EFFECTS OF GRAPHING CALCULATORS ON COLLEGE ALGEBRA STUDENTS' UNDERSTANDING OF FUNCTIONS AND GRAPHS

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Elizabeth Ann Nagel

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THE THESIS OF ELIZABETH ANN NAGEL:

Patrick W. Thompson, Chair Date

Alba G. Thompson

Randolph A. Philipp

SAN DIEGO STATE UNIVERSITY

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CHAPTER I

SIGNIFICANCE OF STUDYING THE GRAPHING CALCULATOR AND MATHEMATICS STUDENTS

The graphing calculator is finding its way into students' hands and into mathematics curricula. It is a resource which both students and instructors are enthusiastic about using in learning and teaching mathematics. According to Hembree and Dessart (1992), "The integration of the calculator into the curriculum where it plays a central role in the learning process is a worthy goal for the research of the 1990s." (p.31) Research into the effective use of the graphing calculator in educational settings is needed to gain insight into claims that the graphing calculator and its associated teaching methodology are of practical use to students.

Cognitive Technology

"A cognitive technology is any medium that helps transcend the limitations of the mind in thinking, learning, and problem-solving activities" (Pea, 1987, p.91). It is the ability of a cognitive technology to directly influence students' thinking that distinguishes it in the students' cognitive environments. This influence allows the computer and graphing calculator to be considered cognitive technologies.

The actual effect of students' use of any cognitive technology may not coincide with the intentions of the designer. Examination of students' use of the graphing calculator may show that students' thinking and learning travels along different paths than those expected by the calculator's designers or by educators.

By externalizing elements of their thought processes students extend their reasoning beyond the limitations of the mind. "A common feature of...cognitive technologies is that they make external the intermediate products of thinking, which can then be analyzed, reflected upon, and discussed" (Pea, 1987, p.91). The designers of the graphing calculator externalized their thinking when they produced the calculator. Students may externalize and analyze their thinking when they verify relationships between functions by entering them into the graphing calculator. Similarly, students may externalize their thinking by writing calculations and interpretations with pencil on paper, or with chalk on a board.

The ever present availability and general use of various cognitive technologies in mathematics instruction shows that instruments of mathematics instruction are neither new nor necessarily singular in nature. The variety of cognitive technologies can be demonstrated in tools of calculation. "From ancient times instruments have been used in calculation. Working on paper is actually a rather recent invention. In antiquity and in the middle ages people worked with counters on a board, or with beads on an abacus" (Freudenthal, 1967, p.69).

These cognitive technologies did not and do not exist apart from the communities in which they are created and used. They are products of the intelligence that created them, as well as, products defined by the communities that choose to accept and use them. "In terms of cultural history, these tools and the practices of the user community that accompany them are major carriers of patterns of previous reasoning. They may contribute to patterns of distributed intelligence configured in activity" (Pea, 1993, p.53).

The graphing calculator was created by the electronics industry and its construction and functioning is dependent upon the mathematical logic used in its creation. The ability of the graphing calculator to persist depends on peoples willingness and ability to recognize it as useful. Thus the graphing calculator is both a reflection of a particular community's mathematics, and a potential stimulus for, a possibly different, community's mathematics.

This role of cognitive technologies, such as the graphing calculator, in affecting not only the mathematical environment, but mathematics itself has a considerable cultural impact. Noss (1988) asserts that "the technology which is at the disposal of a given culture directly influences the kinds of mathematics which are indigenous, spontaneous or frozen into that culture." (p.254) A type of mathematics that seems to be frozen into our culture is paper-and-pencil mathematics. Nine year old students are expected to spend hours of their time practicing the long division algorithm. Later, these same students will be expected to spend time calculating data points and filling in tables from which they can eventually draw graphs. Just as standard calculators offer an alternate to the long division algorithm, graphing calculators are bringing additional choices into mathematics instruction involving functions and graphs.

It is thus apparent that a necessary task of mathematics educators is to examine new, or potential, cognitive technologies to discover how they might effect or alter the mathematical content being taught. Inherent in this examination is the need to carefully consider the teaching approach within which a particular cognitive technology is being employed.

Each cognitive technology used in mathematics has an effect that might build on the current structure of mathematics. "The design of artifacts, both historically by others and opportunistically in the midst of one's activity, can advance that activity by shaping what are possible and what are necessary elements of that activity" (Pea, 1993, p.50). Computers, in general, allow mathematicians to use complex algorithms which otherwise would have been too difficult or problematic to implement. The computer as a cognitive technology has extended the domain of mathematics.

Mathematics is thus cumulative. Yet, at any given time, there exists some criteria for what people consider proper mathematics. The limits of this definition are challenged by the advent of each new cognitive technology, the result often being that the criteria for proper mathematics changes. "An answer to the question 'what is mathematics?' dictates the kinds of problems and methods that are acceptable in the community and determines what parts of the past are included or excluded in the present paradigm" (McCleary & McKinney, 1986, p.51). As long as the long division algorithm is considered as an element of proper mathematics, by influential members of society or by educators who have some degree of influence over schools, then nine year old students will study it.

Due to the prevalence of computers and electronic equipment in industry, today's students will probably be expected to use computers as a routine tool. These students "need to learn a different mathematics than their forefathers. Standard school practice, rooted in traditions that are several centuries old, simply cannot prepare students adequately for the mathematical needs of the twenty-first century" (Steen, 1990, p.2). So, the question is "What mathematics, what teaching approaches, and what use of computers and electronic equipment will be appropriate to these students needs?"

Educators interpret what methods of using computers and electronic equipment in teaching are appropriate. This interpretation effects the way cognitive technologies are used, and this use, in turn, changes the effects the cognitive technologies can have. Studies of the effects cognitive technologies have on education yield similar results. Thus, "we affect computers when we study their use, reflect on what we see happening, and then act to change it in ways we prefer or see as necessary to get the effects we want" (Pea, 1987, p.95). We may examine a method of teaching children to multiply using the computer. The available program might respond with a flashing light when the student answers incorrectly, then supply the correct response. Research might reveal that students easily lose interest in this program. As a result of these findings, educators might then rewrite the program to include sound effects and pictures that attract and keep the attention of students. Alternately, educators may find the need to reevaluate their perceptions of what it means for students to learn multiplication.

"Implementation of new technologies also forces reconsideration of traditional questions about control and the social structure of classrooms and organizational structure of schools" (Kaput, 1992, p.516). The social interaction necessary in cognitive situations created by the presence of new cognitive technologies may prove to be quite different from that required in current educational settings, presuming that teachers' objectives are to change what students can and do learn in that setting. Teachers may have to reevaluate and modify the way they teach, so that they may more effectively communicate and use the new technologies to teach students.

Students will probably need to use computers routinely in most jobs and professions, so it should be an integral part of student environments. "It needs to be thought of as both a tool and a medium for instruction, not as something that is added on to the existing curriculum in appropriate places" (Glass, 1984, p.13). In this case, students should frequently interact, not only with the products of the computer, but also, with the computer itself.

The graphing calculator as a cognitive technology has the potential to help students transcend the limitations of learning. This potential may be dependent on the teaching methodology being employed in any given classroom. Whether or not students gain understanding of functions and graphs by using the graphing calculator may be a direct product of the teaching methodology.

What is Understanding?

Understanding is a goal in any instructional setting. What is meant by "understanding" is thus the key to creating concrete instructional goals. The kind of understanding this researcher will be looking for in students will therefore shape perceptions of students' activities.

Skemp (1978) discusses two kinds of understanding: "instrumental understanding" and "relational understanding". A student with only instrumental or procedural understanding is equipped with rules, but no reasoning to support the rules. A student with relational or conceptual understanding knows what actions to take and why.

Each of these types of understanding have their proponents. By considering mathematics instrumentally and relationally Skemp (1978) points out positive aspects of each. Instrumental mathematics is often based on quick rules and methods. These can be much easier to understand and can produce immediate rewards to students who get correct answers with minimal effort. Alternately, because its methods are used for particular reasons, which the student comprehends, relational mathematics is more adaptable to new tasks and easier to remember.

The differences in the long-term (two years) effects of instrumental and relational learning for two students, were noted by Wearne and Hiebert (1994). The first student "practiced as many as thirty problems each day, usually without a story context. The goal of instruction was efficient, correct computation" (p.273). This student was able to correctly solve a second-grade story problem involving addition of whole numbers, but was not prepared to use her strategy to solve problems involving addition of decimals in the fourth-grade. She had to learn new rules in order to solve the new problem. The second student "spent more time developing place-value ideas, using these ideas to develop procedures for combining numbers, and then sharing procedures with other members of the class" (p.273). This student was able to solve the problem involving decimals by modifying the meaningful strategies she developed in the second grade.

This researcher considers students with a relational/conceptual understanding of mathematics to be better prepared to continue learning mathematics. So, in this context, the goal of instruction is to aid students in constructing a relational/conceptual understanding of mathematics. Not all instructors exhibit this goal in their methods of instruction. Discussions of student behaviors, in this paper, will make specific references to the type of understanding observed in students.

Since students always construct understanding from their experiences, past and present, a consideration of student understanding in the classroom must take into account the multiplicity of materials to which students have access. These include textbooks, the instructor, the graphing calculator, notes, and other students.

This study particularly examines students understanding of functions and graphs. Ayers, Davis, Dubinsky, and Lewin (1988) assert that "understanding the concept of function includes the ability to...form a mental representation of the (possibly mental) action of the function....Thus the process...must be consciously understood or encapsulated into a single, total entity." (p.247) The student's process of constructing understanding can thus be viewed as involving the creation of an internal representation of the concept of function.

This process of students' construction of understanding can be viewed in terms of both external and internal representations. External representations are used in communication and may include pictures, language, physical objects, and written symbols. For example, functions can be represented by graphs and in written form as tables or algebraic equations. The assumption in instruction is that there is some relationship between the internal and external representations. Thus, in order to understand functions students must construct internal representations of functions from exposure to external representations of function definitions, actions, and rules.

Multiple representations within the representation of graphing is recommended by McDermott, Rosenquist, and van Zee (1987). Students were given three different, but identically shaped, motion graphs. They were expected to obtain velocity information from each graph, but the differences between the graphs required that the desired information be extracted from different features of each graph. "Being confronted with all three types of motion graphs at the same time helps impress upon the students the difference in the ways that the same information is conveyed in each graph." (p.511) Students, in this situation, may be able to form multiple internal representations of the situation, then find relationships or connections between the internal representations. These students are building conceptual understanding.

"What evidence we have seems to indicate that it is the need for formalisation, rather than merely the feedback involved, that is seminal in influencing learner's conceptions" (Noss, 1988, p.260). The computer or graphing calculator offers an environment within which the student can communicate only in so far as the student is able to adhere to the rules guiding what the computer or graphing calculator can understand. These rules may be explicit enough that students can procedurally determine how to interact with the computer or graphing calculator. This environment also offers students the opportunity to formalize their intuitions about various applications into the mathematical language of functions and graphs. The student, who may need to be guided to recognize this opportunity, is then able to take advantage of a dynamic and responsive environment, which may inspire learning about functions and graphs with understanding.

When students are learning the formal language of a computer system or program they have the opportunity to consider the actions of the computer. This reflection "can lead to interiorization and can stimulate the construction of a mental representation of this process" (Ayers, Davis, Dubinsky & Lewin, 1988, p.249). This external representation may thus contribute to students' understanding of functions and graphs.

Another perspective of mathematical competence is described by Moschkovich, Schoenfeld, and Arcavi (1993). "Competence in the domain consists of being able to move flexibly across representations." (p.97) A student with conceptual understanding is thus expected to be able to, internally or externally, display this understanding regardless of the representation being accessed.

Students use what they already know and the representations to which they are exposed to construct understanding. Students' constructions are made in the context of their local environments. So, "understanding also can be constructed around representations in conversations that include negotiation of the meanings of symbols and enrichment of the ways in which their reference to concepts can be understood" (Greeno, 1991, p.197). Students may negotiate meanings through conversations with instructors or with other students. So while students' understanding is constructed individually, the source material for this understanding may be negotiated interactionally.

Function - Definitions

A single definition of function does not take advantage of the variety of external representations available to the student. Individual definitions are valuable as descriptions of particular situations and help to describe other representations. "Several representational systems can be used to display a function. These include ordered pairs, equations, graphs, and verbal descriptions of relationships" (Leinhardt, Zaslavsky & Stein, 1990, p.35).

Students' exposure to multiple representations of function is conducive to their gaining understanding. So, researchers and students alike may benefit from an awareness of "the historical development of functions, first as dependence relations describing real-world phenomena, then as algebraic expressions, then as arbitrary correspondences, and finally as sets of ordered pairs" (Cooney & Wilson, 1993, p.146). Although a particular textbook will generally define and use a specific aspect of functions, no textbook contains definitions of or references to all of them.

Functions are sometimes defined as dependence relations. "One variable is a function of a second if at least one value of the first is determined whenever a value is assigned to the second. The variable to which values are assigned is called the independent variable, and the other is called the dependent variable" (Rees & Sparks, 1961, p.66). As the name implies this definition highlights the dependence or relationship between the variables.

Functions can also be defined as algebraic expressions. "An algebraic expression like x+2 [is called] a function of x because its value depends on that of x in such a way that to each value of x there corresponds a definite value of x+2....We call every algebraic expression a function of all the variables which occur in it" (Fine, 1961, p.88). This definition specifies one particular variable as the function, and focuses on the relationship between the other variables and the "function". Dependence between the variables is still apparent. Yet, this definition implies that the specific purpose of creating or examining the dependence relation is to determine values of the function, and that values of the other variables are incidental to this goal.

Correspondences are also used to define functions. "A function from a set A to a set B is a rule of correspondence from A to B which assigns to each element of A exactly one element of B" (Ohmer & Aucoin, 1966, p.115). This definition describes a function as a relationship between two sets, rather than describing the relationship between specific variables.

Finally, a function can be defined as a set of ordered pairs. The textbook used for the class which participated in this study used this definition of function. "A function is a relation with the property: If (a,b) and (a,c) belong to the relation, then b = c. The set of all first entries of the ordered pairs is called the domain of the function, and the set of all second entries is called the range of the function" (Demana & Waits, 1990, p.18). This definition is particularly applicable in the context of graphing. When a function is represented by ordered pairs the language of algebra is easily translated to the visual representation of a two-dimensional graph.

These function definitions contain different words but have a basic commonality. They represent, not stark changes in, but a gradual refining of the concept definition. Vinner (1991) asserts that concept definitions are arbitrary and that "a concept name when seen or heard is a stimulus to our memory" (p.68). What the concept definition brings forth from memory is an associated, non-verbal, "concept image", and each concept image is relative to the individual thinking about the concept.

The specifics of a concept definition are relevant in cognitive tasks, in that, an individual's concept image may be overly generalized or restrictive. Examination of the concept definition might refocus the problem solver's attention on aspects of the concept definition that were not taken into account in the formation of the original concept image.

Students also gradually refine their concept of functions and graphs as they gain experience with them. This dynamic process involves "challenging our old assumptions....They are part of the culture....yet they must give way to more fundamental perspectives if we are to discover what doesn't work -- and why....unwarranted assumptions must be dropped" (Ferguson, 1980, p.28). When students are initially exposed to functions and graphs they perceive them in ways that may have to be reconsidered as they gain experience, knowledge, and understanding. Their concept images are thus refined. In this context, when students use graphing as a method to understanding functions it helps them create concept images more aligned with the graphical representation.

Relationships between variables are fundamental to functions. "To develop understanding required for effective application of algebra, students need to encounter and analyze a wide variety of situations structured by relations among variables" (Fey, 1990, p.70). Since students easily form misconceptions based on the limitations of particular applications, each application they encounter gives them the opportunity to dispel misconceptions acquired from earlier applications. When graphing, students have the opportunity to examine functional relationships between variables. Instruction which focuses on these relationships may help students to form robust concept images. The graphing calculator offers students the facility to quickly graph several similar functions together and examine the results of small changes in variables.

The Importance of Functions and Graphs

There is a close link between functions and graphs. Graphs are visual representations of functions. "Functions and graphs represent one of the earliest points in mathematics at which a student uses one symbolic system to expand and understand another. Graphing can be seen as one of the critical moments in early mathematics. By moments we mean sites within a discipline when the opportunity for powerful learning may take place" (Leinhardt, Zaslavsky & Stein, 1990, p.2).

The goal, then, of instruction is to create an environment which facilitates powerful learning. The graphing calculator provides an environment in which "relationships among functions can be readily observed, conjectures can be made and tested, and reasoning can be refined through graphical investigation" (Dugdale, 1993, p.115). The graphing calculator allows students to advance through this exploratory process much more quickly and accurately than they could using pencil and paper. Kieran (1993) concurs that "the capability of computers to dynamically display simultaneous changes in graphical, algebraic, and tabular representations suggests a mathematically rich environment for learning about functions" (p.189).

Graphs can help students find meaning in particular functional relationships. Students can examine relationships between function variables. They can examine changes in graphs caused by the addition of a constant to a function equation. "Functions establish the conformation of particular relationships between changing entities, and graphs help to display selected portions of the relationship" (Leinhardt, Zaslavsky & Stein, 1990, p.46).

It is critical to student understanding that educators create and use good graphic examples. "The selection of examples is the art of teaching mathematics. Making available for consideration by the student an example that exemplifies or challenges can anchor or critically elucidate a point" (Leinhardt, Zaslavsky & Stein, 1990, p.52). Examples can aid the student in accessing and forming various internal representations. One advantageous feature of graphing calculators is that students can rapidly construct and modify many examples of a family of functions.

The ability of students to constructively use visual information and intuition justifies graphing as a part of the curriculum. Additionally, "one of the motives for studying concepts used in graphing is that it may help us understand the nature of the more general concepts of variable and function and the role that analogue spatial models play in representation" (Clement, 1989, p.77). By discovering what students find easy or difficult about graphing, researchers may come closer to understanding similarities and differences between the representations. These findings may aid instructors in directing instructional methods and content.

Demana, Schoen and Waits (1993) agree that functions are of great importance in the curriculum and, more specifically, define the goal "for students to achieve in-depth understanding about important classes of functions." (p.28) Additionally, they not only advocate, but insist that students use computers or graphing calculators with their pronouncement that "this understanding would need to come from exploring numerous graphs quickly with the aid of technology." (p.28)

The importance of functions and graphs is supported by the National Council of Teachers of Mathematics (NCTM)

"Curriculum and Evaluation Standards for School Mathematics" (1989), hereafter referred to as "the Standards". According to the Standards a curriculum should be designed, in part, to assure that students are able to

> model real-world phenomena with a variety of functions; represent and analyze relationships using tables, verbal rules, equations, and graphs; translate among tabular, symbolic, and graphical representations of functions; recognize that a variety of problem situations can be modeled by the same type of function; and analyze the effects of parameter changes on the graphs of functions (p.154).

Why Graphing Calculators Might Help Students Understand Functions

Students are active learners who build knowledge for themselves. They construct internal representations and understanding from their experiences in the world and in the classroom. Ideally, autonomy and reflection are aspects of this learning process. Instructional activities and the actions of the teacher are also part of the students' experiences.

Personal autonomy is the most important aspect of the student's construction process. Since the graphing calculator is inexpensive and available to all students it offers each student the opportunity to explore mathematically and construct knowledge, at any place or time, in their own way. Yet, the type of knowledge being constructed at any given time is dependent on the goals of the student, and on the student's approach to learning. The student's goals and approach may be effected by the goals and approach of the teacher. While the student has the opportunity to explore mathematically by using the graphing calculator, the opportunity also exists for the student to merely learn the mechanics of operating the graphing calculator.

Reflection is important in that the student can access the constructive process through reflection. The student can consider possible reasons for assuming that certain changes in a function will consistently cause particular types of changes in graphs. The student can create hypotheses about these reasons, then test the hypotheses using the graphing calculator.

Instructional activities and the actions of the teacher are important because they may offer problematic situations from which the student can construct knowledge. "Materials typically characterized as instructional representations are of value to the extent that they facilitate the negotiation of mathematical meanings and thus individual students' construction of mathematical knowledge" (Cobb, 1989, p.39). Similarly, Ruthven (1992) notes that ideal cognitive tools help students by supporting cognitive growth. "Indeed, this is an important element of the rationale for using calculators in the mathematics classroom: that they offer not simply a mechanism for calculating and drawing but a medium for thinking and learning" (p.95). The goal of teaching is then to help the student to understand how to learn, rather than teaching about mathematical structures.

Fey and Heid (1984) also found that "many students who are not good at the manipulative aspects of symbolic algebra can use good quantitative reasoning when interpreting the results of computer-generated computations or graphs." (p.28) Thus, for some students, access to graphing calculators permits them to use modes of reasoning of which they are already capable, while keeping them in an algebraic context.

Studies of the differences in expert and novice behaviors with respect to problem solving show that experts and novices

> differ not merely in [sic] amount of their knowledge but also in the types of conceptions and understandings that they bring to a problem and in the strategies and approaches that they use. Expert-novice studies suggest that the performances of beginning learners often can be understood in terms of the inappropriate or inefficient models that these learners have constructed for themselves (Wilson, 1992, p.215).

These studies thus concur that students' mathematical conceptions evolve with experience.

Instructors teaching about function thus need to recognize that "it is also important to equip students with a capacity for recognizing their own misconceptions, or drifting conceptions, and for learning how to recover from them" (Dugdale, 1993, p.125). Instruction needs to be designed to encourage students to check their results, and to examine topics from multiple perspectives.

Purpose of the Study

The purpose of this study is to examine the use of graphing calculators as an aid to student understanding. This study explores the relationship between college algebra students' use of graphing calculators and their understanding of functions and graphs. The study takes into account the curriculum and the instructional methods within which the graphing calculators are employed.

CHAPTER II

REVIEW OF THE LITERATURE

Students' Understanding of Functions and Graphs

What Students Know About Functions and Graphs

Determining what it is that students actually know and are learning about mathematics is an important part of the educational process. In order to decide what instruction is successful and which students are successful in learning about functions and graphs researchers must know what it is that typical algebra students already know.

Students appear to have little difficulty with graphical manipulation that involves mechanical operations, processes or algorithms. This may be due to the considerable practice they have had in these areas.

Mokros & Tinker (1987) noted that

> there is ample evidence that students even at the college level can have the ability to produce graphs from ordered pairs, while being extremely deficient in their ability to interpret graphs. Yet, observers have noted that students as young as 10 years old accurately use graphs in an MBL [Microcomputer-Based Labs] setting. (p.370)

Thus, while many students have difficulty interpreting some graphical information, there is evidence that instruction can alleviate this problem.

Students' ability to easily identify some information from graphs, yet exhibit difficulty interpreting other aspects of graphs may in part be due to their view of mathematics as a group of procedures and algorithms, and in part due to their level of experience with functions and graphs. Early interpretation of graphs involves "a strong tendency among students to view graphs as pictures rather than as symbolic representations" (Mokros & Tinker, 1987, p.371). This tendency could be due to students' past experiences with pictures as objects rather than as symbolic representations of phenomena.

The difficulty that students have interpreting graphs as symbolic representations and the ability of experience to alleviate the problem has been somewhat verified by Mokros and Tinker (1987). "We know that children have a great deal of trouble with graphing distance and velocity, but that their problems are easily ameliorated through exposure to MBL." (p.379)

The ability of students to overcome their difficulties with graphing may be due to a shift from a quantitative to a qualitative interpretation of graphs. "A qualitative interpretation of a graph in its fullest sense requires looking at the entire graph (or part of it) and gaining meaning about the relationship between the two variables and, in particular, their pattern of covariation" (Leinhardt, Zaslavsky & Stein, 1990, p.11).

Global features of a graphical representation tend to require qualitative interpretation. A student examining a global feature, such as the interval of increase, of a graph representing plant size over time looks at the general trend of the graph rather than at the particular plant size at a given time.

Mokros and Tinker (1987) studied the effects of MBL on students' abilities to communicate using graphs. On a pretest given to students, "easy" items were identified as "those that 75% or more of the students answered correctly on the pretest. [Easy items] were typically those that involved very direct translations from a written description of a phenomenon to a depiction of it on a graph." (p.376) Middle school students were able to easily identify particular data or characteristics given a graph of a situation. These "easy" items probably required quantitative interpretation of local features of the graph rather than qualitative interpretation of global features.

Monk (1992) examines the responses of students presented with Across-Time (global) questions and finds evidence of two weaknesses, both of which reflect quantitative interpretation. "The first is that they have a Pointwise view of the function....The second is that their way of conceiving of the function-as-a-whole is overly naive and perhaps static and monolithic. They tend to think (or hope) that there is, ready-at-hand, a simple governing rule which tells about most of the patterns of behavior they seek." (p.187) This desire to identify a "rule" may be a product of the focus of instruction the students have received. These students are using local aspects of graphs, and seeking procedures and algorithms rather than conceptual understanding.

There is evidence that students tend to gain only a procedural understanding of functions and graphs. Dreyfus and Eisenberg (1983) determined that "while teachers believe students are taught the function concept, they seem, in fact, only to assemble loosely connected mechanical procedures and algorithms which they become more or less proficient in applying." (p.124) This may be partly due to a curriculum that has historically focused on mathematics as a collection of procedures and algorithms, and accepted students' procedural understanding as an adequate indicator of learning.

It is reasonable to assume that students may be able to link their intuitions with the graphical representations with which they are already comfortable. "Findings on enhanced student performance on time-based graphs seem to support empirically the notion that intuitions that are based on students' knowledge of real-world situations operate successfully when reasoning in the graphing domain" (Leinhardt, Zaslavsky & Stein, 1990, p.29). Students presented with a graph of plant growth over time are likely to be able to correlate the rising then leveling off of the graph with actual plant growth. These students may be stepping beyond the realm of procedure and algorithm and demonstrating conceptual understanding.

Students who are presented with graphs with unmarked axes, or who do not have access to algebraic data, seem to make more and better use of their intuition about functional relationships. Goldenberg, Lewis, and O'Keefe (1992) interviewed students attempting to identify functions from a graph and noted that "when numbers weren't in the way, students used visual information and intuition quite insightfully." (p.251) Some of the interviewed students quickly recognized and verbalized general relationships between changes in one variable and concurrent changes in the function under investigation. Alternately, students who were able to extract numerical information from the graph, tended to first create tables, then use the tables to create formulas. Perhaps the difference for students was that, without numbers to work with, they no longer felt constrained by the procedural mathematics they had been taught and were able to use their intuitions and more global aspects of the graph.

Student Difficulties With and Misconceptions About Functions and Graphs

Students appear to be participating in mathematics as observers of what they perceive as many unrelated phenomenon. "They view algebraic data and graphical data as being independent of one another. Moreover, they do not even look for unifying interrelationships between the various mechanical procedures" (Dreyfus & Eisenberg, 1983, p.131). Not only are these students accepting a procedural view of mathematics, they are unable to exhibit competence because they cannot move flexibly across representations which they do not even recognize.

Many students may be satisfied with a procedural approach to algebra, yet others may find problems while attempting to look beyond algorithms. Goldenberg (1988) observed students who may have been searching for conceptual understanding. "Students often made significant misinterpretations of what they saw in graphic representations of functions. Left alone to experiment, they could induce rules that were misleading or downright wrong." (p.137) Thus, instructors must be aware of common misconceptions when presenting students with opportunities to construct conceptual understanding of functions and graphs.

Another example of difficulties which may arise out of students' lack of conceptual understanding of functions and graphs involves matching functions with their graphs. Leinhardt, Zaslavsky, and Stein (1990) observed that "when equations were not presented in the $y = mx + b$ format, students had difficulty matching them with their graphs." (p.36) These students may be exhibiting a lack of procedural understanding, as well as conceptual understanding, in that they appear to be unable to convert the given equation into the form with which they feel most comfortable, or they may not realize that the equation can be written in alternate forms.

Students difficulty matching functions and graphs may be, in part, due to their disconnected notions of functions, graphs, and real-world phenomena. In a study of graphing errors, McDermott, Rosenquist, and van Zee (1987) found "that many are a direct consequence of an inability to make connections between a graphical representation and the subject matter it represents." (p.503) As previously discussed students who are unable to move across various representations exhibit a lack of conceptual understanding.

Students attempting to make a connection between a phenomena and a graph sometimes assume a more literal connection than exists. "There does seem to be an impulse on the student's part to act as if the graph were much more literally a picture than it is. This has come to be called Iconic Translation" (Monk, 1992, p.176). A student exhibiting Iconic Translation may consider a graph representing speed and distance of a car as being a picture of the road on which the car is driven.

Some students exhibit a "Pointwise" view of functions. They "seem to conceive of the information in a function as made up of more or less isolated values, or of input-output pairs" (Monk, 1992, p.183). Students who are looking at a function as a collection of points may have difficulty conceiving of the function's graph as other than a straight line. Dreyfus & Eisenberg (1983) found that "the idea that only a linear function can contain two points was very strong in the students. Yet, these same students, when given three collinear points in the plane, stated that an infinite number of graphs of functions could contain them, and they easily provided examples." (p.130) These students developed this misconception due to an emphasis on "the statement that through two points in the plane there exists one and only one straight line." (p.130)

These students have difficulty interpreting aspects of the graphs which require whole-graph conceptions. "In general, students seem to have a difficult time grasping concepts that arise from variables not actually shown on the graph" (Leinhardt, Zaslavsky & Stein, 1990, p.42). McDermott, Rosenquist, and van Zee (1987) "found that students frequently do not know whether to extract the desired information from the slope or the height of a graph." (p.504) As long as students lack conceptual understanding of graphing they will continue to have difficulty ascertaining which features of the graph contain the information they require.

Students often view graphs in ways which do not take into account individual points. Philip Lewis invented the program RandomGrapher because "students can, in fact, lose track of the points when they face a continuous curve, and that the consequence includes failure at tasks that require consciousness of the points" (Goldenberg, Lewis & O'Keefe, 1992, p.238). Students need to be cognizant of points while being able to recognize global features of functions and graphs as well.

Another difficulty that students have, which is related to scaling, involves their interpretation of parabolas as having "shape". The various shapes that students see are an illusion created by viewing parabolas in windows of varying dimension. Goldenberg (1987) concurs and notes that "students typically use 'shape' of a parabola (on a constant scale and in a fixed-sized window) to determine the A coefficient. Thus, though they learn strategies for solving their problem, the strategies are based on an underlying notion - that parabolas may have different shapes - that is erroneous." (p.203)

Increased experience with graphs may aid students in understanding concepts related to scale. Kaput (1993) recognizes "the fact that the understanding of scaling is normally very limited among today's students and teachers may simply be an artifact of our very limited use of graphs." (p.294) Students who are able to work with numerous graphical examples may develop a more complete understanding of scaling. This quantity of experience could be easily obtained by students with the aid of the graphing calculator.

Students who do not yet possess conceptual understanding of functions also sometimes exhibit difficulty with the concept of translation. "Translating a function is not defined: translation applies to a graph. But the beginning student -- the one for whom function sense is an issue -- is not likely to have a clear sense of what is being translated" (Goldenberg, Lewis & O'Keefe, 1992, p.242). The experience students gain with functions and graphs, perhaps using the graphing calculator, may ease the effects of this potential misinterpretation.

A common functional misinterpretation, made by students, involving graphs, is "the common conclusion that the graph of a linear function moves to the left as the constant term increases" (Goldenberg, 1988, p.162). Students' use of the graphing calculator does not appear to alleviate this particular misinterpretation.

Students who use graphing calculators have the opportunity to modify the constants in an equation as well as the variables. By doing so students can gain insight into the relationship between changes in constants and the resultant graphs. Yet, this feature can lead to misinterpretation of the role of variables in functions. Goldenberg (1988) maintains that "because the variable and the constants switch roles in many graphing packages, the unthinking use of such software may further obscure rather than clarify this difficult concept [of variable]." (p.152) Instruction must take care to establish the differences between these concepts in students' minds.

Students approach learning about functions and graphs with misconceptions and construct misconceptions while constructing understanding. "Educators should be aware of, and attempt to minimize, common misconceptions among students using function-plotting tools" (Dugdale, 1993, p.101). When students avoid common misconceptions they have more time to focus on and gain understanding of interesting mathematics.

What Students Need to Know About Functions and Graphs

Students' understanding of functions and graphs is obtained through experience and through avoiding or reconciling common errors and misconceptions. Typically, instruction in graphing has included drawing simple graphs by hand, based on data in tables, and determining information about particular points. "Various recent efforts to improve students' understanding of graphs have emphasized the need to move beyond plotting and reading points to interpreting the global meaning of a graph and the functional relationship that it describes" (Dugdale, 1993, p.104). Students need increased exposure to graphs with an emphasis on the functional relationships that they represent.

Textbooks have typically included very little graphing content. Demana, Schoen, and Waits (1993) surveyed mathematics textbooks and found that for grades 1-6 only 1-2% of textbook pages included graphing context. For grades 7 and 8 graphing content was found on only 3% of textbook pages.

In grades 1-6 "emphasis is on plotting given pairs on given coordinate systems or naming the coordinates of given points. Students are not expected to generate pairs from given information or to construct and scale their own coordinate system in order to graph certain data....no mention is made of a connection between a numerical relationship and a graph, and no situations involving continuous curves are encountered" (Demana, Schoen & Waits, 1993, p.15).

Demana, Schoen, and Waits (1993) found that even in grades 7 and 8 students are rarely expected to construct graphs. Most of the content in these grades (68.5%) is concentrated on the rectangular coordinate system, and students are exposed to the notion that a line is continuous and infinite. In graphing exercises variables are continuous, but inequalities and nonlinear equations are not discussed.

A study by Bright (1980) on the learning of function concept in college algebra resulted in the following recommendations. "Do not use just algebraic formulas and rectangular graphs as representations of functions but include line-to-line graphs, mapping diagrams, and sets of ordered pairs. Do not graph just algebraic formulas, but also include practice with graphing of a situation for which there is no formula. Do not emphasize just the process of drawing a graph but also the process of analyzing a graph that is drawn." (p.83) This is important because students can draw graphs with a little procedural knowledge of the task, yet by analyzing graphs they have access to the global aspects represented therein.

The instructional requirements associated specifically with functions and graphs reflect the underlying concepts which instructors expect students to understand. Similarly, specific elements of understanding are required for interpretation of particular examples. Clement (1989) identifies what a student must be able to do in order to understand the concept of a bicycle moving at a particular speed at a particular time. "Here, the subject must: (a) have adequately developed concepts for speed and time; (b) be able to isolate these variables in the problem situation; and (c) understand that the specified values occur together." (p.78) The process of learning requires students to construct understanding of concepts based on the understanding they already possess.

Students' prior knowledge is an important aspect of their construction of understanding. Burrill (1992) recognizes the inherent need to possess a certain variety of conceptions in order to comprehend the graph of a polynomial function. "Students must have developed some intuition about the degree of a polynomial, its factors, and their relation to the graph." (p.17) This intuition will help students to recognize the relationship between a polynomials zeros, an equations roots, and a graphs intercepts.

Dugdale (1993) advocates graphing experiences for students developing function concept. "Experiences with graphs are considered important because of their potential for providing a qualitative basis for students' conceptualizations of graphs that describe functional relationships between variables." (p.102) Students with a qualitative view of a concept are demonstrating conceptual understanding.

The graphing calculator allows comparison between functions and answers to questions about graphs. "Because graphs are easily obtained, it is reasonable to emphasize that finding all the real solutions of f(x) = 0 is the same as finding all the x intercepts of the graph of f" (Demana, Schoen & Waits, 1993, p.32). With the aid of the graphing calculator students can make conjectures about the x intercepts of f and compare them with the solutions of f(x) = 0.

Students need to have the this opportunity to experience multiple representations of the concept of function, and multiple graphical examples of functions. Instruction may need to focus on providing this opportunity. In order to promote conceptual understanding "a strong sense of the graphical and algebraic landscape needs to be developed in the student, including a sense of where to look for critical information" (Leinhardt, Zaslavsky & Stein, 1990, p.54).

Computer-Aided Instruction, Computer-Based Education, and Microcomputer-Based Labs

Computer-Aided Instruction (CAI), Computer-Based Education (CBE), and Microcomputer-Based Labs (MBL) all have their advocates in the technological industries that create them and the instructional industries that use them. What are the justifications for this advocacy? How are these media being used to aid in students' mathematical understanding? Are students benefiting from these educational tools and techniques?

The position that computers are able to stimulate learning was put forth by Glass in 1984. Computer technology was gaining popularity with the general public, as demonstrated by the growth in sales of electronic hardware in general. Widespread use of computers had moved into the home and into the schools. According to Glass the active participation of the learner and the man against machine challenge were strong motivational factors supported by computers. Further, when using the computer "a failure is not repeated, does not go unheeded, and is not displayed for all to see. Moreover, the computer opens doors and motivates all students, particularly the gifted, to achieve greater learning independence and greater creativity." (p.13) Thus, the computer was viewed as a stimulus to student motivation and creativity.

The ability of the computer to provide instant feedback on a continuous basis instigated Noss (1988) to consider it as an aid to mathematical education. The "particular facility of the computer to focus the learner's attention and simultaneously to provide feedback seems to provide a promising framework for thinking about teaching mathematical ideas in a computer-based context." (p.263)

The idea of using computers to teach mathematics in the realm of graphing was not new. Mokros and Tinker (1987) found that MBL was a powerful aid in teaching about graphing. They suggested four reasons for the success of MBL in the graphical setting. "MBL uses multiple modalities; it pairs, in real time, events with their symbolic graphical representations; it provides genuine scientific experiences; and it eliminates the drudgery of graph production." (p.369) The interest of a student using MBL is thus stimulated in an environment in which the student has control over, and experiences the mathematics being represented. "This ability of the computer to allow users to interact in a personally powerful way is the common thread that runs through the various cultural manifestations of the computer in society" (Noss, 1988, p.257).

Some of the appeal of MBL resides in the fact that students in the MBL setting are given an opportunity to experience intrinsic feedback. Feedback, according to Thompson (1985), "becomes intrinsic to mental actions only when the outcome of the actions is compared with an expectation." (p.200) In the MBL setting, students can initiate feedback about their way of thinking about a problem or concept by comparing the results of their actions on the computer with their expectations of those results. "Intrinsic feedback is characteristically direct, relevant and diagnostic. It can provide maximum visual feedback to students about their responses to problem solving tasks" (Sfondilias & Siegel, 1990, p.131).

Intrinsic feedback is the guiding factor in an environment, which advocates exploration and problem solving, know as an intrinsic model. An important aspect of this model, intrinsic feedback guides students as they investigate and manipulate the environment. Within this framework "the computer medium can change the character of traditional representations from display representations to action representations" (Kaput, 1993, p.295). The computer offers students the opportunity to quickly modify graphic representations. They can then focus on the results of specific changes to the functions.

In a study by Heid (1988) students believed that using the computer helped them to develop conceptual understanding. They felt that their attention was refocused because the computer alleviated the need to concentrate on manipulation, they gained confidence in the results of their reasoning, and the computer helped direct their focus to global features of problem solving. These experiences aided the students in detaching themselves from a procedural approach, and moving toward a conceptual approach to mathematical problem solving.

Students involved in a study, conducted by Tall and Thomas (1991), using the Dynamic Algebra module exhibited conceptual understanding which exceeded that of a control group. Evidence of differences in understanding of the module users and control students was based, partially, on the following observations. Module users attempted to explain and justify their thinking, while control students exhibited more concern with operations. Module users were able to take a global view of the problem and demonstrated an implicit understanding of processes, while control students often allowed operations in the notation to influence their choice of processes. "The experiments show that the students using the Dynamic Algebra Module are more versatile in their thinking than the students following a traditional course" (Tall & Thomas, 1991, p.144).

This study is particularly significant for two reasons. First, the only difference between the instruction given to the students was a single three week period in which the module users received instruction in algebra using the Dynamic Algebra module and the control students received no algebra instruction at all. Second, after more than a year the module users "were still performing significantly better than those who had not experienced such work" (Tall & Thomas, 1991, p.136). Therefore, the benefit gained by the module users can be directly attributed to exposure to the module and not to a difference in teaching approaches used in separate classrooms.

Not all studies have determined that students using some kind of computer instruction are getting benefits beyond those of a traditional curriculum. Diem (1982) conducted a study which substituted microcomputer instruction for traditional methods in a college algebra course. The computer-aided instruction was "designed to teach the student how to find and graph the solution set of linear inequalities with two variables." (p.iv) Computer programs were created with this goal in mind.

The conclusions of the Diem study did not support the notion that CAI, or more specifically this collection of programs, was of more or less benefit to students than traditional instruction. "The conclusion of greatest importance in this study is that the students using CAI versus those who use traditional methods of study, showed no significant differences in achievement in College Algebra at the .05 level." (p.38) The possibility exists that both forms of instruction were inherently the same and that this caused the close similarity in levels of achievement. This can occur if computer programs are written which effectively mimic the traditional textbook without using the inherent features of the computer which can help instruction transcend the limitations of the traditional methods.

In 1985, Bangert-Drowns, Kulik, and Kulik conducted a meta-analysis of 42 studies in order to determine the effects of CBE on student achievement. The studies involved were chosen based on certain unifying characteristics. All of the studies were conducted within junior and senior high classrooms. Each study examined quantitative data obtained by comparing evaluations of students instructed using the computer with evaluations of students taught by traditional methods. The aptitudes of students being compared were similar. There were no instances of one group being taught specifically to the test which might skew the comparison results. Finally, each of the studies could be easily obtained.

The studies analyzed were conducted in classes on a variety of subjects, and included several different types of computer instruction. More than half (22 studies) were conducted on mathematics classes. "Seventeen studies (or 40%) investigated computer-assisted instruction (CAI). Sixteen studies (38.1%) provided an evaluation of computer-enriched instruction (CEI). Finally, nine studies (21%) examined the effectiveness of computer-managed instruction (CMI)" (Bangert-Drowns, Kulik & Kulik, 1985, p.63).

The results of the meta-analysis showed that, "computer-based teaching raised final examination scores in the typical study by 0.26 standard deviations" (Bangert-Drowns, Kulik & Kulik, 1985, p.65). Results based on the type of computer instruction used reveal some vast differences. Classroom instruction using CAI and CMI resulted in increases in scores of about 0.4 standard deviations. Instruction involving CEI also resulted in increased scores, but this increase (0.07 standard deviations) was not as substantial. Thus, instruction that focuses on, rather than sporadically uses the computer seems to be more effective in teaching.

In a study involving second to sixth grades students, Mehan (1989) found that students' successfully used the microcomputer when it was a "functioning part of the classroom environment." (p.13) Rather than being a separate piece of equipment that students might sporadically learn about, it was integrated not only into the classroom but into the curriculum. This integration effected the organizational nature of classroom instruction and of student-teacher interaction.

It is increasingly apparent that the creators of CAI programs need to be sensitive to the particular needs of the students for which the instruction is created. Goldenberg (1988) came to similar conclusions after conducting clinical studies of perceptual differences between students and mathematically literate adults. These studies revealed that CAI needs to take care not to make difficult topics seem more obscure, while taking into consideration the fact that graphing can increase students' access to significant and challenging mathematics.

Graphing Calculators

How Graphing Calculators May Aid Students Studying Functions and Graphs

The graphing calculator is a specific instructional aid available to students studying functions and graphs. "One of the advantages of graphing calculators over computer-based function graphing software is that every student has access to the tool, both at school and at home" (Kieran, 1993, p.225). This continuous access to the graphing calculator allows students to explore relationships at their convenience and potentially without interruption.

Students exploring graphs are presented with opportunities to make unexpected discoveries. "The unrelenting forcefulness inherent in the character of a good graphic presentation is its greatest virtue. We can be forced to discover things from a graph without knowing in advance what we were looking for" (Wainer, 1992, p.14). Instructors can facilitate constructive discovery by directing students to experiment with specific types of graphing activities.

Kieran (1993) recognizes that newer approaches to graphing basically concentrate on three types of activities. Students may be presented with unscaled graphs and expected to focus on interpretation of global features. Students may investigate the effects of changes in function parameters by studying families of graphs. Graphs may be used to examine applications while problem solving. The graphing calculator offers an environment in which students can easily use each of these approaches. Instruction which directs students to use the graphing calculator in these ways may facilitate students' understanding about functions.

In order to use graphs effectively students must be able to discern the valuable global aspects of the representation. "When automated graphing makes it possible to ask students to induce the effect of the constant by performing many graphing experiments, attention is drawn to the graph as a whole" (Goldenberg, 1988, p.157). Students can use the graphing calculator to quickly create a variety of related graphs that help to specifically bring global aspects into focus.

Students may investigate the effects of changes in function parameters by studying families of graphs. Kieran relates "that one of the greatest benefits of graphing calculators is the feasibility of discovery lessons based on finding patterns in the student-generated graphs of related functions." (p.226) In particular, the instructor can encourage students to consider and discuss how a function's graph changes when a constant is added or subtracted. The instructor then guides the discussion so that students avoid misconceptions and discover the correlation between changes in the function and changes in the graph. The graphing calculator facilitates this kind of discovery by replacing students' need to calculate points with the opportunity to concentrate on relationships between functions.

By using the graphing calculator (or by graphing on the computer) students are easily able to analyze aspects of functions, and to develop intuitions that are not otherwise readily accessible. "Specifically, the [graphing] software allows students to operate on equations and graphs as objects, and that may facilitate the development of the object perspective in ways not possible before the existence of such technologies"

(Moschkovich, Schoenfeld & Arcavi, 1993, p.98). Students can consider a function, then create a second function by adding a constant to the first function. The functions are then treated as objects with the addition of the constant as a transforming operator that converts the first object into the second object.

By viewing the function as a whole students are able to discover meaning in functional situations that are difficult to readily understand. One of these situations is the presence of discontinuities. Dugdale (1992) found that graphic display can aid students in finding meaning in discontinuities. "Instead of being concerned only with one particular x value (where the function is undefined), the student...[uses] the behavior of the entire function, with particular attention to the function near the undefined value." (p.115) By drawing the students attention to global aspects of the function, graphs can bring local aspects effectively into focus.

What Students Need to Know to Effectively Use Graphing Calculators

For students to effectively use the graphing calculator they must have some particular skills. "To use the power of graphing calculators to produce informative graphs of functions and relations, students need skills in algebraic estimation" (Burrill, 1992, p.16). Students need to be able to discern reasonable domain and range values for functions they intend to graph. They also need to be able to estimate plausible scales for the axes.

Without a good estimation of scale students may find themselves looking at the display of the graphing calculator and wondering why the display is empty. They also may be presented with correct graphs that do not appear as expected. Students need to understand possible reasons for these situations.

Graphs which appear other than expected result when the specific requirements of the graphing calculator are not carefully considered. These graphs often display what seem to be visual illusions. "Included among the causes of these illusions are the interaction between the position and orientation of the graph and the shape of its window, and the interaction between the scale of the graph and the scale of the window" (Goldenberg, 1988, p.142).

Scaling factors can cause distinctive changes in the display of a particular graph. Moskowitz (1994) notes that "graphs can be manipulated to be on the verge of deception; the circle equation yields graphs that look nothing like a circle." (p.242) Students need to be aware of the inconsistency of entering the equation for a circle into their calculator and getting the visual image of an ellipse. When faced with this result students have the opportunity to analyze and investigate the source of the discrepancy, or to work toward the realization that what they thought was discrepant in fact reflected an inappropriate interpretation or preunderstanding.

Investigation can lead to the realization that the cause of an unexpected graph often involves scale. This is an especially important consideration for students to acknowledge in that "students seem to prefer symmetrical scales on their axes even when these obscure important features of the function they are viewing" (Goldenberg, 1988, p.168). Perhaps this preference can be overcome if students analyze the consequences. The graphing calculator allows students to investigate the effects of numerous changes in scale and thus in the graphing window more quickly than they could if they were graphing by hand.

The experience of observing the scale dependent change in shape of the graph "creates a 'conceptual demand' that may affect the kind of mental images a student is able to construct" (Leinhardt, Zaslavsky & Stein, 1990, p.17). Students' understanding is affected by this construction. The student should eventually be able to discern which features of the graphical representation are and are not affected by changes in scale.

Another consideration related to scale is that "scale is the only attribute of a graph that raises or lowers the significance of the distinction between a point and a dot" (Goldenberg, 1988, p.165). Some students carefully scrutinize the square pixels on the screen and misinterpret them as representing points of a function. This can lead to the notion that a continuous graph is actually a set of squares or rectangles linked together, or induce students to believe that there are "holes" in a particular graph. Students can "verify" that these holes exist merely by magnifying a particular region of the graph. This kind of misinterpretation needs to be confronted early in instruction in order to equip students with skills at estimation of appropriate scale.

Another skill that students need to acquire early when using the graphing calculator is the ability to rewrite equations in terms of one variable. "With most current graphing calculators, to graph functions students must enter a rule after the 'y =' prompt" (Burrill, 1992, p.17). Instruction which uses the graphing calculator will necessarily focus on this skill early in the curriculum.

Possible Benefits for Students Using Graphing Calculators

When students effectively use the graphing calculator, it aids them in a variety of ways. Research has revealed and verified some of the benefits to students of using the graphing calculator. As has already been discussed the graphing calculator reduces the amount of calculation required by students and enables students to focus their attention on global issues related to functions.

Dugdale (1993) found similar results after performing a study in which high-school students were given the opportunity to explore the "relationships among three fundamentally different graphs and the situation that they collectively describe." (p.111) Students were given direction, size, and speed graphs and created a video game using all three. Students' initial methods basically involved creating the game situation, then debugging. Experienced students were observed working on a small section of the problem, comparing the results with the graphs, debugging, adding another section, checking again, and debugging. "They found building and debugging a series of small sections more rewarding because it provided early and frequent feedback and transformed the larger problem into a series of more easily addressed smaller problems" (Dugdale, 1993, pp.111-112).

It is not surprising that students prefer methods that supply frequent feedback, since "use of feedback from a graphic calculator can reduce uncertainty and thus diminish anxiety" (Ruthven, 1990, p.448). Students who are confident about the results of their investigations are much more likely to engage in mathematical exploration.

Student confidence in the ability of the graphing calculator to aid them in understanding mathematics was seen to be directly related to teacher confidence in the Graphic Calculators in Mathematics development project in Britain. This project supplied students in a two year upper secondary mathematics course with continuous access to graphing calculators. Classes in which the teacher had strong reservations about students using the graphing calculator contained more students who chose previous calculating tools than classes wherein the teacher was supportive of the graphing calculator.

Preference did not have an effect on students' abilities to competently use the graphing calculator. "After one school term, nearly all the project students were making confident and spontaneous use of the calculating and graphing facilities of the advanced [graphing] calculator" (Ruthven, 1992, p.92).

What Instructors Need to Know to Effectively Teach with Graphing Calculators

In order for the graphing calculator to be effectively used by students, educators must be aware of certain considerations and of some of the possible visual illusions that the graphing calculator can produce.

Some computer systems incorrectly create misleading graphs. Demana and Waits (1988) discovered a graphing program that "connects the last point plotted to the left of an asymptote with the first point plotted to the right of the asymptote. This is the classic mistake many students make when graphing rational functions." (p.178) Students and instructors must be aware of the possibility of errors like this occurring when they use graphing calculators. A great variety of obvious or subtle errors can also occur if the graphing calculators batteries are failing.

Instructors must also be aware that students may misinterpret graphs based on, what Goldenberg (1988), describes as "such irrelevant perceptual features as the angle at which a linear function intersects the frame of the graph." (p.152) Similarly, students tend to misinterpret the distance between two curves by considering it to be "in a direction roughly normal to the bisector of the perceived angle between the curves." (p.152)

Students must also consider the constraints put on their graphs by certain elements of real world problems. Difficulty can arise if a student does not take into consideration that the graph of a problem involving growth over time is only relevant for positive values of time. "A complete graph of a problem situation is a graph that indicates all of the points and only the points corresponding to the problem situation" (Demana, Schoen & Waits, 1993, p.27).

Functions, even when properly displayed can lend themselves to graphic illusions. An illusion that can appear in graphs of linear functions involves varying the constant in the function equation. Depending on the angle at which the graphs intersect the window edge and the shape of the window the original graph may appear to move either horizontally or vertically.

Parabolas lend themselves easily to several graphic illusions. Two parabolas placed at different heights within the window appear to have different shapes. As with many graphs, the shape of a parabola also seems to change if the scale in the window is modified. Students viewing the graph of a parabola also may get the impression that the function represented by the graph is bounded, even though it is obvious from the algebraic representation that it is not.

Another visual illusion which may contribute to student misinterpretations involves the pixels on the screen. Students may note that different slopes produce varying amounts of jaggedness to the graph. Students who interpret this jaggedness as an accurate depiction of the points of the function may make incorrect inferences about aspects of the function.

CHAPTER III

METHODOLOGY

Methods of Determining What Students Know

In order to decide what instruction is successful and which students are successful in any given program requires ways to determine, at given points in time, what it is that students know.

Processes for determining what students know are created and executed at varying levels within instructional programs and within research programs. Within the context of the classroom, students are often required to demonstrate proficiency in the subject matter by successfully completing homework assignments and exams. The examination process is also a part of students' experiences when they take standardized tests. Additionally, students are sometimes asked to participate in the examination process, outside of the personal arena, in support of research programs.

Written, standardized tests are not the only source of information on students abilities and progress available to researchers. While these tests are prevalent, they often supply only quantitative data on student abilities. In this situation students who get high percentages of problems correct are considered proficient or as "knowing" the material. These tests are unable to offer succinct, qualitative information about the thought processes students use when problem solving mathematically. Often, researchers gain useful information about the thought processes of students through interview and observation.

In an interview the researcher is given the opportunity to question a particular student, at a particular time, about their reasoning while solving a specific problem. In this way, the researcher can gain insight into the student's thought processes during a problem solving situation, regardless of whether or not the student reasons the problem to a correct solution.

Although student interviews can be very successful in providing researchers with insight into students' thought processes, care must be taken not to unduly influence student responses. This can occur if the wording of the researcher's questions directs the student's thinking toward a particular solution. Students who are accustomed to being in situations where they feel threatened by authority figures may respond with what they believe the interviewer wants.

Another type of study is labeled an "educational ethnography, participant observation, qualitative observation, case study, or field study" (Smith, 1978, p.316). This observation of students, without interaction and questioning, can reduce or eliminate the researchers influence on student responses to problem solving. In an observational setting the researcher attempts to make determinations about a student's thinking based, in part, on information the student volunteers, the student's interaction with peers, the student's interaction with the instructor, and the student's posturing within the classroom setting.

Regardless of which setting the researcher uses to gain information about what students know, "it is absolutely essential that the researcher keep in mind that what he sees as 'the' problem imposes nothing of necessity upon the problem solver" (Thompson, 1982, p.154). When constructing their own knowledge students make determinations about how to interpret the problems presented to them based on their experience. Since researchers experiences have been different than those of the student being observed, their interpretations of the problem may be vastly different. The researcher's task is to continuously examine the student's behavior based on what is known about the student, rather than what is known about the problem.

In order for researchers to effectively determine what effect using the graphing calculator has on students' understanding of functions and graphs it is important for them to keep in mind what is already known about students' understanding in this area. This helps to create a context for observation in general. In particular, it remains important for researchers to remember that each student, even in the same classroom setting, is approaching the subject from a unique, personal perspective.

Relevance of Individual Students' Perceptions

When observing a student in the classroom the researcher must be aware of the differences between the student's and the researcher's perceptions of the problem being solved. It is evident that students, who construct their understanding from their personal experiences, are in no way affected by the observer's interpretation of the problem.

A similar analysis can be applied to the effect of the environment on students' experiences. Students' experiences are affected by their environment, and it follows that their constructions are influenced as well. It is the student's personal perceptions of the environment, as opposed to the researcher's perceptions, that are relevant. The researcher needs to keep in mind that "a student's experience...is wholly inaccessible to an observer and hence that there need be no correspondence between what the researcher and the student see as the student's environment" (Thompson, 1982, p.152).

The researcher's attention must thus continuously return to questions of students' perceptions of problems and of their environments. Analysis of student behavior in problem solving relies on the answers to questions such as: "What was the instructor's intent when presenting this problem?" and, "How did the student interpret this problem?" The researcher is thus required to determine both the problem the instructor intended and the problem the student solved.

In attempting to interpret a student's perception of a problem the researcher must try to understand the conceptual environment from the student's perspective. It is from within this personal environment that the student draws information in order to interpret and solve the given problem. The researcher naturally assumes that the student's "activity is rational given his or her current understanding and purposes at hand. The trick is to imagine a world in which the child's activity does make sense" (Cobb, 1989, p.32).

In order to imagine the conceptual environment in which the student's actions make sense the researcher must attempt to examine the problem from the student's point of view. Barnes (1992) recommends questions which may aid the researcher in determining who the students under observation are: "Who are these individuals, and what are their everyday lives like? How do they understand the world? What matters to them? What kinds of change might they wish for, and what do they need to know?" (p.150) Answers to these questions supply a starting point for researcher interpretation of a student's perceptual environment.

"The process of accounting for students' mathematical activity therefore involves coordinating analyses of their mathematical and social cognitions" (Cobb, 1990, p.205). By gaining insight into a student's understanding and perceptions the researcher prepares to build models of possible student conceptual environments. In this sense, a model "refers to a conceptual system held by a particular knower at a particular time" (Thompson, 1982, p.153).

Beginning with the information known about who a particular student is the researcher attempts to analyze the situation from the student's perspective. By examining the requirements and restrictions of the specific situation, the researcher tries to determine what logical action to take in order to act like the student under observation. By reflecting on the reasoning behind a student's behavior the researcher can build a model of the student's conceptual system. The researcher can then use the model to view the mathematical situation from the student's perspective.

Relevance of the Classroom Environment

Classroom Culture

In an educational setting students' conceptions are formed in the context of the classroom. "Teachers have

to teach, pupils have to learn and the didactic contract determines, mostly implicitly, where in this teaching/learning social relation lies the exact responsibility of each partner as far as the mathematical content is concerned" (Artigue, 1992, p.111).
The cultural environment that develops and is established within the classroom, as a result of this contract, creates a context for classroom communication between instructor and students, and between students.

It is the purpose of the researcher, as observer, to "identify and account for aspects of a culture by analyzing regularities and patterns that arise as, say, a teacher and students interact during mathematics instruction" (Cobb, 1989, p.33). These patterns result from the classroom culture and affect the type and quality of instruction that can occur within the classroom environment.

In order to best describe the effects of a classroom culture on the instructional environment, the researcher must find the foundation of the patterns that arise during classroom interaction. "The implicit rules or social norms that the participants appear to be following can be formulated as a first step in explaining their mutual construction of the observed patterns" (Cobb, 1990, p.207). Some social norms that may direct class discussions include:

> listening and trying to make sense of explanations given by others; indicating agreement, disagreement,

> or failure to understand the interpretations and solutions of others; attempting to justify a solution and questioning alternatives in situations where a conflict between interpretations or solutions has become apparent (Cobb, 1990, p.208).

Communication in the classroom is characterized by the types of interactions which are deemed culturally acceptable by both the students and the instructor. McDermott, Gospodinoff, and Aron (1978) "suggested that it is necessary to determine the adequacy of any description of the form and content of concerted behavior in terms of whether it is (1) formulated, (2) posturally positioned, (3) oriented to, and (4) used to hold members accountable for certain ways of proceeding." (p.267) In some settings a student spontaneously asking a question about the reasoning behind using a particular formula in solving a problem might be considered appropriate. In another culturally oriented setting this same behavior might be considered as an inappropriate disruption, or an inappropriate line of questioning, or both.

Graphing Calculator Effect on Classroom Culture

In examining the effects of the graphing calculator on instruction and students' understanding it is important for the researcher to keep in mind that a graphing calculator "in a classroom is a social practice and not a technology....It is what people do with the machine, not the machine itself, that makes a difference" (Mehan, 1989, p.19). The researcher must thus examine the effect the graphing calculator has on the formation of classroom culture and practices.

Mehan (1989) classifies the "relationship between microcomputer [or graphing calculator] use and classroom organization under two headings: (1) the impact on temporal and spatial arrangements and (2) curriculum - what teachers teach and how they teach it." (p.6) These relationships may potentially bring about significant changes in the mathematics offered to students.

The number of students who have access to the graphing calculator at any given time will greatly effect the way it can be used. The opportunity exists for all students to interact simultaneously with the graphing calculator, making it a tool that students might be able to use in class with minimal disruption.

The amount of class time that the students use or save by using the graphing calculator may also be relevant. Students may spend excessive amounts of time entering data, or may be relieved of trivial calculations by using the graphing calculator.

The change in curricular emphasis which is possible with student access to the graphing calculator could be significant. By devoting more time to graphing instructors might find it necessary to reduce emphasis on or remove other topics from the curriculum. Alternately, instructors might find that emphasis on graphs helps students to assimilate other topics more rapidly.

Relevance of a Calculational or Conceptual Orientation in Instruction

The way in which an instructor teaches has a large effect on the type of understanding that students are likely to strive for and obtain. Thompson, Phillip, Thompson, and Boyd (in press) have recognized and elaborated on two types of orientations in mathematics teaching, "calculational" and "conceptual." An instructor teaching from either one of these orientations is demonstrating that instructor's own conceptions of mathematics.

An instructor teaching from a calculational orientation views mathematics as consisting of calculations and procedures designed specifically to obtain numerical answers. This instructor's goal is to help students gain a procedural understanding of the mathematics being taught. Instruction in this classroom is likely to focus on the processes of computation and algorithmic thinking.

Discussions in a classroom which is calculationally oriented will focus mainly on sequences of calculation. The instructor may ask students to explain the reasoning behind their problem solutions but will accept descriptions of sequences of calculations which do not, in fact, convey reasons. Students may be completely unaware of the instructor's reasons for deciding that a particular solution is correct.

A particularly influential difficulty with a calculational orientation to mathematics instruction stems from the inability of students who lack understanding to gain from explanations of particular problems. Thompson, et. al. (in press) observed "that the only students able to follow a calculational explanation are those who understood the problem in the first place, and understood it in such a way that the proposed sequence of operations fits their conceptualization of the problem." (p.10) Since the calculational explanation describes what was done to solve the problem and not why, students are less likely to gain understanding that may be useful in other problem situations.

An instructor teaching from a conceptual orientation views the problem situation as an opportunity for students to reason and to discuss and analyze their understanding. This instructor expects student explanations to reflect their conceptions of the situation and to be supported by reasons. Although students may describe the calculations involved in their problem solutions they are also expected to explain the reasoning they use in choosing their calculations and procedures. This instructor helps students to focus on the meaning behind the numbers and the quantitative relationships drawn from the problem situation.

Students in a conceptually oriented classroom are guided to reflect on their thinking in a way that promotes conceptual understanding of mathematics. Since classroom discussion focuses on reasoning, students are consistently given the opportunity to gain knowledge of and reflect on their mathematical conceptions. Students with these experiences are likely to gain a conceptual understanding of the mathematics being taught.

Analysis of Qualitative Data

Analysis of qualitative data requires examination of information at a variety of levels. This examination can take the form of notes which reflect on different aspects of the data's content by characterizing the data from specific and limited perspectives.

By examining the data from several different perspectives the researcher can gain insight into the different levels of information contained therein. Smith (1978) refers to a possible first-level interpretation of the data as a "descriptive narrative." At this stage in data analysis the researcher might create highly descriptive, observational notes of events, statements and activities that took place during the study. This level of data interpretation is characterized by factual, non-interpretive description of what transpired during the observation.

Smith (1978) describes the next level of analysis as the "theoretical-analytical-interpretive" level. At this stage in data analysis the researcher might assemble "interpretive" or "theoretical" notes associated with each of the existing observational notes. These notes describe and elaborate on the significance of the observational notes. They also specify what particular interpretations of the data support the theories the researcher has formed. This level of data interpretation is characterized by the interpretation of and significance it lends to the data, which leads to and supports the theories the researcher is creating.

Meta-theoretical issues, which form a basis for the researcher's positions in the data interpretation, comprise the resources for the third-level of data analysis advocated by Smith (1978). At this level the researcher might assemble more general notes which illuminate the rationale for the way the results will be expressed. Some of the metatheoretical issues which Smith considers particularly relevant include:

1. The root metaphor within which one works - mechanical, organic, formal, or contextual.
2. The inner or outer perspective one chooses, that is, a stance from the subject's point of view or the outside observer's point of view.
3. A theory which is more limited in scope and time to a local context versus one that is more general.
4. A level of abstraction that is more descriptive and concrete or more abstract and interpretive.
5. A model of explanation that is more covering law versus one that is configurational or contextual.

6. A theory that is more action oriented and more ethical versus one that is more descriptive and analytical. (p.365)

By examining these issues and their relevance to the data being analyzed the researcher can verify the internal consistency of the viewpoint expressed by the theories which result from the study.

CHAPTER IV

STUDENTS' USES OF GRAPHING CALCULATORS

This study was designed to determine how students' understanding of functions and graphs is affected by their use of graphing calculators.

This chapter will show that students' use of graphing calculators essentially had no positive effect, and some negative effects, on what students learned and understood about mathematics. In regard to negative effects, students were often distracted from class discussions because they fiddled with their calculators, or focused on procedures for operating the calculator when they might have spent their time more productively thinking about a situation or an idea.

This chapter will also suggest that potential benefits to students of using graphing calculators were overwhelmed by the momentum of the instructor's and students' existing conceptions of and orientations to mathematics. Also, though the original intent of this study was to determine the effects of graphing calculators on students' understandings, it quickly became evident that what they thought about in this class was influenced greatly by the instructor's knowledge of and orientations to mathematics and it's teaching.

Students' distractions were not due to their use of calculators per se. Rather, they were supported by the instructor's orientation to rules and procedures, and his general lack of focus on forging conceptual connections among the various activities in which they engaged. Thus, an opening section which illustrates the instructor's stultifying effect on classroom discourse (and hence students' thinking) is included to inform the reader of the general atmosphere in which students' calculator usage occurred.

Students' focus of attention may have been to produce answers, as exhibited by their diligently applying procedures presented in the course toward the goal of finding answers. Even with this focus, they often did not recognize the point at which their manipulations produced answers.

Context of the Observation

Instruction in the class which participated in this study involved the teaching of algebra from two perspectives. Students were taught to manipulate functions algebraically and to examine the intersections of function's graphs. The course maintained the traditional focus on algebraic manipulation, while adding the requirement that students use graphing calculators to find solutions to the same problems by analyzing graphs.

The observation took place in a college algebra class given at a two-year community college. The course was designed for students who planned to take a three semester calculus series. The college strongly encouraged students to use graphing calculators in their mathematics classes, and supported instructors in attempting to integrate graphing calculators into their curricula by conducting seminars for instructors on how to use features of graphing calculators.

Thirty-one students were enrolled in the class which participated in the study. The class was scheduled three times a week for fifty minutes. Other sections of the same course were scheduled twice a week for seventy-five minutes.

Material covered during the observation included parts of a unit on logarithmic and exponential functions, a unit on matrices, and part of a unit covering conic sections. The required text for the course was <u>College Algebra: A graphing approach, second edition</u>, by Demana, Waits & Clemens (1992). The authors of the text assumed that students would "have regular and frequent access to a graphing utility for class activities as well as homework." (p.viii)

The Physical Setting

The classroom environment was stark. Desks were arranged in rows of five, closely packed into a long, narrow room. The linoleum floors guaranteed amplification of any noise in the room--creating an apparent chaos whenever students shuffled their chairs or bookbags. A series of chalk boards extended from the door across the front of the room and ended at a projection screen hanging in the corner. An overhead projector sat in front of the screen, and was available throughout the observation.

The room could accommodate about forty students. Even with ample room to spread out, students consistently crowded into the end of the room closest to the door. This was particularly odd, in that, on occasion, the door was left open creating a glare on the chalk board that made it difficult to see from that side of the room.

The Students

Students at this two-year college generally either acquire vocational training, or earn credits which they will later be able to transfer to the state university. In a survey given by the instructor at the start of the semester, students were asked why they were taking this class. (All references to this survey will include data for thirty of the thirty-one students, since a survey was not available for one student.) Twelve students stated that the class was required for transfer to the state university. Sixteen students made similar responses by stating that the course was the prerequisite needed for their major. Ten of these students stated specific plans to continue on to calculus. One student responded with "I like math", and one student did not respond.

It can be inferred from the students' reasons for taking the class that most of the students planned to continue on for a bachelor's degree. Fourteen students gave their major as undecided. One student indicated an educational goal which did not require a four-year degree.

Nearly all of the students had recently completed coursework in mathematics. Twenty-three students took a mathematics course the previous semester. Six students took a course two semesters before the present one, and one student had not taken a mathematics course in two years. More than half (eighteen) of the students had taken intermediate algebra as their last course, which was the prerequisite course specified by the college. Four students responded as having previously taken this same college algebra course, while only two students listed it as their last course taken.

Students enrolled in the college algebra course were allowed to choose the course by self placement. The only stipulation was that they were required to have previously taken intermediate algebra with a grade of "C" or better. This course may have been taken at the community college, the high school, or another accredited institution.

Students at the college were encouraged to purchase the TI-82 graphing calculator. The intention of the college was that students would continue to use the same graphing calculator throughout their mathematics courses at the college. The college ensured that the TI-82 was accessible to students unwilling or unable to purchase it. The TI-82 was available for students to use at the library. The college also loaned the TI-82 to students on financial aid, or students could rent the TI-82 for a minimal fee.

Student response to the need for graphing calculators was excellent. At the time of the survey (during the first week of class) nineteen students had already purchased a TI-82. Most students had some kind of calculator, and all but three had a graphing calculator. Tom noted that by the end of the first week every student had purchased a TI-82.

Most students in this class were unfamiliar with graphing calculators. In response to the survey, sixteen students said that they knew nothing, twelve students said that they knew little, and two students responded that they knew a "fair amount".

All thirty-one students attended class regularly. Regular attendance in itself is not an indication of students' interest in the course. Students were aware that attendance was required for the course and that more than three absences during the semester was considered excessive. They were also informed that the instructor could drop any student from the course due to excessive absence.

Students were generally on time. Although the class was scheduled for fifty minutes, the first and last five minutes of each class period was generally characterized by chattering and shuffling of papers and course materials, leaving forty minute class periods.

Nearly all of the students consistently had pencils, notebooks, textbooks and graphing calculators on their desks, throughout each class session. This was surprising in that the textbooks were rarely used in class. Graphing calculators were used sporadically, and several students used them throughout parts of the lecture which did not make reference to them.

Students took notes diligently any time Tom wrote something on the chalk board. They generally did not take notes while Tom discussed what he had written or gave a verbal presentation of the material.

The Instructor

Background

The course instructor, Tom, had an undergraduate degree in mathematics. He also had a master of arts degree in mathematics and a master of science degree in computers in education. Tom stated that he had completed more than ninety units beyond the BA and that these included courses in education.

Tom had taught mathematics at the community college for twenty years. He spent the last six years teaching beginning and intermediate algebra in a program involving CAI. Students in these classes were expected to use the computer for drill and practice. The instructors involved in creating this program felt that these courses tended to favor questioning by the more advanced students. The program was intended to allow these advanced students to progress at their own pace, while creating more time in-class for other students to interact one-on-one with the instructor and peer tutors. The introduction of peer tutors into the classroom was intended to encourage students, who did not learn the material on their own, to ask questions. The CAI program also included simulations of statistics applications and some graphing experiments.

After being involved in the CAI program Tom was quite enthused about teaching a class which based it's instruction on heavy student use of graphing calculators. This was his second semester teaching a college algebra course using graphing calculators.

Tom's experience with a CAI program and his coursework in computers in education suggested an interest and commitment to using new cognitive technologies in teaching. Tom's department supported and strongly encouraged the mathematics instructors at the college to develop curricula which made extensive use of graphing calculators.

Knowledge and Orientation

Tom's beliefs and attitudes about mathematics may have affected the students' views about mathematics and/or their focus in this course. The attitudes Tom expressed were, therefore, significant aspects of the students' learning environment.

Tom showed that he did not expect students to understand the meaning of the mathematics he was teaching when he responded to students' questions. It was not uncommon for Tom to respond to students' questions with "just do it for now" or "trust me."

Tom's attitude about what constitutes and is evidence of understanding in a student was illustrated by his statement that "anytime you put down the graph it shows you understand about how things intersect, about how things work." Tom appeared to be referring to a procedural understanding of how to obtain an answer, rather than an understanding of the meaning in the problem. This is in alignment with Tom's focus on rules and procedures which imply his intention to advocate procedural rather than conceptual understanding.

Tom's orientation to teaching and learning mathematics was highly calculational (Thompson, et al., in press). This means that he tended to view mathematics as consisting of calculations and procedures designed specifically to obtain numerical answers. He also tended to provide students with calculational explanations for how to solve problems and did not discuss the meanings behind either the manipulations or parts of the problem situation. Instruction in his classroom reflected a calculational view by focusing on the processes of computation and algorithmic thinking.

Tom's homework assignments were consistently calculationally oriented. They typically consisted of 15 to 25 problems, most of which were practice exercises. Problems in a given assignment often were drawn from distantly-related topics (e.g., logarithms and probability density functions) and were discussed in class only in terms of answer-getting procedures.

Tom's focus on "answer-getting" rules and procedures appears to have affected students' abilities to determine the validity of and understand the underlying meaning behind their problem solutions. Students were often unable to determine what values of a particular function solution were relevant to a problem the class was investigating.

Another prominent characteristic of Tom's teaching was his loose use of mathematical terms and incoherent English. Definitions were sometimes "properties," equations had both solutions and answers (e.g., in

3x+2=5, x is a solution, while 5 in an answer), a solution is found "somewhere" in a matrix, and sentences were often half said when he began a new thought. It was not always clear what effect Tom's language use had on students (they appeared used to it), but it was certainly not conducive to reflective, thoughtful analysis of ideas. Sometimes the relationship between Tom's imprecise use of language and students' learning was clear. When Tom asked students what they were looking for when solving the system of equations 2x+3y=6, x-y=3, a student responded, with certainty, that what they were looking for was the "answer to one of the variables."

Lastly, Tom's orientation to mathematics learning was not only calculational, it was highly prescriptive. For example, students were barred (literally) from using subtraction to eliminate variables in simultaneous equations because "this is the addition method," and to eliminate a variable in any but the second equation was illegal. When eliminating variables in a matrix representation the "x's," which were to always be in the first column, were eliminated first, then the "y's," in the second column, next.

Tom's teaching was in alignment with the textbook, which also represented mathematics as a collection of procedures and rules. Students' inability to gain insight into functions was mainly due to the lack of conceptual analysis of functions within the course.

A vignette illustrates two aspects of Tom's instruction. It illustrates his tendency to direct students to focus on "answer-getting" rules and procedures as the mode of activity he intended students to internalize. It also illustrates Tom's tendency to divert students' away from their thinking and questioning, to his prescribed way of looking at a particular problem.

Vignette - April 13, 1994 - Day two of the study.

This class day was designated, on the weekly schedule for this course, as part of a unit on logarithmic and exponential functions. This unit followed units on polynomial functions and rational functions. Approximately one and one-half weeks was allotted for this unit.

Tom asked for questions on the homework from the textbook. A student asked Tom to demonstrate a solution to a problem that was not assigned as homework. This problem was stated in the textbook with instructions to "solve each equation algebraically. Support your answer with a graphing utility." The specific equation was $\log_x(1-x)=1$.

1. Tom: The rest of them [the homework problems] are pretty easy, if you use the properties [Seven "properties of logarithms" are presented as two

theorems in the textbook.] What would be a starting point for solving this problem?

2. Sue: The rules [referring to the "properties" in the textbook].

3. Tom: Look at your rules [referring to the textbook].

4. Fred: Change the base.

5. Tom: [Writes $\log_a x=y$ on the chalk board. This is not on the list of properties.] Like that?

6. Fred: No, isn't there more than one way?

7. Tom: [Interrupting.] When you change the bases? That's not what I was thinking of. What were you thinking of?

8. Fred: X to the first.

9. Tom: That's like definition...[Writes $x=a^y$ then evidently applies "$\log_a x=y \Leftrightarrow x=a^y$" to $\log_x(1-x)=1$ and writes $1-x=x^1$, and $1-x=x$, then $1=2x$, thus $x=1/2$.] A lot of problems can be solved using definitions. You don't have to use a lot of the properties. You just have to rewrite it without logs.

Tom's focus on properties and definitions illustrates a general calculational approach to teaching. Tom focused students' activities on locating an applicable "rule" which could be used as a starting point to mechanically solving problems. Tom's orientation supported students' inclinations that mathematics is purely mechanical and that the way to get an "answer" is to search for an applicable rule or property.

The vignette also illustrates one of many instances in which students who were trying to pursue a particular line of thinking were diverted by Tom toward a prescribed procedure. Such shifts often left students wondering about Tom's reason for shifting the focus of the discussion. Students might have inferred that they were thinking incorrectly, that they were supposed to use a prescribed method even if they did not understand it's rationale, or that there is only one way to solve particular problems.

Classroom Observations

Instruction and discourse in the classroom revealed students' attitudes, approaches, and orientations to mathematics involving functions and graphs. Although specific conversations did not tend to convey strong evidence of students' understandings and perceptions, several general themes arose. This section is organized into subsections which describe and characterize students' tendencies not to make connections between multiple representations of function, their orientations to applications problems, and their incomplete understanding of the relevance of domain to interpretations of situations. Descriptive examples characterize the quality and focus of students' discourse in regard to these themes.

Multiple Representations

The variety of external representations of function a student is exposed to may aid the student in building an internal representation of function. A student who is in the process of forming an internal representation of function may make use of external representations, such as algebraic equations and graphs. Instruction which provides students with the opportunity to examine multiple external representations of functions may thus be considered desirable as a way to provide students with materials from which to build internal representations. Although t algebraic functions and their associated graphs represented the same situations.

Students generally did not demonstrate an understanding of the effects changes in elements of an equation had on the graph of the function. Even though shifts in graphs had been previously discussed in an earlier unit of the course students were uncertain about the relationship between a shift in a graph and the corresponding change in an algebraic equation. In one instance students could not determine what aspect of an algebraic equation needed to change in order to accommodate the shift of a graph of a parabola in the xy-plane. They did not readily recognize the relationship between the graphic representation of a function and the algebraic representation of the same function.

Students also did not appear to recognize that they could verify the results of their problem solving by comparing results obtained from examining the problem through different representations. Although students often solved the same problem situation by both graphic and algebraic methods they did not "check" their solutions by directly comparing these results. They checked answers only by substituting their answers back into the original equations.

Even after several months of instruction using graphing calculators students were not moving flexibly across representations of function. They were not making connections between function equations and qualities of the equation's graphs. A classroom discussion about the parabola $y=x^2/8$ illustrates one student's inability to use exposure to multiple representations of functions, and experience with functions and graphs to correctly interpret from a function equation what the corresponding graph would look like.

Tom wrote the equation $y=x^2/8$ on the chalk board and asked if the parabola it represented would "be a real thin, narrow one or kind of open out wide." The student, Lena, responded that the parabola would be "wide." She explained her reasoning to a student after class by saying that she knew from having seen a lot of parabolas that they were wide. Lena's understanding of the parabola was based on her experience that equations of parabolas always looked something like $y=x^2$ and that the graphs of parabolas always looked the same. She was confident in her misconception that all parabolas are "wide". She was so confident about her observation that she was not swayed by Tom's offering two options for the shape of parabolas -- thin or opening wide. She also didn't consider other factors that might make any parabola "appear" wide or narrow, such as changing the scale of her graph.

Students also had difficulty seeing the relationship between a system of equations and the associated matrix representation for the system. Matrices were used to represent systems of equations in the format AX=B, where A was the coefficient matrix, X was the variable matrix and B the matrix of constants. Class discussion of matrices, matrix operations, and row elimination spanned several days and included practice on paper and using the graphing calculator. (The graphing calculator was designed so that operations used for row elimination on paper could be simulated on the calculator. This process was actually more confusing to Tom and the students than paper and pencil manipulations.) Even after several days practice with matrices students were unable to describe what was being represented by each matrix in AX=B. In some cases, students believed that the matrix X represented the algebraic expression x-y.

Students often solved systems of equations in two variables by graphing the equations and using the values of points of intersection of the graphs as solutions to the system. Students' inability to see the relationship between function equations and graphs often prevented them from understanding whether all points of intersection were applicable to the problem situation. In one problem, graphs of the equations $x+y=17$ and $xy=52$ intersected at (4,13) and (13,4). Students were unable to determine which or both of these solutions constituted the answer to the problem situation. Once they had started solving the problem by "graphing" they no longer recognized a relationship between their "answer," which applied exclusively to the graph, and aspects of the original equations.

Applications

Application problems are intended to offer students opportunities to use mathematics in situations that occur in the world around them. Although students may not have had personal experiences with each application problem, they may still connect problem situations with the kinds of mathematical problems they experience in class. Applications problems lend themselves to conceptual discussions of the source of mathematical equations and the meanings behind each element of these equations. The relevance of a function's domain to interpretations of situations may also be illuminated for students by their engagement with applications.

Students typically did not recognize that functions could represent real life situations. Although they occasionally were able to create function equations from concrete situations they didn't analyze or interpret aspects of functions as they applied to the original situation. Once they had equations, they forgot about the original situation and applied answer-getting procedures to the "new" mathematics problem.

In a particular application involving tickets to a baseball game sold at differing prices to students and non-students, the class was able to set up two descriptive equations. One equation, $x+y=452$, described the student tickets sold and non-student tickets sold summing to the total number of tickets sold. The other equation, $.75x+2y=429$, described the sum of the money collected for student tickets sold at \$.75 each and non-student tickets sold at \$2 each for a total income of \$429.

Even though students could state facts like x was "the number of student tickets sold" they were unable to describe the equations in terms of the functional relationships between the number of student tickets sold and the number of non-student tickets sold. Instead of describing the dependence relationship between x and y in either equation, students repeated descriptions of the equations such as "x plus y is 452." The class was also unable to recognize that there was a relationship between the two equations which dictated that the equations would only be satisfied simultaneously for particular numbers of students and non-students.

As soon as students had the equations $x+y=452$ and $.75x+2y=429$ they forgot about the original situation and began discussing whether to solve the "new" problem by elimination or substitution.

When students graphed equations $2y+2x=100$ and $xy=301$, then used the values of points of intersection of the graphs as solutions to the system of equations they had difficulty interpreting their solutions. Students either accepted the points as answers or expressed confusion. When the equations were graphed, the intersection of the equations graphs were found at (7,43) and (43,7). Students were unable to create even a simple problem situation that might account for both possible solutions, or allow them to eliminate one or both as possible solutions.

Domain and Range

Domain, range, and scale are closely linked in the graphing environment. Students must be aware of the domain and range of the problem they are solving in order to accurately size the graphing calculator's viewing window. Students who use graphing calculators to learn about functions and graphs should be constantly aware of the importance of using an appropriate window and of the relevance of domain and range, especially in application problems.

Since attention to domain is an important aspect of the problem solving situation, students should be continuously cognizant of the domain of each problem they analyze. This was not consistently the case for students in this class. Students often displayed an inability to analyze problem situations in order to determine the domain relevant to their problem.

In some problem situations students were easily able to eliminate answers which were unreasonable given the problem situation, while in other problems they expressed uncertainty. Students examining the possible lengths for the side of a box with a particular volume were easily able to determine that the length of the side could not be negative. Alternately, students attempted in another problem to justify a negative value for time, because the event had started in the past. Since students often lost track of the original problem, once equations were created and the process of "finding the answer" began, they also tended to forget to analyze their results for meaningfulness to the original problem situation.

The graphing calculator's default window, the ten-by-ten window, was referred to as the "regular window." When students were told to change the size of the graphing calculator's window they were able to do so and spent a lot of time on this task. Yet, while students easily found points of intersection between two graphs within the regular window they tended to guess at changes in the window dimensions when looking for unseen points.

For example, students' were unable to understand the relevance of and determine the domain of this application problem from a takehome test: "The half-life of Antimony 111 is 2.9 hours. If the formula $P=(1/2)^{t/2.9}$ gives the percent (as a decimal) remaining after time t (in hours), sketch P versus t." The problem required students to supply domain, range, and asymptote data and indicate an appropriate window.

Lane asked Tom if the domain and range applied to "just to the sketch of P versus t," in the first quadrant, or to "the whole curve," for the graph of $P=(1/2)^{t/2.9}$. Most students were unable to distinguish which portion of the graph represented the actual problem situation. Only six of 24 students responded correctly to all parts of the quiz question, including domain and range. The rest of the students indicated data that represented incomplete solution of the application problem. A typical student response was to indicate the domain of the entire graph [-_,_] rather than the domain [0,_] of the application problem.

Students' determination that the domain was the domain of the graph, rather than the domain of the problem situation characterized their tendency to overlook elements of problem situations due to preoccupation with solution methods.

CHAPTER V

CONCLUSIONS

This study began in search of a success story, both for students and for graphing calculators. It ends with a more realistic view of the intricacies of the educational environment, the consequences of a particular mode of instruction, and the impact of random usage of a cognitive technology on students' understanding.

The course was directed by an instructor, whose way of teaching algebra was a product of years of experience. That this experience came without the aid of graphing calculators came to be seen as an aside to the teacher's instructional orientation. Rules and procedures were generously doled out to students. Like bread and water, these staples seemed to sustain students, but without flavor or conceptual nourishment.

Students in this classroom used this course as a stepping stone to the next level of mathematics they would need for their various fields of study. Students approached the subject from personal perspectives which often manifested themselves in blind adherence to prescribed procedures. Even when students appeared to be asking for insight and understanding of a problem situation, they were easily diverted back to rules and procedures. This malleability of students' purpose and focus indicates that they may need to be told quite specifically how to examine a subject if they are going to be expected to derive any meaning from it.

The instructor and the college were in the process of enthusiastically searching for the best ways to use graphing calculators. They were hopeful about the ability of the graphing calculator to aid students in understanding graphs and functions. The students wanted the graphing calculator to help them too. This enthusiasm may help both students and instructors to consider and examine more effective ways of using the graphing calculator to teach mathematics.

Some of what the students didn't understand was so basic that it seemed to result from experiences beyond the few months they had spent in Tom's classroom. On the other hand, in many cases their lack of understanding might have been avoided by subtle changes in classroom discourse and the focus of instruction.

Summary of What the Study Looked for Versus What It Found

This study looked for relationships between students' uses of graphing calculators and their understanding of functions and graphs. Observations revealed that students had some procedural understanding of how to use graphing calculators and how to solve problems using the "method of the day," but no conceptual understanding of functions and graphs.

Although cognitive technologies "make external the intermediate products of thinking" (Pea, 1987, p.91), student did not and were not directed to use the graphing calculator in this way. Students often did not know that intermediate results in their problem solution were not "answers". An inability to recognize when the problem solving process was completed seems to indicate that students lacked a conceptual understanding of mathematics and problem solving for some time. The question that follows is "did they ever know, in any mathematics class, how to determine when they actually were finished with a problem other than that they had reached 'the last step'?"

Pea (1987) also pronounced that "a cognitive technology is any medium that helps transcend the limitations of the mind in thinking, learning, and problem-solving activities." (p.91) He may have based his statement on an image of motivated, reflective, knowledgeable students being taught by a knowledgeable reflective instructor. Students in this course did not appear to transcend any mental limitations, even though they used the graphing calculator.

Were students expected to accept and did they accept rules with no reasons, or did they view mathematical problem solving as a series of actions motivated by reasoning? Tom's instruction presented mathematics as a context to do something rather than as a context to reason. Students were affected by Tom's view, even when they wanted to understand. Students generally accepted rules without reasons and sometimes looked for understanding, but were easily directed to accept rules and prescribed methods for solving problems.

Understanding of function means to create an internal representation of function. These students did not appear to create internal representations for function. They expressed knowledge of a variety of disjointed rules and procedures for manipulating equations and graphs, which did not imply an underlying representation of function. To these students a function was something that needed to be rewritten using "rules," "procedures," and "definitions" in order to get an "answer."

"Mathematical competence can be expressed in students being able to move flexibly across representations" (Moschkovich, Schoenfeld & Arcavi, 1993, p.97), such as algebraic and graphic representations of function. Yet, the examples and teaching methods used in this course often tried to force the graphic representation to simulate the algebraic representation in terms of procedures. The two representations then did not elucidate the meaning behind the function but were an avenue on which students had to force-fit rules and procedures in order to get the expected answers. Students used the answers as something to shoot for rather than as pieces of information which, when determined, enriched their understanding of a situation, or as products of methods which themselves could be analyzed in terms of their logic and validity.

"Good graphic examples are critical since they can exemplify or challenge, and thus anchor or critically elucidate a point" (Leinhardt, Zaslavsky & Stein, 1990, p.52). The examples presented by Tom in class seemed to have been chosen somewhat randomly, without conceptual coherence. On many occasions the examples discussed were borrowed from the textbook and thus represented the textbook authors' perspective of what constituted good examples. Other examples discussed in class included homework problems about which students had questions. These examples were haphazard, although some of them offered opportunities (not taken) to dispel misconceptions and discuss meaning.

Even though a goal of instruction is to teach students how to learn, rather than about mathematics, this class did not teach students how to learn. This class focused strongly on students learning rules and following procedures rather than thinking about concepts or about thinking.

Both Tom and the students should be able to check results and examine topics from multiple perspectives.

The opportunity to do so exists within this curriculum and within class discussions. Checking results was rarely brought up in class -- instances of doing so arose sporadically. Most checking involved substituting results into the original equations to verify that answers "worked." A better way to check would have been to work through a problem using more than one representation, then comparing results. Students also were rarely prompted to check their results for reasonableness.

Recommendations

Students taught by an instructor with a calculational orientation to teaching do not build conceptual understanding. They also gain only limited procedural understanding. The focus of instruction needs to be on helping students to understand more than how to get answers.

Instruction must be monitored statement by statement by the instructor to ensure a thoughtful adherence to a conceptual approach to teaching. This is not easy, but a worthy goal. Given the complexity and depth of the topics discussed in any mathematics course, it is important for the instructor to be continuously aware of the needs of the students and the ways in which the students can be aided in learning. Teaching thus requires the instructor to continuously monitor reactions and responses to students' questions and perceived needs.

Students need to know that the best methods for solving a problem arise from understanding it deeply, and that understandings are personal, not prescribed. They also should understand that a solution to a problem is not unique to the particular method used to determine it. In this way students are led to see that there is meaning beyond the procedures and algorithms used to solve problems. They will also have the opportunity to be confronted with and examine the fact that mathematics is not separate from their personal world of sense and understanding. The students in this study did not have these opportunities, and made few connections.

Application problems cannot help students to gain understanding of functions if the students are not taught to analyze and interpret all aspects of the mathematics involved. Students in this study were unable to see the connections between parts of the problem and the mathematics being used to represent those parts, since the course did not focus on these connections. Without this analysis and interpretation, mathematics will become a separate, disconnected topic which students approach procedurally. Discussions which take the real-world aspects of problems into consideration will naturally focus on important elements of problem situations, such as domain, rather than merely steps in solution methods.

Instructors need to be especially careful when using terminology, especially for the first time, and need to develop consistency in their use of language.

Possible Implications of the Study

This study's examination of students' using graphing calculators illustrates the contrast between the hopeful expectations, for students' understanding, submitted by graphing calculator advocates, and the reality of instructional environments and students' understandings. A concise list of insights which may have been implied by this study may proved helpful to instructors, students and researchers.

1. The graphing calculator is only a cognitive technology if students use it to enhance their learning and understanding of functions and graphs. Otherwise, it is only an answer-getting gadget.
2. Students gain neither procedural nor conceptual understanding of functions from using the graphing calculator procedurally.
3. Students often try to understand and find meaning in the mathematics they are learning. It is by repeatedly being directed otherwise that they tend to allow a gradual abandonment of this goal.
4. As long as students are given to believe that mathematics is made up of numbers and letters without meaning they will have difficulty creating conceptual

understanding. Application problems are important at every step of the learning process.

5. Application problems cannot help students to gain understanding of functions if the students are not taught to analyze and interpret the situation itself initially, and to reflect on all aspects of the mathematics involved in relation to the situation being modeled.

6. Students must be helped to see the connections between parts of problems and the mathematics being used to represent those parts. Otherwise, the mathematics will become a separate, disconnected topic which students approach procedurally.

7. Students need to know that they have both the freedom and the responsibility to approach a problem in any way that makes sense to them, and for which they can build justifications for the sense that it makes.

8. Instructors must be especially alert to statements or actions, by themselves or by students, that do not contribute to a conversation's overall conceptual coherence. Constructive contributions are those that contribute to an emerging image of "what is going on here, and what are we trying to do?" and establishing meaningful connections between current and past activities.

These implications may aid instructors, students, and researchers in thoughtful consideration of the implications of instruction involving graphing calculators. If thinking and learning are motivated, even through studying students' use of the graphing calculator, then perhaps the graphing calculator truly can be recognized as a cognitive technology.

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ABSTRACT

ABSTRACT

This thesis examines the use of graphing calculators as an aid to student understanding, by exploring the relationship between college algebra students' use of graphing calculators and their understanding of functions and graphs. The study was carried out as a classroom observation of thirty-one students in a college algebra course at a two-year college. The course emphasized solution methods involving graphing calculators. The curriculum and the instructional methods within which graphing calculators were employed were considered in a qualitative analysis of classroom discourse. The study found that students did not seem to recognize relationships between multiple representations of function, were unable to meaningfully describe elements of equations they created for applications problems, and were unable to determine relevant domain and range information in problem situations. This research implies that procedural, answer-getting uses of graphing calculators do not help students understand functions and graphs.