicycle at a constant rate of twenty miles per hour. One can conceive of riding at the rate of forty miles per two hours, fifty miles per two and a half hours, sixty miles per three hours, and so on. One can conceive of the bicycle as traveling on and on without stopping. The number of miles traveled is accumulating in constant ratio with the number of hours riding, twenty miles per one hour. The number of miles traveled accumulates as each hour passes by. This situation has no bounds, therefore speed is a rate.

<u>Compose ratios.</u> One employs this mental operation when one conceives of a situation in the following manner: One conceives of two ratios in a situation and composes them to make a third ratio. For example, consider a young boy who surveyed the candy he collected while trick-or-treating. He conceived of the following ratios between various kinds of candy: 3 Snickers for every 2 Milky Ways and 5 Milky Ways for every 6 Jawbreakers. If he were to compose these two ratios, he would form the ratio 7.5 Snickers for every 6 Jawbreakers by something like this series of actions: It takes 2.5 groups of 2 Milky Ways to make 5 Milky Ways, and so 2.5 groups of 3 Snickers will be 7.5 Snickers. Therefore 7.5 Snickers go with 5 Milky Ways, and 5 Milky Ways go with 6 Jawbreakers, so 7.5 Snickers go with 6 Jawbreakers.

Compose rates. One employs this operation when thinking of composing two conceived rates in a situation to conceive of a third rate. For example, consider exchanging foreign currency. One conceives of the rate 1 U.S. dollar per 128 Greek drachmas. One also conceives of the rate 1 Egyptian pound per .331 U.S. dollars. From these two rates, one conceives of a third rate, 1 Egyptian pound per 42.368 Greek drachmas.

Generalize a ratio (to a rate). One has generalized a ratio to a rate "as soon as one re-conceives the situation as being that the ratio applies generally outside of the phenomenal bounds in which it was originally conceived…" (Thompson, in press e). Let's say a small fruit basket has six bananas and three kiwis. The ratio of bananas to kiwis is six bananas to three kiwis. One can also apply this ratio to a medium fruit basket with twelve bananas and six kiwis, a large fruit basket with fourteen bananas and seven kiwis, an extra large fruit basket with twenty bananas and ten kiwis, etc. One now conceives of the number of bananas and kiwis changing in a way that the ratio made by comparing them multiplicatively always has the same value.

Instantiate a rate. The first step in engaging this quantitative operation is to conceive of a situation as involving a rate. Instantiating a rate is thinking of a particular instance of the accumulation brought about by an on-going rate. For example, consider again riding a bicycle at the rate of twenty miles per hour. Instantiating a rate is thinking of going at 20 mi/hr for a certain amount of time. In this example, distance is the quantity produced by instantiating the rate with a certain amount of time. One is conceiving of specific points along the continuing bicycle journey.

Orientations Toward Teaching Mathematics

The discussion of quantities and quantitative operations focused on a way of thinking students may engage in while solving problems. The following explanation of two orientations toward teaching mathematics focuses on the characteristics of teachers who do and do not create a classroom environment that is conducive to developing strong quantitative reasoning abilities. Students can also have either orientation. In fact, in this study, Tom and Liz are college students but are students who intend to be teachers.

Thompson, Philipp, Thompson, and Boyd (in press a) identify two types of orientations toward teaching mathematics: conceptual and calculational orientations. The following characteristics describe the driving force behind the actions of a teacher with a conceptual orientation:

- An image of a *system of ideas* and *ways of thinking* that she intends the students to develop,
- An image of *how these ideas and ways of thinking develop*,
- Ideas about *features of materials, activities, expositions, and students' engagement with them* that can orient students' attention in productive ways,
- An *expectation and insistence that students be intellectually engaged* in tasks and activities.

A teacher with a conceptual orientation toward teaching mathematics does not focus on numerical values in a situation and instead focuses on the quantities and quantitative relationships in a situation. Once the students understand these ideas, the teacher addresses the values of the quantities. This approach gives the students the opportunity to associate meaning with the numbers they see in a problem.

Thompson et al. (in press a) identify a teacher with a calculational orientation as "one whose actions are driven by a fundamental image of mathematics as the application of calculations and procedures for deriving numerical results." These teachers stress the importance of "getting the right answer." Teachers with a calculational orientation possess some of the following characteristics:

- A tendency to speak exclusively in the language of *numbers and numerical operations* .
- A predisposition to cast solving a problem as *producing a numerical solution*.
- An emphasis on *identifying and performing procedures*.
- A tendency to *doing calculations whenever an occasion to calculate presents itself,* regardless of the overall context in which the occasion occurs.
- A tendency to *disregard the context* in which the calculations might occur, and how they might arise naturally from an understanding of the situation itself.
- An inclination to *remediate students' difficulties with calculational procedures* independently of the context in which the difficulties manifest themselves.
- A tendency to treat *problem solving as flat* — nothing about the problem solving is any more or less important than anything else, except that the answer is most important, because getting it is why you are solving the problem.
- A narrow view of mathematical patterns as limited to finding patterns in numerical sequences and across problems in terms of sameness of operations. (This is opposed to finding patterns in one's reasoning in the solution of problems.)

Teachers with a calculational approach to teaching mathematics focus on numbers and numerical operations. They do not engage their students in tasks that enable them to conceive of the quantities and quantitative operations in a situation. Consequently, the images students form of a situation are vestiges of having operated numerically rather than quantitatively.

All of the constructs discussed in this chapter provide a framework for interpreting people's understandings of quantitative situations—the general psychological constructs as background for the interpretations, the quantity and quantitative operation constructs for interpreting their constitutions of situations, and orientation constructs for interpreting motives for the general direction people take in solving problems and talking about situations.

CHAPTER III

METHOD

The Course: Computers in Teaching Mathematics

This study took place in the context of an undergraduate/graduate mathematics education course, Computers in Teaching Mathematics, during a unit on quantitative reasoning. The focus of the study was on two students, Liz and Tom, and their quantitative imagery before and during this unit. The course was taught by a professor who had a conceptual orientation to teaching mathematics. (The researcher made this deduction based on informal observations.) That is, he possessed the following qualities as a teacher: an image of a system of ideas and ways of thinking that he intended the students to develop; an image of how these ideas and ways of thinking develop; ideas about features of materials, activities, expositions, and students' engagement that can orient students' attention in productive ways; and an expectation and insistence that the students be intellectually engaged in tasks and activities. Over the course of the semester, he strived to help shape these future teachers into becoming conceptually oriented teachers as well.

The Quantitative Reasoning Unit

The quantitative reasoning unit was one of several units the professor developed to help obtain his goal. His objective of the quantitative reasoning unit was to help students develop an understanding of elementary quantitative problems. That is, he wanted his students to learn to identify quantities in a situation and to recognize and understand the quantitative relationships among them. He conducted the lessons by asking students to try to conceptualize situations (i.e., engage in quantitative reasoning) instead of concentrating on the calculational aspects of the problem which many of the students have been trained to do in the past.

Word Problem Analyst

During the quantitative reasoning unit, the students used the computer program, Word Problem Analyst (Thompson, 1989), or WPA , as a means to externalize their quantitative reasoning. The professor had students use this program to solve problems because it required the students to identify the quantities in a situation and then to represent the appropriate relationships among them. This learning environment constrained the students from thinking calculationally and instead, oriented them towards thinking conceptually.

The students represented their images of a situation using WPA by constructing a network of notecards. They constructed one notecard for each quantity in a situation. Each notecard contained information on various qualities of the quantity such as the quantity type, name, and the unit in which they conceived of measuring the quantity. They chose from four types of quantities: a number of something, a difference, a ratio, or a rate. After they constructed the notecard, the students were able to insert values or letters into the notecards. Students drew notecards to represent quantities and made connections among them to show that some quantities were made from others by a quantitative operation. The students represented the connections among the quantities by drawing arrows to and from notecards in the following manner: draw two arrows from the quantity made by operating, to each of the two quantities that were operated upon to make it. After the students made the connections, the program inserted numerical or algebraic expressions into each notecard according to the types of quantities in the relationships and according to what numerical or formulaic information had already been supplied or inferred.

It is important to note that the students were able to assign values to the quantities only after they assigned the quantity's type, name, and unit. It was as if the values were additional information only to be considered when carrying out the last step of the problem solving task, to find the solution. This feature of WPA, along with the feature of inferring formulas based on the relationships made among the quantities, may steer students away from thinking calculationally and instead force students to focus on understanding the situation quantitatively. Students could not easily use their customary approaches to solving problems—such as writing equations, setting up formulas and plugging in values, when using WPA. Instead, they first needed to explicitly represent the quantities in the situation and stipulate how those quantities were related.

The Subjects

Two students in Computers for Teaching Mathematics, Liz and Tom, (pseudonyms), were the subjects in this study. The researcher and the thesis advisor collaborated on the selection of these subjects. They were chosen because they were highly competent mathematics students. They were also inquisitive and exhibited a desire to express their ideas in the classroom. The researcher and thesis advisor thought these qualities would lead to more lively and insightful discussions.

Both of the subjects were enrolled in the Master of Arts in Mathematics for Teaching Service program at the university. The program prepared students to teach mathematics on the community college level. Both Liz and Tom were completing their second semester in the program. Both subjects had a Bachelor of Arts degree in Mathematics.

The subjects had varied experience teaching mathematics. Liz had no formal teaching experience. She tutored students in mathematics for over six years. The subjects she tutored ranged from seventh grade mathematics to calculus. Tom taught mathematics in a junior high school for one year. He taught eighth grade mathematics and pre-algebra to ninth graders before he decided to enroll in the master's program. He also taught several classes in a remedial mathematics program at the university as a graduate teaching assistant.

The Interviews

The researcher interviewed Liz and Tom before and during the quantitative reasoning unit. The overall intent of these interviews was to discover what imagery these subjects brought to the solution of elementary quantitative problems. The researcher and thesis advisor selected a set of problems for the subjects to solve during the interviews. These problems are contained in the Appendix. The researcher did not rely on a highly structured set of questions to ask while the students solved these problems. Instead, the researcher asked questions that she deemed appropriate as the interview progressed.

The first two interviews, the pre-interviews, took place one week before the quantitative reasoning unit began. The researcher met with each subject separately and asked them to solve two algebra word problems. The reason for conducting these interviews at this time was to get insight into their imagery of quantitative problems before learning about quantities and quantitative structures in the quantitative reasoning unit. These interviews served as a starting point en route to tracing the development of their imagery as the quantitative reasoning unit progressed.

The researcher held the remaining seven interviews during the three weeks the quantitative reasoning unit took place. The researcher conducted the interviews biweekly, one interview before each class meeting, and with both subjects present. The reason for meeting on the class days was to try to gain insight into how their thinking evolved between consecutive class meetings. The researcher was advised by the thesis chair to meet with both subjects together with hopes of the subjects questioning each other and hence generating rich interviews. In each of these seven interviews, the researcher asked Tom and Liz to solve the selected quantitative problems first using paper and pencil and then using the computer and Word Problem Analyst. When they used WPA they had to enter the name, type, and unit for each quantity they conceived of in the given situation and then make the connections among them.

The nature of the conversations in the interviews was jovial, yet very much on-task. Liz and Tom seemed to enjoy being interviewed. They were not hesitant to share any ideas or frustrations. Liz and Tom worked well together. They helped each other clarify their interpretations of the problems and sought to resolve any conflicts in understandings.

Liz and Tom discussed various issues with the researcher. The majority of the discussions revolved around Tom and Liz's thoughts on the selected problems—first impressions, ideas on how to solve them on paper and using WPA, their thoughts while working, and reflections on their work and their conceptions of the situations. They also talked about things such as what happened in a previous class or interview, homework assignments, their orientations toward problem solving, pedagogical issues, and their general impressions of the course.

The Data

The nine interviews were videotaped and transcribed. The researcher focused on two things when she analyzed the interviews: what imagery the students brought to the solution of the selected problems and how their imagery changed as they progressed through the quantitative reasoning unit.

CHAPTER IV

EMERGING IMAGERY WHILE REASONING QUANTITATIVELY

In this chapter, the researcher will discuss what transpired during the interviews. During this discussion, the researcher focuses on three problems that she asked Tom and Liz to solve: the Moisture in the Berries problem in the pre-interview, the Moisture in the Berries problem revisited after working with WPA, and the last problem they solved, the Brothers problem. The researcher also includes a discussion of what transpired between the pre-interviews and the fifth interview when she asked them to solve the Moisture in the Berries problem using WPA.

The researcher will establish several main points in this chapter: Liz and Tom became increasingly concerned with identifying the quantities and quantitative relationships in situations; they formed increasingly elaborated images of situations; and the subjects had difficulty in conceiving of ratios and rates as quantities, but this difficulty diminished over time.

Pre-interviews: Moisture in the Berries Problem

> In a store there were 100 kg of berries. An analysis showed that the moisture content in the berries was 99%. Determine the total weight of the berries when, by a later analysis, the moisture content had decreased by 1% and had become 98%. (Krutetskii, 1976, p. 160)

The "Berries" problem, given above, involves two ratio comparisons at two different moments in time. The berries' moisture content is a multiplicative comparison between the weight of the moisture within the berries and the berries' total weight. The berries' total weight is an additive combination of the berries' non-moisture and moisture portions. To say that the berries' moisture content decreased by 1%, to 98%, means that the 1 kg of non-moisture substance within the berries is now 1/50th of the total, so the total weight would be 50 kg.

Tom

Tom read the Moisture in the Berries problem and immediately expressed confusion. It seemed that he began by understanding "total weight" as referring only to the non-moisture substance within the berries. From Tom's point of view, the non-moisture substance remained the same even after the moisture content changed, so the berries' total weight was still 100 kg at the moment of the second analysis.

The researcher reread the problem and stressed the fact that there were two analyses. Tom tried to explain how he saw the problem. In explaining his image of the situation, he realized that the berries were composed of two portions—a moisture portion and a solid-substance portion. He then debated whether the moisture portion constituted 99% by weight or 99% by volume. Excerpt 1 provides this episode.

<u>Excerpt 1</u>

1. Tom: That's when I get what the problem is asking. *DT nods.* Oh… okay. *Tom cups hands.* So, you have 100 kg here and 99% of it is moisture. Okay, so if 99% ah, oh okay wait a minute, then is this 99% the, of moisture, is this 99% of the weight we're talking about?

2. DT: What do you think? Why don't you read it again.

3. Tom: Okay. *Tom reads problem again*. So it's not necessarily the weight. It's just that whatever makes up the berry, 99% of it was moisture?

4. DT: So you said that it's not necessarily 99% of the weight?

5. Tom: Yeah.

6. DT: The weight being what in this case?

7. Tom: *Tom cups hands.* So here we have the same batch of berries.

8. DT: Uh huh.

9. Tom: And it says 99% um, because you can have 99% of the volume is taken up by water or whatever, right? So we're talking about 99% of the weight of the berries? *Tom laughs.* I think I'm reading too much into this. I could just, just…

10. DT: No, no, that's, that's fine.

11. Tom: Okay.

12. DT: Okay.

13. Tom: Because see…

14. DT: I, I see what you're asking.

15. Tom: Right.

16. DT: But, whatever, however you interpret it, that's how I want you to…

17. Tom: Okay. So here's the total weight, 100 kg. If 99% of it is moisture, then this is 99 kg. This is the water, I guess, of the berries.

In Excerpt 1 Tom continued constituting the situation as involving berries being made of two portions. His debate about whether the salient quality was the berries' volume or their weight was part of his quantification of the situation—he needed to settle on what quality of the objects (berries) to consider as being measured. Once he settled on weight as the salient quality (¶ 17), the matter did not arise again.

Tom identified the remaining 1% in the first analysis and the remaining 2% in the second analysis as the non-water-based substance in the berries. He then realized that the non-water-based substance in the first analysis did not change in the second analysis. This lead to his debate over whether the moisture portion in the second analysis was 98% of the original total weight or 98% of the total weight in the second analysis. Except 2 relates his debate.

<u>Excerpt 2</u>

1. Tom: What am I thinking? Um, because here, if 98% of the same batch is moisture so you lost some water I guess, whatever, moisture, you lost some moisture, and so um, initially I'm thinking well the, the 1 kg that wasn't moisture before is still 1 kg. And then so the 98% of moisture, 98% that's moisture is 98% of… hmm, I, I lost my train of thought. *Tom laughs.*
2. DT: That's alright.
3. Tom: Okay, so I'm inclined to say that if the 100 kg is made up of these then you still have 1 kg of whatever that is not moisture. And 98, 98% is still moisture. *Tom pauses*. You've got one batch of berries that weighs 1 g and the rest of it weighs 99 g. The same batch of berries that weighs 1 g and the rest should just weigh 98 g? Okay. See, this is what I want to say right? But um, I…
4. DT: You want to say that…
5. Tom: I want say that see if you originally have 100 g and if 99% of it is moisture then it must weigh 99 kg, kilograms, yeah. It must weigh 99 kg. Then whatever else that makes up the berry weighs 1 g. Right? Now if you lost some moisture, whatever else that makes up the berry still weighs the same 1 g, right?
6. DT: Uh huh.
7. Tom: And so the 98% moisture… now is that 98% of the original 100 g? Okay see, I'm saying 98 kg, but um, I have reservations about that because when you are talking

about percent, you're talking about percent of the whole thing, and what makes up the whole thing now is the 99 kg not the 100 kg. Any of this make sense?

8. DT: Yes, so then you're thinking about taking a percent of the 99 kg, is that what you're saying? Are you debating whether or not…

9. Tom: Right.

10. DT: … The percent of the 100 kg or the 99 kg?

11. Tom: Right. That's, so I better read this again. *Tom reads problem again.*` Right? Because originally if was 99% moisture. So now is this 98% moisture of the original or is it 98% of what you end up with there? in terms of weight?

12. DT: Uh huh. What do you…

13. Tom: Okay. So, well from reading the question, because it's not very specific, I'm assuming it is 98% of the original content.

It seems that Tom established a complex chain of inferences that ended up with him considering whether he should take 98% of 99kg. He first took 98% of 100 kg to establish the amount of moisture at the second analysis (¶ 3), getting 98 kg, for a total weight of 99 kg at the second weighing (¶ 7). He then considered whether he should think of 98% as applying to the total weight at the second analysis, but took the total weight at the second analysis to be the 99 kg that he had already determined (¶ 7).

In (¶s 1-5) Tom's image of the total weight of the berries in both analyses was that it was comprised of a certain amount of non-moisture weight and moisture weight. Tom thought the moisture content in the second analysis was 98% of the total weight in the first analysis (¶ 3). Then Tom doubted this interpretation and thought the moisture weight in the second analysis may be 98% of the total weight in the second analysis because "what makes up the whole thing now is the 99 kg" (¶ 7). In (¶ 13) Tom settled on the moisture weight in the second analysis as being 98% of the total weight of the berries in the first analysis.

It is important to note that the end of Excerpt 2 came 12 minutes into the interview. That is, Tom had spent 12 minutes trying to conceptualize the basic situation—that the berries were constituted by two portions, that the non-moisture portion's weight did not change over time, and that the berries lost some amount of moisture between the first and second analysis.

It appears that Tom did not think of 99% (99/100) or 98% (98/100) as the value of a ratio made by multiplicatively comparing the moisture weight and the total weight. Instead, it seems that Tom thought of 99% and 98% as operators—telling him how much to take of something (e.g., total weight) to make something else (e.g., moisture weight). As such, instead of being able to quantify the situation as an additive comparison of two multiplicative comparisons where one quantity (non-moisture weight) is common to both multiplicative comparisons, Tom was constrained to think of 99% being changed into 98% by "losing" some of the moisture. Given the way he was thinking of the situation, he had no way to relate 99% and 98% except that 99 is 1 more than 98.

Tom contrasted his debate in Excerpt 2 with his debate over different interpretations of a discount sale at a store. He explained that there are clearance sales where a sales item is 50% off the original price of the item and if someone comes into the store at a certain time they will receive an additional 10% off. He explained that there are two ways to find the total amount of the discount; take 10% off the original price or take 10% off the already discounted price. Tom thought that his latter interpretation, taking

10% off the already discounted price, was similar to taking 98% of the total weight in the second analysis instead of in the first analysis.

When he discussed the percents in the situation of discounted sales he appeared to be thinking of how much money to take off some other amount of money. The significance of this observation is discussed after Excerpt 3, which gives his explanation of how one would determine the cost of an item in the clearance sale.

Excerpt 3

1. Tom: Okay. *Tom scratches head.* But then you know, in these sort of problems from my experience I guess, background sort of thing, I tend to have to worry about that because there are, you can ask questions in which you are saying the percent of the new total not the percent of the original total.
2. DT: What kind of problems were those?
3. Tom: I don't know. *Tom laughs.* What kind of problems were those? Uh, uh let's see, percent, uh, those sales items where you know like… oh, thank you for asking that because I just recall something. I'm talking about sales items say like, you know like a store has, the problem says a store has clearance sale 50% off and if you show up the first two hours you get an additional 10% off. You know, that sort of problem. So you know an additional 10% off does not mean that you get 10% off the original price…it does not mean you get 50% off of the original price and then, and then you get 10% off… okay, so if you get 50% off and then 10% off that's 10% off of the 50% that you already got. Whereas another way to interpret that is you get 50% off and then 10% off of the original price so you, in effect you get 60% off.
4. DT: Uh huh.
5. Tom: Right? Whereas in the other case you get like 55% off in my scenario. So that's what I was… you were asking for what kind of similar problem.

Several times during this discussion Tom used a percent as the "name" of an amount of money. This in itself is not unconventional or invalid, but it oriented Tom away from thinking of percents as values gotten by comparing two quantities multiplicatively. Tom explained, "that's 10% off of the 50% that you already got" (¶ 3). In this statement, Tom used 50% to "name" the amount of money the item cost after applying the 50% discount. In (¶ 3) Tom spoke of taking 10% and 50% off of prices. It appears that, to Tom, "taking a percent" was a procedure; he also used the percent to refer to the amount you obtain by having applied the procedure. Thus, his image of percent as "take a percent" oriented him away from thinking of two quantities in comparison, and instead oriented him toward thinking of doing something to one quantity to get (or perhaps to evaluate) another quantity.

The researcher asked Tom to solve the problem. Tom gave a value of 98 kg for the new moisture weight and 1 kg for the non-moisture weight. He concluded that the total weight of the berries at the time of the second analysis was 99 kg.

The researcher asked Tom to list the quantities in the problem. Tom listed the following items as quantities: 100 kg of berries, 99 kg of moisture, 1 kg of non-moisture in the first analysis, 98 kg of moisture, and 1 kg of non-moisture in the second analysis. Tom did not identify any quantities for which 98% and 99% might be values aside from their use by him as percentages—amounts gotten by "taking a percent" of something.

Liz

Liz read the Moisture in the Berries problem and thought she needed to "use a ratio." She mentioned the term "ratio" but did not explain what she meant by it. She set up the proportion, 99/100=98/x. It appears that she set up a proportion because she knew three numbers and needed to find a fourth one. Since Liz did not mention that the fractions in her proportion were comparing anything, the researcher asked Liz what the 99/100 compared. Liz responded by saying, “if they have 99% moisture they weigh 100 kg. So if they have 98% moisture how much do they weigh?”

Liz proceeded to perform algebraic manipulations to solve for the unknown number. She used a calculator and found that the new total weight of the berries was 98.89 kg. The researcher then asked Liz about the quantities and quantitative relationships in the situation.

As the interview progressed, Liz seemed to continue using percents to refer to amounts of something. When the researcher pointed to 99% in the problem and asked “… this is 99% of what again?”, Liz responded, “The 99% represents the moisture in the berries.” She seemed to be thinking of 99% as the amount of moisture in the berries. She did not think of 99% ($\frac{99}{100}$) and 98% ($\frac{98}{100}$) as the value of the ratio made by multiplicatively comparing the moisture weight and the total weight. Instead, it seems Liz, like Tom, thought of 99% and 98% both as operators—values to use in the procedure “take a percent”, and as percentages—the amounts you get by applying the procedure. The significance of this observation is that Liz’ use of percents in these ways oriented her away from thinking of percents as values of a multiplicative comparison between two quantities. Instead, it seems they oriented her to think of the quantities “total weight,” “moisture weight” and “non-moisture weight” in isolation of each other, except that you do something to total weight to get moisture weight and non-moisture weight.

The researcher asked Liz to explain what makes up the other 1% of the berries in the first analysis. She struggled to give an explanation. This discussion is given in Excerpt 4.

Excerpt 4

1. DT: Okay. What is, so if 99% is the moisture weight, *restating something Liz said moments before,* what is the other 1%?
2. Liz: I don't know. *Liz laughs.*
3. DT: I mean, could you…
4. Liz: Umm…
5. DT: I don't mean a specific answer, but…
6. Liz: Yeah, I mean… so then the other 1% would be… I mean I really don't know. I mean I really have no idea.
7. DT: Does it have something to do with the berries? Or is it there?
8. Liz: Well, I…
9. DT: Does it represent something?
10. Liz: It's got to represent something.
11. DT: Okay.
12. Liz: I mean, 'cause you can't just, you know, 100% is gotta be somewhere. But um, I guess maybe it would represent just the actual weight of the berry itself.
13. DT: Okay. So it's like the…
14. Liz: So if you took out all of the moisture, the substance would weigh that one percent.

Before the discussion in Excerpt 4 took place, seven minutes of the interview had passed and Liz did not have an image of the basic situation. She thought of what makes up the other 1% only after the researcher prodded her. In (¶ 2- 10) Liz had "no idea" what made up the other 1% of the berries. When she formed an image of the other portion of the berries (¶ 12) she explained, "it would represent just the actual weight of the berry itself." Liz referred to the 1% of total weight as the weight of the "berry itself," presumably meaning the solid substance of the berries.

The researcher then asked Liz to discuss what makes up the non-moisture weight in the second analysis. She determined that the non-moisture substance in the berries would be the same in the second analysis as in the first analysis. She continued to refer to percents as amounts of something. Excerpt 5 provides this discussion.

<u>Excerpt 5</u>

1. DT: Okay. So how about after the second analysis after the moisture had decreased 1% and had become 98%, what about that non-moisture weight? That substance right there.
2. Liz: Well the way I see it is as the moisture decreases the amount of weight the berry holds can't increase just because the moisture decreases, so if the moisture decreases a little bit, then the whole is gonna decrease. But now you have that 1% versus that 2%, that 1% and that 2%, the actual weight if you were to figure it out should be the same.
3. DT: Of the…
4. Liz: The actual weight of that 1% versus 2%.
5. DT: Of the berry, of the non-moisture weight?
6. Liz: Yeah, of that non-moisture, whatever it is, the berry itself should actually be the same unless the berry somehow grew. *DT and Liz laugh.*

Liz seemed to be thinking of 1% and 2% as operators (¶ 2). That is, she thought of using the 1% and 2% to tell her how much to take of the total weight to get the non-moisture weights. It appears Liz did not think of 1% or 2% as the value of the ratio made by multiplicatively comparing the non-moisture weight and the total weight. In (¶ 4) Liz said "The actual weight of that 1% versus 2%." Here she referred to the 1% and 2% as the actual weights of the non-moisture substance. She seemed to be alternating her use of percents—as an operator and as an amount of something. After she thought of using the percent as an operator, she used the same percents, 1% and 2%, to refer to the weight she obtained by operating.

The researcher asked Liz to list the quantities in the situation. She identified the following things as quantities: 100 kg that represents the total weight of the berries, 99% that represents the total percent of moisture in the berries, 1% (in the problem statement) as the amount of moisture that was no longer in the berries, 98% as the total percent of moisture in the berries after some moisture has been lost, and the new total weight of the berries with only 98% moisture. Liz stated that this was all the quantities she could see.

Seventeen minutes into the interview the researcher directed Liz's attention to the non-moisture weight hoping she would recognize the non-moisture substance as a quantity. Liz still struggled when trying to describe the non-moisture portion of the berries. She referred to these portions as what is left over after moisture is taken out, like a raisin.

Summary of the Pre-interviews:

Moisture in the Berries Problem

Both Tom and Liz struggled to understand the basic situation. It took Tom 12 minutes to realize that the berries were made of two portions and that the non-moisture portion did not change in weight when the total weight changed. It took Liz 17 minutes to realize that the berries' non-moisture portion was "what is left over after moisture is taken out—like a raisin." Both subjects thought of percents as operators, and also thought of percents as if they were percentages—the amount you get by multiplying a percent and a number.

Tom eventually understood that the berries' total weight diminished between analyses, and gradually understood that the non-moisture weight remained the same between analyses. Yet Tom felt no conflict in determining that the moisture weight at the second analysis was 98% of the total weight at the first analysis.

Liz began by trying to set up a proportion—an equality of two "ratios," but it was not clear that Liz thought of ratio as anything other than the symbolic form X/Y. It is possible that Liz settled quickly on proportion because of knowing proportion as a procedure for finding one of four numbers when you know the three others and/or an association of percents with ratios. She may have had a sense of proportionality in her image of the situation, but if she did have a sense that the situation involved a proportional relationship she did not express it clearly.

When asked to list the quantities in the situation, neither Tom nor Liz mentioned quantities for which 98% and 99% would be values. Instead, it seemed that these were numbers to be used in a "take a percent" procedure to evaluate another quantity. That is, they were thinking of .98 and .99 as given information, to be used, and not thinking of them as values of quantities or where they might come from. At the risk of exaggeration, it seemed as though they were reading the problem like this: *In a store there were* **100** *kg of berries. An analysis showed that the moisture content in the berries was* **99%**.

Determine the total weight of the berries when, by a later analysis, the moisture content had decreased by 1% and had become 98%.

Transition: Pre-interviews to the Moisture in the Berries Problem Revisited

Before the next interview, Liz and Tom were introduced to quantitative reasoning in class. Before addressing the construct, quantitative reasoning, the professor explained calculational and conceptual orientations towards teaching mathematics and contrasted them. He distributed a paper that explicated these orientations. Next, the professor presented the idea of quantity and four kinds of quantities to the class. There was an extensive discussion about quantity, and then, in particular, the quantities ratio and rate. Then the professor introduced WPA. He gradually explained the conventions of WPA while representing a particular problem. He asked the students to list the quantities in the problem while he created notecards to represent them. A discussion ensued about the relationships among the quantities. The professor stressed that WPA is able to evaluate quantities after the user specifies the quantities and the relationships among them. He explained that after a situation is represented, WPA has a basis for reasoning about what operations to use.

During the first interview after this class meeting, Liz and Tom worked on their assignment: to read and work out the examples in the WPA tutorials. As they progressed through the tutorials, they became familiar with some of the conventions of the program. First, they represented simple problems using the program while following step-by-step instructions. Next, they set up simple situations on their own before reading through the step-by-step instructions. The conversations throughout the interview revolved around

Liz and Tom's thoughts while familiarizing themselves with the conventions of the program. Consequently, this interview did not reveal the images they had formed of the given situations.

Liz and Tom continued to work through the tutorials in the next interview. The problems they represented on WPA were slightly more complex. They seemed to be more comfortable with identifying the quantities in a problem. Tom even expressed concern over the preciseness of the names of the quantities. They learned how to change from representing a situation as involving ratios to a representation of the situation as involving rates, and vice versa. When they drew arrows to represent the quantitative relationships, they drew the arrows as the tutorials prescribed and did not talk about what they meant by "ratio" and "rate". Thus, it was difficult to gain insight into their images of ratios and rates.

In the next two interviews, the researcher asked Tom and Liz to solve several problems. When they solved these problems on paper before using WPA, they either asked to use the program first because it helped to organize their thoughts or they solved the problem by manipulating algebraic equations without talking about all of the quantities and quantitative relationships. As they tried to represent the problems using the program, they seemed to gradually became more aware of the need to identify all of the quantities and quantitative relationships in the problems. They also realized that they needed to conceive of all the quantities in order to complete all of the quantitative structures. Their growing concern over identifying the quantities and quantitative relationships took precedence over making use of the values given in the problem statement.

Moisture in the Berries Problem Revisited

Three weeks into the quantitative reasoning unit, the researcher asked Tom and Liz to do the Moisture in the Berries problem again. The researcher's reason for revisiting this problem was to observe any changes in the subjects' imagery of this situation after working with WPA for three weeks. In the beginning of Interview 5, neither subject expressed confidence about his or her understanding of the situation. Tom explained that the berries were made up of two portions, moisture and non-moisture, and Liz agreed. Tom still thought the new moisture weight was 98% of the original total weight. Tom continued to think of the percents as operators.

Liz said that she had an interpretation that was different from Tom's. She began by explaining that the non-moisture weight did not change over time. During this explanation she continued to think of percents as referring to amounts of something. Excerpt 6 includes part of this discussion.

Excerpt 6

1. Liz: Okay? Now you can think of it this way okay, but if this is ninety-eight percent, and this is now two percent, not kilograms okay? *Liz moves down to the second analysis and changes that "1 kg" to "2%" and changes the "98 kg" to "98%."* But this still weighs the same as this did. *Liz circles the "2%" and draws a double ended arrow to connect the "2%" with the "1%" of the first analysis.*
2. Tom: Uh huh.
3. Liz: Okay? So you can't say that, so if this is what, this is the weight that's changed. *Liz circles the "98%".* Do you understand what I'm saying?
4. Tom: Yes. So this two percent of the weight…
5. Liz: Uh huh.
6. Tom: … Is still one kilogram.

7. Liz: If you want to talk in kilograms, I didn't…

8. Tom: Okay, so you want to talk in percent we can talk in percent.

9. Liz: I didn't talk in kilograms at all except for the hundred. This is the only thing I left in kilograms. *Liz circles the "100 kg" of the first analysis.*

10. Tom: Okay.

11. Liz: So how I did the problem was I say well this is always the same. *Liz points to the "1%" of the first analysis and the "2%" of the second analysis.*

Liz stressed throughout the interview that she was thinking in terms of percents and he was thinking in terms of kilograms yet it appears Liz was still thinking of percents as referring to amounts of something. She said in (¶ 3) that the 98% is the weight that changed. She also explained that 1% was the same as the 2% (¶ 11). It appears she was thinking the weights of the 1% and the 2% were the same.

Subsequent to Excerpt 6, Liz set up a proportion similar to the one she set up in the first interview, $\frac{100}{99\%} = \frac{x}{98\%}$. Tom stated that he did not see the connection between the proportion and her discussion of the non-moisture weights being the same. Tom then realized that the moisture weight in the second analysis was 98% of the new total weight. He explained that the problem implied that the moisture weight in the second analysis is 98% of the new total weight. After Liz set up her proportion she never referred to it again. She began talking about her image of the situation. Excerpt 7 shows her reasoning.

<u>Excerpt 7</u>

1. Liz: So ninety-eight percent of the new weight is now the moisture. Whereas before ninety-nine percent of the weight is the moisture. So I mean technically you can say like you were talking in kilograms that this is going to weigh one kilogram okay, the other? *Liz writes "1 kg" next to the "1%" of the first analysis.* And this is going to weigh one kilogram. *Liz writes "1 kg" next to the "2%" of the second analysis.* Gosh,

you don't know what this is, this is not given to you. *Liz scribbles out the second "100 kg".*

2. Tom: Yeah, that's what I was…
3. Liz: And this is ninety-eight percent. *Liz circles the "98%" of the second analysis.*
4. Tom: Of the new weight.
5. Liz: Right.
6. Tom: See I did not interpret it that way.
7. Liz: You don't know what the new weight is. You just know what this is. *Liz circles the "1 kg" of the second analysis.*

Why Liz abandoned her proportion is not clear. The last line in Excerpt 7 suggests a possible reason. Perhaps Liz realized that if her proportion were valid, then the non-moisture weight, moisture weight, and total weight should *all* change. But since the non-moisture weight remained the same, and the total weight's change came by way of a loss of moisture alone, then her proportion could not be valid.

It may have struck the reader that Liz and Tom were extremely close to solving this problem. They had said that 1 kg is 2% of the new total weight, which to many persons would suggest that the new total weight was 50 times as big as 1 kg. However, they had fixated on new total weight as an additive combination of new moisture weight and non-moisture weight, and the fact that they would need to multiply the new total weight by .98 to determine the new moisture weight. This, combined with the condition that they did not know the new moisture weight, appeared to them to be an obstacle to answering the question.

Tom and Liz decided to think of how they were going to set up the problem using the computer program, WPA. To use this program, the user must identify the quantities and the quantitative relationships in the situation. They kept the conventions of the program in mind during the ensuing discussion. As they started to think of the situation and how to represent it using the program, they discussed how to represent the percents using the program. Excerpt 8 provides this discussion.

Excerpt 8

1. Tom: WPA. Analyzer. *Liz and DT laugh.* Okay, well percent. When we are talking about percents, we are talking about ratio, or rate.
2. Liz: Right.
3. Tom: Which one do you prefer now? I know that you prefer to set it up as a rate don't you?
4. Liz: Right. *Liz, DT and Tom laugh.*
5. Tom: I prefer to set them up as ratio, from the first day.
6. Liz: See, rate makes more sense to me in this case again.
7. Tom: Uh huh.
8. Liz: Because the ninety-eight percent never changes. And the ninety-nine percent never changes.
9. Tom: Uh huh.
10. Liz: Maybe it's because that's the way I always think about it.
11. Tom: Yeah.
12. Liz: You know? That's just not changing you know?
13. Tom: Everybody has a preferred way.

Tom preferred to think of percents as ratios and Liz, as rates (¶ 3, 5). It is not clear what they meant by “ratios” or “rates.” Liz gave some explanation when she said that she was thinking of 98% and 99% as rates because they never change (¶ 8). Perhaps they were used to seeing rates and ratios in most of the problems they had set up using WPA up to this point and tried to “fit” them into this set up as well.

They identified "old weight" and "new weight" as quantities and assigned the appropriate units. Next, they struggled when thinking of how the "moisture percentage" could in fact be the value of a quantity and not be just a number you use in a "take a percent" procedure. Excerpt 9 shows the struggle they experienced.

<u>Excerpt 9</u>

1. Liz: We have the moisture, percentage.
2. Tom: Well we have the first analysis and the second analysis moisture too.
3. Liz: Right. Yeah I know. I was, I was going to say moisture of, I was just thinking. After I said moisture I was thinking boy. *Liz laughs.*
4. Tom: Two moisture…
5. Liz: Well no that's not what I was thinking. I was thinking do we want to use, how do we want to use that? If we're going to use the percentage as a rate, okay?
6. Tom: Uh.
7. Liz: What is it a rate of? It is a rate of how much, you see, I see it as a rate of how much the moisture weight per the weight of the whole berries. Do you see what I'm saying?
8. Tom: You have to do it that way if you're going to use the information that says ninety-nine percent because it is ninety-nine percent of…
9. Liz: Right.
10. Tom: … The whole thing. Right?
11. Liz: Right. So I see it as, you know, so in that case, we need the percentage, we need the weight of the moisture also.

In (¶ 5) Liz expressed uncertainty on how to "use the percentage as a rate." She wondered what the percent was a rate of (¶ 7). It seems Liz struggled because she thought of percents as operators and hence did not know how a percent could itself arise by a quantification process. In (¶ 7) Liz identified two quantities, moisture weight and weight of the whole berries (i.e. total weight), to be used when using percent as a rate but did not talk about the situation dynamically, which would be natural if she had actually conceived of 99% as the value of a rate.

Liz wanted to set up the following quantitative relationships on WPA: the old weight as an additive combination of the moisture weight and the non-moisture weight and the moisture weight as being formed by instantiating the weight of moisture per old weight rate. Liz was thinking of old weight being made up of moisture weight and non-moisture weight while at the same time thinking of moisture weight being made up of the old weight and the rate. The program did not allow her to complete this setup—it objected because saying one quantity is made from another implies a temporal precedence. So, the way Liz wanted to set up her representation, she would have to say that each of two quantities preceded the other in her thinking.

Tom stated that he realized he was confused because he was thinking of the percent as a ratio. However, he did not explain what he meant by "ratio." They talked about the difference between the two interpretations but only as they related to the conventions of the program. They continued to identify more quantities: "weight of the other" to represent the non-moisture weight at the time of both analyses, "weight of moisture first" to represent the weight of the moisture in the first analysis, and "weight of moisture second" to represent the weight of the moisture in the second analysis. They formed two rate notecards, "moisture first per old weight" and "moisture second per new weight." It is not clear how they were thinking of these quantities as rates. They attempted to draw connections among the quantities in a way that represented the quantitative relationships. They had difficulties when trying to represent the relationship between the "weight of moisture first" and "old weight" the way Liz had explained it earlier. The interview concluded shortly after they encountered this obstacle.

Two days later, Tom and the researcher met to continue where they had left off in the last interview. Liz was unable to attend. Tom looked at the set-up they constructed. As Tom looked at a rate notecard he said, "The ratio of the weight of the moisture versus the weight of the whole batch is ninety-nine over one hundred." This was the first time Tom appeared to be thinking of a comparison. He seems to have been thinking of 99% ($\frac{99}{100}$) as the value of a ratio that compares moisture weight and the total weight. Upon hearing "ratio," the researcher asked if he would prefer expressing the rate as a ratio instead. Tom explained that he liked to use ratios better than rates but did not explain what he meant by "ratio." He proceeded to change the rate notecards to ratio notecards and to make the appropriate connections among the quantities.

During the next ten minutes, Tom entered known values into the notecards and made a table to solve for the unknown values. The table feature allowed Tom to equate algebraic expressions and vary the value of the unknown to determine when the two expressions had the same value. Tom varied the possible values for the new total weight from 50kg to 100kg. With some hesitation, Tom determined that the total weight in the second analysis was 50 kg. It seemed that he thought the answer was reasonable only after he saw the solution. He then explained that after he realized the moisture weight in the second analysis was 98% of the new total weight he guessed the answer and said, "In my head I said if the old weight was one gram and the one gram was equivalent down here and it's about two percent then you know it's got to be half of it about."

Eight days later, Liz resumed where she and Tom had stopped in the first Moisture in the Berries Revisited interview. She looked at their original set up and tried to recall what the situation was about and what went wrong with the set-up they constructed over a week ago. She thought about the problem with the set-up and thought that changing the rates to ratios would solve her dilemma—the "weight of moisture 2nd" notecard would no longer have an arrow going to and coming from the "old weight" notecard.

Liz proceeded to change the rate quantities to ratio quantities. Liz changed the rate notecard named "moisture first per old weight" to a ratio notecard. Then she drew arrows to denote that the ratio, moisture first versus the old weight, was made by a multiplicative comparison between the quantities named "weight of moisture 1st" and "old weight." A message appeared asking Liz whether she was thinking that "weight of moisture 1st" was a number of times as large as "old weight," or vice versa. Liz struggled as she tried to understand the program's question regarding the direction of her multiplicative comparison between the two quantities. Excerpt 10 provides this discussion.

<u>Excerpt 10</u>

1. Liz: Weight moisture first is a number of times larger than old weight. *Liz reads the information off of the screen.* Weight moisture… no old weight is number of times larger. Right? *Liz clicks that option.*
2. Tom: Old weight is number of times larger than weight moisture first. *Tom rereads the option that Liz checked.*
3. Liz: It's vague.
4. Tom: It's vague yeah but that depends on how you think of it.

5. Liz: Yeah.

6. Tom: I mean how are you thinking of it?

7. Liz: Okay, I am thinking of it as… okay. Ratio, ratio, ratio… *Liz laughs.* We know that the moisture is ninety-eight percent. So that means that… or ninety-nine percent. So that means that it's ninety-nine of it is moisture and one of it is the other.

8. Tom: Yeah but we are not…

9. Liz: Oh, ninety-nine of it is moisture and a hundred of it is, is the weight then.

10. Tom: Right.

11. Liz: The old weight.

12. Tom: Ah ha.

13. Liz: So if I go ninety-nine over a hundred, it's this much. *Liz points to the notecard "Weight moisture 1st."*

Liz was unsure that the berries' original weight was a number of times as large as the moisture weight at the first analysis (¶ 1). She thought the question the program asked was vague (¶ 3). Perhaps Liz thought the question was vague because the program was asking a question that was not relevant to her. If so, this would suggest that even though she had depicted a ratio comparison, she did so largely because the program's design pushed her in that direction. It wouldn't allow her to set up the situation as involving rates, and it wouldn't allow her to use "99%" as an operator. So, she set it up as a ratio because that was her last option.

Tom asked her how she was thinking about the relationship (¶ 6). In (¶ 7) Liz struggled when thinking about the multiplicative comparison between the moisture weight at the first analysis and the original weight of the berries. First, she recalled the relationships between the 99%, the original weight of the berries, the first moisture weight, and the non-moisture weight. She appears to have been thinking of 99% as an operator because she talked about the moisture and non-moisture weights as being a result of applying the procedure when she said, “So that means that it’s ninety-nine of it is moisture and one of it is the other” (¶ 7). In (¶ 9) she focused on just the relationship between the moisture content ratio, the moisture weight at the first analysis, and the original weight of the berries after Tom started to object to her response (¶ 8). However, Liz was not speaking of a ratio comparison. Instead, she was speaking of numbers of things.

Liz proceeded to change the other rate notecard, “moisture 2nd per new weight” to a ratio notecard and to designate it as standing for a ratio made by comparing “weight of moisture 2nd” and “new weight” multiplicatively. Liz entered “w” to represent the value of the total weight at the second analysis. She was curious to see what formulas the program would generate, so she used a feature of the program that made it show one-at-a-time each formula it generated and the quantities and relationships that formed the basis for inferring that formula. As she noticed the new formulas she paid particular attention to the relationship between the three highlighted quantities. Excerpt 11 has part of this discussion.

Excerpt 11

1. Liz: *Liz enters "w" into the formula cell of the notecard "New weight." WPA indicates that it is ready to infer a formula for “weight of moisture second” based on its relationship with “New weight” and “Moisture 2nd per new weight.”* Okay that's the first thing I expected.

2. Tom: Right.

3. Liz: *Liz clicks the mouse. WPA enters the formula “0.98 * w” into the notecard for “Weight of moisture 2nd,” and then indicates that it is ready to infer a formula for “weight of*

other" based on its relationship with "New weight" and "Weight of moisture 2nd." That, I didn't expect. Oh, I wasn't thinking though. That this is related from these two. *Liz points to the notecard "Weight of other" then to the two notecards "Old weight" and "New weight."* So, so in that first step… you got something here. *Liz point to the notecard "Weight of other." Liz clicks the mouse. The formula, w - 0.98 * w, appeared in the "Weight of the other" notecard.* Now how did you get that? Wait a minute. *Liz clears the formula cell of the notecard "New weight."* If I put x here and I hit return… *Liz enters "x" into the value cell of the notecard "New weight." Liz pauses.* It says… I am going to do something here. *WPA highlight the notecard "Weight moisture 2nd." Liz points to the highlighted notecard.* Why did it change this one here? *Liz points to the notecard "Weight of other."*

4. Tom: No, do you see these things highlighted? *Tom points to the highlighted handle bars of the notecards "New weight" and "Moisture 2nd percentage."*

5. Liz: Right, it says it's related to these two and I am going to get something here. *Liz points to the notecards "New weight" and "Moisture 2nd percentage."* Okay. *Liz clicks the mouse. WPA highlights the notecard "Weight of other." Liz points to the highlighted notecard.* Why did it do that? Why did it…

Liz expected that the formula, w, was formed by the quantities, moisture weight at the second analysis with the formula, 0.98 * w, and the moisture content ratio in the second analysis with the formula, $\frac{98}{100}$ (¶ 1) . She did not expect the formula for the non-moisture weight, w - 0.98 * w, to be inferred from the quantities, new total weight of the berries, w, and the moisture weight in the second analysis, 0.98 * w (¶ 3). She said, "Oh, I wasn't thinking though" (¶ 3) and tried to understand the relationship. She explained that the non-moisture weight was related to both the total weight of the berries at the second analysis and the total weight of the berries at the first analysis. She did not seem to think of the total weight of the berries at the second analysis, w, as being formed by additively combining the quantities, non-moisture weight, w - 0.98 * w, and the moisture weight in the second analysis, 0.98 * w.

During the next ten minutes, Liz entered values and used a table to solve for the value of the total weight of the berries at the second analysis which she determined was 50 kg. Liz was surprised the value was so low. Tom explained that 99% of the original total weight of the berries was the moisture weight in the first analysis and there was 1% left over—the non-moisture weight. He then explained that 98% of the total weight of the berries at the second analysis was the moisture weight in the second analysis. He added that there was 2% left over—the non-moisture weight. He continued his explanation and said the non-moisture weight was the same at the time of both weighings. Tom concluded by stating it makes sense that the new weight was half of the old weight. Liz stated that she did not understand his explanation. It appears Liz was surprised and had difficulty understanding Tom's explanation because she was thinking of a number of things rather than percents as made evident in Excerpt 10 (¶ 9) when she said, "99 of it is moisture and 100 of it is the weight." This suggests that, all along, she was speaking of ratios and rates, but she was thinking of numbers of things.

Summary of the Moisture in the Berries Problem Revisited

In the beginning of the interview, Tom and Liz discussed their ideas about the problem. It appears as though their image of the situation did not change because they gave the same explanations of the situation as they did in the pre-interviews. During this discussion, Liz abandoned her proportion perhaps because she realized it was not consistent with her reasoning. Tom realized that the new moisture weight was 98% of the new total weight only after hearing Liz's explanation. At this point, both seemed to have a clearer understanding of how the non-moisture and moisture portions made up the total weight of the berries at each analysis; however, they still did not think of the percents as ratio comparisons. Instead, both thought of percents as operators. Liz also apparently thought of a percent as a number of something.

Tom and Liz isolated the quantities in the situation as preparation for setting up the situation using WPA. They struggled when trying to represent 99% and 98% using the program. This struggle seemed to arise from the fact that they thought of these percents as operators to be used in a procedure instead of as values of ratios or rates. Liz also thought of percents as numbers of something— in this case, a number of kilograms. Their images of percents kept them from thinking of the percents as values for quantities.

When Tom tried to resolve the problem with the original setup on his own, he seemed to be thinking of the situation as involving ratio comparisons and thus made the appropriate changes to the setup. Liz seemed to change the original setup to have ratio notecards only because it was her last option. When she talked about the percents, she was speaking about ratios and rates but she was thinking of numbers of things.

The Brothers Problem

> I walk from home to school in 30 minutes, and my brother takes 40 minutes. My brother left 5 minutes before I did. In how many minutes will I overtake him? (Krutetskii, 1976, p. 160)

The development of Liz's and Tom's reasoning, as it entailed ever more elaborated imagery of the quantities and relationships in the problem, showed up in other settings—one being their discussions of the Brothers problem, given above.

Gagne, Yekovich, and Yekovich (1993) provides Thompson's discussion of the quantities and quantitative relationships in the Brothers problem given below (in Gagne's version of this problem the brother left 6 minutes earlier): "I" takes $\frac{3}{4}$ as long as brother to walk the same distance, so I walks $\frac{4}{3}$ as fast as brother; what matters is the distance between the brothers and how long it takes for that distance to shrink to zero; the distance between the brothers shrinks at a rate that is the difference of their walking speeds and since I walks $\frac{4}{3}$ as fast as brother, the difference between their speeds is $\frac{1}{3}$ of brother's speed; the distance between the brothers shrinks at $\frac{1}{3}$ of brother's speed, so the amount of time in which it shrinks to zero is 3 times the amount of time in which brother walked it; therefore, it will take "I" 15 minutes to overtake brother.

After reading the problem, Liz wanted to use WPA because she thought "she would understand the problem more" but she knew they were expected to solve the problem without using WPA first so they proceeded to do so. They discussed the problem while writing notes on paper keeping in mind they were going to eventually set up the situation using WPA. At the beginning of the discussion, Tom identified an implicit quantity. They also drew a picture when discussing what they thought was going on in the situation. Excerpt 12 provides this discussion.

Excerpt 12

1. Tom: So you're not, we are talking WPA here.
2. Liz: I know.
3. Tom: Obviously there is something implicit there and that is the distance. Right?
4. Liz: Right.
5. Tom: The distance is the same. Right?
6. Liz: Right. No matter what…
7. Tom: I am going to draw a picture for this because I like a picture…
8. Liz: I know that's what I was going to say.
9. Tom: Okay, go ahead.
10. Liz: Okay we got home… *Liz draws a box with the word "home" above it.*
11. Tom: Right. And we got school.
12. Liz: And we got school. *Liz draws another box with the word "school" above it.*
13. Tom: So the…
14. Liz: So this is the distance. *Liz draws a bracket stretching across the length between the "home" box and the "school" box.*
15. Tom: Right, home to school.
16. Liz: And here's "I"… *Liz draws a stick figure and places the label "I" underneath the picture.*
17. Tom: And here's "bro".
18. Liz: And here's "bro". *Liz draws a stick figure and places the label "bro" underneath the picture.* I didn't give "bro" a face. Look, I'll even draw hair.
19. Tom: Okay, so "bro" takes forty minutes to get to school that means "bro" has a rate of… we'll call this d. *Tom points to the distance between "home" and "school."* Do you like this d? D over forty right?

In (¶ 1) Tom seemed to mean that they needed to talk about quantities and quantitative relationships because they were preparing to use WPA. Tom immediately conceived of the implicit quantity, distance (¶ 3). It seems they both agreed that the distance from home to school is the same for each brother (¶ 5, 6). Both Tom and Liz wanted to draw a picture (¶ 7). In (¶ 10-15) they drew two boxes across from each other and labeled them home and school, respectively. They identified the space in between the boxes as the distance between them and labeled it "d" (¶ 19).

Tom identified the quantities, rate of brother and rate of "I", and assigned expressions to each quantity, r_b and r_I, respectively. He did not express what he meant by "rate," but by saying the rates are not equal "because it is the same distance and different times already" suggests that he understood that "equal" rates would imply covering the same distance in the same amount of time.

Tom and Liz briefly discussed the times given in the problem statement. They gave the quantity, the time it took the brother to walk to school, a value of 40. They also explained that it took "I" 30 minutes to walk to school. Tom suggested that they set up an equation in terms of time and then mentioned the fact that the brother left five minutes before "I". This statement caused Liz to think about the rates of the brothers and what they were trying to find. Her inquiry led them to discuss the various distance quantities in the situation. Excerpt 13 gives some of this discussion.

Excerpt 13

1. Liz: I've just had a thought, uh…even though, even though the brother left five minutes before, it's still going to take him forty minutes. And we need to remember that. You know like his rate is still going to remain the same. No matter what, the rates are going to remain the same. So what we're trying to do is we're trying to find out when their times are, not when their times are equal…
2. Tom: When their distances are equal.

3. Liz: When their distances are equal. When they've gone the same distance. When they meet, I don't know if that's right either. Do you know what I'm trying to say.

4. Tom: Yeah. Oh so, see when we say this d then, we're not talking about d as a variable, we're talking about d as a set number.

5. Liz: I was. Yeah.

6. Tom: Okay, okay. No, I mean I wasn't clear on that so I wanted to state that.

In (¶ 1) Liz stated that the rates of the brothers do not change. Taking this into consideration, Liz tried to think about what they were trying to find. Tom helped her and explained that they were trying to find when their distances were equal (¶ 2). Liz restated what Tom said and explained that she thought they were trying to find when they meet (¶ 3). This prompted Tom to explain that the variable “d” was a set value. It seems Tom was referring to the distance the brothers had to walk from home to school as a set distance, d. Tom was distinguishing between the distances the brothers walked until they met and the distance the brothers had to walk to get to school. They seemed to be thinking of the brothers walking the same distance but for different amounts of time to meet along the way to school. Liz expressed her concern over wanting to clarify her understanding of what they were trying to find (¶ 6).

They then addressed the issue of representing the different times in the situation. Tom explained that the 30 minutes and 40 minutes were how long it took “I” and brother to get to school, respectively. Liz stated that “the time it took the brother to get to the point where they meet is 40 minus the time that he has to walk [to get to the meeting point]” and represented this relationship with the equation, $t_b=t_I+5$ where t_I was the time it took “I” to walk to the meeting point.

One minute after the discussion in Excerpt 13 took place, Tom wrote two equations to represent the distance each brother walked until they met: $d_b=r_bt_b$ and $d_I=r_It_I$. As he was writing, he explained that the distance, d, used in the rate expressions, $r_b=\frac{d}{40}$ and $r_I=\frac{d}{30}$, was the total distance between home and school. He added that d_b represented the distance the brother walked until they met and d_I was the distance "I" walked until they met.

At one point during the discussion Liz expressed uncertainty with an equation Tom wrote, $d_b=r_bt_b$, and requested him to explain what the equation represented. Tom added details to the drawing as he explained. Excerpt 14 provides the discussion that followed:

Excerpt 14

1. Tom: This is the variable time. *Tom points to the "t_b" in the equation "d_b = rate$_b t_b$."* And this is d of I equals rate of I times t of I. Right? *Tom writes "$d_I = r_I t_I$."*
2. Liz: Explain that to me. Explain exactly what you're representing.
3. Tom: Okay, I am representing… I'm thinking of, here's, when it's like this. *Tom draws a box around the equation "d_b = rate$_b t_b$."* Right? I am thinking of here's the brother, starting here, starts walking. *Tom points to the box labeled home and moves the pencil along the path from home to school.*
4. Liz: Uh huh.
5. Tom: Okay? So he just starts walking and so say the brother's here. *Tom stops and puts a dot on the path.* Okay? Then I'm saying, you know then the brother has, then he's this far away from home; D_b away from home. *Tom draws a bracket from home to the dot. Then Tom points to the "d_b" in the equation "d_b = rate$_b t_b$."*

6. Liz: Okay.

7. Tom: Has walked this amount of time. *Tom points the "t_b" in the equation "$d_b = rate_b t_b$."*

8. Liz: Okay.

9. Tom: Okay? And the rate we already know.

10. Liz: Right.

11. Tom: Right? And same thing with, with "I."

It seems that Liz was genuinely concerned with wanting to understand the equation Tom wrote (¶ 2). It seems as though she wanted to understand how the equation represented what they had discussed before carrying out the algebraic manipulations. Tom's explanation revealed that he was thinking of the situation dynamically. He began his explanation by moving his finger along the path from home to school to represent the brother walking (¶ 3). He marked a point along the path to represent how far the brother walked. It seems Tom was thinking of this marked point as how far the brother walked until they met. He labeled the distance from home to the mark, d_b, and pointed to the same expression in the equation (¶ 5). He then explained that the time it took the brother to get to this point was represented by the variable, t_b, in the equation. He stated that "I" had a similar situation and hence the reason for the equation, $d_I=r_It_I$. A short time later, they said they wanted to find when the distances each brother walked were equal so they equated r_bt_b with r_It_I.

They performed the appropriate algebraic manipulations on the equation, $r_bt_b=r_It_I$, and came up with a solution of 15 minutes for the time "I" walked until he met his brother. Before they used the program to represent the quantities and the quantitative relationships in the situation, they drew a quick sketch of what they needed to construct using the program. They identified all the quantities they had used in their previous discussion and drew arrows to represent the relationships among them.

Summary of the Brothers Problem

This section revealed Tom and Liz's emerging image of the situation of the Brothers problem. After Tom and Liz read the Brothers problem they began to identify each quantity in the problem. They were no longer content with solving the problem without identifying the quantities. This suggests that Tom and Liz had developed a more conceptual orientation to problem solving. Tom recognized that the total distance the brothers traveled to school was an implicit quantity, that it was important to identify it and to think about its relationship with other quantities in the problem. Tom thought it was important to identify all of the implicit quantities because they (Tom and Liz) needed them to make a complete representation of the situation when using WPA.

Tom and Liz spoke of how they imagined the situation as they continued to identify the quantities. They had an image of two brothers walking to school at different speeds. They also understood that the brothers walked the same amount of distance in different amounts of time when they isolated the distance each brother traveled until they met along the way. They understood that the solution came from finding the amount of time that must pass from some starting time for the two brothers to have walked the same distance at the same moment.

They did not immediately apply the formula d=rt to solve the problem. Tom and Liz thought about all of the quantities and quantitative relationships in the problem before employing numerical operations. He gave a dynamic illustration of the equation, $d_b=r_It_I$. Tom used a diagram and he moved his finger to help Liz understand what he was thinking. He explained that the path from home to school represents Brother walking. He marked a point on the path to show how far Brother walked, labeled it d_b, and later explained that "I" walked the same distance to the meeting point. He also stated that it took the brother a certain amount of time to walk to the meeting point and labeled it, t_b. His discussion of quantities and the quantitative relationships reinforces the researcher's claim that Tom developed a conceptual orientation to problem solving.

Overall Summary

As the unit progressed, Tom and Liz increasingly elaborated their images of situations. Initially, they did not discuss their images until they were prodded by the researcher. Once they were prodded, they mentioned the explicit quantities in the problem as they tried to explain their imagery. However, in the pre-interviews and in the beginning of the Moisture in the Berries Revisited interview, they neglected to assign values given in the problem statement to a quantity of which it was a value. This was true when Tom and Liz thought of percents as operators instead of values of ratios. This seemed to have delayed Tom and Liz from developing an image of the basic situation.

Tom and Liz's thinking of percents as operators revealed their difficulty in conceiving of ratios as quantities. They also struggled conceiving of rates as quantities in several problems during the transition interviews. It is not clear how they were thinking of ratios and rates but their struggle with representing the situation on WPA leads one to believe that their understanding of these concepts may not have been very strong. During the final interview when they solved the brothers problem, Tom and Liz showed more confidence and a deeper understanding of ratios and rates as quantities. They explained that the rate of the brother was how far he walked in a certain amount of time. They identified all of the quantities that made up the rates and made the appropriate connections using both rates and ratios in the setups.

During the pre-interviews, Tom and Liz did not seek to identify any implicit quantities. Tom did not understand why the researcher was asking if there were any implicit quantities in the situation. Several times in the subsequent interviews, Tom mentioned that he understood why the researcher had asked him such a question. Both Tom and Liz began to understand the need to identify all the quantities, explicit and implicit, in order to gain a better image of the situation. That is, they started to develop an orientation to look for quantitative structure. For example, after reading the Brothers problem, Tom and Liz identified an implicit quantity that they thought must be there because without it they would have had an incomplete quantitative structure.

CHAPTER V

DISCUSSION AND IMPLICATIONS

Discussion

In Chapter 1, the researcher discussed the importance of communication in the mathematics classroom. The *Curriculum and Evaluation Standards for School Mathematics* developed by the National Council of Teachers of Mathematics (NCTM, 1989) explained that students must engage in discussions with the teacher and other students about the relationships between concepts and symbols. The NCTM explains that if students do not achieve this level of discourse they may view mathematics as the automation and manipulation of symbols instead of as a well-structured network of concepts.

Thompson et al. (in press a) identified two orientations to teaching mathematics that they claim influence the effectiveness of instructional conversations between students and teachers in the classroom: calculational and conceptual orientations. Teachers with a calculational orientation structure their conversations with students around asking questions such as “What is the answer?” and “What numerical operation do you need to perform?” The goal of these teachers is to teach students how to manipulate symbols and numbers to get the “right answer.” Teachers with a conceptual orientation ask students questions such as “This is a number of what?” and “What are you trying to find?” The goal of teachers with a conceptual orientation is to engage students in conversations that help them to form images of situations. Once they understand a situation the teachers then address the ideas of numerical operations and manipulating algebraic equations.

This study provided insights into Tom and Liz's imagery of elementary quantitative problems and the elaboration of their imagery as the quantitative reasoning unit progressed. Thompson (in press e) stated that "mathematical reasoning is firmly grounded in imagery" and as the study showed, Tom and Liz's reasoning became more elaborate as their imagery and elaboration of imagery became more developed. Consequently, there was a shift in the kinds of things Tom and Liz reasoned about when attempting to solve problems—which is similar to saying that Tom and Liz changed their orientations to problem solving. Since Tom and Liz are both students and future teachers, the researcher will discuss Tom and Liz's imagery and orientation to problem solving and then discuss the implications for their orientation to teaching mathematics.

Before the unit began, Tom and Liz did not talk about their images of situations until the researcher asked them to do so. Instead, they had a calculational orientation to problem solving. That is, they talked about numbers and numerical operations and how to apply them to get the right answer. They did not seem overly interested in making sure they understood the problem. They were primarily concerned with using the given information to form algebraic equations to find a solution. This was evident in the pre-interview when they read the Moisture in the Berries problem and continued to solve the problem without understanding the basic situation. When the researcher inquired about what they thought was going on in a situation, Tom and Liz struggled to try to explicate their images.

Tom and Liz's struggle to talk about their images diminished as they began to develop a more conceptual orientation to problem solving. For instance, upon reading the Brothers problem, Tom and Liz immediately identified the explicit and implicit quantities in the situation. They also talked about their imagery while forming the algebraic equations needed to solve the problem. They were no longer content with performing numerical operations on the numbers in the problem unless they understood the situation. It is important to note that they started to talk about what they thought was going on in the situation only after WPA showed them how to think about quantities and quantitative relationships in the situations. The program did not allow them to perform numerical operations to obtain a solution. Tom and Liz were responsible for setting up a representation of the situation per se and then WPA was responsible for performing the numerical operations to obtain the solution.

It seems reasonable to suggest that Tom and Liz would have primarily had a calculational orientation to teaching mathematics had they not grappled with the problems in the quantitative reasoning unit. It is likely that Tom and Liz's orientations to problem solving would be reflected in their orientations to teaching mathematics. If Tom and Liz had discussed the Moisture in the Berries problem with students before the quantitative reasoning unit, they would have talked about the numbers found in the problem statement and the numerical operations they needed to perform to find the right answer. They would not have had a well-developed image of the situation so they would not have been capable of talking about anything other than the numbers and numerical operations. Consequently, their students would not have a chance to develop an image of a situation because for the students, the numbers and numerical operations had no meaning. They would be constrained to talking about the numbers and numerical operations in a situation as well.

As the unit progressed, Tom and Liz gradually changed from having a calculational orientation to a more conceptual orientation to solving problems. This change, if it persists, would support a change in their orientation to teaching mathematics from a calculational orientation to a more conceptual orientation. During the last interview, the researcher asked Tom and Liz how they would present the Brothers problem to their students. They stated that they would not talk about numbers and numerical operations until the students understood the situation. Instead, they would try to encourage the students to identify the quantities and quantitative relationships in the problem. It seems reasonable to suggest that when discussing a problem, Tom and Liz would be more likely to ask students “This is a number of what?”, “What are you trying to find?”, and “What did this calculation give you?” Consequently, they would help students develop an understanding of what was going on in situations.

Tom and Liz began to develop the skill that teachers must have to conduct conceptually grounded conversations with students: the ability to express their ideas within the context of a given occasion, topic, idea, problem, or, in general, a domain of discourse and to use a language to relate their ideas that students will understand. To express their ideas in the manner described above, Tom and Liz would need to have a deep conceptualization of the problem. Both Tom and Liz were highly competent mathematics students. They were both well on their way to completing their masters degree in mathematics with a specialization in teaching mathematics on the community college level. Despite their competence in mathematics, Tom and Liz had difficulty understanding elementary quantitative problems. This suggests that perhaps they did not have a strong conceptual understanding of particular mathematical ideas (e.g. thinking of percents as ratios).

However, as Tom and Liz became more aware of their images they seemed to gain a better understanding of particular mathematical concepts embedded in the problems. In Chapter I, the researcher suggested that perhaps if teachers with a calculational orientation towards teaching mathematics became more aware of their images they would be able to expand on their language and hence express themselves more clearly. The study showed that the language Tom and Liz used to explain their reasoning gradually became more precise and easy to understand. They expanded their language— their approach to problem solving evolved from first talking about numbers and operations to first talking about the quantities and quantitative relationships in a problem, and then talking about numbers and numerical operations.

Before drawing any implications for this study, it should be noted that the intensive observations which led to the assertions made in this paper were based on only two people, neither of whom had extensive teaching experience.

Implications

Even though some teachers and future teachers may appear to be highly competent in mathematics, they may not necessarily have well-developed images of situations. That is, they may not have an adequate grasp of the quantities and quantitative relationships in a problem. As this study showed, even sophisticated mathematics students have difficulty with some nuances (e.g., rate and ratio). This study showed that such inadequacies may form a barrier in developing a conceptual orientation towards both problem solving and teaching mathematics.

It is important that mathematics teachers possess a conceptual orientation towards teaching mathematics. This study suggests that one way to help attain this goal is to have teachers become aware of their imagery and to develop it while solving quantitative problems. How can we engage teachers so that they develop more refined imagery and so that they develop a conceptual orientation? One way may be to organize both in-service and pre-service training programs to develop these ideas. It may be difficult to encourage more traditional teachers to try to think about quantities and quantitative relationships and to change their teaching style accordingly, but the leaders of these programs must make an effort. This study shows that future teachers can progress in forming well-developed images of situations and changing their orientations to problem solving.

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APPENDIX

PROBLEMS GIVEN IN THE INTERVIEWS

PRE-INTERVIEW PROBLEMS

1) Moisture in the Berries:

In a store there were 100 kg of berries. An analysis showed that the moisture content in the berries was 99%. Determine the total weight of the berries when, by a later analysis, the moisture content had decreased by 1% and had become 98%. (Krutetskii, 1976, p. 160)

2) Some workers were hired for repairs, and they were to do the work in a certain number of days. If there were 3 fewer men, the deadline would have to be moved forward 6 days; if there were 2 more men, they would finish the work 2 days before the deadline. How many workers were hired?

INTERVIEW PROBLEMS DURING THE QUANTITATIVE REASONING UNIT

3) Tutorial Problems

a) Jim gave a cashier $20.00 for groceries that cost $16.23. How much change should Jim get?

b) Mrs. Robertson bought groceries for $18.74. She received $31.26 in change. How much did she give the cashier?

c) John tried to pay for his groceries. He gave all the money he had and still owed $3.72. The groceries cost $17.65. How much did John give the cashier?

d) Make a formula that tells how to compute the amount given a cashier when you know how much the groceries cost and how much change is received.

e) Make a formula that tells how to compute the cost of groceries when you know the amount given a cashier and how much change is received.

f) Tom has $\frac{2}{3}$ as many marbles as Sally. Sally has twice as many marbles as Fred. Fred has 12 marbles. How many marbles does Tom have?

g) Tom has 15 marbles. Fred has 10 marbles. Tom has $\frac{3}{4}$ as many marbles as Sally. What is the ratio of Sally's marbles to Fred's marbles?

h) Joe knew three things: the number of marbles that Tom has, the number of marbles that Fred has, and the ratio of Tom's marbles to Sally's marbles. He wanted to know the ratio of Sally's marbles to Fred's marbles, but didn't know how to compute it.

Joe went to Harriet's house to ask for help, but he forgot the paper that had the numbers, and he couldn't remember them. Harriet still told Joe how to compute the ratio between Sally's number of marbles and Fred's number of marbles. What did Harriet tell Joe?

i) Put numbers in value cells so that every operation inferred by WPA is multiplication.

j) Sally has 6 times as many marbles as does Fred. Tom has 12 times as many marbles as Sally . What is the ratio of Tom's marbles to Fred's marbles?

k) A train accelerated at the constant rate of $0.37\,\frac{\text{(mi/hr)}}{\text{sec}}$. It ended up going 65 $\frac{\text{mi}}{\text{hr}}$. For how long did it accelerate?

l) A boat traveled upstream on a river. The boat's speed relative to the river was 15 mi/hr. It took the boat 2.2 hours to travel along 12 miles of riverbank. What was the speed of the river relative to the riverbank?

4) A biologist released 200 marked fish into a lake. He later captured six samples of fish. On the average, the number of marked fish in each sample was $\frac{1}{60}$ th of the sample size. Approximately how many fish are in the lake? (Assume that the marked fish had become "thoroughly mixed" with the rest of the fish.)

5) While driving, John accelerated at 7 $\frac{\text{ft}}{\text{sec}}$ / sec for 10 seconds. Then, he drove at a constant speed. His speed as he began accelerating was 33 $\frac{\text{ft}}{\text{sec}}$. How far did John drive in the first 5 seconds after he stopped accelerating?

6) Elmer Elementary School has a circular race track. It takes 20 laps to run a mile. What is the diameter of the track?

7) Moisture in the Berries Problem Revisited:

In a store there were 100 kg of berries. An analysis showed that the moisture content in the berries was 99%. Determine the total weight of the berries when, by a later analysis, the moisture content had decreased by 1% and had become 98%. (Krutetskii, 1976, p. 160)

8) An 80 year-old person snapped his fingers. The snap took approximately $\frac{1}{10}$ th of a second in this person's lifetime. The earth is approximately 4,500,000,000 years old. What would be a snap of the fingers in geological time? (given to Tom only)

9) I walk from home to school in 30 minutes, and my brother takes 40 minutes. My brother left 5 minutes before I did. In how many minutes will I overtake him? (Krutetskii, 1976, p. 160)

ABSTRACT

ABSTRACT

A study of the imagery two future teachers bring to the solution of elementary quantitative problems. The study focuses on characterizing their elaboration of imagery as they work in an environment which constrains them from thinking calculationally. This study begins to provide a framework for developing a theory of the reflexive relationships among teachers' orientations, imagery, and language. The context for the study is a unit on quantitative reasoning in a mathematics class offered at a reputable university.

The future teachers attempted to solve elementary quantitative problems during biweekly interviews during the quantitative reasoning unit. Analyses of these interviews revealed that the subjects became increasingly concerned with identifying the quantities and quantitative relationships in situations. The subjects also formed increasingly elaborated images of situations. It was evident that the subjects had difficulty in conceiving of ratios and rates as quantities, but this difficulty diminished over time.

Despite being highly competent mathematics students, the subjects had difficulty solving elementary quantitative problems. This study shows that such inadequacies may form a barrier in developing a conceptual orientation towards both problem solving and teaching mathematics.

It is important that mathematics teachers possess a conceptual orientation towards teaching mathematics. This study suggests that one way to help attain this goal is to have teachers become aware of their imagery and to develop it while solving quantitative problems. This study shows that future teachers can progress in forming well-developed images of situations and changing their orientations to problem solving.