STUDENTS' CONCEPTIONS OF TIME AS A VARIABLE

by

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What is time? Time is a mysterious concept whose definition and meaning has been debated for thousands of years by philosophers and scientists. For instance, St. Augustine said that he knew clearly what time is—until he had to explain it. In this paper, I have researched functions involving that ever present entity of time and students' difficulties associated with those functions. Time is a unique component of our lives, and it presents interesting problems when students are asked to deal with it as a variable.

In the first section of this paper, I summarize three of the main discussions in mathematics education research concerning the topic of students' learning of functions. These three topics, concept images of functions, functions as action and process, and, in particular functions as covariation, have a specific relevance to time-related problems and problems with time as a variable. The previous research gives insight into the difficulties that students have in dealing with functions, and I propose that some of these difficulties can be explained, in part, to be due to preconceptions students hold about time as a quantity.

In order to determine what the basic human preconception of time is, the second section of the paper concentrates on time and its definitions and explanations according to both ancient and modern philosophers. Through their thoughts, I have created a list of preconceptions that summarize the "common" person's interpretations of this abstract subject.

From this list of preconceptions, I propose connections to the difficulties that students show in handling functions and then show these connections through assessments that were given to college algebra students in two classes and interviews that I held with two of the students. By examining the work of these students, as well as by listening to their explanations, I show that there is strong evidence to suggest that students hold these preconceptions about time and make errors that are partly due to them.

1. Functions and Time

Education researchers have engaged in numerous investigations regarding students' understandings of the mathematical concept of a function. Overall, it has been found that students from high school through graduate level mathematics encounter difficulties in interpreting and working with functions. As Marilyn Carlson states in her cross-sectional examination, "Research results show that acquisition of essential aspects of the function concept is extremely complex. Students have difficulty translating between different representations and applying basic concepts at different levels of abstraction" **[4, p.117].** Research regarding the difficulties that students have in "acquiring the essential aspects of the function concept" gives useful background for understanding connections between students' preconceptions of time and their conceptions of functions.

1.1. Concept Images and Functions as Action or Process

One proposed explanation for the difficulty that inexperienced mathematics students face is that they have erroneous concept images. A concept image, as described by Vinner and Dreyfus, is "The set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them" **[9, p.356].** For students to have an erroneous concept images means that their perceptions of functions are not in accord with formal mathematical definitions (concept definitions) of them. The concept images of students are the result of examples they have seen, the images they have retained due to those examples, and, particularly with word problems, their own preconceptions about the situation. As students progress, their concept images become closer to the accepted concept definition, and only then can their intuition guide their reasoning when handling functions.

When approaching a function in which time is a variable, students' intuitive understandings of time will certainly influence their concept image of the function. Due to previous experiences and understandings regarding similar and dissimilar problems involving time, students will view the problem as one "type" of problem, and deal with it accordingly. This seems true of any "type" of problem – particularly those dealing with real-world situations. Students will examine what they are given and believe that it is similar to their concept images. **[9]** They may believe that a particular problem of this "type" follows certain rules, whether it does or not. Thus, when students approach problems involving time, their past experiences with similar problems, as well as their preconceptions of time, will have an effect on their concept image.

Another topic, which has been the focus of numerous research papers, is that of the function as an action, a process, and an object. In Thompson's paper **[8]**, he compares an action view of a function to a recipe applied to numbers. This means that students who view functions as actions see them as rules to apply to individual numbers, and hence they are unable to see a function as mapping numbers in a domain to numbers in a range. The questions students will be able to answer are those that call for a specific method of evaluation, as long as they can "perform" the actions prescribed by the given function's definition. They may not, however, be able to interpret the meaning of the function to understand the entirety of the situation it represents.

When students are able to view a function definition as a “self-evaluating” expression, they can then imagine “running through” the numbers in its domain and thereby generating the functions range. This view will allow them to have much more insight into the meaning and situation of the function and will also be the groundwork of developing an object conception. An object conception is an even deeper understanding of a function. However, the object conception is discussed in multiple other works and has little effect on my research, so I will not explain it here.

Within the scope of my research, the two main topics mentioned above – students having an erroneous concept image and students viewing a function definition as an action – help to illuminate the difficulties of students when they encounter problems with time as a variable. First, students that view functions as an action have difficulties in deeply understanding any problem that is not a simple calculation. The problems involving time as a variable in this research require a deeper understanding of a function’s representation of a situation than an action view provides. Second, students that have erroneous concept images about time will have difficulty creating mathematical models that capture the relationships between time and other variables in a situation.

1.2 Covariation

Another topic that emerges in mathematics education research is that the concept of a function as a correspondence is widely taught to students instead of a function as covariation. According to Thompson, “The current standard definition of function highlights correspondence over variation—elements in one set correspond to elements in another so that each element in the first corresponds to exactly one element in the second” **[8, p.10]**. In fact, covariation, which presents a function as two variables varying at the same time instead of as an "ordered pair" as in correspondence, is rarely taught in schools **[8]**. This is a misfortune, because with covariation, the use of functions and the information obtained from them is less restricted. In the current teaching of functions as correspondences, one variable is labeled "the dependent variable" and the other "the independent variable". In time functions, each time unit corresponds to a change in the other variable in the problem, and so time is labeled as "the independent variable". This labeling creates a biased concept image toward situations involving time since time is expected to correspond to the other variable in a specific manner. Instead, a view of time functions as covariation would allow students to interpret the situation based on the variations of each variable and thus not have a preconceived notion about how time should behave. However, even if a student is taught to regard functions as covariation, a function involving time is still difficult to view as covarying. The main reason for this is that time is viewed as completely separate from every other common variable, and is always thought to vary in the same way – thus the thought of it covarying is antithetical. The reasons for this will be addressed more thoroughly in the next section.

Another issue that arises, particularly when regarding a function as correspondence instead of covariation, is stated by Carlson, "Monk and Kaput both report that students expect the shape of the graph to reflect visual aspects of the situation described by the graph, rather than the representation of the relationship between two variables" **[4, p.117]**. In time functions, students are even more apt to view the graph as a 'pictorial' representation of a situation, due to their comprehensions of the abstract nature of time, which will be further investigated in the next section.

It is clear that there are a multitude of reasons concerning why students struggle to develop deeper understandings of functions. Some of these difficulties are further exacerbated when dealing with time as a variable. Another aspect of the difficulties students have with functions involving time is their preconceptions of time and the preconception of time every 'common' human innately understands. Philosophers have investigated the universal preconceptions, and so, at this point, it would be beneficial to investigate philosophers' views on how humans perceive time and their interpretations of what this entity is that we call "time".

2. Preconceptions of Time

Great past philosophers such as Aristotle, Plotinus, and St. Augustine have speculated on the characterization of time, and all three have had great difficulty in constructing a clear, meaningful definition. It should be mentioned, however, that a specific and deep definition of time is of little interest to this research. Instead, it is important to understand common conceptions of time and how these conceptions, true or otherwise, may influence the "common" student's understandings of functions with time as a variable. For this reason, although there are very deep philosophical, as well as mathematical and physics-based definitions of time, it is more essential to examine philosophies which scrutinize universal human understandings. I will attempt, therefore, to present the philosophers' outlooks that are the most insightful for this particular investigation.

Aristotle analyzed the concept of time in his *Physics*. His attempt to examine this difficult subject is the first on record and was written during the fourth century B.C. The matters that he first examines are what he describes as the "traditional accounts" of time

[1]. One of these traditional accounts is time as "the movement of the whole". He argues that time cannot be movement because time is "present equally and everywhere and with all things" and because "change is always faster or slower, whereas time is not" **[1, p.11]**. Yet, throughout the rest of his writings on time, Aristotle argues that "time does not exist without change" and that time itself is the measure of motion while motion is defined by time. This leads to a rather circular argument in which time and motion depend on one another. Whether this definition is philosophically sound has been contended by other philosophers **[7]**, and yet it offers to this research the view that time and motion are somehow inherently linked, and this idea has significance in explaining our preconceptions of time.

Plotinus, in the third century A.D., takes the argument further and states that "time…is something other than the mere number measuring movement" **[7, p.29]**. Plotinus instead argues that time is much more important, for it is "the Life of the Soul". In one of his most descriptive passages he states:

> We are brought thus to the conception of a Natural Principle – Time – a certain expanse (a quantitative phase) of the Life of the Soul, a principle moving forward by smooth and uniform changes following silently upon each other – a Principle, then, whose Act is (not one like that of the Supreme but) sequent **[7, p.32]**

This poetic description of time is probably not a common understanding as a whole, but it brings up certain aspects of time which seem to exemplify it in the mind's eye. Once again, although he argued that time is not a mere measure of movement, he too, describes it as "moving forward". Also, the words, "smooth", "uniform", and "sequent" seem somehow natural when describing time and help to describe preconceptions that humans hold.

St. Augustine, around 400 A.D., once again asked the question, “What then, is time?” to which he answered to himself:

> If no one asks me, I know: if I wish to explain it to one that asketh, I know not: yet I say boldly that I know, that if nothing passed away time past were not; and if nothing were coming, a time to come were not; and if nothing were, time present were not. **[2, p.40]**

Augustine thus determined that, “time cannot be without created being” **[2, p.53]**. This is a common conception – that time does not occur without something happening and that our actions do not occur without time. St. Augustine contemplates that, “all time past is driven on by time to come, and all to come followeth upon the past” **[2, p.39]** and also:

> If an instant of time be conceived, which cannot be divided into the smallest particles of moment, that alone is it, which may be called the present. Which yet flies with such speed from future to past, as not be lengthened out with the least stay. […] The present hath no space **[2, p.42].**

Both of these passages exemplify another common conception of time – its continuity. When we imagine the present having no space, and past driving the future, it seems apparent that there must be no breaks in time. Its nature is smooth and continuous with the present being nothing more than a divider between the past and the future.

Modern day philosophers also have offered numerous accounts of the concept of time, much of which focuses on a scope beyond the "everyday" conception of time. However, in the writings of Bruzina in the year 2000, a suggestion is mentioned that seems to capture another preconception that most people incorporate **[3]**. While analyzing temporality with a lighted metronome, Bruzina speaks of the characteristic of time "retaining" the "just-having-shown" and "protending" the "not-yet-shining". He states that, "protention and retention comprise the 'horizonality' of time, that is, the dimensions that amplify the otherwise stark and unrelenting nowness of the now (and that indeed make the movement, the passing of time possible)" **[3, p.73]**. This idea exemplifies our natural desire to create a physical dimension for time to "move" through. It seems we, when imagining time, wish to see it moving straight forward ahead of us and straight back behind us. This conception of time gives it a parallel and generally horizontal movement along our own lines of travel.

Another modern era conception of time stems from Piaget's (1970) investigations of children's development of the concept of speed **[6]**. His work took an approach that combines psychological and philosophical methods, and came to the position that time is not a primary intuition. That is, children construct time as a quantity by coordinating changes in objects' locations.

> The relation $v = d/t$ makes speed a relation and makes d as well as t two simple intuitions. The truth is that certain intuitions of speed, like those of outdistancing, precede those of time. Psychologically, time itself appears as a

> relation (between space traveled and speed ...), that is, a coordination of speeds, and it is only when this qualitative coordination is completed that time and speed can be transformed simultaneously into measurable quantities **[5, p. 111]**

3. Generalized Preconceptions of Time

Time is clearly a difficult concept to understand. The above and many other philosophers have attempted to identify what "time" is and how it fits into our world. Through what I have summarized here of the work of Aristotle **[1]**, Plotinus **[7]**, Augustine **[2]**, Bruzina **[3],** and Piaget **[6]**, as well as by contemplating their works in comparison to other philosophers, I have compiled a short list of the most common and encompassing understandings of time that are applicable to this research of time as a variable. These are by no means the only ideas that we retain when we contemplate time, and some of us do not share all of these conceptions (in fact the philosophers themselves have argued that some of these are not in actuality, "true"). However, there is reason to believe that generalized conceptions can encapsulate many people's feelings on the subject. Time is abstract, and so we must add a concrete foundation to understanding it, and these foundations can be captured in four main ideas.

The four basic ideas of time are:

1. Time is inherently related to motion and motion to time **[1] [7].**

2. Time is a made up of indivisible units and is sequential. It is smooth, uniform, and thus continuous **[2] [7].**

3. Time exists but is not directly affected by anything, while at the same time it cannot exist on its own **[2].**

4. Time moves in a dimension parallel to our own, which seems most easily imagined as horizontally moving ahead of and behind us **[3].**

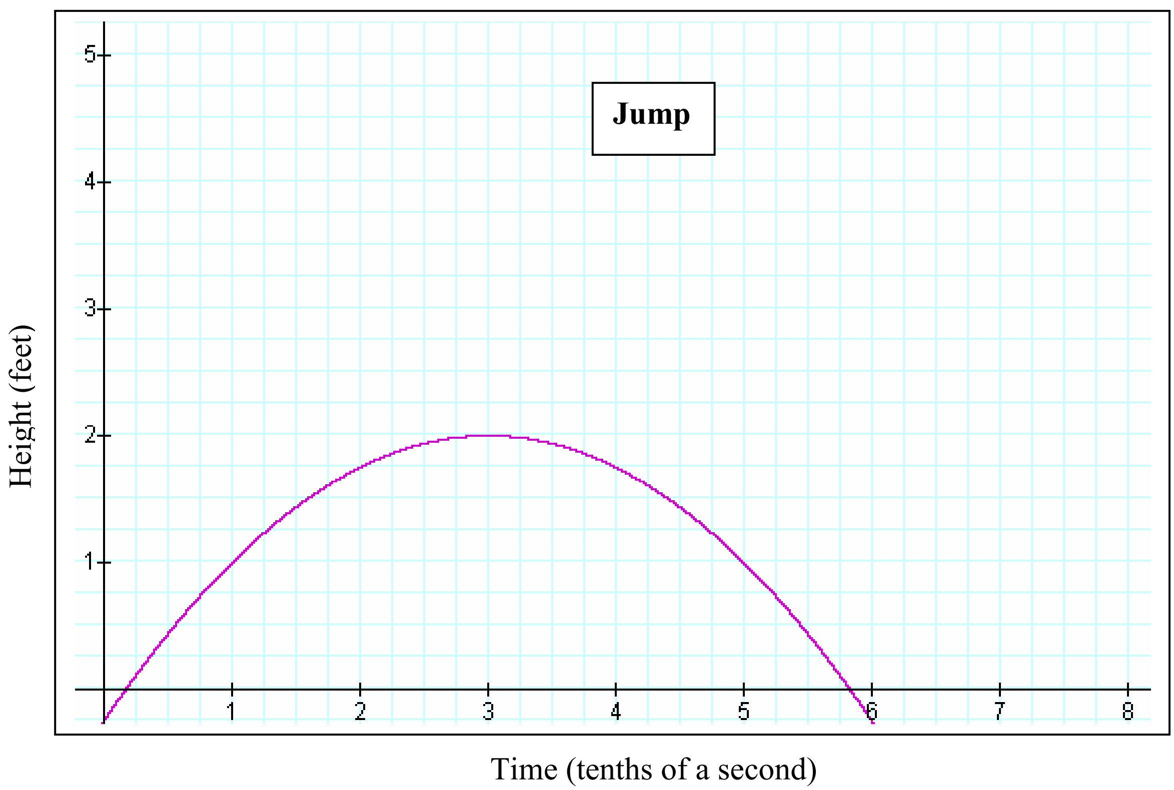
These four ideas, which are an encapsulation of a "regular" human's preconceptions of time, can be linked directly to certain aspects of functions with which students have difficulty as shown above.

From Numbers 1 and 3, we gain an understanding of why students' insist that time must be an independent variable. In fact, in our lives, time is one of the only things we can think to be truly independent from other forces acting around it. While time essentially needs "being" to exist, that state of "being" does not push it or change it in any way. This can explain why understanding time functions as covariation may cause difficulties, since time does not seem to "vary" in the sense that other variables do.

As Number 2 states, time is sequential. It does not vary due to outside forces. In this sense it is what many students deem the "constant" variable. This perception causes students to have a very strong concept image in which time is always the independent variable, and this creates difficulty when dealing with situations where it is represented differently.

Number 2 also explains why students expect a function involving time to be continuous and smooth. Due to this preconception, the students' concept image of graphs involving time may not include graphs with holes and pointed edges or equations in which time is squared or cubed, even though these graphs and equations are perfectly legitimate.

From Number 4, we can explain why students believe they can “see” a situation occurring when time is along the horizontal axes (in a sense, the horizon) of the graph. The graph becomes a miniature representation of reality, and with time along the bottom, students may believe that they are able to follow the situation by looking at its shape over time. In this case, time is a horizontal mover and everything happens as time moves along horizontally, just as it moves along on a graph. So, for instance, if we jump forward, we can watch our height over time move along the graph, just as though we were watching the situation:



Unfortunately, it is not always the case that a time function graph looks like the situation that it represents. This idea, which may be in part due to the preconception involving time, can lead to errors and misinterpretations.

In the following section, I will present examples in which students have made errors in solving certain problems involving time and how these errors can be explained, at least partially, as a result of the above preconceptions.

4. Assessments and Interviews

In order to test students' basic understanding of functions involving time, two College Algebra classes, with a total of 17 students, were given an assessment. This assessment included seven questions, each involving a function in which time played a different role, or in one case, no role at all. By including time in different roles, the study allows us to examine which concepts of time the students grasped and which gave them the most difficulties.

The students were given thirty minutes to complete the assessment, and they were given no help in answering the questions. Most completed the assessment much faster than thirty minutes, which implies that their answers are more intuitive than deeply explored. The students' quick reactions allow us the opportunity to scrutinize the students' instinctive responses and thus their innate perceptions.

Interviews were held with two of the students after the assessments were administered and scored. These interviews were meant to investigate the reasons that students had for answering the way they did and the meanings behind their understandings.

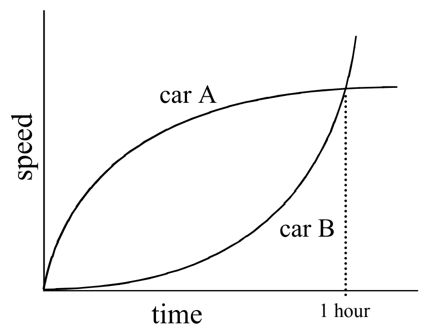
4.1 Results and Analysis

The discussion of each result from the assessment includes a statement of the question, a table indicating the answers given and related statistics, interview transcripts from both of the interviews which revealed the most about how certain students

responded to the question, and a discussion of the results.

Question 1

The following is information for both Question 1 and Question 2: The given graph represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.) Use this information and the graph to the right to answer items 1 and 2.



What is the relationship between the ***position*** of car A and car B at $t = 1$ hr.?

a) Car A and car B are colliding.
b) Car A is ahead of car B.
c) Car B is ahead of car A.
d) Car B is passing car A.
e) The cars are at the same position.

Answer	Number of Students	Percent of Students
a	0	0.0%
b	**1**	**5.9%**
c	0	0.0%
d	2	11.8%
e	14	82.4%

Correct Answer: **b**
Percent Answered Correctly: 5.9%

Most Common Answer: **e**
Percent Answered Most Common: 82.4%

Discussion:

Only one person out of 17 answered this question correctly. This indicates that the students were not correctly envisioning the situation over time. Since the answer of "e" indicates that the cars are at the same position, this shows that instead of analytically thinking of the situation, the students viewed the shape of the graph and saw that it "looked like" the cars were at the same position at t = 1 hour, since their graphs crossed there. It indicates that they expect a graph over a time interval to "show" them the situation. As can be explained by Number 4 of our established preconceptions, this shows that a graph with time along the x-axis is somehow the clearest way to look at a situation, since it is almost like watching the situation happen over that time period. The following excerpt from an interview with Student N shows this very thought process:

Int: [In reference to his answer of "e"] … why did you choose that?

N: Because, ummm, this is the meeting point for them.

Int: Where they cross?

N: Yeah, that's the meeting point for them, so I mean, regardless for the speed for the Car B or Car A, they're going to meet at some point and that point would be that same position. That's given here in the graph.

Int: How does the graph tell you about the position? It's a speed and time graph, right? How does it show you the position?

N: It shows you the time after one hour, which the question asks, like, when are the cars going to be after one hour. The speed doesn't actually matter in this question, because you just look at the graph, I mean, whether it's increasing or decreasing, they're meeting at that point.

Int: Okay, so the graph shape tells you that?

N: Yeah.

It is clear here that Student N believed that the shape of the graph showed him the situation, and thus, since the lines cross at a point in the graph, the cars were believed to be at the same position. Student N gave even more insight into his answer in the following excerpt from later in the interview in response to the same question:

> *Int:* What would happen if we put time along the y-axis and speed along the other axis, would that make sense? Would it be confusing at all?
>
> *N:* Well, you won't be able to, umm, I mean, if the time becomes here [points to y-axis], you won't be able to answer the first question, because the whole graph is going to be a different shape, so you won't be able to. But, it depends on the question, I believe.

Student N's belief that you cannot answer this first question if time is on the y-axis suggests that he believes there is something special that occurs when time is running horizontal in a graph. It exemplifies the preconception that time runs parallel to our world and therefore allows us to see a situation as long as we graph time horizontally.

Question 2

What is the relationship between the ***speed*** of car A and car B at *t* = 1 hr.?

a) Car A is going faster than car B.
b) Car B is going faster than car A.
c) The cars are traveling at the same speed.
d) Car B is catching up to car A.
e) Not enough information.

Answer	Number of Students	Percent of Students
a	1	5.9%
b	3	17.6%
c	**11**	**64.7%**
d	1	5.9%
e	1	5.9%

Correct Answer: **c**
Percent Answered Correctly: 64.7%

Most Common Answer: **c**
Percent Answered Most Common: 64.7%

Discussion:

Since the majority of students answered the problem correctly, this shows that this answer was easier for these students to determine than Question 1. It seems that most students were able to answer a question which was explicitly shown by a point on the graph. An interesting detail is that 9 of the 11 people that answered correctly that the cars were going at the same speed at t = 1 hour, also answered that the cars were at the same position at t = 1 hour. This implies that they viewed the same line on the same graph as representing both speed and position over time. Once again, this is an example of Number 4 that with time along the x-axis, they seem to believe that they are somehow "shown" the answer, even though they were answering two different questions.

Three of the students answered "b" to this question: that Car B is going faster than Car A. Since the slope of car B is steeper at this point than that of Car A, it seems clear that these students may still have been regarding the graph as a time verses position graph. Student N exemplified this line of thought:

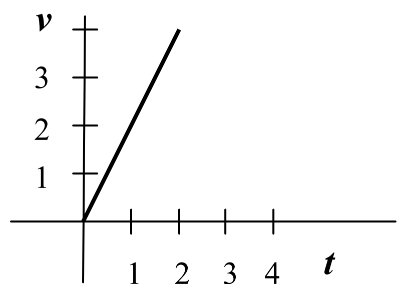
Int: Now, on number 2 you said that car B was catching up to car A. Now, does speed matter in that? We're looking at the relationship of the speed. You said that speed didn't matter here, though [number 1].

N: Yeah, it didn't matter in the question where they asked…they were just asking if it's going to meet or not, so I was looking at the graph at the meeting point, which it came with the one hour, that has to do with the question. The other one here, in the second question, umm…, like how did I know? Because, like, when you go up, the speed increases, and like, from the graph you can see that car A is slowing down, from the graph, and car B is catching up because the speed is increasing.

Question 3

A hose is used to fill an empty wading pool. The graph shows volume (in gallons) in the pool as a function of time (in minutes). Which of the following defines a formula for computing the time, t, as a function of the volume, v?

a) $v(t) = \frac{t}{2}$
b) $t(v) = 2v$
c) $\boldsymbol{t(v) = \frac{v}{2}}$
d) $v(t) = 2t$
e) $t(v) = v - 1$



Answer	Number of Students	Percent of Students
A	3	17.6%
b	4	23.5%
c	**2**	**11.8%**
d	8	47.1%
e	0	0.0%

Correct Answer: **c**
Percent Answered Correctly: 11.8%
Most Common Answer: **d**
Percent Answered Most Common: 47.1%

Discussion:

Since only two students answered this question correctly, it is clear that there was significant difficulty in understanding what needed to be determined and how to do it. The answer of "d" is the answer that would be correct if the graph was showing the situation that they were asked to define. Thus, either the students did not read the question, or they read it and felt that the only correct way to draw a graph of the situation was with time along the bottom, and they gave the formula directly from the graph.

The fact that not even half of the students answered "d" and that the answers have such a wide spread on this question indicates that the students may face an additional difficulty in the question. It seems that the students were unclear whether to write t(v) or v(t). Since no student answered “e”, it is clear that every student either derived the formula from the graph or found the inverse. The majority, even with the discrepancy in t(v) or v(t), seem to have read directly off of the graph, since “b” and “d” both give a variable multiplied by 2, and 12 students answered one of these two answers.

Student K answered “b” originally, but during the interview changed her incorrect answer to another incorrect answer choice, “a”, and said the following:

K: Okay. I got this one by just looking at the point, and it’s two over one, so it’s two.

Int: Okay, so you looked at the graph to figure out what that equation is. But, if you read the question, it says that we have a graph that shows volume as a function of time, right. But, for the answer, we’re actually looking at what is time as a function of volume?

K: Oh, so then it would have been one over two.

Int: Okay, which of these would you change the answer to?

K: Ummm…it’d be like 1 t over v or something like that. Ummm, t over 2. Yeah.

Student K did not seem to understand the notation t(v) and v(t), but was able to find the inverse and draw the graph correctly. Student N also had trouble with the notation. He originally answered “a” to the problem:

Int: Do you remember how you got to this answer. You said that v(t)=t/2.

N: Yeah, I was between two, I believe. I was between, uhh, A and C. Because, I got confused at that point.

Int: About whether it’s v(t) or t(v)?

Int: Is it different than what’s on the graph?

N: I guess so. But, I know that the shape of it is going to be something over 2, but I’m not sure if it’s the time or the [5 second pause].

Int: Okay, well how do you know that it's going to be over 2?

N: Umm, because it goes to half. You know, it's like, I determined that because I was trying to spot a point on them, and I noticed that one of them is double the other, so that's why we can put a relationship between them that one is half the other.

Interestingly, when asked about dependency, Student N decided to change his answer from "a" to "b":

Int: So, is either dependent on the other? Have you heard those words before?

N: Yeah, dependent…yeah. Well, I believe that, uhh, the time doesn't have to do with the volume – it's an independent, and as the time goes, the volume is related to it, because you are taking the time on the [5 second pause] Uhhh, I believe I made a wrong mistake on this one.

Int: Okay, which one do you think it is then?

N: I believe it's B.

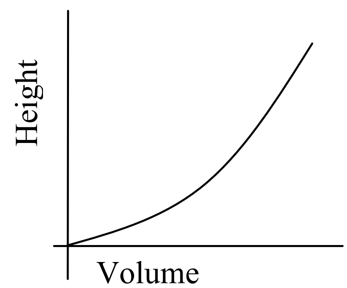
Int: Okay. It's 2 times v?

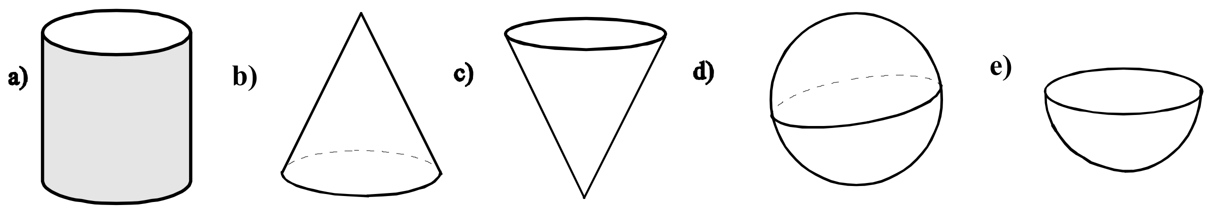
N: Yeah, because you know, this is an independent, and as the time goes, the volume is related to the time. When the volume increases, that means that time is increasing, but it doesn't affect the time. But, the volume is related to the time.

This shows Student N's hesitation to make time a function of volume. He had almost done this originally, but when he realized that his function would seemingly make time depend on volume, he wanted to change his answer. This shows that time being the independent variable is very ingrained in his mind, as explained by preconceptions Numbers 1 and 3. His adherence to this idea caused him to second guess his mathematical calculation.

Question 4

The following graph represents the height of water as a function of volume as water is poured into a container. Which container is represented by this graph?





Answer	Number of Students	Percent of Students
a	0	0.0%
b	**3**	**17.6%**
c	14	82.4%
d	0	0.0%
e	0	0.0%

Correct Answer: **b**
Percent Answered Correctly: 17.6%

Most Common Answer: **c**
Percent Answered Most Common: 82.4%

Discussion

Since only 3 people answered correctly, it is clear that, in general, they are not viewing the situation as "what happens to height as volume increases".

It seems that they are trying to "see" the situation within the graph. They see that the bottom of the container is smaller and the graph is smaller at first. Thus, this graph would be the correct one if we were looking at volume as a function of height. This once again shows the assumption of viewing the graph as the situation itself.

From preconception Number 1, since time is inherently linked to motion, my expectation was that the students to believe that time played a role in this problem, even though it was not a variable. However, when I asked the interviewed students if the graph would look different if the water was poured in more quickly, they both answered correctly that time played no part in the problem. From Student K's interview, I got this response:

> *Int:* What would happen if we poured the water more quickly into the container, would that change the shape of the graph at all?
>
> *K:* It might make … no, because the height would still remain the same, and same amount of volume would still remain the same in the end.
>
> *Int:* Okay, so does time play a part in this question?
>
> *K:* No.

Similarly, from Student N:

> *Int:* If we were pouring the water in more quickly, would that change the shape of the graph? Does time matter?
>
> *N:* Umm, on the graph no.
>
> *Int:* No, okay, so you can pour it in as slow or as fast as you want the graph would look the same?
>
> *N:* Yeah, except if they actually make another graph that has to do with time and with height, that's a different situation.

The response from these two students show that they were able to understand the situation well enough to know that the amount of time it took to pour the water into the container did not matter. This is contrary to the suggestion that they may link time to any motion that occurs (such as pouring water into a container) even when it is not a variable in the problem. Other students may have had this preconception, but there is no conclusive evidence in this sample to that effect.

Question 5

A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 5 cm per second. Express the area, A, of the circle in terms of the time, t, (in seconds) that have passed since the ball hits the lake.

a) $A(t) = 25\pi t$
b) $A(t) = \pi t^2$
c) $\mathbf{A(t) = 25\pi t^2}$
d) $A(t) = 5\pi t^2$
e) $A(t) = 10\pi t$

Answer	Number of Students	Percent of Students
a	4	23.5%
b	1	5.9%
c	**1**	**5.9%**
d	11	64.7%
e	0	0.0%

Correct Answer: **c**
Percent Answered Correctly: 5.9%

Most Common Answer: **d**
Percent Answered Most Common: 64.7%

Discussion:

Only one person answered correctly. Moreover, this student changed his answer during the interview. This implies that it was not clear to the students that they had to square both time and the number “5”.

When pressed, students in the interviews had a difficult time explaining why it is okay to square time. They seemed to believe that squaring time was not allowed because time is “constant”, as in the interview with Student K (who answered “d”):

Int: You got that the area is 5 pi times the time squared. Do you know how you got that?

K: Well, a circular area is Pi*r^2, and…you want to express it in time, so obviously t needs to be a variable in there. And….umm, [5 second pause]. I guess it probably would have been…I guess that's why I put 5*pi*t^2. I don't know. I honestly think I guessed on this problem.

Int: Okay, why do you think you would have guessed on it?

K: Because I've never had to make an equation using those variables.

Int: With time as a variable?

K: [nods] Uh-huh.

Int: What would it mean to square time there?

K: That probably wouldn't be right because you're changing a constant variable.

Int: Changing it how? What do you mean by that?

K: By squaring it, umm, [5 second pause], it's increasing faster than it normally would.

Student N also believed that time should not be squared because it was a "constant":

N: Well, they told that you were creating a circle, so it has to do with the circle formula, and ummm, it was given that the speed like, ummm, like the speed of it, it's five cm per second, so, I was actually in between two things, I was between C and A.

Int: Okay, so you weren't sure if time should be squared or not?

N: Ummm, yeah. Cause, I kind of forgot the formula at that time.

Int: Okay, do you remember the formula right now?

N: No.

Int: Well, I can give you the formula: pi*r^2. Does that help?

N: Pi*r^2? …. Okay, ummm…

Int: Would that help you to determine if you should square t or not?

N: Well, ummmmm, [re-reads question, 5 second pause] uhhhh…..uhhhh…. well the t is not actually…it should be constant.

Int: Okay, what do you mean by constant?

N: Like, no I mean, it doesn't…. it shouldn't be squared.

Int: Okay, why is that?

N: Because, actually, only the r should be squared, which is 5.

From this explanation given by both Student K and Student N, it is clear that for some students, the preconception that time is a “constant” entity, by which it seems that they mean it is predictably sequent, as in preconception Number 2, it should not be changed in a way that will “make it go faster” than it naturally does.

Question 6

Water is dripping from a leaky faucet into a cup at a constant rate all night. At certain points during the night, a man is awakened from his sleep by the dripping. At those moments, he records the height of the water in the glass as well as the time he awoke. In the morning, he wants to know what time it was when the height of water was at 10 cm. What time was it?

a) 12:00am
b) 12:35 am
c) 12:45 am
d) 1:00 am
e) 1:30 am

Height of Water (cm)	Time
1 cm	10:30 pm
3 cm	11:00 pm
9 cm	12:30 am
14 cm	1:45 am
16 cm	2:15 am
24 cm	4:15 am
33 cm	6:30 am

Answer	Number of Students	Percent of Students
a	1	5.9%
b	4	23.5%
c	**10**	**58.8%**
d	1	5.9%
e	1	5.9%

Correct Answer: **c**
Percent Answered Correctly: 58.8%

Most Common Answer: **c**
Percent Answered Most Common: 58.8%

Discussion:

Since the majority of students answered this question correctly, we are shown that this way of presenting a problem allows them to understand the situation and the role of time in the situation. They may have simply viewed the variables as covarying and not even considered that one is dependent on the other. They seemed to handle this problem more easily since the data was not presented along the x-axis or the y-axis. When asked to graph the situation, the interviewed students seemed to prefer time along the x-axis, possibly due to the fact that they could "watch" the situation more easily when it was graphed there. However, unlike problem 1, Student N did not feel that it was necessary for time to remain there.

Student N graphed the situation and stated in the interview:

N: [drawing graph, time along horizontal, height along vertical] Okay, umm, this one should be the height, and this one should be the time.

Int: Okay, why did you choose to put time along the bottom?

N: Well, it doesn't matter – you can just switch it.

Student K also graphed the situation. Her graph showed time along the x-axis and height along the y-axis as well. She was slightly more reluctant to "switch them" but also agreed that it would be okay:

Int: Could you graph this so time was along here, and time was along there? [opposite of what it was]

K: No, cause height isn't a constant variable.

Int: Okay, so you always have to put the constant variable along the x-axis?

K: Uh-huh.

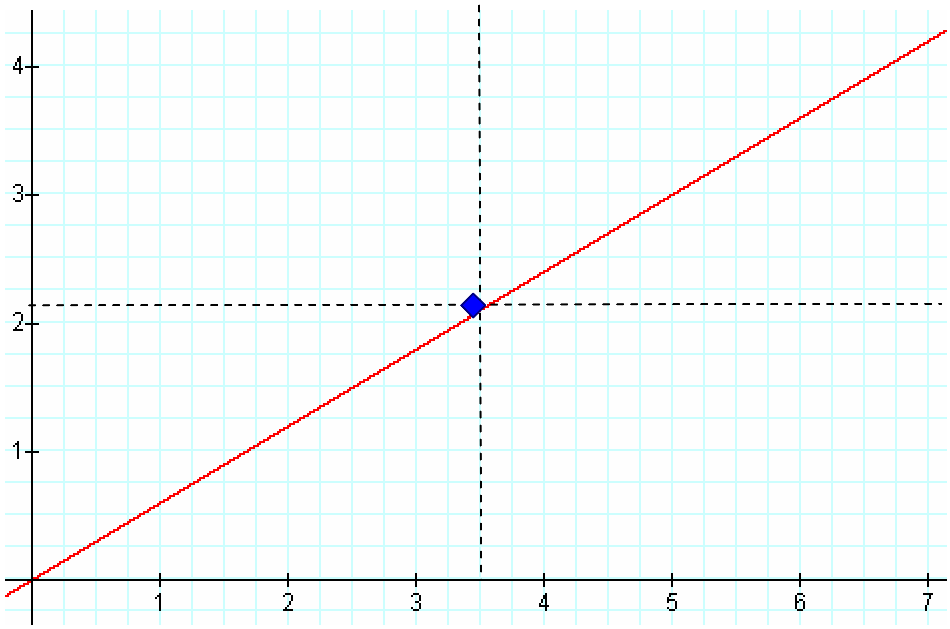
Int: But, we were looking for the time, given a height.

K: Then, I guess you would be able to the graph it that way.

It is clear that students have a strong impulse to place time along the x-axis, which is consistent again with preconception Number 4. However, when a problem is stated in such a way that the variables are covarying instead of obviously correspondent, it seems relatively easy for them to read off the chart and draw the graph opposite of what they originally would have, even when dealing with time.

Question 7

Bill and Fred started from the same spot and walked in the same direction. Bill walked at a rate of 5 ft/sec. Fred walked at a rate of 3 ft/sec. The graph below shows Fred's distance in feet from the start (along the *y*-axis) in relation to Bill's distance in feet from the start (along the *x*-axis). One point on the graph is highlighted.



a) What does this point represent?

b) How many seconds does this point represent?

Question 7(a) Since this is not a multiple choice question, the following key was created to categorize students' answers:

BFASP = Bill and Fred at same place.
NA = No Answer
BFGSD = Bill and Fred gone same distance.
BFP = Bill and Fred's points of ft/sec
AN = "Absolutely Nothing"
B1.5IF = Bill is 1.5 feet in front of Fred

Answer	Number of Students	Percent of Students
BFASP	9	52.9%
BFGSD	3	17.6%
BFP	1	5.9%
AN	1	5.9%
NA	1	5.9%
B1.5IF	**1**	**5.9%**
2/3 Sec.	1	5.9%

Correct Answer: **B1.5IF**
Percent Answered Correctly: 5.9%

Close to Correct Answer: **BFP**
Percent Answered Close to Correctly: 5.9%

Most Common Answer: **BFASP**
Percent Answered Most Common: 52.9%

Discussion:

Since only one student answered correctly, and only one other student's answer was almost correct, it is clear that this problem was challenging for them. Also, since most of the students answered that the point on the graph is the point at which Bill and Fred were at the same place, it is clear that they, first, expected the point to be an important point somehow and, second, since they saw two lines intersecting, they believed that this must represent an intersection of something (even though the "intersecting lines" were merely dotted lines to let them know which x- and y-coordinates were at that point). The second most common answer was that Bill and Fred had gone the same distance. Once again, they expected the graph to visually "show" them something about the situation, and took the 'intersection of lines' to mean something physical about the situation. With regard to "time", the foregoing implies that the students expect a situation to be shown to them as preconception Number 4 illustrates once again.

Question 7(b). How many seconds does this point represent?

Answer	Number of Students	Percent of Students
<1 sec. OR 2/3 sec.	**3**	**17.6%**
1.5 sec.	1	5.9%
2 sec. OR 2.25 sec OR 2.5 sec.	4	23.5%
3 sec. OR 3.5 sec.	5	29.4%
4 sec.	1	5.9%
5 sec.	1	5.9%
17 sec.	1	5.9%
NA	1	5.9%

Correct Answer: **<1 sec. OR 2/3 sec.**
Percent Answered Correctly: 17.6%

Most Common Answer: **3 sec. or 3.5 sec.**
Percent Answered Most Common: 29.4%

Discussion:

Both <1 sec and 2/3 sec. were counted as correct due to the difficulty in correctly reading the graph. If the student realized the answer was less than a second, it can be assumed he or she had a relatively clear understanding of the situation. Additionally, the other answers were grouped by a 0.5 margin due to the same difficulty in viewing the graph accurately and the approximations made when answering the question. Only 3 students gave a “correct” answer, implying that this problem contained a serious difficulty.

The most common answer was either 3 or 3.5 sec. The second most common answer was 2, 2.25, or 2.5 seconds. These answers, which a combined 52.9% of the students answered, imply that the students felt that they could somehow find time on or assign time to one of the axes. It may be that they did not correctly read the question and understand what they were given, yet interestingly, many of them correctly labeled the graph's axes as representing the distance that Bill traveled and the distance that Fred traveled, and yet still somehow felt that time must be represented along one of the axes. This shows that, if time is represented in a graph, students assume it is along an axis. This verifies preconception Number 4, that time is inherently understood as moving along in the situation.

In the interviews, it was clear that Student K did not understand the question. When asked about part (a), she answered:

> *Int:* You got an answer, right? You said that the part here where we have a point is where Fred walks and intersects with where Bill walks. Do you remember how you came up with that answer.
>
> *K:* Ummm, no… I don't remember. [15 second pause, looking at problem]
>
> *K:* So, Fred's on the Y-Axis and Bill's on the X-axis?
>
> *Int:* Yeah, if you want to draw those in that's fine.
>
> *K:* And then, this box is where they meet?
>
> *Int:* Well, that's what you said in your answer, is that what you think that dot means?
>
> *K:* That's what I thought it meant.
>
> *Int:* Okay, why do you think that?
>
> *K:* Ummm…. I don't know, because usually dots on the graph are the point where they intersect.

Student K gave an interesting answer to part (b) in her assessment. She stated that the point represented 17 seconds. She could not explain this either, but interestingly was able to determine the correct answer in the process:

Int: All right, well you did get an answer for this one, though, you said the seconds this point represents is 17, and I'm just curious how you got that answer.

K: I think I just wrote it down, honestly, as a number…ummm…

Int: Did it have to do with the number of feet one of them has walked and the number of seconds it took?

K: I don't see why it should because, they haven't even walked one second if it's less than three feet. It's three feet for one second, and 5 per second. They haven't even walked a second, so I don't see how it could be 17. If I'm doing it the way I was thinking about it.

Student N was also confused by the problem, and could not explain his answer to part (a) that the point on the graph was a meeting point:

Int: Why do you think it's the meeting point? Where do you get that from the graph?

N: Well, they said they started from the same point, so I believe they started somewhere here, and they walked in the same direction, but it's just…you can…I just…I, you know I had to look at the graph more than you know, starting to break out the question.

Int: Okay.

N: So, when I looked at it, it's like, what does that point represent, well they're just walking, that's a point in the middle, it has to be a meeting point. It can't be something different than that. That's what was in my head.

Student N also had difficulty explaining his answer on part (b):

Int: You said here [b] that this point represents 2.25 seconds. Where did you get the 2.25?

N: Umm… because, I considered that [x-axis] as a second.

Int: Okay, well this one [y-axis] looks like it might be at about 2.25, is that where you got it from?

N: Actually, I'm getting confused.

Int: That's okay. You can think for a minute. It's confusing.

N: Yeah…I believe, no no sorry, cause I thought it said 2.5 and I just, I don't know what I was thinking, but this one should be the seconds [y-axis] and this one should be the…cause I tried ummm, figuring out that this is 2.25, I'm not sure if that's exactly accurate, but…

Int: Well, that's fine, don't worry about that part, but you're thinking that the [y-axis] is what?

N: This is like the time [points to y-axis], and this is the distance that they are walking on [points x-axis].

Even after reading and re-reading the question, Student N wanted to put time along the x-axis and assume that the problem allowed him to "see" the situation and show him the meeting point of Fred and Bill. Having time in a different role than the students were used to led to a great deal of confusion and frustration.

5. Reflections and Further Research

This research was an attempt to understand difficulties students had in dealing with time as a variable within a function. It is clear from the problems on the assessment and the statements students gave in the interviews that there are certain things they expect to occur in a function involving time. These difficulties can be linked to other mathematics education research in that they deal with erroneous concept images, action instead of process views, and the inability to view time as a covarying variable. I propose that one reason for some of the assumptions they make and for some of the difficulties they encounter is that they have preconceptions about the entity of time itself.

I compiled a list of what I have found within the work of philosophers to be common conceptions about time. The four preconceptions that I defined linked multiple times to answers given during interviews, particularly: time as a "constant", time moving horizontally, time moving continuously, and time as an "independent" variable. It is clear that students hold these preconceptions, though it is not clear where these strongly-held beliefs originally came from and how they remained ingrained into college students after years of mathematics education.

Whether the common conceptions about time are accurate or inaccurate philosophically is of little consequence. It is, instead, important to realize that these beliefs do exist and that some students shape their concept images of functions involving time around them. For some, these beliefs lead to mistakes, assumptions, and a lack of effort in truly understanding a situation described by a function.

Further research in this area would be beneficial to understand exactly which conceptions most students believe regarding time. An understanding of this could potentially help teachers to emphasize that in a math problem, it is beneficial to set aside certain beliefs and attempt to understand what the function is truly representing.

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7. Plotinus, Time and Eternity, in *The Six Enneads*, Translated by S. Mackenna, in *The Philosophy of Time, A Collection of Essays*, R. M. Gale, Editor, MacMillan, London, 1968, p. 24-37.

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Appendix

The following is the assessment used in this research. Questions 1-5 were composed by Dr. Marilyn Carlson for her research on students' conceptions of functions **[4]**. Kate Mullen created Question 6, and Dr. Patrick Thompson wrote Question 7.

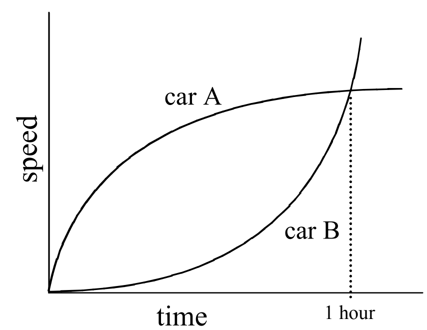
Functions Assessment

Name____

MAT 117 Instructor_____________________________

Please show your work on the following problems.

The given graph represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.) Use this information and the graph to the right to answer items 1 and 2.



1. What is the relationship between the ***position*** of car A and car B at $t = 1$ hr.?

   f) Car A and car B are colliding.
   g) Car A is ahead of car B.
   h) Car B is ahead of car A.
   i) Car B is passing car A.
   j) The cars are at the same position.

2. What is the relationship between the ***speed*** of car A and car B at $t = 1$ hr.?

   f) Car A is going faster than car B.
   g) Car B is going faster than car A.
   h) The cars are traveling at the same speed.
   i) Car B is catching up to car A.
   j) Not enough information.

3. A hose is used to fill an empty wading pool. The graph shows volume (in gallons) in the pool as a function of time (in minutes). Which of the following defines a formula for computing the time, t, as a function of the volume, v?

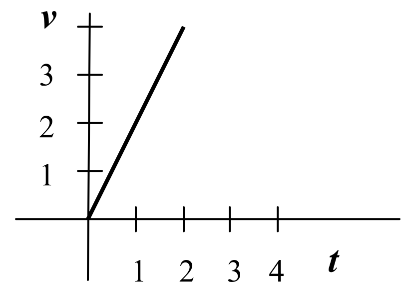
f) $v(t) = \frac{t}{2}$

g) $t(v) = 2v$

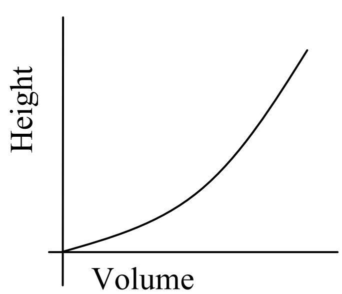
h) $t(v) = \frac{v}{2}$

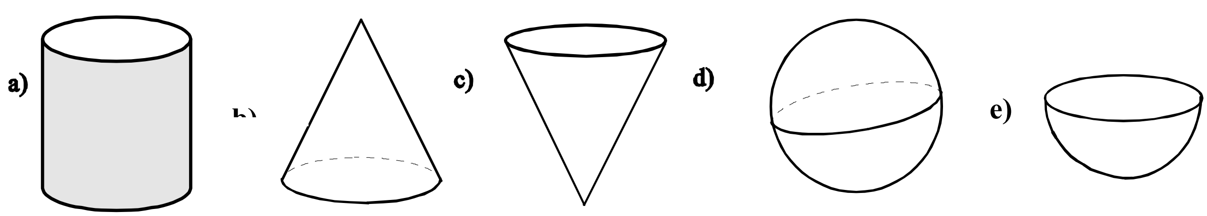
i) $v(t) = 2t$

j) $t(v) = v - 1$



4. The following graph represents the height of water as a function of volume as water is poured into a container. Which container is represented by this graph?





5. A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 5 cm per second. Express the area, *A*, of the circle in terms of the time, *t*, (in seconds) that have passed since the ball hits the lake.

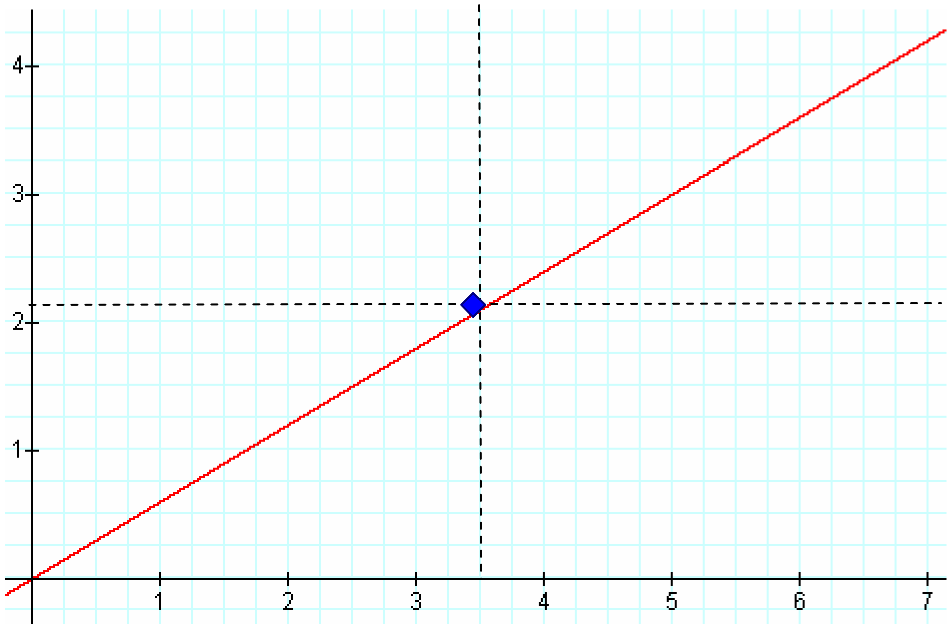
f) $A(\text{t}) = 25\pi t$
g) $A(t) = \pi t^2$
h) $A(t) = 25\pi t^2$
i) $A(t) = 5\pi t^2$
j) $A(t) = 10\pi t$

6. Water is dripping from a leaky faucet into a cup at a constant rate all night. At certain points during the night, a man is awakened from his sleep by the dripping. At those moments, he records the height of the water in the glass as well as the time he awoke. In the morning, he wants to know what time it was when the height of water was at 10 cm. What time was it?

a) 12:00am
b) 12:35 am
c) 12:45 am
d) 1:00 am
e) 1:30 am

Height of Water (cm)	Time
1 cm	10:30 pm
3 cm	11:00 pm
9 cm	12:30 am
14 cm	1:45 am
16 cm	2:15 am
24 cm	4:15 am
33 cm	6:30 am

7. Bill and Fred started from the same spot and walked in the same direction. Bill walked at a rate of 5 ft/sec. Fred walked at a rate of 3 ft/sec. The graph below shows Fred's distance in feet from the start (along the *y*-axis) in relation to Bill's distance in feet from the start (along the *x*-axis). One point on the graph is highlighted.



a) What does this point represent?

b) How many seconds does this point represent?