A CASE STUDY OF A SECONDARY MATHEMATICS TEACHER'S UNDERSTANDING OF EXPONENTIAL FUNCTION: AN EMERGING THEORETICAL FRAMEWORK

by

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ABSTRACT

This dissertation study describes a case study investigating a secondary mathematics teacher's ways of thinking as he worked through a collection of activities related to the concept of exponential function. The research instruments were designed to explore his ability to reason multiplicatively in the context of exponential behavior. Although previous research studies have examined students' multiplicative reasoning abilities, there have been no reported findings on secondary teachers' thinking about exponential behavior from a multiplicative perspective. Furthermore, the current body of research is devoid of a framework for knowing and learning exponential function. The study addresses this research gap by providing an account of one teacher's thinking through the lens of a developed exponential function framework.

Answers to the following research questions were sought: (a) What conceptions does a secondary mathematic teacher hold about exponential growth and decay?; (b) How effective are the current attributes of the developed exponential function framework in explaining the ways of thinking about exponential function?; and, (c) How does an instructional unit – focused on the concept of exponential function with an emphasis on multiplicative reasoning – facilitate the development of an understanding of exponential function? These research questions were approached using a cognitive constructivist perspective through teaching experiment methodology and semi-structured interview techniques.

Analysis of the data yielded varying ability levels for quantitatively representing and qualitatively describing exponential behavior. The ability to reason through exponential situations requires a mature understanding of constant percent

(multiplicative) change, recursion, and language to describe exponential notation. Furthermore, well-formed conceptions of linear function can provide *conceptual springboards* for advancing the primitive understanding of exponential function. Lastly, viewing exponential behavior as a recursive *process*, rather than as recursive *actions*, can lead naturally to ideas of covariation and multiplicative change for building a conceptual understanding of exponential behavior.

Based on the findings, the study calls for an increased emphasis on developing both frameworks and curricular materials that are research-based – and that focus on improving teachers' and students' ability to reason multiplicatively in the context of exponential function. This study proposes an Exponential Function Framework for analyzing conceptions of exponential function.

To my grandparents, Bernice and O.H. Willis, Jr., who provided me the most beautiful set of wings to navigate through life. I love you and miss you dearly.

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CHAPTER 1: INTRODUCTION

This dissertation study describes a case study investigating a secondary mathematics teacher's ways of thinking about exponential function and his development of knowledge as he worked through a collection of activities related to ideas of exponential function. In this chapter, I provide a brief outline of the existing body of research relative to exponential function, along with the guiding research questions and motivation for conducting this investigation. I also discuss the theoretical underpinnings, based on relevant research, which serve as the foundation for this investigation. Lastly, an overview of the study is presented.

Exponential function is an important function concept in algebra because it provides rich learning opportunities for developing students' multiplicative reasoning abilities. The National Council of Teachers of Mathematics (NCTM) *Standards* (2000) advocate that high school algebra students generalize patterns among functions defined both explicitly and recursively. NCTM specifically indicates that understanding exponential behavior, such as constant percent change, is vital for making sense of real-world situations. Confrey (1994) states, "exponential functions offer a unique opportunity to explore the relationship between mathematics and nature and, in doing so, can make mathematics relevant and accessible" (p. 294). Furthermore, Confrey also contends that current instructional treatments often contribute to the general population's deprived understanding of exponential growth and she argues for the significance of understanding the exponential function concept for an educated community.

Statement of the Problem

Despite the significant progress made by NCTM and mathematics education researchers to illuminate the importance of emphasizing exponential behavior (Bartlett, 1976; Confrey, 1994; Confrey & Smith, 1994, 1995), little change in curriculum and instruction has occurred in U.S. algebra classrooms of the 21st century. Changes in our K-14 curriculum must include bridging the gap between viewing multiplication as solely repeated addition to other ways of viewing multiplication in terms of growth, decay, scaling and magnification (Confrey, 1994; Confrey & Smith, 1994) and proportionality (Thompson & Saldanha, 2003). Expanding students' notions of multiplication will advance later notions of ratio and proportion which are foundational for understanding the concept of exponential function (Confrey, 1994; Thompson & Saldanha, 2003).

While researchers and national mathematical organizations agree that understanding exponential behavior is vital for the citizenry of our nation, little evidence exists that these ways of thinking about exponential behavior play an integral part of high school algebra courses. Furthermore, no empirical studies have reported on the ways in which high school mathematics teachers reason about exponential behavior. Research devoted to infusing high school mathematics curriculum with multiplicative change and exponential behavior should be a primary concern for mathematics and teacher educators (Thompson, 1994c; Thompson & Saldanha, 2003). In order to accomplish this task, researchers need to also investigate *teachers'* ways of thinking about exponential function. This gap in research knowledge is addressed in this investigation by studying one teacher's ways of thinking – and emerging understanding of – exponential function.

While the NCTM *Standards* (2000) recognize exponential behavior as an important competency among high school students, some state-level standards, such as Arizona high school algebra standards, ignore this function family entirely[1]. Unless students and teachers include concepts in their curriculum that go beyond the state-level standards, students may exit high school mathematics without any explicit instruction on exponential behavior. Investigating how teachers understand exponential behavior is imperative in working towards building a solid foundation and creating motivation for emphasizing exponential functions in high school mathematics classrooms. This dissertation study addresses the gap in the body of research literature and promotes the necessity for infusing ideas surrounding exponential behavior into secondary mathematics as part of the state standards and, more importantly, into the classroom practices of secondary mathematics teachers.

Statement of the Research Questions

This dissertation study investigates a secondary mathematics teacher's ways of thinking about the concept of exponential function. The main research questions driving this study are:

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[1] Arizona's current high school mathematics standards include topics through the 10th grade level given that Arizona students are required to complete only two years of high school mathematics. Beginning fall 2008, incoming high school freshman in Arizona will be required to complete three years of secondary level mathematics, while the incoming freshman class of 2009 will be required to complete four years of secondary level mathematics. The Arizona Governor's P-20 Council is currently revising the state mathematics standards to accommodate this increase in the number of high school mathematics credits required for graduation. The proposed changes to the standards include the topic of exponential functions with an emphasis on compound interest.

1. What conceptions does a secondary mathematics teacher hold about exponential growth and decay?
2. How effective are the current attributes of the developed exponential function framework in explaining the ways of thinking about the concept of exponential function? What revisions to the framework are necessary for capturing the important ways of thinking and reasoning about exponential function?
3. How does an instructional unit – focused on the concept of exponential function with an emphasis on multiplicative reasoning – facilitate the development of a secondary mathematics teacher's understanding of exponential function?

Motivation for the Study

My motivation for investigating a mathematics teacher's conceptions of exponential function stems from experience in learning and teaching this topic. My personal experience is that many teachers cover exponential functions as a precursory approach to introducing logarithmic functions. Teachers using this treatment generally provide little emphasis on the multiplicative structures of such exponential behavior. This type of instructional treatment focuses on graphing various exponential functions, such as $f(x) = 2^x$ or $f(x) = \left(\frac{1}{2}\right)^x$, without emphasizing the multiplicative structure of exponential functions. Other aspects of exponential behavior, such as developing the ability to reason covariationally or proportionally, are largely ignored in conventional curricula (Confrey & Smith, 1994; Thompson & Saldanha, 2003).

In my own mathematics training, my high school teachers and college professors did not facilitate students' understanding of multiplicative structures and covariation. My educational experience left me ill-equipped for teaching the exponential function concept in a way that promoted deep, conceptual connections of the multiplicative structure. My recently acquired awareness and knowledge of alternative ways of thinking about exponential function have fueled my passion to investigate other teachers' ways of thinking about this topic as a means for enhancing curriculum and instruction.

Theoretical Perspective

This dissertation study is situated within two main theoretical perspectives: pragmatism and cognitive constructivist theory. The pragmatic perspective framed the investigation's quest for *truth* and its relation to knowledge from an empirical as well as a rational point of view. A pragmatic perspective provides a lens for learning about how humans grow through experience and how they obtain knowledge that becomes their own personal reality. The result of knowledge and truth acquisition occurs through a process known as inquiry where learning takes place through action.

The design and implementation of this study adopted a pragmatic worldview in conjunction with a cognitive constructivist perspective. Relative to Piaget's (1970) theory of learning, cognitive constructivism is the philosophical belief that learning occurs through experiences and action, rather than through knowledge passed on by others. Cognitive constructivists focus on describing the mental images of an individual to learn the individual's ways of thinking and approaches to problem situations (Piaget, 1970; Steffe & Thompson, 2000). The assumption on learning is that knowledge is gained through a process involving dynamic events and actions set forth to promote change (Steffe & Thompson, 2000). Learners then create their own meaning through the process of sense-making (Piaget, 1970). Schoenfeld (1987) describes this constructivist perspective by saying, "we all build our own interpretive frameworks for making sense of the world, and we then see the world in the light of these frameworks. What we see may or may not correspond to 'objective' reality" (p. 22).

The methodological motivation for conducting this present study stemmed from Steffe and Thompson's (2000) work on conducting teaching experiments in mathematics education research and Thompson's (2000) insights on conducting conceptual analyses. The purpose of this study was to create models of *teachers' mathematics*[2] based on local hypotheses of teachers' mathematical understanding as they completed tasks involving exponential functions. Teachers' mathematics refers to teachers' individual understanding, which is unknown and obscure to the outsider and "is indicated by what they say and do as they engage in mathematical activity" (Steffe & Thompson, 2000, p. 268). This study attempts to connect the aspects of teachers' thinking to create models that describe teachers' mathematics. In this study, these models are referred to as *mathematics of teachers*.

Background for the Investigation

The present investigation into a secondary mathematics teacher's ways of thinking about – and emerging understanding of – exponential function is guided by

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[2] Steffe and Thompson (2000) discuss their notion of mathematics of students which are models of their interpretations of the students' mathematics. These interpretations of students' mathematical reality become models of student thinking (mathematics of students). This approach to research provided the foundation for the present dissertation investigation where the emphasis was on teachers and developing models of the mathematics of teachers.

research on multiplicative reasoning (Confrey, 1994; Confrey & Smith, 1994, 1995; Thompson & Saldanha, 2003; Vergnaud, 1994), proportional reasoning (Thompson & Saldanha, 2003), and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Carlson, Larsen, & Jacobs, 2001; Confrey & Smith, 1995; Rizzuti, 1991; Saldanha & Thompson, 1998; Silverman, 2005; Thompson, 1994a, 1994c). Researchers have proclaimed that these reasoning abilities are fundamental to understanding the concept of function (Carlson *et al.*, 2002; Confrey & Smith, 1994; Thompson, 1994b, 1994c) in preparation for calculus topics such as the Fundamental Theorem of Calculus (Thompson, 1994a), derivative (Zandieh, 2000), and limit (Cottrill *et al.*, 1996).

The purpose for the present study is to gain knowledge about a teacher's ability to reason multiplicatively through exponential situations. Multiplicative reasoning in the context of exponential function entails the ability to think about (a) the proportionality of the function parameters, (b) the recursive structure of the situation, and (c) the constant proportionality of output values.

Another reasoning ability found to be vital in developing an understanding of exponential function is the ability to reason covariationally. That is, the ability to coordinate changes in the dependent variable in tandem with changes in the independent variable (Carlson *et al.*, 2002). In their study, Carlson *et al.* (2002) found that even high-performing second semester calculus students had difficulty engaging in covariational reasoning when attempting to investigate dynamic function events. "They appeared to have difficulty characterizing the nature of change while imagining the independent variable changing continuously" (p. 373). As a result of these findings, Carlson *et al.*

(2002) suggest that it is important to "monitor the development of students' understandings of function and their covariational reasoning abilities prior to and during their study of calculus" (p. 374). Despite the arguments and research findings, covariational reasoning used as a tool for developing the concept of function has been largely ignored in conventional curricular materials permeating our secondary and post-secondary mathematics classes (Lloyd & Wilson, 1998; Rizzuti, 1991; Thompson, 1994a).

Overview of the Study

This dissertation study investigates the ways in which one teacher thinks about exponential behavior – both quantitatively and qualitatively – and reports on the emerging understanding of this teacher, named Ben. At the time of the study, Ben was participating in a 15-week graduate-level mathematics education course designed to further teachers' knowledge of the concept of function. The investigation began with a pre-test assessment followed by task-based interviews and an in-depth teaching experiment focused on the concept of exponential function. The study focused on Ben's ways of thinking about exponential behavior as he worked through a collection of activities. This analysis facilitated the development and refinement of a framework designed to describe the notation, language and reasoning abilities for knowing the exponential function concept.

The main conclusions resulting from this study influenced the development of the exponential function framework. This refined framework provides a lens for exploring the concept of exponential behavior and for developing meaningful curricula designed to emphasize multiplicative reasoning. This discussion provides details of the framework, including initial renditions of the framework, which guided the implementation and analysis of the study.

The data suggest that a mature understanding of linear function serves as a *conceptual springboard* for making sense of multiplicative situations and it provided powerful avenues for building strong conceptions of exponential function. In addition, the data suggest that thinking about exponential situations as *recursive processes* requires the ability to mentally imagine the indefinite actions placed on the *recursive objects*[3] along a continuum of values. The ability to act upon a finite number of recursive objects does not translate to the ability to reason multiplicatively throughout the problem situation. Finally, the data suggest that conceptualizing b^x as x factors of b for rational values of x provides powerful opportunities to think about fractional exponents as partial factors (i.e., $3^{1/2}$ represents ½ factors of 3).

Chapter 2 provides a detailed overview of the research literature on multiplicative reasoning, proportional reasoning, and covariational reasoning, which guided the implementation and analysis of this study. Chapter 3 presents the initial exponential function framework along with a brief discussion of the Exploratory Study and findings. Chapter 4 outlines the theoretical perspective and revised exponential function framework. Chapter 5 follows with a discussion on the methodology for data collection and data analysis for the study. Chapter 6 provides the results and analysis of the pre-test

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[3] Recursive object is defined by Thompson (1985) "any object which contains an instance of itself as a component" within the recursive process. He defines recursion to indicate the collection of recursive processes and recursive objects.

assessment and interviews, while chapter 7 presents the results and analysis of the teaching experiment conducted with Ben. Finally, chapter 8 closes with a discussion about the general and specific conclusions of the study, along with a description of the limitations of the study and the contributions to research, curriculum and instruction resulting from this investigation.

CHAPTER 2: REVIEW OF THE LITERATURE

Introduction

This chapter presents a synthesis of the relevant research literature on knowing and learning exponential function. Three main areas of research relative to the exponential function concept provided the theoretical and philosophical motivation for this investigation: (a) multiplicative reasoning, (b) covariational reasoning, and (c) exponential function reasoning. First, the chapter begins with research ideas related to multiplicative reasoning, including ratios and rate of change. Second, the chapter continues with a discussion on covariational reasoning as a tool for understanding exponential function. Finally, the chapter concludes with a brief review of literature involving linear function with a focus on reasoning additively and proportionally as a way for building connections to exponential function.

While this body of research focuses on students' ability to reason multiplicatively and covariationally, there is no research devoted to investigating teachers' ways of thinking about exponential function using these reasoning constructs. This dissertation builds upon the research provided in this literature review by investigating a teacher's reasoning abilities and emerging understanding of exponential function relative to multiplicative, proportional, and covariational reasoning. Furthermore, the chapter discusses linear function understanding, which lays the groundwork for findings of this investigation.

Multiplicative Reasoning

This dissertation investigation is grounded in two main research areas related to multiplicative reasoning: (a) multiplicative conceptual field as a theoretical perspective on multiplicative reasoning and (b) multiplicative constructs (i.e., splitting, unitizing, rate of change, and partial-interval reasoning) as a framework entailing proportional reasoning and recursion for investigating understanding of exponential behavior. The next sections discuss each of these research areas.

Multiplicative Conceptual Field

Rooted in the theory of pragmatism, the core idea of conceptual field theory is that knowledge has many local facets that are molded and connected by mastering problems that are related conceptually (Vergnaud, 1994). One theory, in particular, that has captured the attention of mathematics education researchers is multiplicative conceptual field (MCF) theory, which addresses the ability to reason multiplicatively. Multiplicative reasoning plays a significant role in the development of conceptual understandings of multiplication, division, fractions, proportions, ratios, rates, rational number, and linear functions (Vergnaud, 1994). It also serves as a gateway for building a mature understanding of exponential function. Vergnaud refers to these topics as ingredients for the MCF which he defines as a complex theory that is "simultaneously a bulk of situations and a bulk of concepts" (p. 46) where concepts (tools for analyzing situations) become meaningful through situations (things that require multiplicative operations) and situations are analyzed through these multiple concepts. This reflexive relationship between concepts and situations provides the basis of the conceptual field theory. MCF is a broad theme that extends from developing multiplication skills to building ratios and rates to understanding multiplicative growth or decay patterns. The theme is complex and Vergnaud argues that the operations of multiplication and division are "only the most visible part of an enormous conceptual iceberg" (p. 47).

Greer (1994) points out that a disproportionate amount of research has focused on the development of multiplication, the earliest stage within the conceptual field, rather than on the more advanced elements within the MCF, such as exponential growth. In contrast, Greer advocates investigations at every stage within the MCF. He argues that mathematics educators and researchers should attend not only to the vertical mathematization of concepts, but also to the horizontal extension where mathematics instruction exposes children to a variety of multiplicative situations. Other research calls for more emphasis on the development of students' multiplicative reasoning structures and de-emphasis on algorithmic approaches and formulas (Confrey & Smith, 1995; Kieren, 1994; Thompson & Saldanha, 2003). This perspective on research related to multiplicative situations informed this dissertation study and provided the motivation for addressing the research gap on knowing and learning exponential function.

Multiplicative Constructs

A common (mis)conception among children as they learn multiplication is the assumption that "multiplication makes bigger and division makes smaller" (Greer, 1994; Simon & Blume, 1994; Sowder *et al.*, 1998). Greer (1994) maintains that overcoming such a conceptual obstacle requires a major conceptual reconstruction. There are instances where this fallacy complicates further development of multiplication and other components within the conceptual field. Nesher (1987) offers a possible explanation for the existence of this fallacy by commenting that counterexamples of "multiplication makes bigger and division makes smaller" are rarely presented in mathematics textbooks. For example, many mainstream algebra textbooks focus on examples of exponential growth while minimizing the amount of exponential decay situations they provide for students (see Larson & Hostetler, 2004). This lack of emphasis on exponential decay could potentially further students' misunderstanding of multiplication and exacerbate the (mis)conception that "multiplication makes bigger and division makes smaller." As a means for developing a mature understanding of the conceptual field, researchers have argued for alternative ways of thinking which promote sophisticated ideas of multiplication and develop powerful ways of conceptualizing multiplicative structures, such as splitting, unitizing, and rate of change (Confrey, 1994; Confrey & Smith, 1995; Lamon, 1994; Thompson, 1994c). The next sections discuss these three multiplicative constructs.

Splitting

Many researchers have argued that thinking of multiplication as solely repeated addition is limiting for students as they encounter multiplicative situations requiring more powerful conceptions, such as exponential growth (Confrey & Smith, 1994, 1995; Thompson & Saldanha, 2003). As Confrey and Smith (1995) maintain, another way of viewing multiplication as an extension of the notion of repeated addition is to consider the idea of *splitting* as a primitive model for multiplication and division which also serves as a precursor to ratio. Confrey (1994) defines splitting as "an action of creating equal parts or copies of an original" (p. 300). Confrey further argues that the idea of splitting is a natural construct for children when structuring problem situations around "magnification, similarity, sharing, halving, and doubling" (p. 67).

One example of exponential behavior, described by Confrey and Smith (1994), is cell population growth. If the population of cells is counted every hour and it is observed that the population each hour is three times the population of the previous hour, then it is imagined that each individual cell splits into three cells after one hour. This split repeats each hour where each new cell continues to split into three cells. Confrey refers to this idea as a 3-split (see Figure 1).

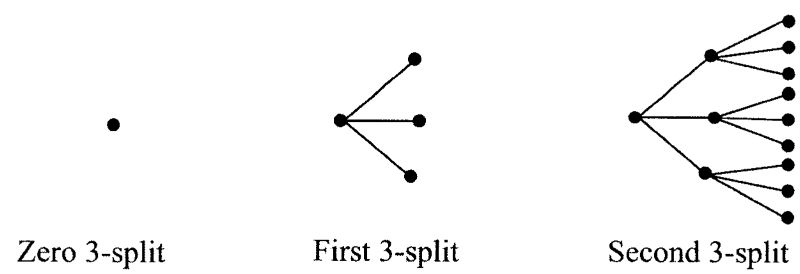

Figure 1. Tree diagrams illustrating 3-splitting.

The multiplicative unit in this example emerges as "3 times as many" which is an invariant quantity between the previous value and the successive value and remains invariant throughout the situation. The next section expands on this notion by focusing on unitizing as an extension of splitting.

Unitizing

Lamon (1994) advocates the concept of unitizing as a cognitive foundation for learning multiplicative structures. She found that "the ability to construct a reference unit or a unit whole, and then to reinterpret a situation in terms of that unit, appears critical to the development of increasingly sophisticated mathematical ideas" (p. 92). Lamon believes that unitizing necessarily involves advanced thinking since it requires knowledge of the whole and of the individual parts composing the unit.

Confrey and Smith (1995) extend this argument of unitizing in the context of exponential function as creating a multiplicative unit, or a unit created from an *n*-split where the unit is a split by the value *n*. In essence, unitizing is an extension of the primitive notion of splitting. In the example given earlier about cell population growth, the multiplicative unit is three given that the situation involved a 3-split or a tripling of the number of cells each hour. They contend that the repeated action of splitting generates a *recursive view* of the multiplicative growth process where the object of multiplication represents the unit. Confrey and Smith claim that the idea of splitting and multiplicative units for introducing function is most effective for building a mature understanding of exponential behavior.

The splitting and unitizing model provided insight into this investigation for exploring a teacher's ways of thinking about the progression of output values for a given exponential function situation. However, limitations emerged from the splitting model when considering situations involving rational values of splits (e.g., ½-split). For example, conceptualizing the expression $2^{1/2}$ requires a more advanced way of thinking beyond what the splitting model provides. A thorough review of the research literature illuminated a knowledge gap for understanding ways of thinking about fractional exponents. This dissertation study investigated the usefulness of an extended model for thinking about fractional exponents: partial-interval reasoning. Later sections of this chapter discuss partial-interval reasoning in more detail. For now, the focus continues with the notion of unitizing as a way of thinking about rate of change and exponential behavior.

Rate of Change and Quantitative Operations

Knowing and learning exponential functions hinges upon the ability to reason multiplicatively when grappling with the notion of rate of change (Confrey, 1994; Confrey & Smith, 1994). The concept of rate has been the focus of several empirical studies and theoretical manuscripts by mathematics education researchers (Adamson, 2005; Coe, 2007; Thompson, 1994a, 1994c). Thompson's (1994c) work on the concept of rate, for instance, provides a way of thinking about rate as emerging from the development of ratios. He discusses the process for building ratios as beginning with a mental operation of conceiving one quantity in relation to another, which he calls a *quantitative operation*. "A quantitative operation is nonnumerical; it has to do with the comprehension of the situation" (pp. 187-188). His motivation for emphasizing the nonnumerical characteristic of quantitative operations is two-fold: (1) to distinguish this operation from a numerical operation involving arithmetic and (2) to point out the complexity of mentally imagining the situation of the given context. Thompson also points out that "A quantitative operation creates a structure – the created quantity in relation to the quantities operated upon to make it" (p. 185). The mental structure created as a result of a quantitative operation ultimately supports images of other numerical operations (e.g., ratio).

An example that illustrates this idea involves recognizing the pattern of doubling given a set of data. After examining the data, the existence of a common ratio of 2:1 was apparent and it was determined that the data represented an exponential function. Thompson defined a quantitative operation for this situation as "the scheme of thinking that supports having an image of successive pairs of successive states of something that grows 'continuously' so that its successive states are always in the ratio 2:1" for an interval of size one unit (personal communication, March 3, 2006). Successive states represent successive intervals whose ratios of output values are *proportional* to the ratio of output values of another interval of the same size. The graph in Figure 2 illustrates Thompson's idea by using the values obtained from the function $f(x)=2^x$. For every 1.5 units added to the input value, a multiplicative unit of $2^{1.5}$ is multiplied to the previous output value to obtain the successive output value. The interpretation of building a ratio to describe the exponential behavior exhibited in this example requires proportional reasoning of the situation, which begins an assembly of the ability to reason covariationally.

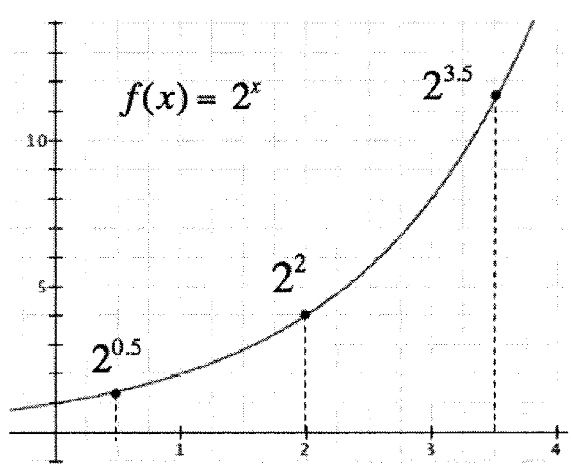

Figure 2. Graph of $f(x)=2^x$.

This idea represents the invariant quantity that proportionally relates one output value to the preceding output value *and* is simultaneously proportional to the invariant quantity relating a second successive set of output values. This idea illustrates that, for an exponential function of the form $f(x) = ab^x$, then

$b^{x_m - x_n} = b^{x_p - x_q} \Leftrightarrow \frac{f(x_m)}{f(x_n)} = \frac{f(x_p)}{f(x_q)}$ where $|m-n| = |p-q|$ where the ratio of two output values remains in a constant proportional relationship with successive output values for uniform interval sizes.

Rate, as defined by Confrey and Smith (1994), is a "unit per unit comparison" (p. 153) with emphasis given to the idea that the unit of both numerator and denominator of the rate quantity expresses some repeated action. The notion of a repeated action gives rise to more advanced notions of recursion (Confrey & Smith, 1995). In essence, Confrey and Smith's image of rate of the exponential function concept emerges as a multiplicative unit constructed from the output values compared multiplicatively with the additive unit constructed from the input values. Previous investigations by Confrey and Smith led them to view rate of change of exponential function as a gateway for students as they developed their understanding of exponential behavior. In addition, they advocate a more qualitative approach – specifically, a covariation approach to function – for developing a rate of change understanding.

Contrasting Thompson's (1994c) view of ratio as a comparison of two quantities multiplicatively, Confrey and Smith (1994) assert that Thompson's notion of ratio restricts the concept of rate to additive situations (e.g., speed as distance per unit time). Confrey and Smith's definition of rate offers alternative ways of constructing units in the multiplicative world. While Confrey and Smith agree with Thompson that the notion of rate is essential for developing an understanding of function, their images of rate are rather different but both powerful when illuminating attributes of exponential behavior. Thompson's perspective on rate provides the motivation for conceptualizing exponential behavior as a *proportional relationship*. Conversely, Confrey and Smith's perspective on multiplicative unit derived from primitive notions of splitting and unitizing provide the foundation for conceptualizing exponential behavior as a *recursive process*. Both images of rate as a proportional relationship and multiplicative unit comparison lead naturally into building the necessary tools for reasoning covariationally in function situations. Another reasoning ability that facilitates the development of a mature understanding of exponential function is the ability to reason multiplicatively for fractional input values. This discussion continues with the notion of partial-interval reasoning as a construct for developing the ability to reason multiplicatively.

Partial-Interval Reasoning

Returning to the exponential function, $f(x) = 2^x$, it is also important to discuss output values for the function where the inputs are non-integer rational values (e.g., 0.5). Thompson (personal communication, March 3, 2006) has argued than an understanding of quantitative operations in exponential situations should entail a "scheme of thinking that supports having an image of successive pairs of successive states of something that grows 'continuously' so that its successive states are always in the ratio 2:1" for an interval of size of one unit. However, the question remains: what is the nature of the scheme of thinking for understanding this idea when the interval size is not equal to one unit (e.g., interval size of 0.5 units)?

Table 1 illustrates the multiplicative unit constructed by multiplying the previous output value by $\sqrt{2}$ for every 0.5 units increased in the input values. This action produces successive states that remain in constant ratio of $\sqrt{2}:0.5$. Since there seems to be no research evidence – neither for teachers nor for students – about how they come to understand this particular notion of exponential behavior, the present dissertation study investigated the various ways of thinking within this construct to fill the gap in the current body of research.

Table 1

Ratio Table of $f(x) = 2^x$

ΔInput	Input	Output	First Ratios ®*f(x)*
	0.0	1.0	
+ 0.5			$\times \sqrt{2}$
	0.5	$\sqrt{2}$	
+ 0.5			$\times \sqrt{2}$
	1.0	2.0	
+ 0.5			$\times \sqrt{2}$
	1.5	$2\sqrt{2}$	
+ 0.5			$\times \sqrt{2}$
	2.0	4.0	
+ 0.5			$\times \sqrt{2}$
	2.5	$4\sqrt{2}$	
+ 0.5			$\times \sqrt{2}$
	3.0	8.0	

The ability to reason multiplicatively must encompass the ability to consider multiplicative comparisons for input intervals other than one unit. Developing a mature understanding of exponential function entails the ability to fluently imagine the function

$f(x) = 2^x$ growing by a multiplicative factor of two for every one unit added in the input. Furthermore, a mature understanding of exponential function extends to imagining that some other constant multiplicative factor relates two output values for intervals other than one unit (e.g., a multiplicative factor of $\sqrt{2}$ corresponds to an interval of size 0.5 units). The guiding philosophy for coordinating the decreasing interval size with the newly formed multiplicative factor allows for the emergence of covariational reasoning to facilitate in developing deep understandings of exponential behavior.

This discussion now turns to the notion of covariational reasoning as a tool for making sense of exponential function.

Covariational Reasoning

The present investigation encompasses covariational reasoning as a tool for developing a mature understanding of exponential behavior. The next section focuses on research organized in two main categories: (a) the role of covariational reasoning in learning *function* and (b) the role of covariational reasoning in learning *exponential function.*

The Role of Covariational Reasoning in Learning the Function Concept

Early definitions of function included the dependency relationship among variables and constants. First published in 1748, Euler's *Introductio in analysin infinitorum* defines function as an analytic expression composed of variables and constants:

> A function of a variable quantity is an analytic expression composed in any manner whatever of this variable and constants (attributed to Leonhard Euler, 1748, by C. B. Boyer, 1946, p. 12).

Over time, this definition evolved into the more set-theoretic modern perspective. This adaptation of the function definition builds upon set theory and focuses on the correspondence relationship between the domain values and the range values. In the early 1800s, Dirichlet revised the definition of function to the succinct correspondence-laden definition used in traditional curriculum in the United States:

> y is a function of x if for any value of x there is a rule which gives a unique value of y corresponding to x (attributed to P. L. Dirichlet, 1837, by M. A. Malik, 1980, p. 491).

Several researchers, whose focus has been concentrated on the concept of function, have argued the powerfulness of covariational reasoning for developing robust conceptions of function (Carlson *et al.*, 2002; Confrey & Smith, 1995; Cottrill *et al.*, 1996; Rizzuti, 1991; Thompson, 1994a, 1994b; Zandieh, 2000). Confrey and Smith (1995) argue that covariation, as an alternative way of thinking about the concept of function, offers a more qualitative approach to describing in general the changes of one quantity in relation to another quantity.

While the definition of covariation during Euler's time was loosely associated with a relationship of variables and constants (Boyer, 1946), researchers have expanded upon this original conception to focus on the interplay of variables in relation to one another. Saldanha and Thompson (1998) extend previous notions of covariational reasoning (Confrey & Smith, 1994; Thompson, 1994a) to include ways of thinking that are more imagistic:

> Covariation is of someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (Saldanha & Thompson, 1998, p. 298)

Their notion of covariation is developmental, beginning with covariation as discrete images and then building to more mature images of covariation involving time as a continuous quantity. According to Saldanha and Thompson, "In the case of continuous covariation, one understands that if either quantity has different values at different times, it changed from one to another by assuming all intermediate values" (p. 299). This evolution of the definition of covariation from the ideas of Euler in the 18th century facilitates in building a more powerful way of thinking about changes in one variable relate to changes in another variable than emphasizing the set-theoretical definition of function.

Carlson *et al.* (2002) drew upon the work on covariation by Confrey and Smith (1994, 1995) and Saldanha and Thompson (1998) by defining covariational reasoning as the "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). This extension of the definition of covariation to entail the ability to reason covariationally is an elucidation of the mental actions that students experience when learning function. Carlson *et al.*'s (2002) study of covariational reasoning in the context of modeling dynamic events proposes a covariation framework encompassing detailed mental actions that students exhibit as they engage in mathematical activities. Each mental action is associated with a level of covariational understanding. Students obtain higher levels of mental actions once they exhibit the current level as well as lower levels of mental actions. For example, students reason covariationally at Level 4 when they exhibit the mental actions associated with Mental Action 1 (MA1) through Mental Action 4 (MA4). Table 2 illustrates the mental actions of the covariational framework.

Table 2

Mental Actions of the Covariation Framework (Carlson *et al.*, 2002)

Mental Actions	Description of Mental Actions	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other variable	Labeling the axes with verbal indications of coordinating the two variables (e.g., *y* changes with changes in *x*
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	Constructing an increasing straight line Verbalizing an awareness of the direction of change of the output while considering changes in input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	Plotting points/constructing secant lines Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	Constructing contiguous secant lines for the domain Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	Constructing a smooth curve with clear indications of concavity changes Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

The present study employed Carlson *et al.*'s (2002) definition of covariation (i.e., the coordinated image of two variables changing in tandem) and covariational reasoning (i.e., the resulting mental actions and reasoning abilities that students experience and employ while grappling with covariation tasks). These two notions strongly motivated the use of covariation and covariational reasoning in this dissertation investigation.

The Role of Covariational Reasoning in Learning the Exponential Function Concept

Focusing more specifically on exponential function, this discussion now turns to research findings which illustrate that students experience difficulty developing a mature understanding of exponential function and that these difficulties may be connected to impoverished understandings of rate and multiplicative relationships (Coe, 2007; Confrey & Smith, 1994, 1995; Thompson, 1994c; Weber, 2002). Despite the importance of exponential functions and their applications within scientific environments, mathematics curricula in the United States typically present this topic by first introducing formulas and rules of exponents while emphasizing conventional algorithmic methods devoid of context (Confrey & Smith, 1995). Confrey and Smith (1994) argue that this conventional correspondence approach to learning functions, specifically exponential functions, is counterintuitive for students given the emphasis of curricular materials on creating a correspondence between x and y without understanding the progression of the output values in relation to the input values. Confrey and Smith (1994) advocate a more covariational approach to learning function where students are "able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1}" (p. 137) where $y_m = f(x_m)$ and $y_{m+1} = f(x_{m+1})$. This covariational approach is crucial in students' development of a mature understanding of function, specifically linear and exponential functions (Confrey & Smith, 1994, 1995; Wilhelm & Confrey, 2003) and advances students' ability to grasp calculus topics such as the Fundamental Theorem of Calculus (Thompson, 1994a, 1994b), limit (Cottrill *et al.*, 1996) and derivative (Zandieh, 2000).

Saldanha and Thompson's (1998) perspective on covariational reasoning connects to exponential functions and translates to the notion that one quantity has a value which is *proportionally* related to the other quantity. This characterization of simultaneous continuous variation is more descriptive of what students need to imagine as they grapple with the multiplicativeness of output values in relation to the additiveness of input values in the context of exponential function. Moreover, the continuity of covariation allows students to view accumulations of two quantities as accruals over time intervals without discretely imagining the progression of these accumulations.

Confrey and Smith's (1994, 1995) observations of how children wrestle with multiplicative structures motivated their claims that covariation yields an effective approach to learning exponential function. Through the use of data tables, the ability to reason covariationally evolves naturally when making sense of situations represented by data (Confrey & Smith, 1994). As Confrey and Smith proclaim, building the idea of rate of change covariationally based on data tables is a natural progression from intuitive notions of function to more formal conceptualizations. Confrey and Smith (1994) further state:

> We believe that the covariation approach is central to the rate concept. One recognizes that constructing rate involves recognizing the repeated action in each column ([for example] +1 in the *t*-column; *9 in the *c*-column) as a 'unit', understanding that one wants to compare these units as one ®-unit of cells for one

Δ-unit of time, and seeing that as one moves down the unit columns (Δ*t* and ®*c*) one is always comparing one unit to one unit. (p. 153)

The current dissertation study utilized this approach for creating multiplicative units based on data tables exhibiting exponential behavior. The discussion now shifts to relevant research literature on knowing and learning linear function as a means for developing sophisticated notions of exponential behavior.

Linear Function Understanding

This dissertation investigation incorporates the philosophy that a mature understanding of exponential function emerges from comparing and contrasting exponential behavior with linear behavior. Thus, the next section of this chapter focuses on reviewing the relevant research on linear function and rate of change as these ideas connect to knowing and learning exponential function.

Linear Behavior versus Exponential Behavior

Studies have reported that students experience difficulty developing a mature understanding of exponential function that can be attributed to impoverished understandings of rate and multiplicative relationships (Bartlett, 1976; Confrey & Smith, 1994, 1995; Weber, 2002). Upon finding similar results relating to rate of change of linear function, Wilhelm and Confrey (2003) argued for placing more curricular emphasis on the covariational aspect of slope instead of the steepness idea that does not apply to nonlinear situations.

For students to make powerful connections between linear and exponential functions, researchers and reform textbook authors have advocated that teachers should present these two functions consecutively and interchangeably (see Connally *et al.*, 2004; Jacobs *et al.*, 2000). The motivation of this study is to influence the ways in which secondary mathematics teachers situate the concept of exponential function within their curricula and to advance their preparedness to promote conceptual understanding of exponential behavior among their students.

The next discussion focuses on research literature relative to linear behavior. The perspective presented highlights linear behavior as an agency for developing exponential behavior.

Important Aspects of Linear Behavior

One purpose of this dissertation study was to investigate a teacher's ability to reason through multiplicative situations by way of first thinking about the situation in comparison to linear situations (i.e., exploring for exponential growth by considering linear growth). Many researchers have studied students' understanding of linear function (Buck, 1995; Johari, 1998; Lobato, Ellis, & Munoz, 2003), constant rate of change (Stump, 2001; Wilhelm & Confrey, 2003) and rates in general (Thompson & Thompson, 1996; Thompson, 1994a, 1994c; Thompson & Thompson, 1994). Lobato *et al.* (2003), for example, found that students often only attributed changes in the dependent variable to the m value in $y = b + mx$ of a linear relationship, rather than coordinating these changes with changes in the independent variable. Further, their study showed that when students were provided a linear function represented by a table, they tend to view the progression of output values as isolated differences independent of the progression of input values. Their students thought of slope as a *difference* rather than a *ratio.* According to Lobato *et al.*, "One concept fundamental to the understanding of slope is the creation of a ratio of the variation of one quantity to the associated variation of another quantity, where the two quantities covary" (p. 4). Their finding and assertions resonate with Thompson's (Thompson, 1994c; Thompson & Saldanha, 2003, personal communication, December 1, 2007) argument for emphasizing the proportional, rather than additive, aspects of function, specifically linear function. A curricular emphasis on (a) rate of change as a proportion and (b) covariational reasoning is critical for students to "negotiate the quantitative complexity of unfamiliar situations" (p. 2).

Using Linear Behavior to Build an Understanding of Exponential Behavior

A review of research studies in the two areas of linear and exponential functions revealed that few investigations have looked at linking the two function families by comparing additive change and multiplicative change. Most studies, instead, have focused on one specific function family – either linear or exponential – and a few of these studies have offered suggestions on designing curricular activities that promote deeper understandings of these functions (Johari, 1998; Klanderman, 1996; Lobato *et al.*, 2003; Stump, 2001; Vlassis, 2002; Wilhelm & Confrey, 2003).

Chapter Summary

In this chapter, the discussion focused on various constructs for building a mature understanding of exponential function. This mature understanding is crucial in constructing a comprehensive understanding of the function concept, as advocated by NCTM (1989, 2000) and mathematics education researchers (Carlson, 1998; Carlson *et al.*, 2002; Confrey, 1994; Confrey & Smith, 1994, 1995; Rizzuti, 1991; Saldanha &

Thompson, 1998; Thompson, 1994a, 1994b, 1994c; Thompson & Saldanha, 2003; Vergnaud, 1994). Researchers have called for extending the notion of multiplication beyond simply repeated addition (Thompson & Saldanha, 2003) and denouncing the fallacy "multiplication makes bigger and division makes smaller" (Greer, 1994; Simon & Blume, 1994; Sowder *et al.*, 1998).

The importance of emphasizing multiplicative reasoning in students' mathematical development has been echoed by many mathematics education researchers (Confrey, 1994; Confrey & Smith, 1994, 1995; Greer, 1994; Thompson & Saldanha, 2003; Vergnaud, 1994). Nonetheless, current instructional treatments neglect to develop students' ability to reason multiplicatively (Confrey, 1994; Thompson & Saldanha, 2003). Moreover, little research has been conducted to facilitate the establishment of mature understandings of the more advanced components of the multiplicative conceptual field (Confrey & Smith, 1995; Greer, 1994).

Research-based arguments have also highlighted the importance of proportional reasoning when making sense of function contexts (Thompson & Saldanha, 2003). Thompson and Saldanha contend that reasoning multiplicatively facilitates students' image of rate of change along with their ability to reason covariationally in function situations.

The historical evolution of the concept of function provides insight for an alternative way of thinking about function. While modern-day mathematics curricula tend to emphasize the set-theoretical correspondence perspective of function, many researchers have provided guidance in developing covariation as a tool for building an understanding of function (Carlson *et al.*, 2002; Carlson *et al.*, 2001; Confrey & Smith, 1995; Rizzuti, 1991; Thompson, 1994b). These researchers have also argued that covariation should be a theme of mathematics instruction to allow students the opportunities to build covariation constructs as they prepare for calculus.

Emerging from the research literature review are various mathematical constructs. This dissertation study sought to gain knowledge of the ways of thinking about the exponential function concept through the lens of the following constructs:

- Reason multiplicatively and covariationally through exponential situations.
- Reason proportionally as a way of relating two successive output values.
- Understand exponential behavior relative to linear behavior.

The next chapter outlines the initial framework used for investigating these ways of knowing and learning the concept of exponential function conducted in the Exploratory Study.

CHAPTER 3: EXPLORATORY STUDY

This chapter describes an exploratory investigation conducted with four secondary mathematics teachers along with an initial exponential function framework that guided the implementation and analysis of this preliminary investigation. The first section of this chapter discusses the initial framework. The second section of this chapter presents the results and subsequent analysis of the Exploratory Study.

The analysis presented in this chapter focuses on the four teachers' ways of thinking about exponential function as they worked through a collection of tasks focused on exponential behavior. Emerging from the data were three clusters of difficulties that teachers experienced when reasoning through exponential tasks. These clusters of difficulties entail: (a) conceptualizing exponential decay, (b) unpacking ideas about multiplicative growth and decay, and (c) categorizing exponential functions.

Initial Exponential Function Framework

The Research Gap

A synthesis of the research literature (Confrey, 1994; Confrey & Smith, 1994, 1995; Greer, 1994; Kieren, 1994; Thompson, 1994c; Thompson & Saldanha, 2003; Vergnaud, 1994) along with analysis of the Exploratory Study revealed that many learners have not developed a robust understanding of exponential function potentially due to their inability to reason multiplicatively. This impoverished understanding may be due, in part, to the treatment of this topic in K-12 curriculum (Confrey, 1994; Confrey & Smith, 1994, 1995); regardless it leads one to suspect the presence of potentially impoverished understandings of exponential behavior held by mathematics teachers.

Despite these assertions by researchers, little empirical research evidence exists to help curriculum developers and mathematics educators identify the critical components that are foundational for building robust and dynamic conceptions of exponential function. Greer (1994) points out that a disproportionate amount of research has focused on the development of multiplication as opposed to more advanced notions, such as exponential behavior, within the multiplicative conceptual field. Thus, the lack of research knowledge on knowing and learning exponential function motivated this dissertation study.

Furthermore, a review of the relevant research exposed the lack of a theoretical framework for investigating secondary mathematics teachers' (a) understanding of exponential function and (b) ways of thinking about them. Existing frameworks for investigating students' understanding of the function concept have contributed to the research knowledge on important reasoning abilities necessary for function understanding (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson *et al.*, 2002; Carlson, Oehrtman, & Thompson, 2007). These function frameworks guided the development of the initial framework for knowing and learning exponential function. Thus, one goal for this dissertation study was to provide a theoretical framework for exponential function to lay the groundwork for developing curriculum that emphasizes multiplicative reasoning.

Drawing from the research literature discussed in the previous chapter, a framework emerged to characterize the notation and language, reasoning abilities and conceptual understandings needed to posses a mature understanding of exponential function. Initial renditions of the framework guided smaller studies before this investigation and the subsequent analysis of these studies provided initial findings for refining the framework. The initial framework guided the development of curricular activities on exponential function. In addition, this framework directed data collection and analysis of the Exploratory Study. The next sections discuss each of the three main components within the framework.

Notation and Language

Function notation and language have been reported to be enormous obstacles for students and teachers as they develop and refine their understanding of the function concept (Carlson, 1998). These documented obstacles provided the foundation for thinking about important characteristics of exponential function relative to notation and language. Table 3 provides a summary of these characteristics.

Table 3

Initial Exponential Function Notation and Language Framework

ENL1	Interpret b^x as representing x factors of b.
ENL2	Represent symbolically the multiplicative processes of exponential function.
ENL3	Interpret the definition of exponential function in the form of $f(x) = a \cdot b^x$ with the multiplicative growth factor b.
ENL4	Interpret the definition of exponential function in the form of $f(t) = a \cdot e^{kt}$ with the continuous growth factor e.
ENL5	Interpret contextual situations of constant percentage change as representing exponential behavior.

Reasoning Abilities

The ability to reason through function situations necessitates a dynamic view of the function f as a process of accepting input values, x, and producing output values,

$f(x)$ (Carlson, 1998). Also important for possessing a mature understanding of exponential behavior, in particular, is the ability to reason multiplicatively using notions of doubling, tripling, and halving (Confrey & Smith, 1994, 1995). Moreover, viewing f as a process of recursion where output values are produced by multiplying the previous output value by a constant factor facilitated in developing strong conceptions of exponential behavior (Confrey & Smith, 1994). Table 4 summarizes the reasoning abilities for knowing and learning exponential function.

Table 4

Initial Exponential Function Reasoning Abilities Framework

ER1	View f as a process that accepts x as input and produces output, $f(x)$.
ER2	View f as a process of repeated multiplication for incremental increases in the input (or repeated division for incremental decreases in the input).
ER3	View f as a process of recursion where output values are produced by multiplying the previous output value by a constant factor.
ER4	View family of exponential function as a process resulting from changes in parameters.
ER5	View properties of exponents and exponential function as equivalent phenomena.
ER6	Describe exponential patterns by distinguishing from linear patterns.
ER7	Describe exponential patterns using covariational reasoning. • Observe changes in the input as it relates to changes in the output • Observe changes in the output as it relates to changes in the input
ER8	Describe exponential patterns using multiplicative reasoning. • Conceptualize ideas of doubling, tripling, and halving • Conceptualize ideas of constant percentage growth
ER9	Describe exponential patterns using partial-interval reasoning. • Growth: Change in first part of interval is less than change in second part of interval • Decay: Change in first part of interval is more than change in second part of interval

Conceptual Understandings

Essential for building an understanding of exponential function is the ability to interpret and conceptualize the dynamic function situation (Carlson, 1998). The idea of multiplicative growth is central to this concept. Emphasis is also on building this understanding through comparing and contrasting multiplicative growth of exponential functions with additive growth of linear functions (Confrey & Smith, 1994, 1995). The focus of this investigation included using data from real-world situations and building a structure that supports the phenomenon within the situation. Table 5 summarizes the conceptual understandings included in the exponential function framework.

Table 5

Initial Exponential Function Conceptual Understandings Framework

EC1	View b^x as a multiplicative process of multiplying x factors of b for integer values of x.
EC2	View b^x as a multiplicative process of multiplying x factors of b for non-integer values of x.
EC3	Understand how changes in the parameters a and b alter representations of the function.
EC4	View e as representing continuous growth as the number of compounding periods increase to infinity.
EC5	View x factors of the growth factor as a proportional relationship between the initial value and the function value, $ab^x = f(x) \Leftrightarrow b^x = \frac{1}{a} \times f(x)$.
EC6	View growth factor as remaining in constant proportion throughout function's domain: $b^{x_2 - x_1} = b^{x_3 - x_2} \Leftrightarrow \frac{f(x_2)}{f(x_1)} = \frac{f(x_3)}{f(x_2)}$.
EC7	Connect exponential behavior to real-world phenomenon.
EC8	Connect multiplicative patterns of exponential functions to geometric sequences and series.

Exploratory Investigation

The purpose of conducting an exploratory investigation was to assess the design of the research instruments (i.e., exponential function framework, interview tasks, and teaching experiment activities) and to gain knowledge of teachers' ways of thinking about exponential function. In addition, the Exploratory Study provided an opportunity to gather a wide range of teacher responses to the instrument tasks to allow for a more comprehensive description of the important components for knowing and learning exponential function. The findings from this exploratory investigation contributed to the refinement of the exponential function framework.

Methods

During the summer of 2006, four secondary mathematics teachers participated in task-based interviews. The teachers selected for this study had registered for a graduate-level mathematics education course in the fall of 2006. The course, titled *Functions: Mathematical Tools for Science*, was the first course in a sequence of four courses these teachers completed as part of their participation in a funded project. The four teachers[4] participated in two interview sessions, each lasting 50-90 minutes. In addition to the interviews, one teacher participated in an individual teaching experiment consisting of 5 two-hour episodes. Analysis of the videotaped interviews and teaching experiment episodes provided insights to the emerging categories of exponential thinking. All four teachers also completed a pre-test which was used to evaluate their knowledge of the concept of function and rate of change.

Interviews

The teachers completed their interviews within approximately one week. In order to complete the remaining tasks, one teacher required a third session, which lasted for

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[4] One of the four teachers primarily taught high school science courses. However, this teacher was included in this study because he also taught mathematics courses on a regular basis.

approximately 30 minutes. After the teachers addressed each initial task, they answered additional pre-planned and unplanned questions in response to their verbal and written responses. Appendix A provides the complete list of interview tasks and pre-planned follow-up questions.

Based on the relevant research, the following mathematical ideas were conjectured as foundational to building an understanding of exponential functions and they guided the development of the interview protocol: (a) variable as a varying magnitude, (b) rate of change, (c) average rate of change, (d) covariational reasoning, and (e) multiplicative reasoning.

Teaching Experiment

One of the four teachers, named Andy, participated in an individual teaching experiment for the Exploratory Study. The videotaped experiment consisted of five 2-hour episodes completed within two weeks. Consistent with Steffe and Thompson's (2000) notion of a teaching experiment, each episode included the teacher, the researcher and an observer. After each episode, the researcher and the *observer* reviewed the videotape data and compiled an analysis of the teacher. This analysis helped inform and refine the tasks for the next episode of the experiment. The goal of this process was to continuously mold the teaching experiment to address the conceptual needs of the teacher. As a result, this process contributed to the refinement of the activities and tasks for assisting the teacher in overcoming any cognitive obstacles.

Results of the Exploratory Investigation

The following three categories emerged from the data analysis: (a) exponential decay seemed to be more difficult for teachers to reason through and conceptualize than exponential growth, (b) teachers had difficulty unpacking their ideas about the multiplicative structure and rate of change of exponential functions, and (c) teachers held persistent (mis)conceptions that exponential functions entailed all functions containing an exponent or that were concave up. The next section describes the results from each of these categories.

Difficulties with Exponential Decay

The data from the pre-test and interview sessions revealed that teachers experienced more difficulty in making sense of exponential decay tasks than exponential growth tasks. All three exponential decay tasks were answered incorrectly; however, none of the exponential growth tasks were missed. Two tasks, the Half of the Half-Life and the What is the Half-Life (see Appendix B, question #5 and #6, respectively), were included on both the pre-test assessment and the interview.

The *Half of the Half-Life* task provided an opportunity for teachers to reason multiplicatively as they grappled with finding the amount of substance remaining after 30 minutes with half-life of one hour. One conceptual pitfall for this question was the tendency to think about this problem in terms of additive, rather than multiplicative, behavior when considering the halfway values for both the time and the amount of substance remaining. Macy, for example, correctly determined the amount of substance remaining after zero hours, one hour, and two hours. However, she incorrectly determined the halfway values of the remaining substance for time of 0.5 hours and 1.5 hours. Her interview response to this task was that the amount of decay is ¼ of the substance and this was consistent with her response on the pre-test assessment. The following excerpt highlights her thinking:

Exploratory Excerpt 1

Macy: So one hour, so half of the half. So if it was 100 and it takes an hour to get to 50, in 30 minutes it should be at 75. So a fourth of it has decayed.

AS: Can you explain this using a graph?

Macy: I can try [draws graph and begins to plot values]. We start out at 100 so in 30 minutes…we are at 75, 50 at one hour we are at the half-life, and then 90 minutes we're at 37.5 and then two hours would be at 25. So, it goes like this [connects the points on the graph using a decreasing concave up curve as opposed to a line]

AS: So it's not a straight line? It's curving?

Macy: Um, hm. It's curving down.

In lines 1-2, Macy conceptualized the partial half-life situation as being "half of the half" and thus leading to her conclusion that the amount of substance remaining after 30 minutes was exactly ¼ of the initial amount of substance. This suggests that Macy is thinking that "halfway between input values" translates to "halfway between output values" without consideration for the function's behavior. This way of thinking supported Macy's initial response that the amount of substance would decay exactly ¼ after 30 minutes. Yet, her thinking for partial hours (i.e., 30 minutes and 90 minutes) was inconsistent with her calculations for full hours (i.e., zero, one and two hours). At the end of this excerpt, Macy further reveals her thinking by drawing a correct exponential decay curve despite the incorrect intermediate values, which did not appear on the curve she sketched. After graphing the half-life situation, Macy still asserted that the values halfway between half-lives produced values halfway between output values despite her ability to correctly calculate the output values for integer values of time. Macy's interview excerpt illuminates her ability to consider the multiplicative decrease in the amount of substance remaining after one hour and two hours. However, when calculating the output values for hours 0.5 and hours 1.5, she found the midpoint between the output values at integer values of time. This way of thinking suggests that Macy reasoned additively when calculating the halfway values for hours 0.5 and 1.5, while reasoning multiplicatively when calculating the integer values for hours zero and one. In summary, this excerpt revealed the difficulty in making sense of exponential decay for values between intervals of one unit.

Difficulties with Unpacking Ideas about Multiplicative Structure

All teachers experienced difficulty in unpacking their ideas about the multiplicative structure of exponential function. One example occurred with Kay, a predominately strong mathematics teacher, who commented that the *What is the Half-Life* question stumped her and she could not discriminate between the choices provided. However, with probing by the researcher in the interview session later, she was able to provide a correct and detailed response to the *What is the Half-Life* question. Based on her comments during the interview, it was evident that she had thought about the problem after completing the pre-test, but she was unaware that this question would also appear during the interview later. The following excerpt illustrates Kay's thinking of this task during the interview:

Exploratory Excerpt 2

Kay: This is the one question on that test from last week that I just said I don't know where to begin with this and even though it looks familiar...(long pause)

AS: Would you be able to tell me what $A(t)$ means?

Kay: That's going to be the amount after a certain amount of time, the amount that's left after a certain amount of time.

AS: Okay. What about the 100?

Kay: Well, that should be the portion you are starting with. And the ½ I think is, well let's see, one minus ½ would leave you ½ so it must be decaying at a ½ per whatever that time divided by 5 is.

AS: Tell me more about the t over 5.

Kay: Well if it was just time, t is measured in years so let's see if it was one year every fifth year it's going to decay ½? Is that? I don't know. That would be...(pause) I guess I could plug in some values. (pause) Okay, it would take five years then, would be my guess. Yeah because if I put in a five for the time, if time is years, that would be five over five or one, ½ to the first power is ½ so

100 times the ½ is going to bring me to 50 which would be ½ of the original substance if 100 represents the original amount of substance. [laughs] Maybe I know a little more about this than I think I do.

Despite the fact that Kay was unable to choose an answer for this question when it appeared on the pre-test, she was nonetheless able to answer this problem correctly with prompting from the researcher. All four teachers, including Kay, eventually reasoned through this task in similar ways. They all provided correct explanations for how the exponent represented the number of half-lives and how the value of t must be 5 for an exponent of 5/5 or 1 in order to obtain a 'whole' half-life. This level of reasoning seems to point to an understanding of proportionality given that when the value of $t/5$ is not a whole number it serves as a *portion* of the half-life and thus the function's value is not precisely ½ of the original value. This finding also facilitated the development of a category called "partial-interval reasoning" which was incorporated in the framework. The finding also revealed the need to consider conceptualizing fractional exponents as partial factors of the base where the full factor of the base is defined when the exponent is exactly one. Through a process of deductive reasoning of the situation, Kay provided a coherent argument for thinking about which portion of the function represented the half-life for the function of the form $f(x) = a\left(\frac{1}{2}\right)^x$.

Difficulties with Categorizing Exponential Functions

Not only did teachers exhibit difficulty in reasoning multiplicatively through exponential tasks, especially exponential decay tasks, they also experienced difficulty distinguishing exponential functions from quadratic functions. The conceptual difficulty entailed the (mis)conception that the exponential function family includes "all things with exponents" or "all things concave up." One of the teachers, Andy, exhibited this (mis)conception as he worked through a collection of exponential activities during the teaching experiment. In general, he provided the strongest mathematical arguments of all the teachers in the group through the Exploratory Study, and in particular, he outperformed them on the pre-test assessment. Nonetheless, the data illuminated Andy's deep-seated (mis)conception that quadratic functions and other power functions exhibited exponential function properties because of the existence of an exponent or due to the concavity of the graph.

The first appearance of Andy's thinking about exponential function as encompassing quadratic function was when he discussed the distinction between rate of change of linear function with rate of change of exponential function. Andy connected his thinking to the context of gravity, as illustrated in the next excerpt.

Exploratory Excerpt 3

Andy: I usually refer to, like when we deal with gravity and stuff, an object only falls this far and then falls this far and then this far and then this far [draws motion diagram of a falling object with increasing distances between snapshot images of the ball] over equal time intervals, that as being an exponential rate, not necessarily a constant rate. That's kind of the distinction I make in science.

In the second episode of the teaching experiment, Andy continued the discussion about his interpretation of an "exponential rate" as opposed to a "linear rate." Speaking with self-confidence, he was assertive about his understanding of "linear rate" as compared with his understanding of "exponential rate" in the following excerpt:

Exploratory Excerpt 4

Andy: I kind of understand the difference between what I consider as an exponential rate and a linear rate. It kind of has to do with what the graph looks like, where here...if we are looking at velocity versus time, I would consider that to be a linear rate because this [points to the graph of a line] we can describe by a linear function. That the velocity equals acceleration times time, where we don't have any exponents so it's just a line so I consider it to be a linear rate. Here [points to a curved graph and the formula $d = \frac{1}{2}at^2$ he wrote on the board] where we do have exponents I call it an exponential rate where...both of them being constants being in there, the a or the ½a. But mathematically understanding that if the students mapped out the linear rate, on a piece of paper or something, given a bunch of data points they would be able to see if it's a linear rate [draws a line with his finger] or an exponential rate [draws an increasing curve with his finger] where it's curving up, use that to try to come up with what the equation would be.

Based on Andy's description of his interpretation of exponential rate, it appears that he viewed curves and functions with exponents as having an exponential rate. He did not expand upon his understanding of exponential rate in the same way as he had revealed his understanding of linear rate. The data suggests that he labeled "things with exponents" as an exponential rate without any consideration for the mathematical meaning of exponential behavior.

The next excerpt reveals additional insight into Andy's thinking about quadratic and exponential situations.

Exploratory Excerpt 5

Andy: Here [pointing to $f(x) = 2^{x-1}$] you have the x up here inside the exponent and here [pointing to $f(x) = \frac{1}{2}bx^2$] you have the x down here when you try and take the differential of this [pointing to $f(x) = \frac{1}{2}bx^2$] then you're going to get the $f(x) = bx$ which is the equation we would get when velocity is equal to acceleration times time. But taking the differential of this [pointing to $f(x) = 2^{x-1}$] if you have the variable inside of the exponent...here [pointing to $f(x) = \frac{1}{2}bx^2$ and $f(x) = bx$] the number on the exponent dropped, here [pointing to $f(x) = 2^{x-1}$] dropping the number on the

10 exponent is going to be the same as doing like $f(x) = 2^{x-2}$.

Andy experienced difficulty making sense of why the linear function $y = 3x + 2$ had constant first and second differences while the exponential function, $f(x) = 2^{x-1}$, generated from the penny activity did not possess the same characteristic. After thinking about this issue during the time between teaching episodes, he realized that the function $f(x) = \frac{1}{2}bx^2$ produced a curve and its derivative function $f'(x) = bx$ produced a straight line. He realized that the exponential function $f(x) = 2^{x-1}$ and its derivative function did not behave in the same way as the quadratic function and its derivative. It is important to note here that Andy's derivative function of $f(x) = 2^{x-2}$ is incorrect; he was relying on limited knowledge of calculus to find what he referred to as the "differentials."

The data suggests that Andy holds a more structural, rather than conceptual, image of function. He held an image of the mathematical relationship among gravity, velocity and acceleration and he attempted to use the same decomposition pattern with exponential functions. Since he grouped exponential and quadratic functions together as possessing an exponential rate, he expected the rate of change function of an exponential function to behave in similar ways as the rate of change function of the quadratic function $f(x) = \frac{1}{2}bx^2$. The redesigning of two teaching episodes ensured that other teachers confronted this conceptual issue. As a result, the curriculum for this study promoted similar states of disequilibrium in other teachers.

At this point in the teaching experiment, Andy understood that his thinking was in contradiction with the mathematics of the exponential function. He realized that the placement of the variable x was different in the quadratic function than in the exponential function. Thus, Andy arrived at the conclusion that the exponential function "doesn't differentiate the same way." While he realized this issue, he was still unable to articulate why the rate of a quadratic function is not an exponential rate, as he had previously conjectured. Although he was well aware of the conflict in his thinking, he was still unable to rethink the relative growth of these two functions from a multiplicative perspective.

Chapter Summary

This chapter presented the initial exponential function framework that guided the Exploratory Study, along with the results and analysis from this preliminary investigation. The initial framework focused on the notation and language abilities, reasoning abilities, and conceptual understandings – grounded in relevant research and teaching experience – that emerged as important for building a mature understanding of exponential function. This original framework incorporated the following reasoning perspectives: (a) covariational reasoning, (b) exponential function reasoning, (c) multiplicative reasoning and (d) partial-interval reasoning.

This chapter also described an exploratory investigation conducted with four secondary mathematics teachers who participated in two interview sessions and one teacher who participated in an individual teaching experiment. These four teachers also completed a pre-test assessment on the concept of function and rate of change. The final section of this chapter presented the results and analysis of these teachers' ways of thinking about the concept of exponential function as they worked through a collection of tasks focused on exponential behavior. Analysis of teachers' utterances – both written and verbal – illuminated the following three categories of difficulties: (a) conceptualizing exponential decay, (b) unpacking ideas about multiplicative growth and decay, and (c) categorizing exponential functions.

The next chapter presents the theoretical perspective and revised framework that guided the dissertation study.

CHAPTER 4: THEORETICAL PERSPECTIVE AND FRAMEWORK

This chapter contains two major components of the dissertation study: (a) the theoretical perspective of cognitive constructivism from a pragmatic view and (b) the revised exponential function framework that guided the data analysis of the study. Analysis of the Exploratory Study yielded deeper insights into teachers' ways of thinking which prompted revisions to the framework. Furthermore, the theoretical perspective, discussed in the next section, served as the lens for improving the cohesiveness and comprehensiveness of the framework.

Theoretical Perspective

Two main theoretical perspectives guided this dissertation study: pragmatism and cognitive constructivist theory. The pragmatic perspective framed the investigation's quest for *truth* and its relation to knowledge from an empirical as well as a rational point of view. A pragmatic perspective provides a lens for learning about how humans grow through experience and how they obtain knowledge that becomes their own personal reality (Piaget, 1970).The result of knowledge and truth acquisition occurs through a process known as inquiry where learning takes place through action (Piaget, 1970). Maxcy (2003) provides the following illustration of this process:

> … pragmatists are convinced that human thought is intrinsically linked to action. Theory was joined with practice. Ideas operate as instruments rather than ideas. Reality is in process, undergoing change at every turn of events. The universe is seen as evolving rather than static. (p. 63)

The design and implementation of this study adopted a pragmatic worldview in conjunction with a cognitive constructivist perspective. Cognitive constructivism is the philosophical belief that learning occurs through experiences and action, rather than through knowledge passed on by others (Steffe & Thompson, 2000). Individuals gain knowledge through making sense of the world by exploring, interacting, reacting, and communicating with elements within this world (Piaget, 1970). Cognitive constructivists focus on describing the mental images of an individual to learn one's ways of thinking and approaches to problem situations (Piaget, 1970). The assumption about learning is that knowledge is gained through a process involving dynamic events and actions set forth to promote change (Steffe & Thompson, 2000). Learners then create their own meaning through the process of sense-making. Schoenfeld (1987) describes this constructivist perspective by saying, "we all build our own interpretive frameworks for making sense of the world, and we then see the world in the light of these frameworks. What we see may or may not correspond to 'objective' reality" (p. 22).

The methodological motivation for conducting this present study stemmed from Steffe and Thompson's (2000) work on conducting teaching experiments in mathematics education research. The purpose for conducting the present study was to create models of teachers' mathematics[5] based on local hypotheses of teachers' mathematical understanding as they completed tasks involving exponential functions. Teachers'

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[5] Steffe and Thompson (2000) refer to the notion of *students' mathematics* and *mathematics of students* as a research focus and research outcome. The discussion of this dissertation study, however, focuses on *teachers*' mathematics and mathematics of *teachers*. Nonetheless, this present study utilized Steffe and Thompson's perspective for developing models of *teachers*' mathematics.

mathematics refers to teachers' individual understanding, which is unknown and obscure to the outsider and "is indicated by what they say and do as they engage in mathematical activity" (Steffe & Thompson, 2000, p. 268). The primary philosophical underpinning of this dissertation study is the belief that understanding teachers' mathematics involves a process by which action exemplified by the teacher provides access to various aspects of their thinking. Consistent with Steffe and Thompson's notion of constructivism, this study maintained the perspective that teachers' mathematics could be contradictory to the conceptual definitions accepted by the mathematics community. The purpose of this study was to connect the aspects of teachers' thinking to create models to describe *teachers' mathematics*. These models became descriptions for *mathematics of teachers*.

The next section discusses the exponential function framework, including details of the various categories and components within the framework.

Exponential Function Framework

Drawing upon the findings of the Exploratory Study and using the theoretical perspective as a more focused lens, the exponential function framework characterizes the notation, language and reasoning abilities for developing a mature understanding of the exponential function concept. An initial rendition of the framework was tested in the Exploratory Study conducted prior to the current investigation (discussed in Chapter 3). These initial findings prompted further revisions of the framework. The purpose of this framework was to provide guidance in developing curricular activities for teaching the concept of exponential function. In addition, this framework served as the lens for analyzing and interpreting the data in the dissertation study.

The remainder of the chapter lays out in detail the two main categories of the framework: (a) Notation and Language and (b) Reasoning Abilities.

Notation and Language

The Notation and Language category of the framework entails six components that relate to the following aspects of thinking about exponential function and processes: (a) notation of b^x [ENLN], (b) language of b^x [ENLL], (c) multiple representations [ENLM], (d) parametric changes [ENLP], (e) implicitly-defined exponential function [ENLI], and (f) explicitly-defined exponential function [ENLE]. The next sections present each of the six components of the framework.

Notation [ENLN] and Language [ENLL] of b^x

Function notation and language have been reported to be obstacles for students and teachers as they develop and refine their understanding of the function concept (Carlson, 1998). Complicating matters further, exponential notation can prove to be an even larger obstacle due to impoverished understandings of the exponentiation process (Weber, 2002). Weber's study found that many students who had simply memorized rules for exponents were unable to provide meaningful responses to the following questions:

- Is $\left(\frac{1}{2}\right)^x$ an increasing function or a decreasing function? Why?
- Is $(-3)^{10}$ a positive or negative number? Why?
- Is 5^{14} an even number or an odd number?

Weber (2002) suggests that students may benefit from a more conceptual understanding of exponent operations as opposed to requiring them to memorize the rules of exponents. Weber's motivation for his study was to facilitate students' understanding of exponentiation as a process of producing x factors of b from the expression b^x. He also found that interpreting b^x as representing x factors of b was foundational for viewing exponentiation as a process. Therefore, the framework incorporates both the notation [ENLN] and the language [ENLL] for representing b^x as x factors of b. Understanding the notation and the language of exponents and exponential functions can facilitate the development of further understandings of multiplicative structures to build stronger reasoning and conceptual abilities within this topic (Confrey, 1994; Confrey & Smith, 1994; Weber, 2002).

Multiple Representations [ENLM]

Past research studies and theories have illuminated the necessity of using multiple representations when developing the concept of function (Carlson, Oehrtman, & Thompson, 2007; Lobato & Bowers, 2000, April; Rizzuti, 1991). Carlson *et al.* (2007) maintain that curricular materials should allow students the opportunities to "make and compare judgments about functions across multiple representations" (p. 161). These ideas informed the development of the framework along with the creation of the tasks and activities incorporated into the present investigation. The framework for this dissertation study guided the creation of the exponential activities with a focus on the tabular, graphical, algebraic and contextual representations.

Parametric Changes [ENLP]

The framework included the notion of parametric changes as a tool for understanding characteristics of the exponential function. The perspective incorporated in the framework focused on describing the effects of parametric changes of the initial value and the growth/decay factor relative to changes in the table, graph, algebraic formula, and contextual representation.

It is important to interpret the parameters and variables in the algebraic form of an exponential function in order to distinguish these same parameters and variables with other function formulas, such as linear functions in the form of $f(x) = b + mx$ and quadratic functions in the form of $f(x) = ax^2 + bx + c$. Understanding the initial value, a, in the exponential function $f(x) = a \cdot b^x$ along with knowing how the function grows or decays from this value (i.e., repeatedly multiplied by b x number of times) is foundational to building conceptions of exponential behavior.

Implicitly-Defined Exponential Function [ENLI]

In addition, thinking about exponential situations implicitly as recursive processes facilitates the ability to mentally imagine the indefinite actions placed on the recursive objects along a continuum of values. Possessing the ability to fluently carry out recursive processes on the recursive objects entails recursion (Thompson, 1985). The ability to act upon a finite number of recursive objects does not translate to the ability to reason multiplicatively throughout the problem situation. Instead, the idea of considering the multiplicative progression of output values as representing an indefinite sequence determined by multiplying the previous output value to obtain the next output value was considered as a recursive process. This idea can be represented as $\text{NEXT} = \text{NOW} \cdot b$ or more formally as $f(x_{n+1}) = f(x_n) \cdot b$.

Explicitly-Defined Exponential Function [ENLE]

The framework also encompassed the explicit definition of an exponential function in two forms: (a) $f(x) = a \cdot b^{\frac{x}{t}}$ where t represents how often the function is increased by a factor of b and (b) $f(t) = a \cdot e^{kt}$ where k represents the constant of proportionality. Moreover, bridging ideas of growth/decay to ideas of continuous growth/decay in the form of $f(t) = a \cdot e^{kt}$ is important for students to learn in preparation for calculus courses. Nonetheless, previous explorations have shown that secondary mathematics teachers have impoverished understandings of the number e (Strom, 2007). The data from the Exploratory Study (discussed in Chapter 3), however, also revealed that teachers have weak notions of continuous growth/decay. These teachers had difficulty explaining the effects of increasing compounding periods, n, for monetary investments relative to the limit of the growth factor (i.e., as the number of compounding periods goes to infinity, the growth factor goes to e).

Table 6 summarizes the characteristics that comprise the Notation and Language category of the exponential framework.

Table 6

Revised Exponential Function Notation and Language

ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function. ENLM1. Tabular ENLM2. Graphical ENLM3. Algebraic ENLM4. Contextual
ENLP	Use *parametric changes* to alter representations of an exponential function. ENLP1. Describe effects of changes in initial value relative to the table, graph, algebraic formula, and contextual representation. ENLP2. Describe effects of changes in growth/decay factor relative to the table, graph, algebraic formula, and contextual representation.
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively: ENLI1. $\text{NEXT} = \text{NOW} \cdot b$ ENLI2. $f(x_{n+1}) = f(x_n) \cdot b$
ENLE	Use *explicit definition* of exponential function to represent exponential situations explicitly: ENLE1. $f(x) = a \cdot b^{\frac{x}{t}}$ where t represents how often the function is increased by a factor of b. ENLE2. $f(t) = a \cdot e^{kt}$ where k represents the constant of proportionality.

Reasoning Abilities

The underlying premise of this Exploratory Study was to capture teachers' ways of thinking as a means for describing their reasoning abilities while working through a collection of exponential function tasks and activities. Emerging from the exploratory data were four themes for describing the reasoning abilities associated with exponential function: (a) covariational reasoning, (b) exponential function reasoning, (c) multiplicative reasoning, and (d) partial-interval reasoning. The common mathematical thread connecting all four reasoning abilities is the concept of rate, both as a quantitative measure of change and as a qualitative description of the dynamic function situation.

The conception of rate used in the present study was most analogous to Thompson's (1994c) image of (a) rate as the covarying of two quantities proportionally and (b) ratio as the comparison of two quantities multiplicatively. This image of rate involves a more dynamic view that supports the image of something growing continuously and simultaneously as opposed to Confrey and Smith's (1994) definition of rate as a "unit per unit comparison" which entails a more static view of the progression of output values in relation to input values. Thompson's image of rate also engages the notion of accumulation of accrued intervals of time. The proportionality in this image of rate gives rise to the covariational aspects of the quantities being compared and ultimately provides the foundation for developing an image of an exponential rate as being proportional to the amount. It is within the construct of Thompson's image of rate that the present study was situated. Bearing this viewpoint, the following discussion presents each of the four categories for describing the reasoning abilities of knowing and learning exponential function.

Covariational Reasoning [ERCR]

While several researchers agree that covariational reasoning is important in developing an understanding of function, the researchers differ in their perspective of covariation. The perspective of covariation used for this investigation was most analogous to Carlson *et al.*'s (2002) definition of covariational reasoning as the "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). This notion of covariation drew upon the work on covariation by Confrey and Smith (1994), Thompson (1994a) and Saldanha and Thompson (1998).

Reasoning through function situations necessitates the ability to view f as a process of accepting input values, x, and producing output values, $f(x)$ (Carlson, 1998). Carlson *et al.* (2002) argued that covariation plays an integral role in building a profound understanding of the function concept. Other researchers have offered similar justifications for recommending an increased curricular emphasis on covariation when developing an understanding of function because it offers a more qualitative approach to describing the changes of one quantity in relation to another quantity (Carlson, Oehrtman, & Thompson, 2007; Confrey & Smith, 1994, 1995; Rizzuti, 1991; Saldanha & Thompson, 1998).

Using Carlson *et al.*'s (2002) covariation framework as a guide for developing the exponential function framework, the Covariational Reasoning sub-category entails four components (levels) that relate to the following aspects of reasoning about exponentials function and processes: (a) coordinate output changes, (b) coordinate amounts of change, (c) coordinate constant percent change, and (d) coordinate multiplicative change. The next sections describe each of these four components.

Level 1: Coordinate Output Changes [ERCR1]. This initial level of covariational reasoning entails the mental action of coordinating output changes with changes in the exponent (i.e., changes in the input). Utterances include descriptions about the exponent as a varying quantity while also considering the output as a varying quantity.

Level 2: Coordinate Amounts of Change [ERCR2]. This second level of covariational reasoning entails the mental action of coordinating amount of change of output with amount of change in the exponent. This level is higher than Level 1 because the focus is on amounts of change in addition to variation. The difference in this level of covariational reasoning in the context of exponential function encompasses the ability to unpack notions of amounts of change for both input and output quantities. Utterances include descriptions about increasing or decreasing differences of outputs for increases in the exponent.

Level 3: Coordinate Constant Percent Change [ERCR3]. This third level of covariational reasoning entails the mental action of coordinating constant percent change of outputs with constant additive change in the exponent. This level is higher than Level 2 because the focus is on constant percent change in addition to amounts of change. The subtle difference in this level of covariational reasoning in the context of exponential function encompasses the ability to unpack notions of amounts of change for both input and output quantities with references to constant percent change. Utterances include descriptions about constant percent increases or decreases for increases in the exponent and may not include support for multiplicative reasoning.

Level 4: Coordinate Multiplicative Change [ERCR4]. This fourth level of covariational reasoning entails the mental action of coordinating multiplicative change (changing by a constant *factor*) of outputs with constant additive change (changing by a constant amount) in the exponent. This level is higher than Level 3 because the focus is on the multiplicative comparison of outputs rather than solely on percentage change. The subtle difference in this level of covariational reasoning in the context of exponential function encompasses the ability to unpack notions of constant percent change with references to percent as a multiplicative factor. Utterances include descriptions of multiplicative change either in terms of increasing/decreasing at an increasing rate (by a constant factor) or in terms of concavity.

Exponential Function Reasoning [ERFR]

Essential for building an understanding of function is the ability to interpret and conceptualize the dynamic function situation (Carlson, 1998). The dynamic function conceptualization is central to the multiplicative growth/decay concept. Emphasis is also on building this understanding through comparing and contrasting multiplicative growth/decay of exponential functions with additive growth/decay of linear functions (Confrey & Smith, 1994, 1995).

Exponential function reasoning was included in the framework as a means for capturing utterances of teachers' spontaneous and solicited comparisons between linear and exponential functions. Data from the Exploratory Study revealed that a mature understanding of linear function served as a conceptual springboard that provided a powerful catalyst for making sense of exponential situations. When grappling with an exponential context, the teachers in the Exploratory Study often examined tabular data, graphical characteristics or algebraic representations for the presence of linear aspects as a means for differentiating between exponential and linear function behavior. The rate of change concept emerged as one of the most powerful constructs for teachers in the Exploratory Study as they grappled with exponential function situations. Yet, evidence also revealed that some teachers inappropriately applied characteristics of function (e.g., rate of change) to situations describing exponential behavior. As a result of this evidence, the exponential function framework was further refined. The Exponential Function Reasoning sub-category of the framework entails four components that relate to the following aspects of reasoning about exponential function and processes: (a) inappropriate application of function characteristics, (b) comparison of amount and percent change, (c) comparison of constant and changing rates, and (d) comparison of constant and multiplicative rates. The next section details each of the four components.

Inappropriate Application of Function Characteristics [ERFR0]. This initial level of exponential function reasoning entails the inappropriate application of function characteristics to exponential contexts. Utterances include the use of additive reasoning when describing exponential behavior with no indication or consideration of the multiplicative structure of the situation.

Comparison of Amount and Percent Change [ERFR1]. This second level of exponential function reasoning entails comparing amount of change of outputs for incremental increases in inputs for linear functions with percent change of outputs for incremental increases in inputs for exponential functions. Utterances include comparing growth/decay of outputs by constant amount for linear functions with growth/decay of outputs by constant percent for exponential functions.

Comparison of Constant and Changing Rates [ERFR2]. This third level of exponential function reasoning entails comparing constant rate of change for incremental increases in inputs for linear functions with changing rate of change for incremental increases in inputs for exponential functions. Utterances include comparing rates of change but these utterances are limited to references about changing rate of change for exponential functions without descriptions of *how* the rate of an exponential function changes.

Comparison of Constant and Multiplicative Rates [ERFR3]. The fourth level of exponential function reasoning entails comparing constant rate of change for incremental increases in inputs for linear functions with multiplicative rate of change for incremental increases in inputs for exponential functions. Utterances include comparing rate of change for linear and exponential functions with indications that exponential functions have multiplicative rate of change while linear functions as having constant rate of change.

Multiplicative Reasoning [ERMR]

Another important characteristic of possessing a deep understanding of exponential behavior is the ability to reason multiplicatively using notions of doubling, tripling, and halving, for example (Confrey & Smith, 1994, 1995). In addition, conceptions of multiplicative growth/decay are distinguished from linear behavior through comparisons of the respective function characteristics and patterns, such as rate of change and covariation. Using linear behavior as a conceptual springboard to develop conceptions of exponential behavior proved to be a powerful mechanism for making sense of exponential situations. The Multiplicative Reasoning sub-category of the framework entails three components that relate to the following aspects of reasoning about exponential function and processes: (a) proportional parameters [ERMR1], (b) recursive change by constant factor [ERMR2], and (c) constant proportionality of output values [ERMR3]. The next sections present each of these components of the framework.

Proportional Parameters [ERMR1]. This first level of multiplicative reasoning entails describing the relationship between the growth/decay factor, b, and the initial value, a, of an exponential function as a multiplicative (proportional) relationship. Utterances of this level include the ability to consider x factors of the growth/decay factor as a proportional relationship between the initial value and the function's value. Thinking of exponentiation multiplicatively requires the understanding that an exponential function's value is precisely the product of an initial value and x factors of the growth/decay factor while also simultaneously realizing that the value of x factors of the growth/decay factor is $1/a$*th* of the function's value (where a represents the initial value). Understanding this proportional relationship is vital for building a conceptual understanding of exponential behavior.

Recursive Change by Constant Factor [ERMR2a]. This second level of multiplicative reasoning entails describing the changes in outputs as the result of *multiplying* the previous output value by a constant factor (or percent). Utterances of this level of reasoning include emphasis on finite successive pairs of output values without consideration for the infinite recursive process of carrying out the repeated multiplication.

The *Principles and Standards* (National Council of Teachers of Mathematics, 2000) maintain that constant percentage growth/decay is crucial for students to comprehend in high school algebra courses. The language of percentage growth/decay in contextual situations tends to lead many novices to consider the situation as linear behavior as opposed to exponential behavior due to impoverished notions of percents. The ability to interpret and recognize percentage growth/decay as behaving exponentially allows for deeper conceptions of exponential function.

Recursive Change by Constant Factor [ERMR2b]. This next level of multiplicative reasoning entails describing the change in outputs as the result of *repeatedly multiplying* the previous output value by a constant factor (or percent). Utterances of this level include the ability to recognize the infinite recursive process underlying the exponential function.

Conceptualizing exponentiation as a process of repeated multiplication, contrasted with repeated addition for linear function, provides a solid foundation for building an understanding of how exponential situations operate (Confrey, 1994; Confrey & Smith, 1995; Weber, 2002). The findings from the Exploratory Study suggest that this foundation acts as a springboard for advanced notions of proportional reasoning and compound growth/decay. Moreover, viewing f as a process of recursion where output values are produced by repeatedly multiplying the previous output value by a constant factor (Confrey & Smith, 1994) was also found in the Exploratory Study to be a common way of thinking about exponential behavior.

Constant Proportionality of Outputs [ERMR3]. This third level of multiplicative reasoning entails describing the output values as remaining in constant proportion for any successive pairs of intervals of uniform size. Utterances of this level focus on the ability to fluently imagine the multiplicative growth/decay factor as the invariant quantity that proportionally relates one output value to the preceding output value for any interval in the function's domain. This idea can be illustrated as $b^{x_2-x_1} = b^{x_3-x_2} = \frac{f(x_2)}{f(x_1)} = \frac{f(x_3)}{f(x_2)}$ where the ratio of two output values remains in a constant proportional relationship with successive output values for uniform interval sizes.

Partial-Interval Reasoning [ERPIR]

During the Exploratory Study, another important reasoning ability that emerged as critical was the ability to conceptualize and describe the function's growth/decay behavior for domain intervals less than one unit. In the exponential framework for the present study, this reasoning ability is called "partial-interval reasoning." The partial-interval reasoning sub-category of the framework entails two components that relate to the following aspects of reasoning about exponential function and processes: (a) Partial

Change [ERPIR1a, ERPIR1b] and (b) Partial Factors [ERPIR2]. The next sections present these two sub-components of the framework.

Partial Change [ERPIR1a, ERPIR1b]. This reasoning ability describes exponential behavior by focusing on amounts of change within sub-intervals of an exponential function for equal intervals of inputs. For exponential growth [ERPIR1a], utterances focus on describing the amount of change in the first part of the interval as less than the amount of change in the second part of the interval, where the sum of the partial intervals represents a full interval of one unit. For exponential decay [ERPIR1b], utterances focus on describing the amount of change in the first part of the interval as more than the amount of change in the second part of the interval. This idea of partial-interval reasoning is applicable to any function with a changing rate of change. The data from the Exploratory Study, however, suggest that partial-interval reasoning is necessary for expanding this idea to more sophisticated notions of proportionality between amounts of change in the second part of the interval in relation to amounts of change in the first part of the interval (i.e., exponential functions change multiplicatively while other functions do not).

Partial Factors [ERPIR2]. Weber's (2002) study provided motivation for conceptualizing b^x as representing x factors of b for rational values of x. This idea is consistent with the notion that 3^2 denotes 2 factors of 3 or 3 multiplied by itself two times. Furthermore, the framework of the present study included this notion to investigate how conceptualizing $3^{1/2}$ as representing "½ factors of 3" facilitated the development of a mature understanding of exponential behavior. The findings from the Exploratory Study revealed that emphasizing the term *factor* as a tool for multiplication provides an opportunity to consider ½ factor as a *partial factor* that when multiplied by itself twice reveals the *full factor* of 3^1. The premise of this understanding relies on thinking with flexibility about $3^{1/2}$ as the factor which multiplied by itself twice results in the full factor of 3 while also holding in mind that $3^{1/2}$ stems from the operation of root (i.e., $\sqrt{3}$). Thus, $\sqrt{3} \cdot \sqrt{3}$ yields a full factor of 3 just as $3^{1/2} \cdot 3^{1/2}$ also produces the same result. The Exploratory Study found that thinking about fractional exponents as representing a smaller part of a factor, and also simultaneously considering how multiples of this smaller part of a factor produces a full factor, was a powerful tool for understanding exponential behavior.

Table 7 summarizes the Reasoning Abilities for exponential function contexts found to be critical for broadening conceptions of exponential behavior.

Table 7

Revised Exponential Function Reasoning Abilities

ERCR: Covariational Reasoning

Use *covariational reasoning* to describe exponential behavior by attending to incremental changes in the dependent variable in tandem with incremental changes in the independent variable.

ERCR1. Coordinate *output changes* with changes in the exponent.

ERCR2. Coordinate *amount of change* of output with changes in the exponent.

a. Growth: Utterance of increasing differences in output for equal increases in exponent; the longer it goes, the faster it grows.
b. Decay: Utterance of decreasing differences in output for equal increases in exponent; the longer it goes, the slower it decays.

ERCR3. Coordinate *constant percent change* of output with changes in the exponent.

a. Growth: Utterance of constant percent increases in output for equal increases in exponent.
b. Decay: Utterance of constant percent decreases in output for equal increases in exponent.

ERCR4. Coordinate *multiplicative rate of change* of output with changes in the exponent.

a. Growth: Utterance of increasing at an increasing rate or graph as concave up.
b. Decay: Utterance of decreasing at an increasing rate or graph as concave down.

ERFR: Exponential Function Reasoning

Use *exponential function reasoning* to compare and contrast exponential function with other functions (e.g., linear function).

ERFR0. Inappropriately use additive reasoning when describing exponential behavior with no indication of multiplicative reasoning.

ERFR1. Compare amount of change for incremental increases in input for linear function with percent change for incremental increases in input for exponential function.

ERFR2. Compare constant rate of change for incremental increases in input for linear function with changing rate of change for incremental increases in input for exponential function.

ERFR3. Compare constant rate of change for incremental increases in input for linear function with multiplicative rate of change for incremental increases in input for exponential function.

Table 7 (continued). Exponential Function Reasoning Abilities

ERMR: Multiplicative Reasoning

Use *multiplicative reasoning* to describe exponential behavior.

ERMR1. Proportional Parameters: Describe relationship between growth/decay factor, b, and initial value, a, of an exponential function as a multiplicative (proportional) relationship.

ERMR2. Recursive Change by Constant Factor:

a. Describe changes in output as the result of multiplying the previous output value by a constant factor (or percent).
b. Describe changes in output as the result of repeatedly multiplying by a constant factor (or percent).

ERMR3. Constant Proportionality of Outputs: Describe output values as remaining in constant proportion for any successive pairs of intervals of uniform size.

ERPIR: Partial-Interval Reasoning

Use *partial-interval reasoning* to describe exponential behavior.

ERPIR1. Partial Change: Describe partial change of output values for b^x as x factors of b for non-integer rational values of x.

a. Growth: Change in first part of interval is less than change in second part of interval for any given interval.
b. Decay: Change in first part of interval is more than change in second part of interval for any given interval.

ERPIR2. Partial Factors: Conceptualizing b^x as representing x factors of b for non-integer rational values of x.

Chapter Summary

This chapter has presented the various components of the exponential function theoretical framework used for this dissertation study. The ideas embedded in the framework are rooted in findings that are reported in the literature as well as revealed in the Exploratory Study conducted prior to this dissertation study. The framework provided the principal tool used for the design and analysis phases of the present study. Table 8 illustrates a summary of the framework, which includes both the Notation and Language category and the Reasoning Abilities category.

Table 8

Revised Exponential Function Framework

	Notation and Language
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.
ENLP	Use *parametric changes* to alter representations of an exponential function.
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.
ENLE	Use *explicit definition* of exponential function to represent exponential situations explicitly.
	Reasoning Abilities
ERCR	Use *covariational reasoning* to describe exponential behavior by attending to incremental changes in the independent variable. ERCR1: Output Changes ERCR2: Amounts of Change ERCR3: Constant Percent Change ERCR4: Multiplicative Rate of Change
ERFR	Use *exponential function reasoning* to compare and contrast exponential function with other functions (i.e., linear function). ERFR0: Inappropriate Use of Additive Reasoning ERFR1: Compare Amount of Change with Percent Change ERFR2: Compare Constant Rate with Changing Rate ERFR3: Compare Constant Rate with Multiplicative Rate
ERMR	Use *multiplicative reasoning* to describe exponential behavior. ERMR1: Proportional Parameters ERMR2: Recursive Change by Constant Factor ERMR3: Constant Proportionality of Outputs
ERPIR	Use *partial-interval reasoning* to describe exponential behavior. ERPIR1: Partial Change ERPIR2: Partial Factors

The next chapter focuses on methodology, which guided the design and implementation for the dissertation study.

CHAPTER 5: METHODOLOGY

This chapter provides an overview of the essential research design elements of the case study. The discussion includes a full description of the methods of inquiry, setting and participants, and methods of data analysis employed in this study. Also provided is a rationale for the decisions regarding the investigation's design and implementation. In short, this chapter includes the research techniques used to explore a secondary mathematics teacher's ways of thinking about and emerging understanding of the exponential function concept.

Research Design: Case Study

This case study focused on one secondary mathematics teacher who enrolled in a graduate-level mathematics education course, called *Functions: Mathematical Tools for Science*. The case study design provided an empirical inquiry into exploring the ways of thinking and understanding of exponential behavior held by one teacher. This in-depth examination provided rich information about the robustness of the exponential function framework categories and components. The case study also presented a holistic portrayal of the teacher's thinking relative to the framework.

This investigation employed both qualitative and quantitative data collection methods. The timeline for data collection spanned 20 weeks, starting approximately five weeks before the beginning of a 15-week course and continuing until the end of the course. All data were analyzed qualitatively for the purposes of describing in detail teachers' ways of thinking about exponential function. Qualitative analysis included retrospectively analyzing interviews and teaching experiment sessions, coding teachers' utterances, and journaling about the meaning of teachers' utterances.

Methods of Inquiry

The investigation incorporated multiple approaches of data collection: (a) pre- and post-test assessments; (b) semi-structured, task-based interviews; (c) classroom observations of the *Functions* course; and (d) an individual teaching experiment. Multiple methods of inquiry facilitated in triangulating the data to increase the viability and reliability of the findings from this study. Figure 3 provides the chronology of data collection.

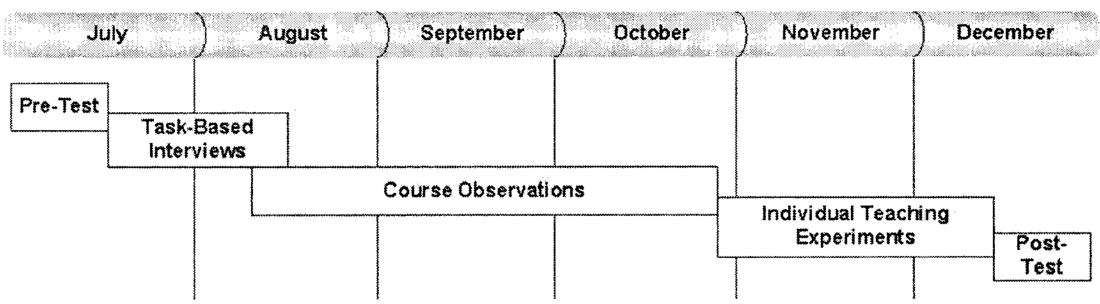


Figure 3. Timeline of data collection for this study.

The analysis process focused on the qualitative data (i.e., interviews, observations and teaching experiment). The quantitative data (i.e., pre- and post-test assessments) provided additional support for generated conjectures based on the qualitative data. These scores also provided criteria for selecting teachers for this investigation.

The current body of research offers limited knowledge of the important characteristics for knowing and learning exponential function. Therefore, the Exploratory Study focused on building and strengthening categories and components of the initial exponential framework. The Exploratory Study provided an opportunity to revise the framework through conducting and analyzing semi-structured, task-based interviews with four mathematics teachers. The findings from these interviews informed (a) the refinement of the interview tasks, (b) the construction of the teaching experiment activities of the present study, and (c) the categories and components of the framework. The next sections discuss the roles of each of the four methods of inquiry.

Pre- and Post-Test Assessments

The quantitative portion of this investigation utilized test results from 18 mathematics and science teachers enrolled in the *Functions* course. These teachers completed a pre- and post-test assessment, called the Precalculus Concept Assessment (PCA), consisting of 25 multiple-choice questions focused on the function concept and rate of change. Of the 25 questions, six encompassed exponential behavior. The PCA pre-test results provided a metric to categorize performance levels of all teachers in the course. These performance levels included: (a) teacher as *high performing* for PCA scores of 19 or higher (out of 25), (b) teacher as *above average* for scores between 16-18, (c) teacher as *average* for scores between 13-15, and (d) teacher as *below average* for scores below 13. The PCA pre-test scores informed the selection process of the subject for this dissertation study. A later section provides details of the selection process.

Research has shown that student success in first semester calculus can be predicted with a high level of accuracy based on PCA scores (Carlson, Oehrtman, & Engelke, 2007). In one study with 125 students from seven different sections of first semester calculus, Carlson *et al.* found that (a) 82% of students who received a score of

11 or below on the PCA failed or withdrew from the course; and (b) 83% of those who received a 13 or higher on the PCA received an A, B, or C in the course. These results suggest that the PCA provides a positive correlation between scores of 13 or higher and success in first semester calculus. Thus, the present investigation incorporated the PCA to assess teachers' understanding of precalculus concepts, such as exponential function.

Post-test scores from the PCA were collected to gain further insights on teachers' shifts in multiplicative reasoning and in understanding the function concept, including exponential function. These post-test scores were analyzed by calculating normalized index scores, also known as Hake gains (Hake, 1998). These figures provided a ratio comparison of the actual gain in scores to the possible gain that the individual teacher could have achieved.

Classroom Observations

Research data for this study entailed classroom observations of teachers participating in the *Functions* course. Observation fieldnotes provided documentation focused on teachers' utterances relative to rate of change, multiplicative reasoning, and covariational reasoning. Teachers' classwork and homework artifacts were included into the data corpus for purpose of data triangulation. The classroom observations also informed the selection process of teachers for the teaching experiment. Later sections of this chapter describe the classroom setting and participants.

Semi-Structured, Task-Based Interviews

The present study incorporated semi-structured, task-based interviews as a method for investigating the teacher's cognitive processes when working through exponential function tasks. The task-based interviews provided opportunities to explore the teacher's ways of thinking relative to the following mathematical ideas: (a) variable as a varying magnitude, (b) rate of change, (c) average rate of change, (d) covariational reasoning, and (e) multiplicative reasoning.

The teacher participated in two interview sessions, each lasting approximately 75-90 minutes. Both interview sessions took place after the teacher completed the PCA pre-test. These sessions were videotaped, transcribed and coded using the revised exponential function framework. The interviews incorporated 10 mathematical tasks, which included pre-planned follow-up questions based on the teacher's verbal and written responses to the tasks. This strategy provided a more semi-structured environment adding flexibility to the discussion. Appendix A provides the complete list of interview tasks and pre-planned follow-up questions.

The design of the interview protocol was guided by Goldin's (2000) principles for task-based interviews. Goldin's four stages of interview principles include: (a) posing the question, (b) suggesting heuristics minimally, (c) guiding the use of heuristics, and (d) questioning using exploratory questions. Table 9 illustrates an example of an interview task designed using Goldin's principles.

Table 9

Example of Interview Question Using Goldin's (2000) Principles

Posing the question	A radioactive substance decays according to the function $A(t)=100\left(\frac{1}{2}\right)^{\frac{t}{5}}$ where t is measured in years and A is the amount of the unstable portion of the substance in micrograms. How long does it take for half of the substance to decay? In other words, what is its half-life?
Suggesting heuristics minimally	Can you draw a graph of this function to find the half-life? Can you construct a table of values to find the half-life?
Guiding the use of heuristics	What aspects of the graph provide information about half-life?
Questioning using exploratory questions	Would the half-life of this function change if 100 was changed to 200? Explain. Would the half-life of this function change if $\frac{1}{2}$ was changed to $\frac{1}{3}$? Explain.

The data from the Exploratory Study revealed that the mathematical ideas included in the interviews were vital for developing a mature understanding of exponential behavior. Data collected from the interviews informed the teacher's hypothetical learning trajectory and teaching experiment activities. The next section discusses the individual teaching experiment.

Individual Teaching Experiment

The teaching experiment consisted of five two-hour sessions spanning approximately four weeks. Each teaching episode included the teacher and the researcher as they interacted together with the mathematics. Each episode was videotaped. The individual teaching experiment served two purposes: (a) to gain further insights about the teacher's emerging understanding of exponential function; and (b) to study the ways in which an instructional unit, focused on multiplicative reasoning, facilitated the development of the teacher's understanding of exponential behavior. Steffe and Thompson's (2000) theoretical perspective of teaching experiment methodology provided the foundation for conducting the present study's teaching experiment. The teaching experiment activities promoted mathematical discourse and evoked mathematical thinking between the teacher, the researcher and the mathematics.

As discussed in chapter 4, the present study incorporated teaching experiments as a method for creating models of *teachers' mathematics* based on local hypotheses of teachers' mathematical understanding as they completed tasks involving exponential functions. Initial conjectures of teachers' understanding of exponential function informed the activities in the teaching experiment and the hypothetical learning trajectory for studying teachers' mathematics. Establishing models to holistically describe teachers' thinking allowed for the development of models of *mathematics of teachers*.

The teacher participated in an individual teaching experiment, which consisted of eleven activities segmented into five teaching episodes. Each teaching episode encompassed a hypothetical learning trajectory tailored to the conceptual needs of the teacher. The results and findings from the teacher's performance on the PCA pre-test and task-based interviews informed the teaching experiment episodes. The analysis of the teacher's utterances obtained from the interviews facilitated in successively re-evaluating and refining the trajectory of the teaching experiment.

Another component of the teaching experiment involved the participation of an informed expert observer throughout all teaching episodes. The researcher of this study collaborated with an informed observer after each individual episode to create initial hypotheses of the teacher's utterances. The observer for this study was a mathematics education doctoral student with approximately four years of teaching experience at the community college level. The observer also had experience in facilitating teacher development and education programs. The observer provided feedback on interpretations gathered from each teaching episode. These observations and initial hypotheses informed the activities of the subsequent teaching episode.

The next section discusses the hypothetical learning trajectory, which guided the implementation of the teaching experiment.

Hypothetical Learning Trajectory for Learning the Concept of Exponential Function

Many researchers advocate for the use of developing hypothetical learning trajectories (HLT) when planning teaching experiments (Cobb, 2000; Simon, 1995, 2000; Steffe & Thompson, 2000). An HLT is a pre-planned roadmap of the learning process, which the implementer designs to guide the learning of subjects. Since the nature of this roadmap is pre-planned, the trajectory is merely hypothetical with the assumption that obstacles along the path may arise and an alternative course of action may need to be executed. Consistent with Simon's (1995) perspective of hypothetical learning trajectories, the HLT for this investigation included three components: (a) learning goals for the teachers, (b) an instructional set of activities, and (c) a conjectured learning process where the researcher projected how the individual teacher's understanding might unfold and evolve throughout the experiment. The next section describes these components relative to learning exponential function.

Learning Goals.

The primary learning goal of the teaching experiment was to enhance the teacher's understanding of exponential behavior with each session designed to promote important characteristics of reasoning and conceptualizing exponential growth and decay. As a result of the interview data and analysis conducted prior to implementing the teaching experiment, one major characteristic weaved into the teaching activities included the idea of viewing b^x as the multiplicative process of multiplying x factors of b for non-integer, rational values of x. Data gathered from the interviews indicated that the teachers in the Exploratory Study held impoverished notions of fractional exponents. Therefore, the goal of the teaching experiment sessions was to specifically enhance the teacher's understanding of fractional exponents relative to exponential function through meaningful activities and discussion.

Learning Activities.

Each of the five teaching experiment sessions included one or more activities. The activities facilitated an understanding of teachers' mathematics through conceptual activities, which promoted multiplicative reasoning. The activities emphasized important ideas about exponential behavior such as doubling, tripling, and halving for rational domain intervals. The activities chosen for this teaching experiment included conceptual ideas consistent with multiplicative reasoning, such as (a) building knowledge of doubling and half-life, (b) comparing linear and exponential functions using multiple representations, (c) developing deeper understanding of fractional exponents, and (d) developing understanding of continuous growth and the number *e*. Chapter 7 presents the discussion about the teaching experiment activities along with the results and analysis.

Conjectured Learning Process.

Research related to knowing and learning exponential behavior (Confrey, 1994; Confrey & Smith, 1994; Weber, 2002), multiplicative reasoning (Thompson, 1994c; Thompson & Saldanha, 2003) and covariational reasoning through dynamic function situations (Carlson *et al.*, 2002; Thompson, 1994b) facilitated in the development of the HLT for the present study. National calls for infusing exponential behavior into high school and college curriculum influenced this study (American Mathematical Association of Two-Year Colleges, 2006; National Council of Teachers of Mathematics, 2000). In addition, the researcher relied upon her eight years of teaching experience with community college level mathematics students to guide the initial creation of the exponential function activities. The researcher also worked closely with secondary mathematics teachers in previous implementations of curricula focused on exponential function, which guided the final refinement of the instructional unit. In short, the final teaching experiment and HLT incorporated a fusion of ideas and perspectives gathered from the research literature, the Exploratory Study, previous informal research investigations, and the researcher's teaching experience.

Figure 4 summarizes the methods of inquiry for the research investigation, which highlights the chronological phases of this study and the cyclic patterns of conjecturing, testing, analyzing and revising within each phase. The HLT, represented by the dotted curve, indicates a general path of learning with the endpoint vertically higher than the initial start point. The actual learning trajectory, represented by the solid curve, illustrates the fluctuating path of the learner in relation to the HLT initially conjectured.

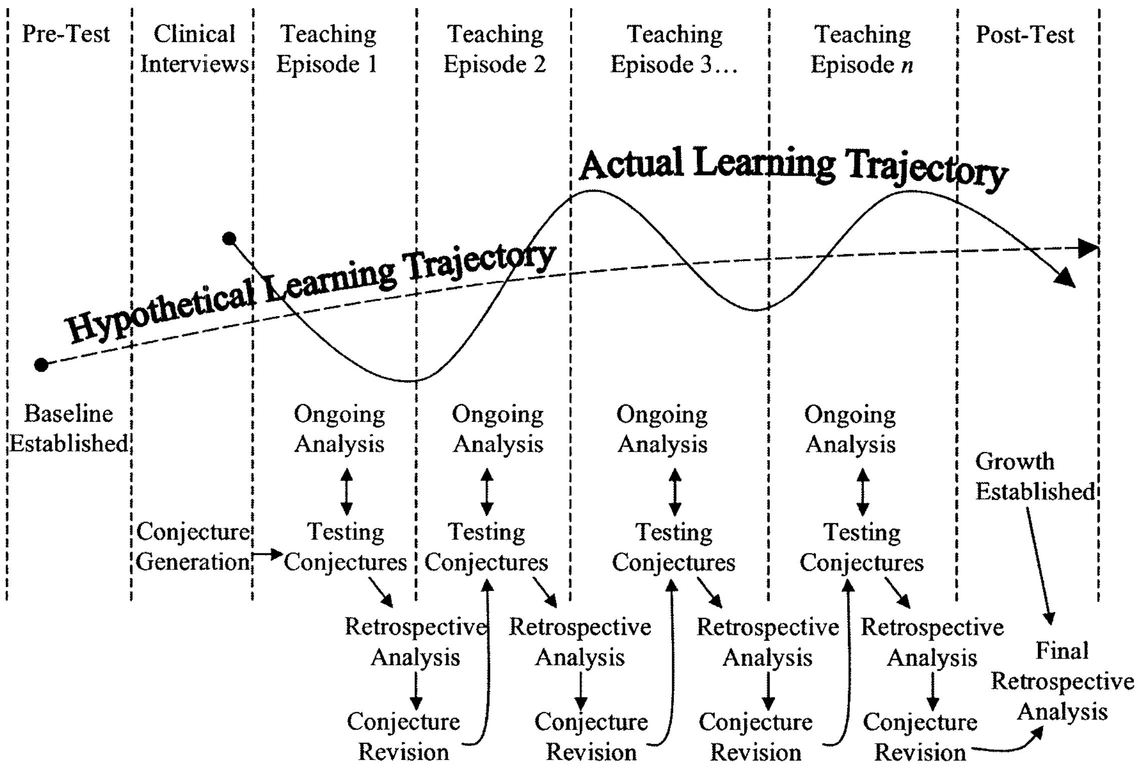


Figure 4. Methods of inquiry and cyclic process of conjecturing, testing, analyzing, and revising (Middleton, Carlson, Flores, Baek, & Atkinson, 2004).

After each teaching episode, the researcher and observer retrospectively analyzed the video to revise initial conjectures. These revised conjectures informed the next episode's activities. The cyclic pattern of conjecturing, testing, analyzing and revising continued through the duration of the teaching experiment to align the conjectured learning trajectory of the teacher with the actual learning trajectory. A final retrospective analysis was conducted to establish growth in the teacher's understanding of exponential behavior and to analyze the effectiveness of the teaching experiment in connection to the *actual learning trajectory.*

Setting and Participant

The next sections detail the setting for the present study, including the classroom, interview, and teaching experiment settings. This section also describes the participating teacher in the study.

The Classroom Setting

The classroom setting for this study was situated during the 15-week *Functions* course co-taught by a mathematics education professor and researcher along with a local mathematics and science high school district coordinator. The mathematics education professor also served as a consultant for this study. The *Functions* course was the first course these teachers completed in a sequence of four courses which (a) promoted effective habits of science, technology, engineering and mathematics (STEM); and (b) emphasized strong understanding of and ability to use the function concept. The *Functions* course focused on developing mathematics and science teachers' covariational reasoning abilities while building a profound understanding of proportional reasoning, linear function, quadratic function and exponential function through highly conceptual and collaborative activities. The concept of function remained a common thread in the three additional graduate courses this cohort of teachers completed.

Teachers in the *Functions* course met weekly for a three-hour session. Of the 18 secondary mathematics and science teachers, five expressed interest in pursing a Master of Natural Science degree. The *Functions* course was the first graduate-level mathematics education course for 15 of the 18 teachers. Classroom instruction was a combination of direct instruction, whole class discussion, collaborative learning and class presentations. Most class sessions utilized all of these modes of instruction during each three-hour session.

The classroom environment promoted and emphasized effective STEM habits. During the first class meeting, teachers negotiated "rules of engagement" between the professor and the teachers to facilitate the learning process. These rules were:

- *Speaking with meaning:* Teachers must speak meaningfully when engaging in conversation with their colleagues and the professor.
- *Persist in sense-making:* Teachers must not only persist in sense-making of their own problems and thinking but also their colleagues' thinking.
- *Exhibit mathematical integrity:* Teachers must base their conjectures on logical foundations and must not pretend to understand when they do not.
- *Respect the learning process of their colleagues:* Teachers must allow their colleagues to have the opportunity to think, reflect and construct their own understanding.

These rules of engagement continued to be reinforced and emphasized throughout all class sessions and research inquiries. While it is beyond this investigation to report the effects of implementing these rules of engagement, describing them here provides an image of the classroom environment and expectations of interaction among the teachers and professor. These negotiated expectations among the teachers were also emphasized in the interview and teaching experiments sessions of this current investigation.

As part of the *Functions* class culture, teachers were expected to utilize multiple technological resources during the semester. The most integral part of their learning involved the use of laptop computers with appropriate computer software, such as Fathom Dynamic Data™, Texas Instrument's TI-Connect™, and Pacific Tech's Graphing Calculator to aid in course instruction and facilitate their learning outside the classroom. Teachers were expected to bring their laptop computers to each class session and oftentimes they were given opportunities in class to analyze data using their computer. Class activities also incorporated Texas Instrument's Calculator-Based Laboratory TI-CBL 2™ and Calculator-Based Ranger TI-CBR™. The teaching experiment episodes incorporated all of the above technology to investigate teacher's understanding of exponential function.

The Interview and Teaching Experiment Setting

Each interview and teaching experiment session was videotaped using two cameras. One document camera was mounted in the ceiling and focused on the teacher's written responses. Another camera was mounted in the corner of the room and focused on the teacher to capture hand gestures and facial expressions. All interview and teaching experiment sessions were conducted in a small interview room containing a round table and a whiteboard. The table was equipped with a microphone to capture all audio correspondence.

All interview tasks and teaching experiment activities were provided to the teacher in print form during the sessions. The teacher was encouraged to think aloud during the tasks and to illustrate his thinking on the tasks provided. Furthermore, the teacher were reassured at the beginning of the session that his performance would not be measured in terms of the number of correct responses he provided to various tasks. Instead, the purpose of the session was to understand more of his thinking while working through exponential tasks.

The Participant

Non-probabilistic means of sampling, such as *extreme case sampling* where participants are selected based on their unique characteristics or special features was used to select one mathematics teacher for this investigation (Morrow & Smith, 2000). This type of purposeful sampling, defined by Patton (1990), provided an opportunity for special consideration to be given to a select participant who represented interesting characteristics.

The participant for this study was a secondary mathematics teacher who enrolled in the *Functions* course at a large southwestern university. The teacher received compensation at a rate of $35 for each hour he participated in research conducted outside the normally scheduled class meetings. After completing the PCA pre-test and the interviews, the teacher agreed to participate in an in-depth teaching experiment investigation on exponential behavior. The criteria used to select the participant for the teaching experiment were: (a) teacher must primarily teach secondary level mathematics, (b) teacher must agree to participate in all teaching experiment sessions, and (c) teacher earned an average score (between 13-15) on the PCA pre-test. The decision to focus on a teacher who performed at an *average* level afforded the opportunity to investigate the potential cognitive obstacles encountered when working through tasks involving exponential function.

The next section highlights the methods of data analysis chosen for this investigation.

Methods of Data Analysis

The data analysis process involved data from all sources previously described. This process provided an opportunity to make sense of the data corpus as a whole and as individual pieces of a larger story. The objective of data analysis of this study was two-fold: (a) to discover patterns and generate categories of teachers' thinking relative to exponential function (generative purpose) and (b) to establish connections between these categories as a tool for enhancing convergence of findings of the investigation (convergent purpose). Clement (2000) supports these two purposes for conducting a study involving interviews. Generative purposes, according to Clement, tend to lead to interpretive analysis where the purpose is to generate new observational categories and new elements of a theoretical model. Furthermore, Clement contends that convergent studies tend to lead to coded analysis where predetermined categories exist to guide the research and analysis of data. The present investigation embraced both the generative and convergent perspective of investigation the teacher's ways of thinking about of exponential function to provide reliable, empirically-based models for knowing and learning exponential function.

Given the motivation for investigating the ways of knowing exponential function, it was important to study and describe the ways in which the teacher knew this concept. .

Certainly, the knowledge of the teacher provides opportunities for the teacher to do mathematics using his/her own sensible approach. The nature of one's knowing is best captured through a *conceptual analysis* of one's thinking. Thompson (2000) states that one motivation for conducting a conceptual analysis is to "devise ways of understanding an idea that, if students had them, might be propitious for building more powerful ways to deal mathematically with their environments than they would build otherwise" (p. 428). Thus, this dissertation study employed a conceptual analysis to describe a mathematics teacher's ways of thinking about exponential function.

Furthermore, the approach to data analysis in this investigation was most consistent with the Miles and Huberman (1994) approach which provided a structured method for data analysis. The initial exponential function framework served as the lens throughout the data analysis process. This investigation followed three main stages of data analysis suggested by Miles and Huberman: (a) reduction of data into categories, (b) concept map development, and (c) warranting of assertions. The next section discusses these stages relative to the present investigation.

Stage 1: Reduction of Data into Categories

The first stage of analysis in this investigation involved the reduction of data into categories by partitioning the data into subsets and attaching codes to sections of the data. The use of coding software, called Studiocode (Sportstec), facilitated this process. This initial stage of analysis occurred in three phrases: (a) Phase 1: Analysis of PCA pre- and post-test scores, (b) Phase 2: Analysis of interview sessions, and (c) Phase 3: Analysis of teaching experiment episodes. The next sections detail the analysis for each phase of Stage 1.

Phase 1: Analysis of PCA Pre-Test and Post-Test Scores

The first phase of this stage of data analysis involved analyzing the PCA pre-test and post-test data for evidence of teacher's understanding of the concept of function and exponential behavior. All 18 teachers participating in the *Functions* course were ranked, from highest to lowest, based on their pre-test score. The participating teacher ranked eighth out of 18 teachers.

An evaluation of the teacher's performance on the six pre-test exponential tasks contributed to the analysis. This evaluation contributed to the characterization of the teacher's thinking of exponential function relative to other function tasks on the assessment. This categorization guided the trajectory of the interview tasks and the teaching experiment activities. Finally, post-test scores were analyzed to further quantify the conceptual growth of the teacher, both for overall performance of the assessment items and for specific exponential tasks. The analysis included the computation of Hake gain scores for the overall PCA test and for exponential items specifically (Hake, 1998). This analysis provided information about conceptual gains relative to the exponential items on the PCA assessment. Chapter 6 provides a detailed discussion of the results and analysis of the PCA scores.

Phase 2: Analysis of Interview Sessions

The next phase of data analysis focused on examining interview video data which provided the opportunity for the researcher to begin the initial open coding process, as suggested by Strauss and Corbin (1998). Strauss and Corbin's open and axial coding techniques facilitated the development of the coding structure for the interview data. Studiocode was used to construct the coding structure and attach codes on appropriate video clips marked as evidence of important characteristics for understanding exponential behavior. While the initial exponential function framework (discussed in chapter 3) guided the initial coding, the analysis remained receptive to new, emerging codes and categories.

In this phase, the analysis process involved multiple passes through the video to code segments, write memos about the nature of teacher's meaning, and construct categories by grouping the coded data. The purpose of analyzing the interview data was to become intimately aware of the ways of thinking about various function tasks intended to promote covariational reasoning, multiplicative reasoning, variable as a varying magnitude, rate of change and average rate of change. Because these mathematical ideas are central to understanding exponential function, it was important to develop knowledge of how teachers reason through these tasks before conducting the teaching experiment.

In an effort to build coder congruence, three reviewers independently coded transcript excerpts from the interviews using the initial exponential function framework. The first reviewer (R1), who had also served as the teaching experiment observer, provided an informed perspective to the coding process based on the teaching experiment observations. A second reviewer (R2) was a mathematics education doctoral student with over seven years of teaching experience at the community college level. R2 was involved in early versions of the exponential function framework and collaborated with the researcher in past preliminary investigations pertaining to teachers' understanding of exponential function. Lastly, the third reviewer (R3) holds a doctorate in mathematics education and has extensive research experience along with over 20 years teaching experience at the community college level. Moreover, R3 co-authored instructional materials focused on building strong conceptions of exponential behavior. All of the reviewers were experienced in coding using Strauss and Corbin (1998) coding techniques and they were familiar with the exponential function framework.

Once all three reviewers had coded portions of the interview transcripts, they collaborated with the researcher to reflect on the framework and the teacher's utterances in working through the exponential function tasks. The purpose of these collaborations was two-fold: (a) to increase inter-rater agreement of the coding structure using the initial exponential function framework, and (b) to improve the robustness of the framework through the lens of multiple expert researchers and educators. This technique also strengthened the research claims of the present study, which provided increased credibility and trustworthiness to the findings. These discussions facilitated in the refinement of the framework utilized in the teaching experiment analysis described next.

Phase 3: Analysis of Teaching Experiment Episodes

The analysis process of the teaching experiment episodes began by retrospectively analyzing each episode to (a) gather initial analyses of the teacher's ways of thinking about exponential function while working through the teaching experiment activities, and (b) refine the HLT for the teacher. Following the last teaching episode, the data were coded and analyzed using the revised exponential function framework. The retrospective analysis conducted during this phase further refined the conjectures generated about the teacher's ways of thinking about exponential behavior. Chapter 7 provides the teaching experiment results and analysis of the teacher's ways of thinking about and emerging understanding of exponential function.

Figure 5 provides an illustration of the analysis phases for the Exploratory Study and the present study. This diagram illuminates the various research influences that informed the framework.

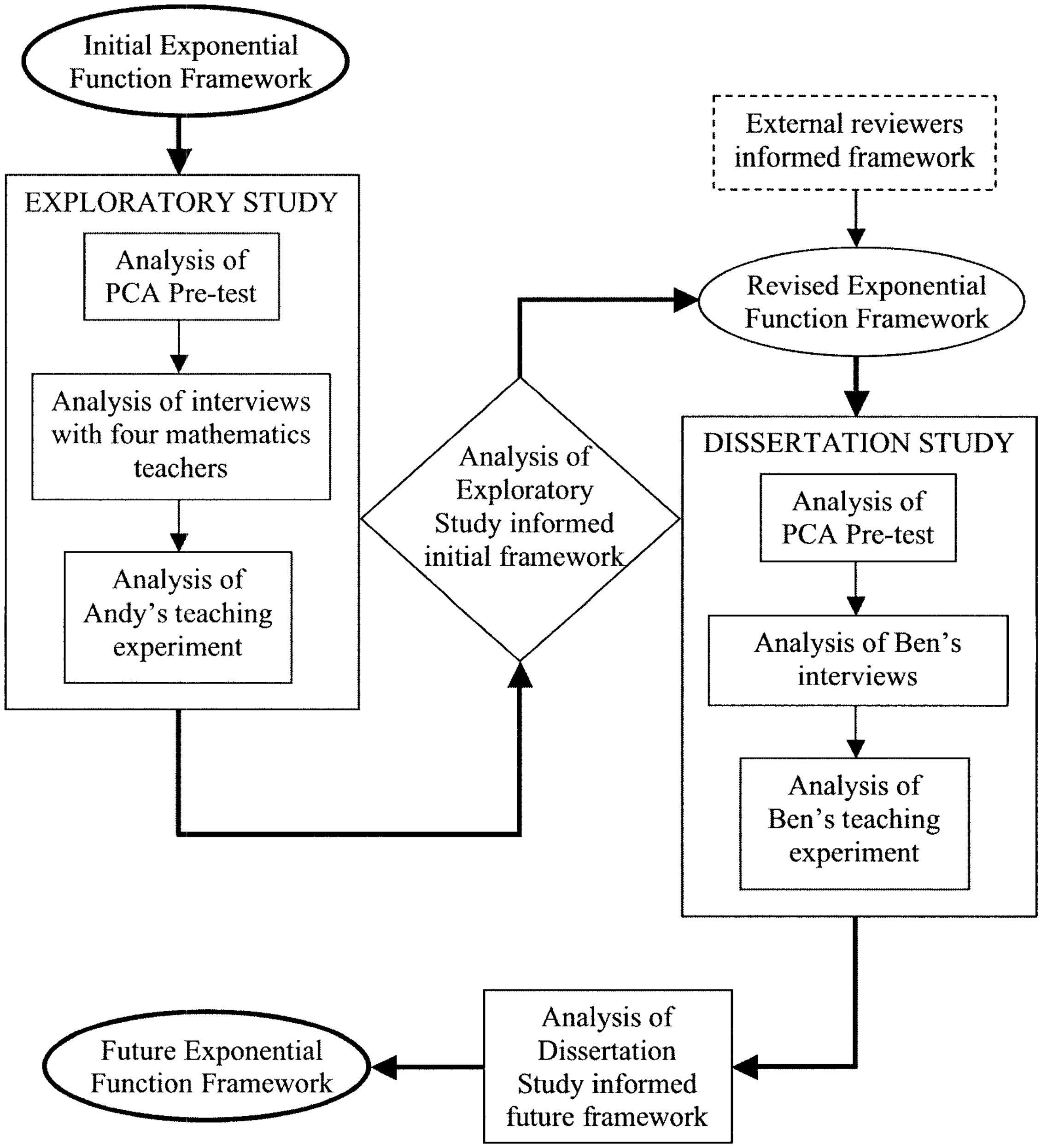


Figure 5. Framework revision process based on phases of analysis.

Stage 2: Concept Map Development

The second stage of data analysis involved diagramming the data in the form of a concept map. This visual aid served as a tool for making sense of the data. The process of arranging the data into a graphic organizer provided opportunities for the relationships and patterns to emerge from the data. The goal was to encapsulate the data by organizing events in chronological order and re-categorizing the data. This stage contributed to continuous refinements of the emerging categories.

This phase of analysis also involved writing memos to facilitate the decision-making process. The act of writing memos included journaling about interesting ideas, reflections, and assertions generated while in the process of analyzing the data. Most importantly, these memos included conjectures about the meaning of the teacher's utterances at various stages of the interview and teaching experiment episodes. These reflective memos became part of the data corpus.

Stage 3: Warranting Assertions

The third stage of the analysis process focused on drawing conclusions and verifying assertions constructed during the first two stages of analysis. It was at this stage of the process that assertions were warranted by finding evidence, both confirming and disconfirming, for the conclusions. Participant and peer checks took place at this stage to provide feedback and criticism to add robustness and credibility to the study. In this third stage of analysis, triangulation of data provided further evidence for warranting assertions. Data triangulation entailed surveying the entire data corpus for prevalence of findings among the multiple methods of inquiry. Any assertions remaining after this process became the final research findings for this investigation.

Chapter Summary

This chapter outlines the methodology for investigating a secondary mathematics teacher's ways of thinking about and emerging understanding of exponential function. The present study incorporated case study methodology and multiple methods of inquiry, including (a) pre- and post-test assessments, (b) semi-structured, task-based interviews, (c) classroom observations, and (d) an individual teaching experiment. Utilizing Steffe and Thompson's (2000) approach to teaching experiments, the study incorporated an in-depth teaching experiment of one teacher's understanding of exponential function. A hypothetical learning trajectory, including the learning goals, the learning activities, and the conjectured learning process, provided a structure for each of the teaching experiment episodes (Simon, 1995).

The settings for this study included the *Functions* course classroom, interview sessions and teaching experiment episodes. The participant for this present study included one mathematics teacher who participated in the *Functions* course and received an average score (between 13-15) on the PCA pre-test assessment.

The methods of data analysis for the present study occurred in three stages: (a) Stage 1: Reduction of data into categories, (b) Stage 2: Concept map development, and (c) Stage 3: Warranting assertions. Stage 1 encompassed three phases of focused on analyzing data from each of the three methods of inquiry: (a) PCA pre-test and post-test, (b) interviews, (c) classroom observations, and (d) teaching experiment.

The next chapter presents the results and analysis of the PCA pre-test and interviews conducted with the participating teacher.

CHAPTER 6: RESULTS AND ANALYSIS OF PRE-TEST AND INTERVIEWS

This chapter provides an account of Ben's ways of thinking about exponential function as he responded to a sequence of interview tasks. This account focuses on addressing the first research question of this study: What conceptions does Ben hold about the notions of exponential growth and decay? This chapter begins with background information about Ben and descriptions of the setting in which he teaches. Next, an evaluation of Ben's performance on the PCA pre-test is given. The main body of the chapter discusses in detail Ben's responses to five interview tasks that focus on exponential growth and decay.

Ben's Story

Ben is in his mid-twenties and is a new high school mathematics teacher who teaches second year algebra and mathematics-standards courses designed to help high school sophomores prepare for the Arizona's Instrument to Measure Standards (AIMS) test. Within the past year, Ben received his Bachelor of Arts in Secondary Education (Mathematics) from a local state university. He immediately accepted a teaching position at the same high school he graduated from six years prior. Ben's school, which enrolls approximately 2800 high school students, is located in an established middle-class neighborhood.

In addition to his teaching duties, Ben is also the head coach of the school's wrestling team and serves as an assistant football coach. He has a shy but friendly personality. His personal philosophy is "if you quit, you'll always regret it." This quote seems to appropriately explain Ben's personality. He has shown immense persistence when working on mathematical tasks that he found to be frustrating. Ben strives to deepen his understanding of the mathematics he teaches and struggles with how best to present topics to his students.

Throughout the investigation, Ben often mentioned that he prefers to mentally grapple with mathematics content before attempting to explain concepts to his students or colleagues. His persistence through problem-solving situations was strong, yet he demonstrated low confidence in his ability to reason through challenging mathematical tasks. He also frequently second-guessed his own explanations. When working with other teachers in the *Functions* course, Ben commented that he preferred to listen to others' explanations while he quietly absorbed the material. He stated that he enjoys working and learning in a group environment "where everyone is comfortable working with one another and feels free to voice their opinions/concerns with their instructor." He continued by saying that he is always "enthused and interested in learning more ways to better instruct math as well as deepen (his) knowledge and understanding of the subject."

PCA Pre-Test Results and Analysis

Ben was enrolled in the *Functions* course during the time period in which the research data for this study were collected. Scores from the PCA pre-test, which was administered at the beginning of the semester to all teachers participating in this course, were analyzed (see Table 10 for teachers' ranking). Although slightly above the class mean and median, Ben's PCA score was the lowest of the mathematics teachers' scores and his exponential pre-test score was among the lowest of the mathematics teachers.

Table 10

Results of the PCA Pre-Test

List of Teachers	Primary Discipline	Overall Pre-Test Score (out of 25 items)	Exponential Pre-Test Score (out of 6 items)
1	Science	21	6
2	Math	20	4
3	Math	19	3
4	Science	18	3
5	Math	16	4
6	Math	15	4
7	Math	15	3
Ben	Math	14	3
9	Science	13	2
10	Science	10	1
11	Science	10	1
12	Science	10	4
13	Science	9	2
14	Science	8	0
15	Science	6	1
16	Science	6	1

As illustrated in Table 10, Ben's overall pre-test score of 14 resulted in a ranking of 8 out of 16 teachers. The class scores ranged from 6 to 21 out of a possible score of 25. Scores were calculated by awarding one point for each correct response and zero points for each incorrect response for all 25 items assessed. The overall median score was 13.5 and the class mean was 13.31 with an overall standard deviation of 4.88. Table 11 summarizes these results and includes statistics based on the group of mathematics teachers as compared with the group of science teachers.

Table 11

Statistics of the PCA Pre-Test Results

	n	Low Score (max 25)	High Score (max 25)	Mean	Standard Deviation
Overall	16	6	21	13.31	4.88
Math	7	14	21	17.14	2.79
Science	9	6	18	10.00	3.71

Despite the small number of participants ($n = 16$), these results are comparable to other sections of the *Functions* course taught during the same semester as this investigation, with similar descriptive statistics findings for the individual mathematics, science and overall groups. As a result of the analysis and initial findings from the PCA pre-test, the exponential decay tasks seemed to pose the most difficulty for these teachers. Of the six exponential tasks, three were categorized as exponential growth and three as exponential decay. Ben correctly answered all three exponential growth tasks on the pre-test. However, he incorrectly answered all three decay tasks. Similar trends were found with other mathematics and science teachers in the course. These results suggest, therefore, that the exponential decay tasks posed more difficulty than the exponential growth tasks for this group of teachers. Results from the post-test assessment are discussed later in chapter 7. For now, the results from the interview sessions conducted with Ben are discussed.

Interview Results and Analysis

This section describes Ben's responses to five interview tasks and questions. Two of the three exponential decay tasks were also questions from the PCA assessment (i.e., Half of the Half-Life and What is the Half-Life?). Minor revisions were made to these two items to convert them from multiple-choice questions, as seen on the assessment, to more open-ended questions suited for the semi-structured interview.

Exponential Growth Interview Tasks

The exponential growth tasks presented here are the Population Growth task and the Salary Problem. These two tasks shed considerable light on Ben's ways of thinking about exponential growth.

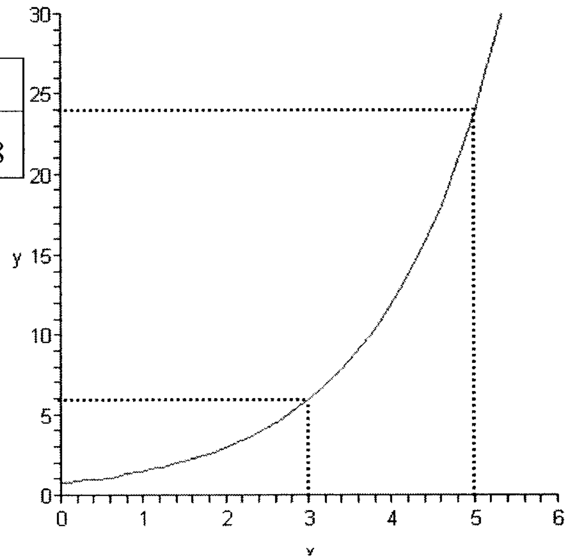
The Population Growth Task

The Population Growth task (Coe, 2007) called for Ben to consider average rate of change for an exponential situation. The follow-up questions probed Ben's thinking of changing rate of change and the idea of doubling. To answer these follow-up questions, Ben was prompted to reason multiplicatively to make sense of the task. The Population Growth task follows.

Population Growth Task

Imagine that you provided both a table and a graph illustrating population growth as a function of time (measured in hours) and you ask a student to find how fast the function is changing (growing) between $x = 3$ and $x = 5$. The student responds 18. What might the student have been thinking?

x	0	1	2	3	4	5	6
$f(x)$	0.75	1.5	3	6	12	24	48



The following questions were also asked for the purpose of gaining insight into Ben's multiplicative reasoning abilities:

- How would you like your students to interpret the table? [ENLM1]
- How would you like your students to interpret the graph? [ENLM2]
- What would the population be at eight hours? [ERMR]
- How would you describe to the student how fast the function is changing between $x = 3$ and $x = 3.5$? [ERCR, ERMR, ERPIR]

Implicit in this task was the necessity to recognize the multiplicative growth, namely the doubling pattern, which is modeled by the table of values and graph. The purpose of the first two follow-up questions was to prompt Ben to consider the doubling pattern relative to the values and graph displayed.

The third follow-up question was used to elicit Ben's ability to recognize and use the doubling pattern to make predictions about future output values. Requisite for answering this question was Ben's identification of the multiplicative factor of two and his action upon this factor as an object using recursion [ERMR2a] and repeated multiplication [ERMR2b]. Ideas about recursion and repeated multiplication were implicit in this question because of the necessity to repeatedly multiply by two for the sixth hour and again for the seventh hour to find the population after eight hours.

The final follow-up question was intended to learn how Ben reasoned about exponential behavior for interval sizes smaller than one unit for values of x not presented in the table [ERPIR]. In this example, the discussion focused on the half-interval to study Ben's description of exponential growth from hours 3 to 3.5 in relation to how he also previously described the overall growth pattern depicted by the table and graph.

The Population Growth task provided information about various components of the exponential function framework. Table 12 outlines the mapping of the Population Growth task to the framework.

Table 12

Matrix Mapping of Exponential Function Framework and Population Growth Task

Code	Framework Description	Population Growth Questions
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.	How would you like your students to interpret the table/graph?
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.	What would the population be at 8 hours?
ERCR	Use *covariational reasoning* to describe exponential behavior.	How fast is the function changing between $x = 3$ and $x = 5$? How would you describe to the student how fast the function is changing between $x = 3$ and $x = 3.5$?
ERFR	Use *exponential function reasoning* to compare and contrast exponential function with other function.	How fast is the function changing between $x = 3$ and $x = 5$? How would you describe to the student how fast the function is changing between $x = 3$ and $x = 3.5$?
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	What would the population be at 8 hours? How would you describe to the student how fast the function is changing between $x = 3$ and $x = 3.5$?
ERPIR	Use *partial-interval reasoning* to describe exponential behavior	How would you describe to the student how fast the function is changing between $x = 3$ and $x = 3.5$?

The next part of this discussion focuses on Ben's responses to the Population Growth task follow-up questions, with an emphasis on the last two follow-up questions.

Analysis of Ben for the Population Growth Task.

Ben was able to articulate how much the population changed for an interval of time between hours 3 and 5. He stated, "The 18 represented how much that population grew in that amount of time. From three hours to five hours, the population grew by 18" [ERCR2, ERLM4]. Ben's utterance provides evidence of his ability to describe the amount of population change in relation to the amount of time elapsed.

To investigate Ben's ability to recognize patterns among data tables, he was prompted to describe how he wanted his students to interpret the table. He stated that he wanted his students to recognize that the table provided the population value for a specific time. However, Ben did not mention anything about the doubling pattern exhibited in the table, nor did he mention doubling in his response to the next question, in which he was asked to explain how he would determine the population at eight hours. Ben's response is illustrated in Excerpt 1.

Excerpt 1

Ben: I would do it like if I were uh finding the function of anything [ENLM3]. The way I'd tell my students is if you're given a table like this, take the first values on the table, take the last values, find that slope and then also find out if there's any y-intercepts and then that way I can create a function for this [ENLM3]. Then I will be able to plug that x in to determine what my y or what my $f(x)$ will be at any point so I'm not just limited to what's in the table I can find it at any point in time.

Ben's response to the initial question about what he would like his students to acquire from the table of values demonstrated his knowledge of the dependency relationship among input and output values of a function. He stated that the amount of population depends upon the hours. When probed to predict a future value, Ben determined the rate of change of the line through the two end points. Ben's thinking, which seems to be deeply rooted in the concept of linear function, was to calculate the average rate of change function based on data provided in the table. Ben later said he would use his function to predict the output value when time was eight hours by plugging the value of eight into his function. Figure 6 illustrates Ben's work for producing a function to describe the data between the first and last data points. He neglected to consider that his linear model created from these points would underestimate the actual value of the exponential function at eight hours.

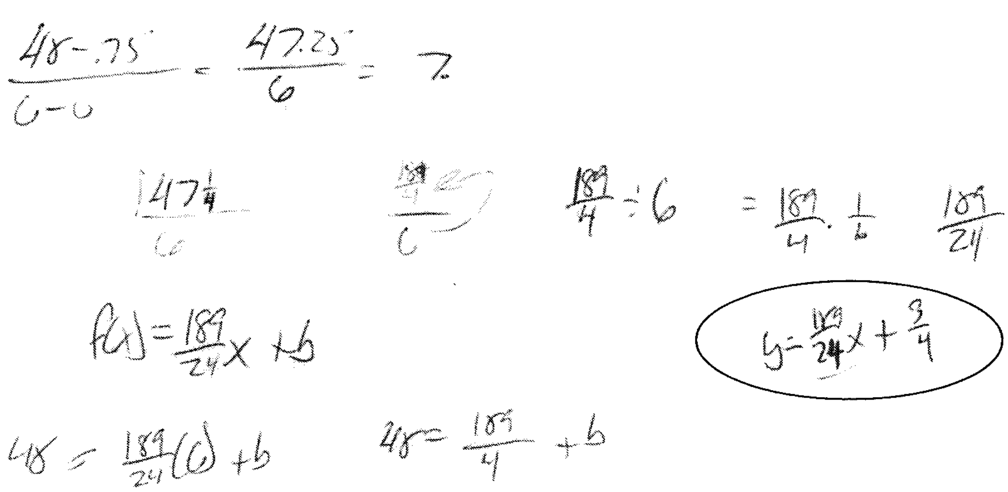


Figure 6. Ben's written work for finding his linear equation.

In Excerpt 1, Ben revealed his conception of data extrapolation which involved using the average rate of change function to calculate the population at eight hours. Ben viewed this linear strategy as sufficient for predicting future population values, even though the context was exponential. The reader will see in other interview excerpts that Ben seems to inappropriately rely on his understanding of linear function when grappling with non-linear situations, yet in some instances Ben's ability to reason about linear function helped facilitate his understanding of exponential behavior. This idea is discused in more detail throughout the chapter as the evidence emerges from Ben's performance on various tasks.

When prompted to discuss what he would want his students to glean from the graph, Ben explained that he wanted students to see the "jumps" as the population increased. Ben determined that the amount of change in population was increasing for incremental changes in time. Excerpt 2 reveals Ben's thinking when analyzing the graph from this task.

Excerpt 2

Ben: I would want them to see a jump from zero to one hours gave me,
you know, increased me by a certain amount but one to two hours I
increase by a lot more [ERCR2], two to three I'm increasing my
population even more [ERCR2] and then looking up as the graph
keeps going up, I'm not going up at a steady rate [ERFR2] I'm
going up a lot quicker so even more time I take the faster that
population is going to grow [ERCR2].

Ben referred to the time increase from one to two hours as constituting a larger increase than the population increase from zero to one hour [ERCR2]. He further explained how this pattern continued throughout the graph at a non-steady rate [ERFR2]. This utterance suggests that Ben was able to use his understanding of linear function to facilitate his thinking of the situation by distinguishing non-linear (non-constant rate) behavior from linear (constant rate) behavior. Ben's conceptualization of changing rate of change was illuminated when using the graph as a tool for making sense of the function [ENLM2], yet Ben did not mention any of the same changing rate of change characteristics when analyzing the table of values. For Ben, the graph rather than the table afforded him the means to express his conceptualization of changing rate of change [ENLM2]. This finding seems partially consistent with Rizzuti's (1991) findings that "students' concept images of functions as pairs of co-varying or corresponding values may be rooted in the tabular and graphical representations of functions" (p. 77).

As seen in Excerpt 1, Ben applied a procedure to find a linear function to describe the data table using the average rate of change to predict future values not listed in the table. His approach to making sense of the graph was conceptually different than his approach to making sense of the data table. He embraced the idea of exponential growth displayed on the graph as having a changing rate of change, but he did not connect this notion with the table when he viewed the data as representing linear growth. Additional evidence of Ben's notion of changing rate of change surfaced in another follow-up question in which the researcher asked Ben to determine how fast the same function was changing between $x = 3$ and $x = 3.5$. Ben's response to this question is revealed in Excerpt 3.

Excerpt 3

AS: Looking close at the table that we have here...[6]would you be able to...describe for me how fast the function is changing between 3 and 3.5, even though it's not provided there on the table?

Ben: I'd give an estimate...I would just use the formula that I just found and just plug it in there. So I'd take my y equals 189 over 24 and then for my x since that's my hours I would put in 3.5 hours.

Despite the researcher's prompting to use the table of values provided, Ben explained that he would use his average rate of change function to determine the population value after 3.5 hours. These data illustrate evidence of Ben's use of procedural thinking when working through this task. Ben did not realize that his average rate of change function was an overestimate of the true function value at 3.5 given that his linear graph is above the exponential curve at that point. Interestingly, Ben used the graph as a tool when he explored changing rate of change but ignored the graph instead preferring his average rate of change function when explaining how fast the function was changing. The data suggest that Ben holds a disconnected image of multiple representations of the same function. He ignored the connection between the provided graph and table of values as though they represented different function phenomena.

---

[6] Ellipses are used in transcript excerpts to represent intentionally omitted text.

Ben was asked to determine the population value at 3.5 hours using the graph instead of the average rate of change function. The purpose of this question was to prompt him to think about multiplicative behavior for interval length less than one [ERPIR]. Ben's approach to this question was to use the graph to determine the output value for an input value of 3.5. He decided that the approximate population value would be 8 given an input of 3.5.

Ben did not recognize his own disconnected thinking when making sense of information provided in the graph, table of values and his average rate of change function. He was unable to recognize the doubling pattern exhibited in the table and graph and he did not reason covariationally to make sense of the situation; that is, he did not consider *how* the output and input values were changing in tandem.

After additional probing by the researcher, Ben began to offer deeper insights into his own thinking about this task relative to his previous utterances. The next part of the discussion started out by comparing the population value of 8 that Ben obtained from the graph with the midpoint value of (3.5, 9) obtained from the points (3, 6) and (4, 12) provided in the table. This discussion centered on the observation that while the input of 3.5 was halfway between 3 and 4, the output of 8 was not halfway between 6 and 12. Excerpt 4 follows.

Excerpt 4

AS: So it would be nine, okay. Why do you think that happened, you
know the way the graph is drawn, that's kind of interesting that
halfway in the x's wasn't halfway in the y's. Why do you think

that is the case in this situation?

Ben: Well because even though I have an equation for this it's not exact because if I were to write this equation just going from zero to one I'd get a totally different slope, so my slope is constantly changing [ERFR2], it's going steep because this graph is going steeper and steeper and steeper and steeper and steeper heading up and up and up [ERCR2, ERFR2, ENLM2] so my difference going from zero to one is going to be different...the amount I increase from zero to one is going to be different from the amount I increase from like three to four [ERFR2] so my y's aren't going up at the constant rate [ERFR2] so that's right there what it deals with we're not going at a constant rate [ERFR2] so we're going halfway in between my hours is not going to give me halfway in between my population because it's not going at that constant rate [ERFR2].

At this point in the discussion, Ben was able to begin articulating his conceptual distinctions between constant rate of linear functions and changing rates for non-linear functions [ERFR2]. Based on lines 5-7 in Excerpt 4, Ben realized that his linear function previously created would not be "exact" due to his image of the changing slopes of secant lines as he moved from left to right on the graph [ERFR2]. In lines 11-13, Ben was able to explain that the increase obtained between hours zero and one would be different than the increase obtained between hours three and four, thus providing evidence that Ben can distinguish between constant rate and changing rate [ERFR2]. These data suggest that

Ben's ability to use partial-interval reasoning is not robust. That is, Ben only described the change in the first half of the interval as being *different* from the second half of the interval with no indication for his interpretation of *different*. As a result, the researcher could not code Ben's utterances as reasoning through partial-intervals. Instead, his reasoning was more aligned with the ability to view changing rate of change as different from constant rate of change [ERFR2].

During Ben's argument, he consistently referred to the graph as his tool for making sense of the situation rather than considering values of the table to help aid his understanding. When using the graph as the primary sense-making tool, Ben was better equipped to consider aspects of changing rate of change relative to the population growth situation. This finding further suggests that Ben's understanding of this function situation was best constructed from the graphical representation of the function while the table of values was seen as the medium for producing a workable algebraic function for finding more exact values of the function, even though he realized that the values obtained would not be accurate.

Summary of Ben's Thinking About the Population Growth Task.

After coding the transcript for this task, Ben's utterances were compared with the exponential function framework to categorize Ben's thinking. Table 13 illustrates the final analysis for the robustness of Ben's thinking in connection to the framework. An open circle indicates that Ben demonstrated no evidence of understanding of this component throughout the task. A partially highlighted circle suggests that Ben was able to demonstrate a particular ability in a non-robust manner, thus exemplifying weak evidence for that ability. For example, Ben received a partially highlighted circle for using multiple representations to represent the multiplicative process of exponential function (ENLM) because while he was able to use the graph to propel his thinking, Ben did not demonstrate evidence of recognizing similar patterns in the table of values. In this instance, the researcher could not say with certainty that Ben was able to connect multiple representations of exponential function together. A closed circle represents strong evidence of Ben's ability to demonstrate robust knowledge of this component of the framework.

Ben's inability to recognize the doubling pattern of the table of values provided in the Population Growth task suggests that Ben was unable to define the function recursively in this context using the tabular representation. The researcher coded this as exhibiting no evidence of an ability to view exponential behavior recursively. Ben's incomplete understanding revealed in this task provides a basis for discussing Ben's future progress in this research investigation.

Throughout this task, Ben demonstrated certain strengths as well as certain weaknesses to make sense of the situation. Relative to covariational reasoning, for example, Ben was able to demonstrate his image of the increasing slopes of secant lines as he varied the input values [ERCR2]. He also demonstrated partial evidence for his ability to reason about linear function. Additionally, he was able to articulate that the changes in output values were not increasing at a constant rate. However, Ben did not reveal that he understood that the output values were changing multiplicatively. He also did not demonstrate an ability to reason multiplicatively; that is, he did not consider the multiplicative growth pattern of doubling for this situation despite his realization of the changing rate. Further, Ben was unable to demonstrate the ability to reason through partial-intervals. He was able to describe that the change in the first part of the interval was less than the change in the second part of the interval. However, Ben only described these changes as being “different” without describing *how* the amounts of change were different. Table 13 summarizes these findings based on Ben’s utterances.

Table 13

Ben's Reasoning Model for Population Growth Task

Code	Framework Description	Analysis of Ben
ENLM ◍	Use *multiple representations* to represent the multiplicative processes of exponential function.	When using the graph, Ben was able to consider the changing rate of change aspect of the situation. However, he did not recognize the changing rate in the table of values.
ENLI ○	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben was unable to recognize the doubling pattern in the table. As a result he did not consider the recursive nature of the situation.
ERCR ◍	Use *covariational reasoning* to describe exponential behavior.	Ben was able to articulate how the slopes of secant lines would increase steeper and steeper as the input increased. He also provided evidence of knowledge of increasing differences.
ERFR ◍	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	Ben was able to articulate that the changes in output values were not increasing at a constant rate. However, he did not provide evidence that the output values were changing multiplicatively.
ERMR ○	Use *multiplicative reasoning* to describe exponential behavior.	Ben was unable to consider the multiplicative growth pattern for this situation despite his realization of the changing rate.
ERPIR ○	Use *partial-interval reasoning* to describe exponential behavior.	Ben was unable to articulate that the change in first part of interval was less than change in second part of interval. Ben only described these changes as being "different."
Evidence of *no* understanding ○	Evidence of *weak* understanding ◍	Evidence of *strong* understanding ●

The Salary Problem

The purpose of the Salary Problem task was to elicit Ben's conceptions of the relationship between linear growth and exponential growth. The idea of growing at a constant *percentage* rate as opposed to a constant *average* rate was incorporated in this task. The Salary Problem follows.

> *The Salary Problem*
>
> Sheena just graduated from college and recently received two job offers. The first job, an economist at a local firm, has a salary of $45,000 per year with a guaranteed raise of $1000 every year. The second job, a teaching position at a local college, has a salary of $35,000 per year with a guaranteed raise of 7% every year. Which position is the best job offer? Explain.

In addition, the following questions were posed:

- Describe the meaning of $1000 as a rate of increase. [ERCR, ERFR]
- Describe the meaning of 7% as a rate of increase. [ENLN, ENLL, ENLI, ERCR, ERMR]
- Provide a graph for each of the two situations described. [ENLM2]
- Write an algebraic representation for each salary situation. [ENLM3]
- Describe how the economist's salary is changing over time. [ERCR, ERFR]
- Describe how the teacher's salary is changing over time. [ENLI, ERCR, ERMR]
- Predict the economist and teacher's salary after 10 years. [ERMR]

The purpose of the follow-up questions was to (a) reveal Ben's thinking about the patterns in the changing values of the two coordinated quantities [ERFR, ERMR] and how the salaries grew over time [ERCR] and (b) provide insights about Ben's ability to reason covariationally and multiplicatively. These questions prompted Ben to use notation and language for describing the exponential growth factor for the teacher salary [ENLN, ENLL]. The interview probed Ben's ability to construct multiple representations of an exponential function [ENLM]. In addition, Ben was asked to create a graphical and algebraic representation for both salaries [ENLM2, ENLM3].

The Salary Problem provided opportunities to investigate these components of the exponential function framework. Table 14 illuminates the mapping of the Salary Problem to the components of the framework.

Table 14

Matrix Mapping of Exponential Function Framework and Salary Problem

Code	Framework Description	The Salary Problem Questions
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.	Can you write an algebraic representation for each salary situation?
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.	Describe the meaning of 7% as a rate of increase.
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.	Can you provide a graph for each of the two situations described? Can you write an algebraic representation for each salary situation?
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.	Describe the meaning of 7% as a rate of increase. Describe how the teacher's salary is changing over time.
ERCR	Use *covariational reasoning* to describe exponential behavior.	Describe how the economist's salary is changing over time. Describe how the teacher's salary is changing over time.
ERFR	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	Describe the meaning of $1000 as a rate of increase. Describe the meaning of 7% as a rate of increase.
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	Describe the meaning of 7% as a rate of increase.

Analysis of Ben for The Salary Problem.

Ben's response to this question was rooted in his observation of how the amount of raise each year was growing for the teacher's salary in comparison to the constant amount of increase for the economist's salary. He noticed that the economist's *salary* increased by a constant $1000 each year while the teacher's *amount of raise* also increased for each year that elapsed. These data provided evidence of Ben's ability to consider the first differences of output values as a means of sense-making. Based on this observation, Ben was able to determine that the teacher's salary was increasing at an increasing rate. Ben's sense-making strategy in this task was to conceptualize the amount of increase based on the previous salary amount and use this knowledge to analyze how the output values were changing. In Excerpt 5, Ben continued to explain his thinking relative to the amounts of increase for both salaries.

Excerpt 5

Ben: Yeah, it [teacher salary] is bigger than a 1000 per year raise [ERFR1]...you're still not starting at that $45,000 they're [economist salary] just going up to 46 already and you're still stuck at 37 [ENLM1] whatever, but the next year you're getting a raise on top of that [ERCR2] so you're getting the 37,450 but your 7% is coming off that 37,450 [ENLM1, ERCR3] so your amount that you just got raised up even more [ERCR2] so as the years continue to progress this jump's [points to the economist job] going up just $1000 each year [ERFR1, ERCR2, begins to air draw discrete points] just going to be here, here, here, here whereas this one [points to the teaching position] you start out a little lower but you're gonna start out go here it's going to be a little more of a raise then a little more of a raise then a little more of a raise

14 [ENLM4, ERCR2].

Ben described the teacher's increase as being "7% raise of whatever your previous year was." This utterance provides evidence of his ability to describe the exponential situation recursively [ENLI] while also considering the constant percentage growth by describing the changes in the output as the result of multiplying the previous output by a constant percent [ERMR2a]. Ben was able to describe the economist position as increasing by $1000 every year which illustrates evidence of his understanding of constant increase [ERFR1]. He was able to distinguish this behavior from that of the teacher salary which increased at 7% per year [ERMR]. In lines 10-12, Ben articulated how the teacher's actual raise each year will be "a little more" than the previous year's raise when increasing by 7% [ERCR3]. These statements provide evidence of Ben's ability to reason through situations that *increase* by a constant percent versus those which increase by a constant amount [ERCR3, ERFR1, ERMR2a].

Ben initially computed the salary amounts for various units of time to help him make sense of the task. The meaning of the situation emerged from Ben's procedure of computing output values and this process facilitated Ben's thinking in considering the effects of constant percentage growth. This evidence seems to suggest the need for Ben to convince himself of the pattern resulting from repeatedly taking 7% of the previous value by procedurally calculating successive output values before he was able to confidently conceive of the overall behavior of the function.

During this investigation, Ben consistently relied upon his thinking about exponentiation as a recursively-defined function: $\text{NEW} = \text{NOW} \cdot b$. At times, thinking about exponentiation as recursion provided powerful insights for him (as shown in Excerpt 6). However, this model of thinking also restricted him in carrying out the recursion to predict future output values (discussed later in Excerpt 7). The next excerpt illustrates Ben's thinking about the actual salary relative to amounts of increase.

Excerpt 6

Ben: 1 Your raises are going to continue to increase as your salary

2 continues to increase [ERCR2] because it's dependent upon

3 whatever that previous year's salary was [ENLI, ERMR2a].

Ben's statements in Excerpt 6 provide an account of his understanding of increasing by a constant percentage amount which seems to be deeply embedded in his ability to view exponentiation as a process of recursion [ENLI, ERMR2a]. He recognized that salary amounts were increasing due to the increasing amount of raises [ERCR2] and was able to coordinate this perspective based on the previous year's salary [ERMR2a]. He also drew a graph of his understanding of the teacher's salary compared to the economist's salary. Figure 7 shows Ben's graph.

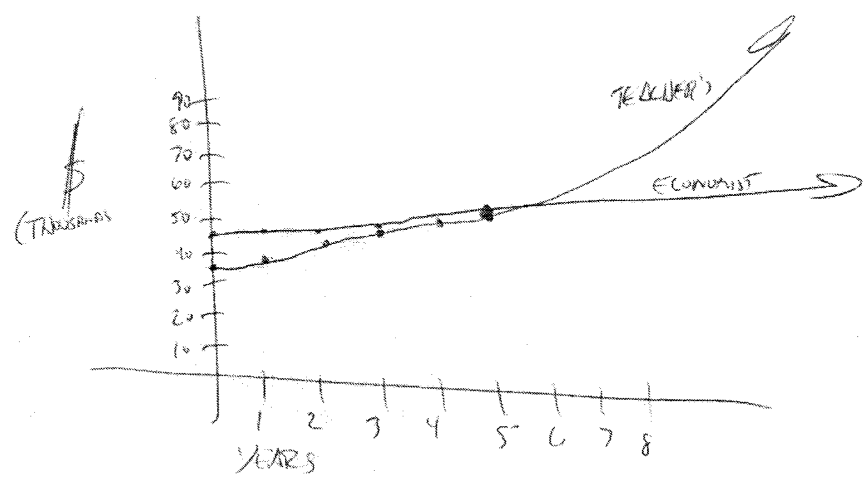


Figure 7. Ben's graph for the teacher and economist salaries.

Ben described the graph of the teacher salary as "arching up because of that jump each year, it's not the same amount each year." This statement illustrates his ability to coordinate increasing increases of the amount of raise with the graphical representation of "arching up." Ben also relied on his ability to reason about linear function by describing how the amount of change for the economist is constant while the teacher salary jumped more and more each year due to the constant percentage growth [ERFR1]. His utterance of the increasing jumps for each additional year also illustrates his ability to reason about the direction of change for the covarying quantities, but only for discrete values in outputs [ERCR2]. Ben was able to translate these ideas into a graphical representation of a concave up curve [ENLM2].

The next discussion focuses on Ben's response to predicting future salary values. In Excerpt 7, Ben articulated how he would predict the salary values for year 10 for these two situations.

Excerpt 7

Ben: You can easily predict what your [economist] salary's going to be in 10 years because that means your salary went up $10,000 [ERFR1] whereas with the 35 [teacher salary] it's a little more difficult to predict where 7% is going to take you in 10 years because it's 7% on top of whatever your previous salary was [ENLI, ERMR2a].

AS: So in order to get what year 10 was you…

Ben: You would have to know what year nine was [ENLI, ERMR2a].

This interview passage illuminates how deep-seated Ben's notion of exponential (constant percentage) growth is within the recursive model of how a future output value is calculated based on the previous output value [ENLI, ERMR2a]. The notion of exponential behavior as recursion seemed to restrict Ben from thinking about exponential behavior in terms of an exponentiation model where the growth factor is multiplied by itself by the number of compounding periods (i.e., the number of years passed). Based on exponentiation using a dynamic recursive model, the teacher's salary at 10 years can be found by taking the initial salary of $35,000 multiplied by $(1.07)^{10}$ where the growth factor of 1.07 represents the multiplicative object that proportionally connects the initial value and the desired output for a given input value. This dynamic model provides the opportunity to think about exponential behavior beyond recursion when previous output values are unknown. In further discussion, Ben was able to connect the growth rate of 7% as being 1.07 times the previous value as evidenced by his reasoning where he explained "I was taking the previous year and multiplying it by 1.07 instead of 0.07...because it's giving me that whole with an extra 7%, whatever that 7% is." However, he did not realize how to jump from an initial salary to the tenth year salary without knowing the intermediate salaries through year nine. As Ben stated in Excerpt 7 line 8, he would need to know the salary for year nine to find the salary in year 10. These data suggest evidence that Ben did not consider the process of repeated multiplication and connect this process with exponentiation at this point in the discussion. As a result, Ben's model of exponential behavior as recursion restrained him from considering how this process could be continued indefinitely. In a later chapter of this study, the researcher presents evidence of Ben's ability to begin moving beyond exponential behavior as static recursion to thinking about exponential behavior more as a process of exponentiation.

Continuing on with the discussion of this task, the researcher prompted Ben to further describe the 7% as a rate of increase. Excerpt 8 illuminates Ben's explanation.

Excerpt 8

Ben: It's a changing rate [ERFR2], I think that's the term I want to use because one year it's not going to be the same as the next year, it's going to be even more [ERCR2], so maybe ah (long pause) a type of exponential rate in that...the rate of change is always going up and up and up and up and up [ERFR2], I mean in this case it's not super drastic but if you went down far enough which it would just continue to keep rising and rising to the point where you are jumping up by leaps and bounds [ERCR2] but that would be after many, many years and most people have retired at that point.

After discussing this task for approximately nine minutes, Ben described the teacher's salary of increasing by 7% as being an exponential rate. His ability to describe this situation as exponential did not appear well-connected to his conceptions of how the salary would increase over time given his utterance in lines 3-4 of Excerpt 8. Ben understood the percentage growth pattern without linking to exponential notions until he grappled with what to call his changing rate idea. Labeling the teacher's salary increase as an exponential rate was not in the forefront of Ben's thinking.

The excerpt also points to additional evidence that Ben was able to reason through situations involving a changing rate of change [ERFR2]. Supported by Ben's response to the Population Growth task, an increasing amount of change in output values indicates a changing rate which Ben viewed as different from a constant rate [ERFR2].

Ben also demonstrated his ability to reason about the amounts of change of the covarying quantities. In lines 2-3 and lines 7-8 of Excerpt 8, he described the teacher's salary has having a changing rate where the salary would eventually be "jumping up by leaps and bounds" at some point in time [ERCR2]. Ben's ways of thinking about exponential growth seem to be embedded in his understanding of the increasing increases in salary with each additional year [ERCR2].

Ben was able to write the correct algebraic functions for both the economist's salary and the teacher's salary. He created the exponential equation, $\text{Teacher} = 35{,}000(1.07)^y$, to describe the teacher salary and the linear equation, $\text{Economist} = 45{,}000 + 1000y$, to describe the economist salary, where y represented the input in number of years for both situations. The data suggest that Ben was able to algebraically distinguish between situations that required repeated addition versus situations that required repeated multiplication [ERMR2b]. This idea points to the significance of conceiving exponential growth as repeated multiplication as a tool for making sense of constant percentage growth. However, Ben did not provide evidence previously of his understanding of repeated multiplication until he was considering how to write an algebraic relationship between the quantities of years and salary. Ben's ability to craft the exponential situation in the form of $f(x) = a \cdot b^x$ illustrates his interpretation of the explicit definition of an exponential function [ENLE].

The next section is focused on Ben's descriptions of the difficulties that students have in understanding and making sense of these two situations. In Excerpt 9, Ben articulated how students might attempt to use linear function reasoning to describe the exponential growth situation.

Excerpt 9

Ben: Just realizing that the 7% will change [ERCR3], like I think they might picture just 7% of 35,000 every year so like I could see them going ah 35,000 maybe plus 35,000 times 0.07 times however many years or setting it up like… $35{,}000 + 2450y$ [ERFR3], so seeing that the 7% each year is applied to the year you are at not the year you started [ERFR1, ERMR2a].

In line 1 of Excerpt 9, Ben stated that the "7% will change." This explanation was Ben's way of thinking about the effects that compounding by 7% will have on the salary [ERCR3]. In line 2, he stated "7% of 35,000" and subsequently computed consecutive salary values appropriately. While he uttered that the 7% would change, his calculations demonstrated his ability to hold the 7% constant as the number of years increased. In this explanation, Ben maintained how the progression of output values for the teacher salary was different than the progression of output values for the economist's salary of constantly adding $2450 for each year. Ben was able to articulate a common mistake made by students when thinking about exponential behavior, that is additively calculating the percent raise based on the initial value for all successive years. Rather, as Ben highlighted, the 7% is applied to the previous year and not just the year you started [ERFR1, ERMR2a].

Ben further explained (in Excerpt 10) how he would help his students overcome this (mis)conception of reasoning additively when grappling with an exponential situation.

Excerpt 10

Ben: Well, I'd say okay you took 7% of that salary which was 2450 after that one year they made the 2450. But how much is this person making now [ENLI]? Are they still making $35,000? No, they're not. They're making the 35 with their raise so next year in year two are they still going to be making a 2450 raise or did their raise go up [ERCR2]? Is the 7% raise still based off of their starting $35,000 salary or their salary they're at now [ENLI]?

Excerpt 10 provides further evidence of Ben's thinking about the progression of output values as a recursive process. The argument illustrates Ben's conception of exponential behavior as a static recursive process [ENLI]. Ben's image of recursion as a static, rather than dynamic, process seems to restrict him from thinking of exponentiation as repeated multiplication. In lines 5-6, Ben describes that the salary in the second year will bring a larger raise than the previous year [ERCR2]. This explanation is consistent with Ben's previous utterances discussed in this chapter (see Excerpt 6 and 7) and promotes the idea of recursion for snapshots of values with no indication of how the recursion process can be carried out indefinitely.

Summary of Ben's Thinking About The Salary Problem.

Similar to the Population Task summary, Ben's responses to the Salary Problem were reviewed to determine any emerging categories based on his utterances while working through the tasks. The components of the exponential function framework supported by the Salary Problem were compared to the codes used in characterizing Ben's utterances as he worked through the task.

As a result, Ben's utterances were coded to determine the robustness of his notation and language abilities relative to components of the exponential function framework. While notation and language for representing b^x as x factors of b for rational values of x was conjectured as important for responding to the Salary Problem, Ben was able to make sense of exponential growth without the need to demonstrate evidence of his ability to consider b^x in this manner. For example, Ben was able to consider that $3^2 = 3 \cdot 3$ but he did not provide evidence for his ability to transfer this knowledge of a multiplicative situation to exponentiation such as

$35{,}000 \cdot \underbrace{1.07 \cdot 1.07 \cdot 1.07 \cdot \ldots}_{x \text{ times}} = 35{,}000(1.07)^x$ even though he was able to write the algebraic function. His utterances suggest a more procedural approach to notating the exponential function without utterances of the multiplicative pattern.

Ben was able to provide evidence of his ability to use multiple representations as a sense-making tool for analyzing this task. He provided both a graphical and algebraic representation of both the teacher and economist's salary while also explaining this situation in context. Consistent with his way of thinking about the Population Growth task, Ben provided additional evidence of exponentiation as a recursive process. However, Ben did not use this knowledge to express how to extrapolate the teacher's salary for year 10 given that he did not know the salary value for year nine. This finding seems to be consistent with Ben's inability to recognize the doubling pattern revealed in the table of values from the Population Growth task. This inability to recognize the pattern progression seems to limit Ben's dexterity in holding stronger conceptions of exponential growth.

Ben was able to demonstrate evidence of his ability to reason through the Salary Problem using covariational and multiplicative reasoning tools. He was able to demonstrate his ability to consider the amounts of change of the covarying quantities as evidenced by his utterances that the teacher's monetary raise increased with each passing year as opposed to being held constant like the economist's raise [ERCR2]. Based on the evidence, Ben seems to be consistently reasoning covariationally at ERCR2 where he is able to coordinate the amount of change of the output values with changes in the input values by taking on the notion of exponential growth as "jumping by leaps and bounds" (see Excerpt 8).

During the discussion, Ben provided evidence of his ability to reason additively when comparing the economist's salary to the teacher's salary. Ben was able to compare the additive rate of change of $1000 per year for the economist salary with the multiplicative rate of change of 7% per year for the teacher salary as resulting in difference rates of increase [ERFR3]. Ben's thinking is deeply rooted in conceiving of exponential behavior as the result of multiplying the previous output value by a constant factor [ERMR2a]. Table 15 provides a summary of Ben's reasoning model relative to targeted components of the exponential function framework.

Table 15

Ben's Reasoning Model for the Salary Problem

Code	Framework Description	Analysis of Ben
ENLN ○	Use *notation* to represent b^x as x factors of b for rational values of x.	While Ben was able to express the situation as $35000(1.07)^y$, he did not provide evidence of this notation as representing x factors of b.
ENLL ○	Use *language* to describe b^x as x factors of b for rational values of x.	Ben did not provide evidence of this language.
ENLM ●	Use *multiple representations* to represent the multiplicative processes of exponential function.	Ben was able to construct a graphical and algebraic representation to describe both salaries.
ENLI ●	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben's thinking is deeply rooted in his conception of exponential behavior as recursively-defined (output value is based on previous output value).
ERCR ◍	Use *covariational reasoning* to describe exponential behavior.	Ben consistently reasoned covariationally at a ERCR2 level by articulating increasing differences for the teacher salary with some utterances about constant percent increases [ERCR3].
ERFR ●	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	Ben was able to compare the additive rate of change of $1000 per year for the economist salary with the multiplicative rate of change of 7% per year for the teacher salary as resulting in difference rates of increase [ERFR3].
ERMR ◍	Use *multiplicative reasoning* to describe exponential behavior.	Ben's thinking is deeply rooted in conceiving of exponential behavior as the result of multiplying the previous output value by a constant factor [ERMR2a].

Evidence of *no* understanding ○ Evidence of *weak* understanding ◍ Evidence of *strong* understanding ●

The data suggest that Ben holds a strong understanding of exponential behavior when compared to linear behavior in the context of the Salary Problem. The researcher conjectures that the context of money was helpful for Ben as he grappled with making sense of exponential behavior. In addition, the data suggest that using linear behavior as a conceptual springboard for making sense of exponential behavior proved to be beneficial in Ben's ability to successfully conceive this task.

A subtlety occurred in Ben's explanation of how he conceived of exponential behavior as a recursive process. He repeatedly stated how output values of an exponential function can be found by multiplying the previous value by the growth factor of 1.07. In Excerpt 7 Ben mentioned that in order to find the teacher's salary after 10 years it was necessary to know the value of the salary after nine years. Ben's understanding of the exponential progression was firmly rooted in his understanding of recursion, as further evidenced by his work on the Population Growth task. In Excerpt 1, Ben discussed how to extrapolate the data to find the amount of population after eight years (recall that data were provided for population values up to six years). Ben's approach was to simply craft a linear function using the first and last data point provided. Ben said he would use the function he created to plug in eight years as a way of finding the population value at that time. While Ben has a deep understanding of exponentiation as recursion, he is unable to move beyond this conception to think of exponentiation as a process. In the context of finding the teacher's salary after 10 years, thinking of exponentiation as a process would involve thinking about the salary as the initial salary times the growth factor multiplied by itself 10 times. Ben did not demonstrate this conception when grappling with exponential growth tasks.

Exponential Decay Interview Tasks

The next part of this discussion focuses on Ben's conceptions of exponential decay. The PCA pre-test data revealed that exponential decay was more difficult for the teachers in the *Functions* course than exponential growth. Recall that Ben was unable to answer correctly the three exponential decay questions that appeared on the PCA.

Because the exponential decay seemed to pose the most difficulty for these teachers, the researcher included both the Half of the Half-Life and the What is the Half-Life? tasks in the interview protocol. Both of these questions were altered from the multiple-choice format on the PCA to an open-ended format for the interview session.

Initial Thoughts About Half-Life

Prior to beginning the exponential decay tasks, Ben was asked to describe his initial thoughts about half-life and how he would approach this topic with his students. The goal of this question was to investigate whether he could describe half-life as representing the amount of time it takes for a substance to decay to half of an amount.

Analysis of Ben's Initial Thoughts About Half-Life.

Ben's initial thoughts about half-life focused on the connection with radioactivity, such as carbon dating. He stated, "I'm not entirely sure how that works" but he was able to offer the idea that half-life is "just figuring out how much it's decayed to figure out how old something is." This suggests evidence of his ability to connect exponential behavior (e.g., half-life) to real-world phenomena [ENLM4]. Ben's conception of half-life involved the notion of time and quantity of decay. Ben did not provide knowledge of half-life as representing the time it takes for a substance to decay to half of the previous (or original) amount.

Ben provided additional insight into how he would approach this concept in his teaching. Excerpt 11 illustrates his comments.

Excerpt 11

Ben: 1 ...after we put x amount of years times whatever the half-life is
2 and that's going to give us our, however old something is or how
3 long it'll date or something like that I guess would apply, you
4 know that would be our y...when you have what the half-life is
5 how we're gonna use that in figuring out how old something is
6 [ENLM4].

In lines 1-2, Ben explained how he would compute the age of something, which is by taking the input number of years multiplied by the half-life which produces the age. Based on Ben's utterances, the data imply that Ben holds an impoverished image of half-life and an incoherent image of the decay function. He also struggles with the appropriate language of explaining half-life behavior. It is unclear exactly what he meant by "amount of years times whatever the half-life is." Ben seems to be suggesting that the half-life of something is a value that only gets multiplied by the amount of years to return the output value, y. Part of his comment is true, however. Consider the exponential half-life function of the form $y = a \cdot \left(\frac{1}{2}\right)^{t/h}$ where a represents the initial value, t represents the number of years and h represents the known half-life time of a substance. In this form, it is true that the number of years is multiplied by the reciprocal of the half-life, yet this value does not solely produce the output value of y or the age of something as indicated by Ben. The researcher does not make the claim that Ben even considered such a function of this form, but this argument is used to illustrate the impoverished and imprecise nature of Ben's utterances.

Excerpt 11 illustrates Ben's attempt to explain how he would approach the concept of half-life with his students. His utterances suggest that Ben is confused about the meaning of half-life and is unable to articulate the input and output units of a half-life function. This confusion translates to an inability to fluently talk about half-life as the amount of time it takes for a certain amount of substance to reduce to half of the original amount of substance. Instead, the data suggest that Ben considers half-life as the age of something (see lines 2 and 5 of Excerpt 11).

The Half of the Half-Life Task

The analysis presented next focuses on the two interview tasks involving half-life. The first of these tasks, called Half of the Half-Life, highlights the rate of change aspect of half-life situations between integer half-life intervals. Covariational, multiplicative and partial-interval reasoning were conjectured as the most important reasoning abilities for this task.

Half of the Half-Life
The half-life of a radioactive substance is 1 hour. In 30 minutes, how much of the substance will decay?

This question proved to be the most challenging question for the teachers in this study based on the data from the PCA pre-test assessment. Ben answered this question incorrectly on the pre-test. For these two reasons, this task was included in the interview sessions and also became a major focus for the teaching experiment conducted with Ben. In addition to the main question, the following questions were asked:

- Do you think it will be less than ¼? More than ¼? Or exactly ¼? [ERCR, ERFR, ERMR, ERPIR]
- Can you explain your thinking using a graph? How? [ENLM2]
- Can you explain your thinking using the idea of repeated multiplication? How? [ERMR2b]

The main question required Ben to consider how much of the substance had decayed in half the time of the original half-life while the follow-up questions probed Ben's ability to use the graph [ENLM2] and repeated multiplication [ERMR2b] to justify his reasoning. This question targeted two predominate concepts: (a) half-life as involving as exponential decay process and (b) the exponential decay process as requiring multiplicative reasoning (by halving). This task facilitated the researcher in investigating specific components of the exponential function framework. Table 16 illustrates the mapping of the task with the framework.

Table 16

Matrix Mapping of Exponential Function Framework and Half of Half-Life

Code	Framework Description	Half of Half-Life Questions
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.	In 30 minutes, how much of the substance will decay?
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.	In 30 minutes, how much of the substance will decay?
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.	Can you explain your thinking using a graph? How?
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.	In 30 minutes, how much of the substance will decay?
ERCR	Use *covariational reasoning* to describe exponential behavior.	In 30 minutes, how much of the substance will decay? Do you think it will be less than ¼? More than ¼? Or exactly ¼?
ERFR	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	In 30 minutes, how much of the substance will decay? Do you think it will be less than ¼? More than ¼? Or exactly ¼?
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	Do you think it will be less than ¼? More than ¼? Or exactly ¼? Can you explain your thinking using the idea of repeated multiplication? How?
ERPIR	Use *partial-interval reasoning* to describe exponential behavior.	In 30 minutes, how much of the substance will decay? Do you think it will be less than ¼? More than ¼? Or exactly ¼?

Analysis of Ben for Half of the Half-Life Task.

Ben's response to this question on the PCA pre-test assessment was that the amount of decay would be exactly ¼ of the substance. He maintained this way of thinking when the researcher provided Ben with the same question during the interview. Ben's responses to this question, both from the pre-test and the interview session, involved additive reasoning with no indication of multiplicative reasoning. His explanation was "half of one hour is half of that half-life, which is just a quarter of that half-life" [ERFR0].

Ben's reasoning at this point in the discussion was deeply rooted in additive reasoning with no consideration for the exponential decay behavior inherent in this task. He considered the situation of half of the half-life to literally represent ½ of ½. According to Ben's reasoning, if 50% decays in one hour, then 25% decays in half an hour [ERFR0]. This type of additive reasoning at the beginning of the discussion was common among teachers from the *Functions* course.

Ben created a graphical representation of this situation (see Figure 8) as a means for verifying his thinking about half of the half-life. Excerpt 12 illustrates the evolution of Ben's thinking as he compared his initial thoughts to his graph.

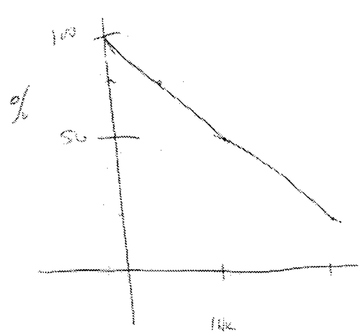


Figure 8. Ben's graph of half of the half-life.

Excerpt 12

Ben: I'm starting at 100, just saying like maybe just a % that's left over, so you start at 100% and then after one hour you're gonna only have 50% [ERMR1] of that substance left and go down and down, in 30 minutes there's not going to be 25% it's going to be a little bit different and ah, because um I'm trying to picture how it's going because in an hour it's going to be at 25 in another hour, in 2 hours, it's going to be at 25% [ERMR1], my graph is off but because it's exponential [ERMR1] it doesn't necessarily imply that in 30 minutes we're going to be halfway down [ERFR1] because that would mean, that would imply it's linear [ERFR1].

In lines 1-4 of Excerpt 12, Ben articulated his thinking about his image of the output as percents in relation to the input as time, rather than considering the relationship as a function of the amount of substance remaining as time elapsed. According to Ben, the outputs (represented as percents) decreased by a constant amount (i.e., by 50% for each hour that elapsed) for given intervals of time. In the beginning, Ben ignored the covariation of amount of substance remaining as a function of time and instead covaried the percentage as a function of time. Ben's utterances suggested that he was treating the percents as a quantitative measure of output (i.e., starting at 100%, then after one half-life the output is 50%, then 25%, then 12.5%, etc.) and then he coordinated this with the varying time.

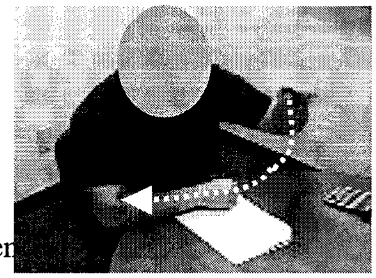
By using the graph as a tool for making sense of this situation [ENLM2], Ben was able to realize his incorrect thinking at the beginning of this discussion. In lines 8-10 of Excerpt 12, Ben states that the situation is exponential which means he cannot consider the halfway point for the dependent variable as literally representing half of the half-life, or 25% decay, since that would indicate a linear context [ERFR1]. The graph illuminated Ben's realization that the half of the half-life for this task could not be exactly 25% of the remaining substance. Ben recognized his disconnected thinking and decided that exactly halfway down would constitute linear decay.

This discussion provides support for the researcher's assertion that first thinking about half of a half-life in a linear context allows the subject to build a foundation from which to springboard conceptually to more advanced thinking, (i.e., thinking of the context as exponential). In essence, making sense of multiplicative contexts by first postulating about the situation as a potential additive process provided a powerful tool. In Ben's discussion, after thinking about half of a half-life as a linear context he was able to abandon this initial idea and move forward to thinking about half of a half-life as not representing linear behavior. This notion provided Ben the opportunity to consider the context as representing exponential behavior. Furthermore, it provided an opportunity for him to consider the exponential context as "going down at a different rate" as evidenced in the next discussion (see Excerpt 13).

Excerpt 13

Ben: 1 So at directly half an hour I know I'm going to be halfway in
2 between my halfway, but that's not what we are dealing

with here, we're dealing with exponential [ERFR1] where it's just going down at a different rate (see picture above), at differe



ERFR2] after every so many minutes the rate it's going down is increasing [ERCR2], the rate it's decaying is increasing [ERCR4].

In line 1-2 of Excerpt 13, Ben described the situation again with additive reasoning undertones where he mentioned that he was going to be halfway between his halfway. He then explained that this description was not indicative of the half-life situation and began to verbalize his thinking using his conception of rate within this exponential context. The notion of changing rate of change [ERFR2] prompted Ben to consider this situation as exponential by considering that the rate of change for this exponential situation is increasing as time goes on [ERCR4]. Ben's thinking of the rate of decay as increasing illuminates his ability to conceive of the dynamical relationship between the amount of decay and the rate of decay. Despite the evidence of Ben's utterances about decreasing at an increasing rate, he did not verbalize his thinking about the multiplicative process of how the decrease was occurring. Ben described the decrease qualitatively with no indication of an understanding of the quantitative decrease. Recall that in the Population Growth task, Ben described how fast the function was changing between $x = 3$ and $x = 3.5$ by saying the change in the first part of the interval was *different* than the change in the second part of the interval. He did not provide evidence of *how* the changes were different. Both instances of the Population Growth task and the

Half of the Half-Life task illuminate Ben's ability to think about exponentiation as a qualitative comparison of rates. The evidence suggests that he views linear behavior as having a constant increase or decrease while exponential behavior involves a different (or changing) increase or decrease with no indication for how the increase is different (such as multiplicative increase or constant percentage increase).

In the next portion of the discussion, the researcher asked Ben to comment on whether the amount of substance remaining in 30 minutes would be exactly 75%, between 100% and 75% or between 75% and 50%. Excerpt 14 illustrates Ben's comments to this question.

Excerpt 14

Ben: 1 In 30 minutes, I think it's going to be somewhere between 75 and
2 50 [ERPIR1b]...just because that rate it's changing is increasing
3 and increasing [ERCR4] so at 30 minutes...it's going to be
4 increasing a little bit more so that it will get down to the 50%
5 [ERPIR1b], so I'm thinking that when it, it's going to hit 75%
6 somewhere before that 30 minutes [ERPIR1b].

Following this discussion, Ben was able to articulate that the amount of decrease would be more than 25% at 30 minutes. His conception of the rate of this exponential decay situation as increasing facilitated his notion that more than ¼ of the substance will have decayed in half of the half-life. Thus as Ben stated in line 5, the graph will hit 75% (or ¼ of the substance has decayed) before 30 minutes has elapsed. A comparison of Ben's initial response to this question on the PCA pre-test and during the interview session illustrates an important evolution of his thinking. Initially, he responded "*exactly* 25%" of the substance had decayed, but after making sense of the context during the interview, he finally responded, "*more than* 25%" of the substance had decayed in 30 minutes.

Ben's initial response to this question, both from the PCA pre-test and interview session, of *exactly* 25% of the substance has decayed to his final response of *more than* 25% of the substance has decayed in 30 minutes illustrates the evolution of Ben's thinking while making sense of the context through reasoning [ERPIR1b] and utilizing the graphs of both linear and exponential contexts [ENLM2].

As an extended investigation, the researcher asked Ben to explain the idea of half-life and half of a half-life as repeated multiplication, with an emphasis on the factor that would be repeatedly multiplied. Excerpt 15 illustrates Ben's response.

Excerpt 15

AS: Would you be able to explain this concept, the idea of half-life, using the idea of repeated multiplication? First of all, what would we be repeatedly multiplying by?

Ben: ½

AS: How often do we do that?

Ben: Well, so every one hour it's going to be at ½ [ERCR1].

AS: It's going to be at ½?

Ben: Well, it's gonna be, well you're using the ½, at one hour it'll be ½, at two hours it'll be ¼, at three hours 1/8th [ERCR1] and just going down and down and down like that [ENLI, ERMR2b].

AS: So we would say that every hour we multiply by?

Ben: We would multiply it by itself, multiply ½ by itself [ERMR2b].

AS: So, what would we be repeatedly multiplying by every 30 minutes,
if every hour we multiply by ½?

Ben: Um, if I split that hour into 60, that would be ½ to the ½ [ENLN],
so it would be... one over whatever the $\sqrt{2}$ is (see Figure 9)
because $2^{1/2}$ [ENLN] is because ½ in the exponent is the same
thing as saying square root of something and the $\sqrt{2}$ is definitely
not the same as ¼ ... it's going to take more because $\sqrt{2}$ is like 1
point something something, it's going to be a lot closer to taking
that much away I guess I want to say. It's going to be a bigger
percentage [ERMR1], I suppose, than just taking away 25%, it will
be taking away a little more than that [ERMR1, ERPIR1b].

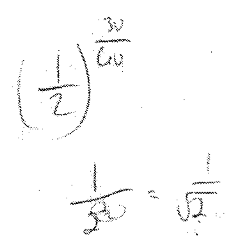


Figure 9. Ben's expression for half of the half-life.

Using the idea of repeated multiplication, Ben was able to explain in lines 8-10 that every hour he would multiply by ½ thus after two hours the calculation would be ½ times ½ and then after three hours the calculation would be ½ times ½ times ½ [ERMR2b, ERCR1]. Consequently, Ben was able to extend his knowledge of this repeated multiplication to the situation of half-intervals by first conceiving of one hour as being 60 minutes and then considering the exponent of $\frac{30}{60}$ reducing to ½. Ben was able to translate half of a half-life notationally as $\left(\frac{1}{2}\right)^{1/2}$ and reduced this expression to $\frac{1}{\sqrt{2}}$ [ENLN]. Ben struggled in articulating his understanding that since $\frac{1}{\sqrt{2}}$ is not equal to $\frac{1}{4}$ the substance could not decay 25% in 30 minutes. These data provided evidence of Ben's ability to use notation to represent b^x as x factors of b [ENLN], but experienced difficulty in using the language to describe this idea [no ENLL]. Ben's explanation seems to suggest that he was able to view $\frac{1}{\sqrt{2}}$ as representing a number larger than $\frac{1}{4}$ and thus the amount of decrease would be more than $\frac{1}{4}$ or 25%. In lines 22-23, Ben settled on thinking that the percentage of decay was more than 25% in 30 minutes. This discussion further suggests that Ben is beginning to reason multiplicatively [ERMR2b], but only after initially applying his knowledge of linear behavior to this context [ERFR0] and ultimately abandoning linear thinking as a way of reasoning through this task. At the end of the discussion, Ben seemed to be thinking about the task using partial-interval reasoning, although he had trouble articulating his thoughts. In lines 18-23, Ben described how the amount of decrease in the first half of the interval would be more than the amount of decrease in the second half of the interval. The researcher coded this utterance as the beginning of Ben's ability to use half-interval reasoning [ERPIR1b]. Ultimately, Ben was unable to provide evidence of conceptualizing half of a half-life of $\left(\frac{1}{2}\right)^{1/2}$ as representing ½ factors of ½. This indicated possible conceptual barriers of Ben's ability to use appropriate language to represent b^x as x factors of b for rational values of x.

Summary of Ben's Thinking About the Half of the Half-Life Task.

In the Half of the Half-Life task, Ben was able to overcome his initial additive thinking to identify the situation as exponential. By the end of the task and with probing by the researcher, Ben was able to use partial-interval reasoning to articulate how the difference of the first half-interval was *more* than the difference of the second half-interval [ERPIR1b]. The Half of the Half-Life task required Ben to consider the amount of decay after only half of a half-life and thus implicitly demanded that he consider changing rate of decay as opposed to constant rate of decay.

Ben ability to view half of a half-life as $\left(\frac{1}{2}\right)^{1/2}$ notationally [ENLN], while expressing understanding that the exponent of ½ referred to the amount of substance remaining after 30 minutes, seemed to facilitate his ability to finally reason through the task successfully [ERMR2b]. Viewing half of a half-life as the expression $\left(\frac{1}{2}\right)^{1/2}$ provided the platform for extended reasoning opportunities of conceptualizing $\left(\frac{1}{2}\right)^{1/2}$ as the factor which multiplied by itself twice equals the full factor of ½ and also as representing ½ factors of ½ [ENLN, ENLL, ERPIR, ERMR]. These notions of multiplicative reasoning for exponential function situations were conjectured as vital for making powerful connections among the context of the situation and the mathematics involved. Based on the findings from the interview sessions, the researcher focused on the notion of half-interval and partial-interval reasoning in the teaching experiment sessions conducted with Ben. Table 17 provides a summary of the analysis of Ben relative to the Half of the Half-Life task.

Table 17

Ben's Reasoning Model for Half of the Half-Life

Code		Framework Description	Analysis of Ben	
ENLN	●	Use *notation* to represent b^x as x factors of b for rational values of x.	Ben was able to notate the amount remaining at half of the half-life as $\left(\frac{1}{2}\right)^{1/2} = \frac{1}{\sqrt{2}}$.	
ENLL	○	Use *language* to describe b^x as x factors of b for rational values of x.	Ben did not use appropriate language to describe $\left(\frac{1}{2}\right)^{1/2}$ as ½ factors of ½.	
ENLM	◍	Use *multiple representations* to represent the multiplicative processes of exponential function.	Ben was able to illustrate his understanding of exponential decay use a graph. However, his graph did not display concavity.	
ENLI	●	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben was able to use recursion to describe repeated multiplication of ½.	
ERCR	●	Use *covariational reasoning* to describe exponential behavior.	Ben was able to use covariational reasoning to describe increasing increases in the outputs relative to increases in the inputs. He also described the rate of decrease as increasing for increases in time.	
ERFR	●	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	Initially, Ben conceived of this task as an additive rather than multiplicative situation. After constructing the graph, Ben realized his (mis)conception and described the decay behavior as exponential.	
ERMR	●	Use *multiplicative reasoning* to describe exponential behavior.	After probing by researcher, Ben was able to use repeated multiplication to describe the progression of output values.	
ERPIR	◍	Use *partial-interval reasoning* to describe exponential behavior.	Ben attempted to describe the decay behavior as decaying more in first part of interval than in second part.	
Evidence of *no* understanding	○	Evidence of *weak* understanding ◍	Evidence of *strong* understanding	●

The What is the Half-Life? Task

The What is the Half-Life? task appeared on both the PCA pre-assessment and interview protocol. Ben's responses to this task are presented along with his way of thinking about this problem during the interview setting.

What is the Half-Life?

A radioactive substance decays according to the function $A(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5}}$ where t is measured in years and A is the amount of the unstable portion of the substance in micrograms. How long does it take for half of the substance to decay? In other words, what is its half-life?

In addition to the main question, the researcher probed Ben with these questions:

- Will it take as long to lose half of its substance when it is 35 kg as when it is 0.25 kg? Why? [ENLP, ERCR, ERMR]
- What difficulties do students encounter when identifying half-life? [ERMR]
- How do you help students understand the concept of half-life? [ERMR]

This task provided an opportunity to explore Ben's understanding of half-life relative to the algebraic representation [ENLE, ENLM]. For Ben to make sense of this task, the researcher conjectured that he would need to reason multiplicatively by considering the frequency of multiplying by ½ [ERMR]. Also, the researcher probed Ben's thinking about the effects of the half-life time when changing the initial value [ENLP, ERCR, ERMR]. The discussion explored about Ben's awareness of the difficulties with half-life that students encounter and how he helps them build an understanding of half-life [ERMR]. Table 18 displays this task mapped with the exponential function framework.

Table 18

Matrix Mapping of Exponential Function Framework and What is Half-Life?

Code	Framework Description	What is the Half-Life? Questions
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.	How long does it take for half of the substance to decay?
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.	How long does it take for half of the substance to decay?
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.	Will it take as long to lose half of its substance when it is 35 kg as when it is 0.25 kg? Why? What difficulties do students encounter when identifying half-life? How do you help students understand the concept of half-life?
ENLP	Use *parametric changes* to alter representations of an exponential function.	Will it take as long to lose half of its substance when it is 35 kg as when it is 0.25 kg? Why?
ENLE	Use *explicit definition* of exponential function to represent exponential situations explicitly.	How long does it take for half of the substance to decay?
ERCR	Use *covariational reasoning* to describe exponential behavior.	How long does it take for half of the substance to decay? Will it take as long to lose half of its substance when it is 35 kg as when it is 0.25 kg? Why?
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	How long does it take for half of the substance to decay? Will it take as long to lose half of its substance when it is 35 kg as when it is 0.25 kg? Why?

Analysis of Ben for What is the Half-Life? Task.

On the PCA pre-test assessment, Ben selected the response of $\left(\frac{1}{2}\right)^{\frac{1}{5}}$ years as the half-life of the substance. When this half-life question arose in the interview setting, Ben altered his response for the half-life as being five years and offered insight into his new thinking. Ben's reasoning is revealed in Excerpt 16.

Excerpt 16

Ben: 1 I would say it's half-life is five years because since it's $t/5$ in the
2 exponent and we have 100 and I'm saying that 100 is representing
3 100% of that substance[7] [ENLE], then we have ½ which is, if we
4 just multiplied by ½ would give half of what's left over [ERMR1]
5 but since it's $t/5$ in five years that would give us 5/5 which $\left(\frac{1}{2}\right)^{1}$
6 [ENLN, ENLL] is just ½ so 100 times ½ [ENLE] gives you half of
7 that so the substance's half-life would be five years [ERMR1].

Even though Ben initially answered this question incorrectly on the PCA pre-test, he was able to provide the correct answer and justification during the interview for the half-life as being five years. Using his knowledge of the explicit definition of an exponential function [ENLE], Ben offered his reasoning that entailed the notion of finding when the exponent of $t/5$ would be exactly one. Thus, he was able to determine that if $t = 5$, then the reduced exponent would be one [ERMR]. This would allow for a full factor of ½ and when multiplied by 100 would obtain the desired 50 or half of the

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[7] It is not clear why Ben conceived of the initial value of 100 to represent 100%.

original amount of the substance [ERMR1]. This way of thinking provides evidence for consideration of thinking about the exponent of one as representing a full factor of the base [ENLL]. For example, if $\left(\frac{1}{2}\right)^1$ was conceived of as one full factor of ½ [ENLN, ENLL] then this notion provides the opportunity to consider fractional exponents as representing partial factors of the base number. In this situation, the full factor of ½ is obtained from multiple partial factors of ½, or in essence $\left(\frac{1}{2}\right)^{1/5}$, and in this case the number of partial factors multiplied together would be five to create a full factor of ½. In Excerpt 17, Ben describes this idea further after the researcher prompts him to explain what ½ represents in the provided algebraic function of $A(t) = 100\left(\frac{1}{2}\right)^{t/5}$ [ENLE].

Excerpt 17

Ben: That was just giving me where that ½ of the 100 would come from [ENLE], where that half-life is because where there's only going to be 50% of the original [ERMR1], yeah it's just used to determine when half that substance is left [ERMR1, ENLL] but then you also have the $t/5$ which is...more going to tell you exactly when it is so it's letting you know the ½ is there to let me know at what point whenever I get to that ½ [ENLL] that's where I'm finding half that substance [ERMR1] so it's saying whenever this $t/5$ just gives me the ½ [ENLL] that's when I'm going to have ½ that substance left [ERMR1] after however many t it takes to get just to be one up

11 there [refers to the exponent, ENLL], something like that.

According to Ben, the ½ as a decay factor is used to determine "when half that substance is left." Although Ben's choice of words to describe his thinking seems to be imprecise in the first half of this excerpt, he is nonetheless able to begin using the language to describe how he views the role of the exponent. In lines 4-8, Ben describes the exponent $t/5$ as the indicator for the length of time it takes to get half of the substance[8]. This explanation suggests that Ben is beginning to think about the exponent as the number of multiplied factors [ENLL]. In lines 10-11 of Excerpt 17, Ben describes his thinking about "however many t it takes to get just to be one up there [in the exponent]" which illustrates his observation that a fractional exponent represents a partial factor [ENLL] and reveals his interest in obtaining a full factor of ½ [ERMR]. In this excerpt, Ben seems to struggle with finding the language to express his thinking for describing what b^x means when x is a fraction. The data suggest that conceiving of fractional exponents in this exponential context as representing partial half-lives could provide a powerful catalyst for building more advanced conceptions of fractional exponents (i.e., thinking of $\left(\frac{1}{2}\right)^{\frac{1}{5}}$ multiplied together as representing $\frac{1}{5}$ factors of ½ while simultaneously holding in mind that five factors of $\left(\frac{1}{2}\right)^{\frac{1}{5}}$ would produce a full factor of ½. This idea is similar to Thompson and Saldanha's (2003) notion of thinking about multiplication multiplicatively in which they state "To think of multiplication as

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[8] This type of thinking is aligned with the conceptualization of half-life in the form of $f(x) = a\left(\frac{1}{2}\right)^t$.

producing a product and to think at the same time of the product in relation to its factors entails proportional reasoning" (p. 120). Correspondingly, to think of exponentiation as producing a multiplicative product and simultaneously thinking about the product in relation to the exponent (number of factors) is the essence of proportional reasoning in combination with multiplicative reasoning.

The next portion of the interview discussion was focused on Ben's understanding of how changes in the initial value affect the progression of output values [ENLP]. The researcher probed Ben to discuss whether the half-life would be the same if the initial value changed from 0.25 kg to 35 kg. As Excerpt 18 shows, Ben was able to articulate how the amount of substance would not impact the time it takes for the substance to lose half its amount.

Excerpt 18

Ben: So every five years is when that substance is getting split in half [ERCR3, ERMR2b], that's not going to change because it's like that example you gave where 35 grams as opposed to two grams it doesn't matter which one I have it's still the same substance [ENLP, ERMR], it's still decaying at the same rate [ERMR2b, ERCR4], so decaying whether you're starting with the 100 and going after that every five years [ENLP, ERCR3] whatever it was previously the five years is going to be half of what that was [ENLP, ENLI, ERMR1, ERMR2a].

In lines 1-2 of Excerpt 18, Ben explained his thinking about how the substance decaying by one half every five years would remain constant as time progresses. This utterance suggests his ability to coordinate the constant percent change of the covarying quantities by considering the constant multiplication by ½ every five years [ERCR3]. Illustrating his point with varying initial values, Ben was able to articulate that the rate of decay would remain constant despite the value of the initial amount of substance thus illustrating his ability to express how parametric changes in the initial value affects the situation [ENLP]. Ben's conception of half-life seems to be consistent with his notion of recursion: namely, the amount of substance remaining after a half-life is half of the previous amount of substance remaining five years prior [ENLI, ERMR2a]. Lines 6-8 in Excerpt 18 provide evidence of Ben's thinking of exponentiation as recursion based on the *previous* amount [ERMR2a].

The next part of the discussion focused on coordinating the decreasing amount of remaining substance with increasing time [ERCR]. The researcher prompted Ben to use the "finger tool" (Silverman, 2005) to describe what was happening to the amount of the substance as time progressed (see Excerpt 19).

Excerpt 19

Ben: So, this time at zero it's going to start up here [ENLM]....this would represent 100% of the substance (Figure 10, first frame), after one year it's dropping down a bit after two years it's dropping down a little more [second frame], three, four years it's about halfway there [third frame], five years, six years [fourth frame] and it's going on and on and on [ERCR2] but it's never going to, it'll approach zero but it will never actually get to zero [ERMR].

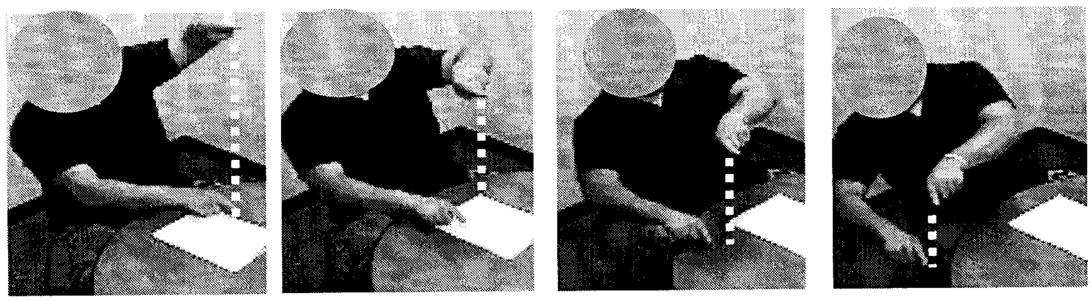

Figure 10: Ben's use of the finger tool to illustrate half-life.

The exercise prompted Ben to coordinate the decreased amount of substance (left hand) with an increase in time (right hand) [ENLM]. In the third frame of Figure 10, Ben said this height represented halfway indicating that he was approaching the point where the remaining substance was half of the original amount [ERCR2, ERMR1]. Comparing the first and third frames of Figure 10, the height of the substance displayed in the third frame is roughly half of the height Ben displayed in the first frame. The researcher then asked Ben to explain how he knew where to position his left hand after five years. His response is revealed in Excerpt 20.

Excerpt 20

Ben: Well, I was just looking at where I started at and I wanted to go down, I didn't want to go down at a straight line [ERFR] I wanted to go down at a sloping line, yeah a sloping line [ERFR2], so I was just trying to slope it down and wherever I got to where I imagined the half would be is where I stopped at [ERMR2a].

This discussion highlighted Ben's conception of how to imagine the curve as a result of decaying by ½ after five years [ERMR2a]. More importantly, Excerpt 20 further illuminates the existing language issues apparent in Ben's ability to articulate his understanding of the mathematics. In line 3, Ben referred to the curvature of his graph as a "sloping line" which might lead others to think that Ben drew a series of straight lines to connect the points of his graph when, in fact, he drew a concave-up decreasing exponential curve. In line 2, Ben said he did not want "to go down at a straight line" which demonstrates his ability to consider this exponential situation as not producing a constant amount of change [ERFR2].

The data suggest that while Ben is able to work through many of the exponential concepts, he lacks well-developed language skills to articulate his understanding. This impoverished ability to verbalize his thinking clearly and concisely could negatively impact his classroom performance as a teacher.

Summary of Ben's Thinking About the What is the Half-Life? Task.

Conceptualizing fractional exponents as being partial factors of the base and also the factor, which multiplied by itself the number of times equal to the denominator of the exponent, was important in making sense of this task. Even though Ben incorrectly answered this task on the PCA, he was able to reason through this task successfully during the interview session by considering the proportional relationship between the function's value and the output value as being a multiplicative factor of ½ [ERMR1]. Ben was able to consider that the function's value in this half-life task can be found by taking ½ of the original amount of substance [ERMR1]. Once this idea was clear to Ben, he was able to focus on the exponent of $\frac{1}{5}$ to think about the value of t that would create a full factor of ½ or $(\frac{1}{2})^1$ [ENLL, ENLN, ERMR2a]. This ability proved to be one of the most successful constructs in helping Ben reason through this problem situation.

Ben demonstrated evidence of his ability to coordinate the constant percent change of the covarying quantities by considering how often the factor of ½ should be applied. Since the exponent was $\frac{1}{5}$, Ben expected a full factor of ½ to be applied every five years [ERCR3]. These data suggest his ability to coordinate the decreasing amount of remaining substance with increasing time.

Moreover, the data indicate that exponential decay posed more difficulty for Ben than exponential growth situations. The researcher conjectures that conceptualizing the half-life expression of $\left(\frac{1}{2}\right)^{\frac{1}{5}}$ as the factor which multiplied by itself five times equals the full factor of ½ and also as representing $\frac{1}{5}$ factors of ½ provides a deeper understanding for fractional exponents, as well as the partial progression of half-life (i.e., for a half-life of five years, a partial progression would be an amount of time less than five years). Given that Ben struggled with the appropriate vocabulary for describing fractional exponents as representing partial factors, the researcher focused on these ideas during the teaching experiments conducted with him. Additional activities were created to facilitate Ben's knowledge of conceptualizing $\left(\frac{1}{2}\right)^{\frac{1}{5}}$ as representing a partial factor of ½.

Table 19 provides a summary of the analysis of Ben relative to the What is the Half-Life task.

Table 19

Ben's Reasoning Model for the What is the Half-Life?

Code	Framework Description	Analysis of Ben
ENLN ◍	Use *notation* to represent b^x as x factors of b for rational values of x.	Ben was able to use notation of $\left(\frac{1}{2}\right)^1$ to guide his understanding of half-life.
ENLL ●	Use *language* to describe b^x as x factors of b for rational values of x.	Ben used language to explain that the role of the exponent was to let you "know at what point…I get to that ½."
ENLM ◍	Use *multiple representations* to represent the multiplicative processes of exponential function.	Ben was able to use the algebraic function to make sense of the situation. In addition, he used the "finger tool" to provide a hand graph of the decay curve.
ENLP ◍	Use *parametric changes* to alter representations of an exponential function.	Ben was able to articulate that changing the initial value of the substance would not impact the half-life.
ENLI ◍	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben's conception of half-life seems to be rooted in recursion (i.e., amount of substance remaining after a half-life is half of previous amount after five years.
ENLE ●	Use *explicit definition* of exponential function to represent exponential situations explicitly.	Ben was able to use his knowledge of the algebra function provided to generate ideas about half-life and repeated multiplication.
ERCR ●	Use *covariational reasoning* to describe exponential behavior.	Ben used his ability to reason covariationally by explaining that the amount of substance would continue to split by half every five years.
ERMR ●	Use *multiplicative reasoning* to describe exponential behavior.	Ben was able to articulate his understanding of the proportional relationship between the initial value and the decay factor of ½. He also provided utterances of half-life representing repeated multiplication by ½ every five years.
Evidence of *no* understanding ○	Evidence of *weak* understanding ◍	Evidence of *strong* understanding ●

Chapter Summary

During five interview tasks of exponential growth and decay, Ben demonstrated various strengths and weaknesses in his understanding. The Population Growth task, which provided a tabular and graphical representation of an exponential function, illuminated Ben's inability to recognize the doubling pattern among the provided data. This inability led to several difficulties: namely, he did not utilize multiplicative reasoning or partial-interval reasoning when describing multiplicative behavior. He was, however, able to recognize the changing rate of change aspect of the exponential function using the graph as his primary source of insight. This finding also further supports the claim that Ben is uncomfortable with numerical representations as a tool for reasoning about contextual situations.

In the Salary Problem, Ben was able to provide evidence of his ability to reason about linear functions while comparing and contrasting with multiplicative growth. The context of an additively increasing salary (growing by a constant amount) and a multiplicatively increasing salary (growing by a constant percent) seemed to provide a more comfortable avenue for Ben. He was able to construct a graphical representation to illustrate both salaries and provided evidence of his ability to discriminate between constant rate of change and changing rate of change. He also consistently reasoned covariationally at an ERCR2 level by articulating increasing differences for the teacher (multiplicative) salary with a few utterances about constant percent increases. However, this task also illuminated Ben's difficulty in extending his notion of exponentiation as recursion to a more dynamic notion of recursion. When asked to determine the two salary values after 10 years, Ben was unable to use the exponentiation process of repeatedly multiplying the previous value by the given percent for the number of years necessary even though he recognized that "your raises are going to continue to increase as your salary continues to increase because it's dependent on whatever that previous year's salary was." Ben's inability to extend his static image of exponentiation as recursion to more dynamic images of repeated multiplication constrained him in his ability to think deeper about the connections between repeated addition as multiplication and repeated multiplication as exponentiation.

The two exponential decay tasks included in the interview, Half of the Half-Life and What is the Half-Life?, seemed to have elicited stronger conceptions of exponentiation and abilities to reason covariationally and multiplicatively than the other tasks. Both of these tasks were included in the PCA pre-test assessment and were multiple-choice questions. Ben incorrectly answered both of these questions on his pre-test assessment. However, when Ben was provided these tasks in open-ended format during the interview sessions, he was able to determine the correct responses by reasoning through the task.

During the interview for the Half of the Half-Life task, Ben was able to use appropriate notation to illustrate the half of the half-life as $\left(\frac{1}{2}\right)^{1/2} = \frac{1}{\sqrt{2}}$. Ben did not use robust language to describe $\left(\frac{1}{2}\right)^{1/2}$ as representing ½ factors of ½. However, Ben was able to use recursion to describe the repeated multiplication of ½ but only after the researcher prompted him to specifically describe this task using repeated multiplication. These data suggest that Ben had a strong ability to notate the mathematics but held impoverished notions of how to qualitatively describe the mathematics. Consistent with previous tasks, he was able to coordinate the amount of change of the covarying quantities given his ability to describe increasing increases over time. His ability to reason through this situation using covariation broadened his ability to begin using partial-interval reasoning. He successfully attempted to describe the decay behavior as decaying more in the first part of the interval than in the second part.

The What is the Half-Life? task provided Ben an opportunity to reason proportionally and begin using robust language to describe b^x as x factors of b for rational values of x. He also demonstrated evidence of his ability to reason multiplicatively by articulating the proportional relationship between the initial value and the decay factor of ½.

In summary, the interview analysis revealed that Ben held a deep-seated conception of exponentiation as limited recursion with static images of how one output value can be found by multiplying the previous output value by some factor. Ben did not provide evidence of his ability to extend this notion by articulating how this process of multiplication can be repeated throughout the domain until probed by the researcher in the Half of the Half-Life task. During the initial discussion about half-life, Ben did not provide knowledge of half-life as meaning the time it takes for something to decay to half of the previous (or original) amount. However, he was able to successfully reason through half-life tasks despite his inability to articulate the definition of half-life. The data also suggest that Ben's conception of exponential behavior is dependent upon whether the differences in output values increase or decrease but he did not provide consistent evidence of knowledge of how these differences were changing.

The results from this analysis of Ben's initial understanding of exponential behavior provided the motivation for developing the teaching experiment activities designed to build Ben's ability to describe exponential behavior in more robust and meaningful ways. The next chapter presents the results and analysis of Ben's ways of thinking about the concept of exponential function while completing a set of activities in a teaching experiment.

CHAPTER 7: RESULTS AND ANALYSIS OF TEACHING EXPERIMENT

This chapter presents the findings and results from the teaching experiment conducted with Ben. The goals of this chapter are to (a) report on Ben's ways of thinking about exponential function in the context of a particular intervention, (b) report on Ben's emerging understanding of exponential function to gauge the effectiveness of the intervention activities, and (c) demonstrate the effectiveness of the Exponential Function Framework in interpreting the data. The chapter concludes with a presentation of data from the PCA post-test assessment.

The Teaching Experiment

The set of activities used in the experiment were designed to facilitate Ben's understanding of exponential behavior and to reveal how his understanding of exponential behavior evolved over the course of the experiment. Ben participated in the teaching experiment by working through the activities with the researcher observing, posing questions, and providing clarifications as necessary. All episodes were videotaped and transcribed for future analysis. Table 20 displays the agenda and time durations for each teaching experiment episode.

Table 20

Chronological Overview of Teaching Experiment Episodes

Episode	Activity	Purpose	Duration
1	Concept Map	Assessment	10 min
	Cheerios® Problem	Multiplicative growth (doubling)	32 min
	Pennies on a Chessboard	Multiplicative growth (doubling)	43 min
2	Properties of Exponents	Review of rational exponents	40 min
	Filling in the Missing Pieces Using Tables	Multiplicative growth & decay	58 min
3	Filling in the Missing Pieces Using Graphs	Multiplicative growth	58 min
	Compare Linear, Quadratic and Exponential Functions	Multiplicative growth, additive growth, linear growth	50 min
4	Light Intensity Activity	Multiplicative decay	115 min
5	Investment Activity	Multiplicative growth; compound interest	72 min

Episode 1: Exponential Growth Using Doubling

The first teaching episode included three activities designed to develop students' notion of exponential *growth*. The three activities were (a) the Concept Map activity, (b) the Cheerios® problem (Klanderman, 1996), and (c) the Pennies on a Chessboard activity. The goals for this episode include (a) assessing Ben's knowledge about exponential function, (b) investigating his conceptual development relative to exponential growth, and (c) facilitating his development of multiplicative reasoning of exponential growth. These activities provided opportunities for him to verbalize his understanding of exponential function relative to the context of doubling. The next discussion first presents the activity followed by a conceptual analysis of Ben's thinking while working through the activity. The first activity presented is the concept map activity.

The Concept Map Activity

Ben was prompted to create a concept map of exponential function prior to Episode 1. He created this detailed graphic organizer as a means of diagramming his knowledge of exponential function by starting with the central theme of exponential function and then branching out with ideas connected to this theme. The concept map activity provided the opportunity for the researcher to assess Ben's knowledge of exponential function and his connections among the main ideas associated with this function family. Once Ben completed all episodes of the experiment, he created a revised concept map of his thinking, which is presented at the end of this chapter. The concept map activity follows.

The Concept Map Activity

Thank you for agreeing to participate in a teaching experiment designed to build knowledge of exponential function. Before we meet for our first session, I would like you to create a concept map (a type of graphical organizer, also called mind-mapping) to summarize your knowledge and understanding of the exponential function concept.

Concept mapping can be a powerful tool for organizing your thoughts about concepts. If you are not familiar with concept maps, you can learn more about them from the following web site: http://en.wikipedia.org/wiki/Concept_map

You are not limited to any maximum or minimum size of your map. Make it neat and easy to read. Be as detailed as possible. Bring your concept map to our first teaching experiment session. We will use it as part of our discussion at our first session.

Ben stated that he had previous knowledge of concept maps, which facilitated his ability[9] to construct his map prior to Episode 1. The episode began with Ben describing the components of his concept map. This discussion provided the researcher with insight into his thinking of the concept of exponential function. Figure 11 illustrates Ben's concept map of exponential function.

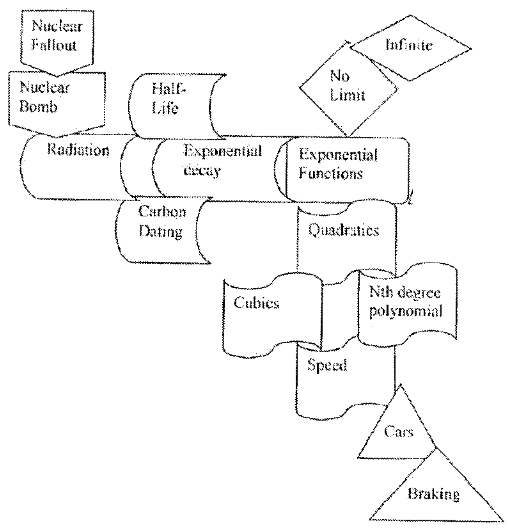

Figure 11: Ben's initial concept map of exponential function.

In the discussion, Ben immediately pointed out that he wanted to include quadratics because this was a topic that he was currently teaching in his high school mathematics courses. His utterances suggested that his thinking was rooted in the concepts he was currently teaching without considering the mathematical meaning

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[9] Concept maps include the connections and descriptions of components included in the map. The absence of these connections in Ben's map revealed that his knowledge of concept maps was unknown. Upon further reflection, it appears that additional detailed directions should have been provided to aid Ben's concept map development.

underlying exponential behavior. Ben described his view of the connection between quadratic function and exponential function by saying, "When I'm thinking of an exponential function, I'm thinking anything that has the power of something." His notion of exponential function simply involved the appearance of the algebraic representation of the function. According to Ben, an exponential function was anything that had an exponent, which included quadratic functions. Predictably, Ben also stated that linear functions could be considered exponential, but admitted that most people would consider "quadratics on up" to be exponential functions. Excerpt 21 illustrates Ben's thinking about the relationship between linear, quadratic and exponential functions.

Excerpt 21

Ben: When I'm thinking of an exponential, I'm thinking anything that has the power of something. So the quadratic...it's just a variable to the power of 2, I guess a linear function could almost be exponential just in the fact that it's not doing anything very fancy or curving or whatever. It's still to the power of something.

The data suggest that Ben did not consider rate of change or covariation as a means for distinguishing between linear, quadratic, and exponential functions. In lines 3-4 of Excerpt 21, he stated that linear functions "could almost be" exponential, thus apparently indicating that since a linear function has an exponent of one, the function could be categorized as exponential. Later sections of this chapter present additional evidence of Ben's categorization of a polynomial function as a special type (subset) of exponential function.

The next section presents the Cheerios Problem followed by a conceptual analysis of Ben's thinking as he worked through this problem.

The Cheerios Problem

The Cheerios Problem (Klanderman, 1996) provided the opportunity for Ben to describe and generalize the relationship between the number of coupons, the number of boxes purchased, and the number of visits to the store. The process of learning within this task started with Ben describing the number of coupons after the first visit to the store, then the second visit, sixth visit, tenth visit and nth visit. Ben also made sense of the multiplicative situation by analyzing the data using differences of output values, ratios of output values, and graphical, tabular and algebraic representations of the context. The Cheerios Problem follows.

The Cheerios Problem

When you purchase a box of Cheerios cereal, there is always a coupon inside the box, which can be applied toward your next purchase. Specifically, the coupon offers you a discount if you purchase TWO boxes of Cheerios cereal. Suppose that you initially have one of these coupons. Also, suppose that you go to the grocery store once every week. Finally, assume that you wish to maximize your discount by using all available coupons each time you go to the store.

- How many boxes will you purchase during your 1st visit? 2nd visit? 3rd visit? 6th visit? 10th visit? *n*th visit?
- How many coupons will you bring to the store during these same visits?
- Describe what happened during the first and second weeks in terms of the number of coupons brought to the store and the number of boxes purchased.
- Describe what happened during the first and second weeks in terms of the number of coupons brought to the store and the number of coupons taken back home.
- Describe a general relationship (using a table, graph, equation, etc.) between the visit number (or week number) and the number of boxes purchased.
- (If above yields an appropriate exponential function) Can you describe a relationship between the week number and the number of coupons brought to the store?
- (If above does not yield a correct function) Can you describe a relationship between the number of coupons brought to the store (in a given week) and the number of boxes purchased?

The Cheerios Problem highlighted various components of the exponential function framework. Table 21 provides the matrix for the Cheerios Problem relative to targeted components of the framework.

Table 21

Matrix Mapping of Exponential Function Framework and Cheerios Problem

Code	Framework Description	Cheerios Problem Questions
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.	Describe a general relationship between the number of visits and the number of boxes purchased.
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.	Describe a general relationship between the number of visits and the number of boxes purchased.
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.	Describe a general relationship between the number of visits and the number of boxes purchased.
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.	Describe what happened during the first and second weeks in terms of the number of coupons brought to the store and the number of boxes purchased.
ENLE	Use *explicit definition* of exponential function to represent exponential situations explicitly.	Describe a general relationship between the number of visits and the number of boxes purchased.
ERCR	Use *covariational reasoning* to describe exponential behavior.	Can you describe a relationship between the number of weeks and the number of coupons brought to the store?
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	How many boxes will you purchase during your 1st visit? 2nd visit? 3rd visit? 6th visit? 10th visit? *n*th visit? Describe what happened during the first and second weeks in terms of the number of coupons brought to the store and the number of boxes purchased.

Ben was successful in recognizing the doubling pattern in the Cheerios Problem. His approach was to calculate the number of boxes purchased and the number of coupons used relative to the number of visits. Excerpt 22 reveals Ben's thinking after he had calculated the values of various quantities.

Excerpt 22

Ben: I'm doubling each previous week [ENLI1, ERCR1, ERMR2a]. So…I went from 2 boxes to 4, 4 to 8, 8 to 16, 16 to 32, 32 to 64 [ENLI1, ERMR2b]. So, pretty much each week I'm just doubling the previous week's amount [ERCR1, ERMR2a] which works into the way I see it is just something like 2^x or 2^n I guess, the n being whichever visit it is [ENLN, ENLM3].

These data provide evidence of Ben's ability to recognize the doubling pattern after constructing his own quantities based on the situation provided. The process of constructing these quantities facilitated his ability to make sense of the situation and recognize the patterns emerging from the data [ENLM4]. The data also suggest that Ben is now able to continue the repeated multiplication of two throughout the domain of visits [ENLI1, ERMR2b]. (Recall that during his interview while completing the Salary Problem Ben was unable to find the salary in year 10 given that the salary in year nine was unknown (see Excerpt 7 in chapter 6)).

The following paragraphs discuss Ben's ability to describe the relationship between the number of coupons, the number of boxes purchased, and the number of visits to the store (see Excerpt 24).

Excerpt 24

Ben: So my first week I bought, I had one coupon which meant I had to buy two boxes and...since I bought those two boxes and each box had one coupon so going into the second week I doubled my amount of coupons from the first week [ERMR2b]. And that meant I can now double the amount of boxes of cereal I buy going from the first to the second week because if I'm doubling the coupons [ERMR2b], then the coupons are like doubling themselves [ERMR2b], so like one coupon buys the number of boxes double that amount of coupons [ERMR2b].

Ben was able to articulate how the number of boxes purchased was double the number of boxes purchased in his previous visit to the store. He utilized this pattern of repeated multiplication [ERMR2b] to find the number of boxes purchased for an increasing number of visits. The emergence of the doubling pattern through calculating values of quantities and making sense of the context appeared to facilitate Ben in building a more robust conception of the situation.

The Exponential Framework for this study includes two components for representing/describing b^x as x factors of b: the notation component [ENLN] and the language component [ENLL]. Excerpt 25 illustrates Ben's thinking about the meaning of exponents relative to his generalized expression of 2^n.

Excerpt 25

Ben: The n is how many sets of two's of boxes I'm buying. So in the second week it's 2^2 so I'm buying two sets of two boxes which is four and the third week I'm buying three sets of two...two and two

4 and two is eight when we multiply that together [ENLN, ENLL].

Here, Ben's understanding of exponents entailed the notion of exponents as representing "sets" of boxes of cereal. Ben appeared to view the operation between the sets as multiplication based on lines 3-4 of Excerpt 25. According to Ben, each additional week resulted in buying an additional set of two boxes. In lines 3, Ben stated, "the third week I'm buying three sets of two" which can be interpreted as three factors of two. This utterance suggests that Ben was thinking of sets to represent factors.

Ben created a graph (see Figure 12) relating the number of boxes purchased as a function of the number of visits. He immediately focused on the first quadrant saying that "since we're not going to have negative boxes at all or negative visits, we can just go with all the positives." Ben initially hesitated when he was asked to scale both the horizontal and vertical axes. He then explained that the "hardest thing" for him to do with his high school students is to decide how to appropriately scale the vertical axis given that the output values "were getting bigger." This utterance referred to the increases in output values. He decided to scale the horizontal axis by a unit of one and the vertical axis by a unit of ten to accommodate for the increasing values. Emerging from this discussion was Ben's thinking about the *direction of change* between number of boxes purchased and number of visits. Excerpt 26 reveals his ability to coordinate the amount of change of the covarying quantities using the context of the graph.

Excerpt 26

Ben: 1 I mean…just looking at the spaces between the boxes [points to the

2 vertical axis], look what happened here, I rose only 2 here and then

3 I rose 4, and then 8 spaces, and more and more so the gap between
4 however many boxes [ERMR2b]...vertically is where the change
5 is happening because I'm getting more and more boxes each time
6 [ERCR2] I'm not getting the same amount of boxes each time I'm
7 getting more with each visit I take [ERCR2].

In the context of the graph, Ben was able to recognize that the change between the number of boxes purchased was increasing relative to the constant increases in the number of visits [ENLM2]. Ben's graph is provided in Figure 12.

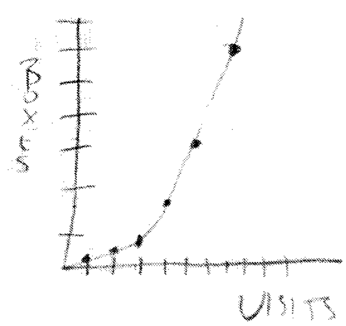


Figure 12: Ben's graph for Cheerios Problem.

In lines 6-7 of Excerpt 26, Ben stated that the number of boxes is not growing at a constant pace. As he stepped through successive increases of one box, he expressed that the number of boxes would increase "more and more." This utterance suggests that Ben was able to reason covariationally by coordinating the *amount* of change of the output with successive changes in the input. He expressed that there are increasing differences in the number of boxes purchased for increases in the number of visits. He also stated the values resulting from the doubling pattern as revealed by his utterances in line 3 of Excerpt 26.

Ben followed by stating that the number of boxes doubled for each unit increase in the number of visits (see Excerpt 27).

Excerpt 27

Ben: It's doubling each time [ERMR2a] so I started out with just a space of 2 here and then I went to a space of 4 [ERCR2] and...then to get to 8 I have a difference of 4 right here [ERCR2] and then from 8, I went to 16, so from 8 to 16 is a difference of 8 [ERCR2], which is double the amount of 4 which is double the amount of 2 [ERMR2a]... From 1 to 2 I'm doubling however many boxes I get or however much the space is in each coordinate [ERMR2a].

This excerpt provides evidence of Ben's thinking about (a) the doubling pattern of the number of boxes as the number of visits increased [ERMR2a], and (b) the emergence of the doubling pattern embedded when considering the differences between the values of boxes [ERCR2, ERMR2a]. In line 6-7 of Excerpt 27, he revealed his thinking about both doubling patterns that emerged from this context. In addition to describing consecutive outputs as doubling, Ben also articulated his revelation that the space between values (i.e., the consecutive differences between two output values) was also doubling. These utterances suggest that Ben views the doubling pattern as (a) a doubling of the output values (i.e., the number of boxes doubles), and (b) a doubling of the *change* between output values (i.e., the change in the number of boxes doubles). These dual images of recognizing the doubling pattern of the function value *and* the change in the function value facilitated Ben's making sense of exponential behavior.

The discussion for this activity ended with Ben writing the algebraic formulas for (a) the number of boxes purchased as a function of the number of visits as $P(v) = 2^v$, and (b) the number of coupons used as a function of the number of visits as $C(v) = 2^{v-1}$. In both functions, he notated the input as v for the number of visits while he notated the output as $P(v)$ for the number of boxes purchased and $C(v)$ as the number of coupons used. Ben compared both sets of output values and he concluded that while both sets were doubling, the number of boxes "had an extra double" when compared with the number of coupons used. Furthermore, he offered insight into how he distinguished between these two functions by explaining that the number of coupons used is always half the number of boxes purchased. As a result of this insight, Ben created two additional functions to compare the number of boxes and the number of coupons. His functions were $B = 2C$ and $C = B/2$. He also commented that these two functions were linear. Table 22 provides a summary of analysis for Ben's thinking of the Cheerios Problem.

Table 22

Ben's Reasoning Model for the Cheerios Problem

Code	Framework Description	Analysis of Ben
ENLN ◍	Use *notation* to represent b^x as x factors of b for rational values of x.	Ben first described the situation as $2 \cdot 2 \cdot 2 \cdot ...$ and then expressed the doubling pattern as 2^n.
ENLL ◍	Use *language* to describe b^x as x factors of b for rational values of x.	Ben described the situation as "sets of two" meaning factors of two.
ENLM ●	Use *multiple representations* to represent the multiplicative processes of exponential function.	Ben demonstrated an understanding and ability to use all four representations of the function situation.
ENLI ●	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben described his thinking of the situation of the number of boxes of cereal as double that of the previous week's amount of boxes.
ENLE ◍	Use *explicit definition* of exponential function to represent exponential situations explicitly.	While Ben expressed the doubling pattern as 2^n and then later as $f(n) = 2^n$.
ERCR ◍	Use *covariational reasoning* to describe exponential behavior.	Ben articulated that the number of boxes was "doubling each time."
ERMR ●	Use *multiplicative reasoning* to describe exponential behavior.	Ben described the doubling pattern as (a) the doubling of the number of boxes and (b) the doubling of the change in the number of boxes.

Evidence of *no* understanding ○ — Evidence of *weak* understanding ◍ — Evidence of *strong* understanding ●

The next section presents the Pennies on a Chessboard activity followed by a conceptual analysis of Ben's thinking as he worked through this task.

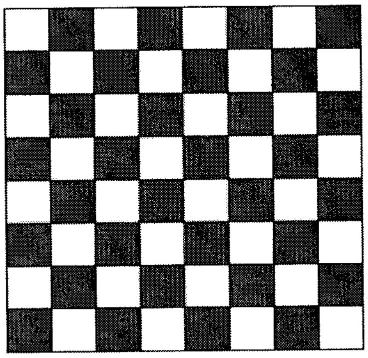
The Pennies on a Chessboard Activity

The final activity in Episode 1 was designed to promote exploration of doubling and multiplicative growth in the context of a physical model. Using a chessboard as a physical model, the Pennies activity prompted Ben to explore the pattern of doubling for various intervals of squares (e.g., doubling every square or doubling every half square) where movement from one square to the next corresponded to multiplying by another factor of two. This idea facilitated Ben's thinking about the effects on the exponent of advancing by a fraction of a square while attending to the multiplicative factor resulting from this partial move. The discussion centered on Ben's utterances that described doubling as multiplicative growth for whole squares as a means for describing multiplicative growth for partial squares. This understanding proved to be critical for facilitating his multiplicative and covariational reasoning as an approach for making sense of exponential function situations.

The next section summarizes the Pennies activity along with the pre-planned questions.

Pennies on a Chessboard Activity

Once upon a time there was a king who loved to play chess. One day the king proposed that if anyone in his kingdom could beat him at chess, the king would grant them any reasonable wish. A poor farmer stepped forward to meet the king's challenge. Much to the king's surprise, the farmer beat the king quickly and with seeming ease. True to his word, the king agreed to grant the farmer's wish. Wanting to wish for something that seemed reasonable, the farmer suggested the following: "I propose that you place on the first square of the chessboard one penny, and on the second square, two pennies, on the third, four pennies, and so forth, until the last square is reached." After a moments thought, the king granted the request, as it seemed that the farmer had, in fact, asked for very little money. As the king began to place the money on the chessboard, however, he soon realized his terrible mistake. Let's see why.



- Describe the general pattern that emerges from this situation. What does this pattern look like?
- How can we find the number of pennies on the 10th square? What process do we need to consider?
- How many pennies will be on the last square?
- How many squares must be filled to obtain exactly half of the number of pennies on the last square?
- If it were possible to move over ½ square, how would this be represented? What does it mean to move over ½ square in the context of this problem?

Table 23 provides the matrix for the Pennies activity relative to targeted components of the exponential function framework.

Table 23

Matrix Mapping of Exponential Function Framework and Pennies on a Chessboard

Code	Framework Description	Pennies on a Chessboard Questions
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.	If it were possible to move over ½ square, how would this be represented? What does it mean to move over ½ square in the context of this problem?
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.	If it were possible to move over ½ square, how would this be represented? What does it mean to move over ½ square in the context of this problem?
ENLP	Use *parametric changes* to alter representations of an exponential function.	Describe the effects of starting with 10 pennies on the first square instead of one penny.
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.	Describe the general pattern that emerges from this situation. What does this pattern look like?
ENLE	Use *explicit definition* of exponential function to represent exponential situations explicitly.	Write a function to represent the number of pennies on an individual square relative to the number of squares.
ERCR	Use *covariational reasoning* to describe exponential behavior.	Describe the general pattern that emerges from this situation. What does this pattern look like?
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	Describe the general pattern that emerges from this situation. What does this pattern look like?
ERPIR	Use *partial-interval reasoning* to describe exponential behavior	If it were possible to move over ½ square, how would this be represented? What does it mean to move over ½ square in the context of this problem?

Throughout this task, Ben consistently explained his thinking by describing the recursive nature of exponential growth by saying it's "doubling the previous amount each time" [ENLI1, ERMR2]. Figure 13 illuminates Ben's attempt to organize his thinking for the chessboard context.

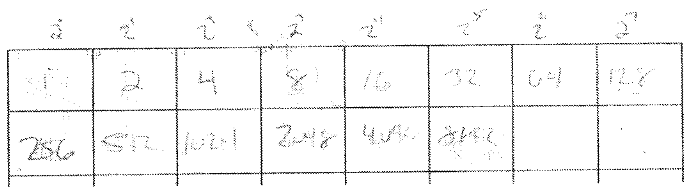


Figure 13: Ben's Chessboard.

Using his thinking about doubling and the values he constructed in Figure 13, Ben was able to create an algebraic representation of the function, which he called $P = 2^{s-1}$ where P represented the number of pennies and s represented the square on the board [ERLM3]. This ability suggests that Ben's attempt to make sense of the output values of an exponential function facilitated his recognition of the patterns in the data. He effectively utilized the exponential growth formula to find the number of pennies on the tenth square by saying "we'd just plug in whichever number square that is here" [ENLE1]. When prompted to describe how he would find this value without the formula, Ben explained that he would continue the doubling pattern. Excerpt 28 provides Ben's response to this question.

Excerpt 28

Ben: I'd just double [ENLI, ERMR2a], I'd go from 7 to 8 and double that amount [ERMR2b], 8 to 9 and double that amount [ERMR2b], 9 to 10 and double whatever 9 was [ERMR2b].

Ben's response in Excerpt 28 provides further evidence of the evolution of Ben's thinking about doubling. Initially, during the Salary Problem (see chapter 6), Ben believed that the salary in year 10 could not be found without knowing the salary for year nine. In other words, he was unable to repeatedly multiply by 1.07 until he reached year 10. In the Pennies activity, however, he made his understanding explicit: while completing the Pennies activity, he was able to repeatedly multiply by two until he arrived at the 10^{th} square.

Ben also discussed his thinking about the effects of changing the initial value of the situation to a positive integer other than one. This discussion corresponded to the framework element of using parametric changes to alter representations of an exponential function [ENLP1]. In this case, Ben described the parametric change for an initial value of 10 pennies relative to the output changes. Excerpt 29 provides a selection of this discussion.

Excerpt 29

AS: 1 Let's say that instead of starting with one penny I say you start
2 with 10 pennies. How does this situation change?
Ben: 3 You're going to get a lot more money because you are going to go
4 from 10 to 20 to 40 to 80 to 160 [ENLP1]. So each of these
5 [pointing to his table] is going to be multiplied by 10 [ENLP1].

Ben was able to articulate how increasing the initial value from one to ten pennies would have the multiplicative effect of increasing the number of pennies on each original square by a factor of ten. The researcher then questioned Ben about the situation of starting with 14.5 pennies. Excerpt 30 displays Ben's response.

Excerpt 30

Ben: I should still be able to multiply these by 14.5 [ENLP1]. Ah…because I'm still performing that same operation I guess [referring to doubling, ERMR]. This is just the base of that operation…Starting at one, then doubling from there [ERMR2a] is going to give me a base for you know if started at two. Well, if I multiply each of these by two or if I start with one and keep going up I can just multiply by whatever that number is [referring to the initial value, ERMR].

This excerpt suggests that Ben was comfortable in considering how a change in the initial value would result in a corresponding change in output values when the multiplicative factor is constant [ENLP1, ERMR]. Excerpt 30 also illuminates the difficulty Ben had in verbally articulating the mathematics involved in this situation and in speaking with meaning. As previously illustrated by Ben's utterances during the interview items (see chapter 6), natural language is Ben's primary obstacle for conveying his thoughts verbally and attaching mathematical meaning to his mental images.

The final task within the Pennies on a Chessboard activity involved a discussion about the meaning of moving over a half square as opposed to moving over a whole square on the chessboard while maintaining consistency in doubling for each whole square move. Ben's immediate response to this task was that the output values would still increase, but by "a square root of two." Excerpt 31 provides an illustration of Ben's reasoning.

Excerpt 31

Ben: Because I can think of that half step thinking if…I started at 2^3 and took a half step, I am at three point five. Now if I thought of that in terms of fractions, two to the seven halves and the seven halves being the 2^7, but anything in a fraction means I'm taking…the root of that number [ENLN]. So, in this case it's the square root of that number. So I could split that again to be in two to the seventh times one half…maybe $\left(2^7\right)^{\frac{1}{2}}$ or $\left(2^{\frac{1}{2}}\right)^7$ [ENLN].

This final task appeared to enable Ben to express his thinking about the relevance and meaning of moving over a half square on the chessboard. His thinking was embedded in considering the notational effects of moving a half square if he started with the square containing the expression 2^3. More specifically, he was able to consider a partial-square movement as resulting in multiplying the previous output by a partial factor of two, and in this case, the partial factor was $\sqrt{2}$, even though he did not specify that $\sqrt{2}$ was a factor. Ben's initial explanation for the meaning of moving over a half square on the chessboard involved a more procedural approach of using roots to demonstrate partial moves. He was able to articulate his knowledge about how multiple partial factors result in a full factor and, ultimately, he was able to demonstrate partial-interval reasoning when justifying his thinking. Excerpt 32 provides a discussion between the researcher and Ben about where $2^{3.5}$ would be located relative to the other squares on the chessboard.

Excerpt 32

AS: If you go way back you started with a power of three point five, which would put you where on the grid?

Ben: If I started here and all of this is my increase…[pointing to the chessboard] so here is my two to the first power starting here it increased. Here is my 2^2 starting and am increasing until I get to my 2^3 , increasing until I get to 2^4. Um, maybe somewhere like down here not, not even halfway [ERMR]. Even though its ahh, half a step I am not halfway in that increase [ERMR].

Ben's explanation of the increase of 2^1 to 2^2, from 2^2 to 2^3, and then from 2^3 to 2^4 reveals that he was able to coordinate unit increases in the inputs with increases in the outputs. He abandoned the idea that the quantity of $2^{3.5}$ "was halfway in that increase" (i.e., the arithmetic mean) between the quantities 2^3 and 2^4. This utterance in lines 7-8 provides evidence that Ben relied upon his understanding of linearity, which facilitated his description of this situation as something that did not follow an additive structure. Excerpt 33 illustrates his evolving reasoning patterns as he continues to reveal his thinking about the meaning of the quantity $2^{3.5}$ in relation to 2^3 and 2^4.

Excerpt 33

Ben: Now instead of increasing by another 2 I'm increasing by a $\sqrt{2}$ [ERMR2b]…I'm at 8 here and I am only increasing by one point whatever so…some number multiplied by itself to get to whatever $2^7$was, which is 128. Well we need to determine what number multiplied by itself gets to 128 [ERPIR].

Ben's thinking about the value of $2^{3.5}$ was to first consider the meaning of $\left(2^7\right)^{\frac{1}{2}}$. This idea lead Ben to reason about the value of this number by conceptualizing $\left(2^7\right)^{\frac{1}{2}}$ as a partial factor which multiplied by itself twice would result in a full factor of 2^7 or 128. Ben's utterance in lines 3-4 of Excerpt 33 provided evidence of his ability to view fractional exponents as representing partial factors of a full factor of the base number. In Excerpt 34, Ben provided additional insight into his thinking about the value of $\left(2^7\right)^{\frac{1}{2}}$; it also shows him posing questions to himself as he makes sense of the situation.

Excerpt 34

Ben: Is that greater than, less than, or equal to the halfway between 8 and 16? Is 12 the halfway between 8 and 16 [ERFR]. Umm, is the $\sqrt{128}$ again being a number that needs to be multiplied by itself [ERPIR]. So this number times that number again to equal 128 [ERPIR] is it going to be greater than or less than 12 [ERMR]?

This discussion in Excerpt 34 illuminates Ben's thinking relative to additive reasoning where he begins his line of questioning by engaging in discourse about whether the value of $\left(2^7\right)^{\frac{1}{2}}$ is halfway between 2^3 and 2^4 [ERFR]. In line 5 of Excerpt 34, he provides the question, "is it going to be greater than or less than 12?" Similar situations emerged during the interview sessions (see chapter 6; Population Growth task and the

Salary Problem). Ben's use of this questioning style provides evidence of his conceptualization of the notion of positive fractional exponents as representing some kind of growth in comparison to additive growth [ERFR].

Ben continued with his explanation by saying that it is important to get students to understand that "we're not increasing by the same amount each time." Excerpt 35 illustrates his ideas.

Excerpt 35

Ben: 1 If I was going to include half steps and partial steps, the difference
2 between this partial step and this partial step is a bigger number
3 [ERPIR1] because now I'm taking...the root of a bigger number
4 now [ERMR].

In this discussion, Ben used the idea that "partial steps" produce an increase in the output value. However, his explanation is somewhat overshadowed by his misused and imprecise language. In line 3 of Excerpt 35, Ben justified his thinking by saying that he is now taking "the root of a bigger number" and he expands on this idea by describing how $\sqrt{8.1} > \sqrt{8}$ and $\sqrt{8.2} > \sqrt{8.1}$. His utterances seem to confuse the idea of root as the multiplicative undoing of exponentiation with his notion of multiplying the initial value by a factor which happens to be in root form. He struggled to describe his thinking, and language continued to be an obstacle when he stated "the difference between those real tiny small numbers keeps increasing by little tiny smaller numbers that are that much bigger than the amount difference within the previous one." The intended meaning of Ben's argument is that the difference in output values – while quantitatively small –

continues to increase as the input values continue to increase. These utterances suggest Ben's ability to reason covariationally at Level 2 given that he was able to coordinate the amount of change of output values with the changes in the input values (exponents) [ERCR2]. In addition, Ben's explanation seems to imply partial-interval reasoning with his articulation that the change in the first part of the interval is less than the change in the second part of the interval [ERPIR1]. However, he did not provide evidence of his understanding of the quantitative nature in which the output values were growing.

Table 24 provides a summary of the analysis of Ben's thinking when responding to the Pennies on the Chessboard activity.

Table 24

Ben's Reasoning Model for the Pennies on a Chessboard Problem

Code	Framework Description	Analysis of Ben
ENLN ◍	Use *notation* to represent b^x as x factors of b for rational values of x.	Ben was able to articulate the notational effects of moving over a half square if he started with the square containing the expression 2^3.
ENLL ○	Use *language* to describe b^x as x factors of b for rational values of x.	No evidence was provided.
ENLP ●	Use *parametric changes* to alter representations of an exponential function.	Ben was able to describe the effects of changes in the initial value and the growth factor.
ENLI ●	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben described the number of pennies on a square as double the previous amount.
ENLE ●	Use *explicit definition* of exponential function to represent exponential situations explicitly.	Ben was able to construct an algebraic representation of the function, which he called $P = 2^{s-1}$.
ERCR ◍	Use *covariational reasoning* to describe exponential behavior.	Ben described the situation as doubling the number of pennies of the previous amount for every square increase.
ERMR ●	Use *multiplicative reasoning* to describe exponential behavior.	Ben described the number of pennies on a square as double the previous amount.
ERPIR ◍	Use *partial-interval reasoning* to describe exponential behavior.	Ben was able to reason about the value of $(2^7)^{1/2}$ by conceptualizing this expression as a partial factor which multiplied by itself twice resulted in a full factor of 2^7.

Evidence of *no* understanding ○   Evidence of *weak* understanding ◍   Evidence of *strong* understanding ●

Episode 2: Exponential Growth and Decay Using Tables

The second teaching episode included two activities focused on reviewing properties of exponents and infusing multiplicative reasoning when the input interval was less than one unit (i.e., the discussion focused on fractional exponents). The activities implemented in Episode 2 were (a) Properties of Exponents and (b) Filling in the Missing Pieces Using Tables. The four goals for Episode 2 were the following: (a) to develop Ben's ability to reason through multiplicative situations for interval sizes less than one unit; (b) to develop an understanding of fractional exponents and language proficiency for describing the meaning of $3^{1/2}$, for example, beyond the obvious definition of $3^{1/2}$ as $\sqrt{3}$; (c) to contrast multiplicative behavior with linear behavior; and (d) to develop Ben's ability to articulate the multiplicative changes in output with respect to the corresponding additive changes in the input. The ability to describe fractional exponents in meaningful ways provided a powerful reasoning tool for making sense of exponential contexts, including situations where the multiplicative unit was described in terms of intervals not equal to one unit. Next, the discussion focuses on describing the Properties of Exponents activity, which is followed by a conceptual analysis of Ben's thinking as he worked through this activity.

The Properties of Exponents Activity

The first activity prompted Ben to think about fractional exponents as partial factors as a tool for building upon his knowledge of exponents. The ability to think about exponentiation with a fractional number of factors – described as partial-interval reasoning – provided the foundation for reasoning multiplicatively in other situations in the teaching experiment. The Properties of Exponents is described next.

Properties of Exponents

Students often have difficulty understanding and utilizing the properties of exponents. How would you explain the problems below to struggling students?

1. $3^2 \cdot 3^5$ 2. $\left(3^2\right)^5$ 3. $\frac{3^2}{3^5}$ 4. $3^{1/2} \cdot 3^{1/5}$ 5. $\left(3^{1/5}\right)^5$ 6. $\frac{3^{1/2}}{3^{1/5}}$

- What does it mean, notationally, for $3^2 \cdot 3^5 = 3^7$?
- Describe the meaning of $3^2 \cdot 3^5 = 3^7$.
- What does it mean, notationally, for $3^{1/2} \cdot 3^{1/5} = 3^{7/10}$?
- Describe the meaning of $3^{1/2} \cdot 3^{1/5} = 3^{7/10}$.

This study included activities that emphasized an extension of thinking about exponents as the *number of factors of the base* by considering that $3^{1/2}$ could be described as ½ factors of 3. Thinking about quantities, such as $3^{1/2}$, using this model provided a powerful tool for conceptualizing exponential behavior. Table 25 provides the matrix for the Properties of Exponents activity relative to targeted components of the framework.

Table 25

Matrix Mapping of Exponential Function Framework and Properties of Exponents

Code	Framework Description	Properties of Exponents Questions
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.	What does it mean, notationally, for $3^2 \cdot 3^5 = 3^7$? What does it mean, notationally, for $3^{1/2} \cdot 3^{1/5} = 3^{7/10}$?
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.	Describe the meaning of $3^2 \cdot 3^5 = 3^7$. Describe the meaning of $3^{1/2} \cdot 3^{1/5} = 3^{7/10}$.
ERPIR	Use *partial-interval reasoning* to describe exponential behavior.	Describe the meaning of $3^{1/2} \cdot 3^{1/5} = 3^{7/10}$. How would you explain to students how to simplify these expressions when you have fractional exponents? What does it mean to conceptually say that $\left(a^{1/2}\right)^2 = a$ for $a \geq 0$?

Consistent with the initial conjectures, Ben was able to describe the positive integer exponents as the number of factors of the base. He first described his method of expanding the exponential notation and then described his thoughts of counting the number of base numbers to rewrite the expression in exponential form (e.g., $3^2 \cdot 3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^7$) [ENLN, ENLL]. In Ben's words, "I always tell my students whatever the number you have in the exponent is telling you how many of this number is here" [ENLL]. Excerpt 36 provides Ben's way of thinking about simplifying the expression $3^2 \cdot 3^5$.

Excerpt 36

Ben: 1 We have a total of two threes and we are combining them with a
2 total of five threes…when I am combining these through
3 multiplication it's still just that three and the thing that is changing
4 is the amount of threes being multiplied together [ENLL].

During this example, Ben grappled with finding the correct terminology to describe his thinking. In lines 3-4 of Excerpt 36, he was able to describe the change in the exponent as "the thing that is changing is the amount of threes being multiplied together" which reflected his thinking of b^x as representing x factors of b [ENLL]. With probing by the researcher, an ensuing dialogue enabled Ben to consider the term *factor* to describe the number being multiplied by itself. It was important for Ben to think about the term factor – given its connotation of multiplication – in preparation for future exponential tasks in this teaching experiment.

Ben provided similar reasoning patterns with questions #2 and #3 from the Properties of Exponents activity. However, Ben experienced more difficulty when he attempted to explain questions #4-6 that involved fractional exponents. Excerpt 37 illustrates this difficulty.

Excerpt 37

AS: 1 How would you explain to students how to simplify these
2 expressions when you have fractional exponents?

Ben: 3 I have actually had questions like this come up and I am trying to
4 think of what I have said and I don't even remember what I said…I

have had a lot of trouble trying to explain it to them the easiest way.

During this discussion, Ben paused for several seconds to try to find a way to explain his thinking about $3^{1/2} \cdot 3^{1/5}$ (see question #4). It seemed as though he experienced a mental block when attempting to describe how to simplify this expression. In lines 4-6 of Excerpt 37, he explained that he could not remember what he had previously explained to students. After another pause, he stated that he would refer students back to the problem with the positive integer exponents where the procedure for adding exponents could be utilized to carry out the simplication. This excerpt reveals additional data that supports the claim that teachers often have impoverished notions of fractional exponents and thus they rely on their procedural understanding when explaining to students how to think about expressions involving fractional exponents.

In the next excerpt, Ben offered his thinking about how the power of ½ related to the square root when thinking about the quantity $3^{1/2}$.

Excerpt 38

Ben: So in terms of explaining how $3^{1/2}$ gets down to the $\sqrt{3}$, I've drawn a lot of blanks on that so, so the biggest thing I would do… to explain it to my students is to just say lets just go back to one of these with whole numbers.

It was clear, even from previous conversations with Ben, that he remembers the rule between fractional exponents and the operation of root. Yet, he was unable to provide any further explanation for fractional exponents other than to return to the procedures developed with integer exponents. Relative to explaining why fractional exponents and roots are synonymous, Ben stated "I draw a blank on explaining it…I know I've tried and I think I did it successfully once but I don't remember what I said."

The evidence of Ben's thinking presented thus far illuminates his impoverished ability to conceptualize fractional exponents in ways that promote meaningful understanding for his students. While he was able to connect the notation of fractional exponent with the operation of root, Ben could not provide further explanation for the meaning of a quantity containing a fractional exponent.

Excerpt 39 reveals Ben's thoughts about the relationship between quantities containing an integer exponent and quantities containing a fractional exponent.

Excerpt 39

Ben: The only thing we changed here [referring to $3^{1/2} \cdot 3^{1/5}$] is now instead of being 3^2 and 3^5 we put a one over these numbers. Nothing has changed we are still…multiplying these same base numbers, but into different powers the only difference is those powers are fractions.

This excerpt suggests further evidence of Ben's thinking about the expressions 3^2 and 3^5 as representations similar to $3^{1/2}$ and $3^{1/5}$, respectively. These data uncover the notion that Ben is considering not the *quantities* represented by 3^2, 3^5, $3^{1/2}$ and $3^{1/5}$, but rather only the procedural operation. Thus, he emphasizes the procedural approach of adding the exponents when multiplying the base numbers. In line 4-5 of Excerpt 39, Ben states, "the only difference is those powers are fractions"; by this he means that the only difference between using the rule of exponents for simplifying the expression $3^2 \cdot 3^5$ and $3^{1/2} \cdot 3^{1/5}$ is that the latter requires students to add fractions.

One of the central goals of this episode – and for the entire teaching experiment – was to find a way for thinking multiplicatively about fractional exponents so as to provide coherent transition from thinking about integer exponents to thinking about fractional exponents without solely relying on previously defined procedures. Although the research in this area is scant, the researcher conjectured that one way of conceptualizing b^x was to consider it as x factors of b even when x was a fraction [ENLN, ENLL]. At this point in this discussion, the researcher illustrated to Ben that since $3^2 = 3 \cdot 3$, one could think of $3^1 = 3^? \cdot 3^?$ where the two factors have equivalent value[10]. Ben described his thinking by saying "if we are multiplying the threes together to one power it would have to be three to one half times three to one half."

Using the idea of 3^2 expressed as a product of its factors, the researcher probed Ben to explain the meaning of 3^1. Ben was able to explain, "$3^{1/2}$ is a number that when I multiply by itself by another $3^{1/2}$ gives me just 3" [ENLL]. This way of thinking led to the notion of reasoning by using partial intervals where the number 3^1 is thought of as the *full factor* of 3. This idea emerges from thinking about fractional exponent factors being multiplied together to create the full factor of 3. Thus, $3^{1/2}$ is a *partial factor* of 3^1.

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[10] Ben had previously made this claim by explaining that the exponent determined the number of threes that were multiplied together.

Ben also discussed that there were other factors of 3 besides the partial factor of $3^{1/2}$. As an example, Ben explained in Excerpt 40 how $3^{1/3}$ is another partial factor of 3.

Excerpt 40

Ben: If I am looking for three things you know these numbers here still have to be the same and these numbers [referring to the exponents] all have to…add up to one…I was going to say these are the only three numbers that when you add them together will add up to one [ENLL, ERPIR].

In this excerpt, Ben was able to describe that other possibilities existed for determining the partial factor, which produced a full factor of 3^1 [ENLL, ERPIR]. However, the exponent for each partial factor depended on the number of partial factors needed to create the full factor. In this example, Ben was prompted to determine the partial factor if the full factor was split into three partial factors. Ben pointed out that it was necessary for the exponents to sum up to one, thus in the case of finding three partial factors for 3^1 the exponent of the partial factor must be $1/3$ [ENLL, ERPIR].

Prior to the teaching experiment, these ideas were conjectured as vital for developing the ability to reason through exponential situations using covariational, multiplicative and partial-interval reasoning. Understanding exponents meaningfully provided the foundation for describing and notating exponential behavior. When Ben attempted to explain his thinking for question #5 on the Properties of Exponents activity, he was able to use this idea of partial factors. Excerpt 41 illustrates his new approach.

Excerpt 41

Ben: 1 Ok, well so $\left(3^{1/5}\right)^5$. Well multiplying $3^{1/5}$ together five times...now
2 that I am kind of thinking in terms of this [points to $3^1 = 3^{1/2} \cdot 3^{1/2}$
3 on the paper] I can give them a better explanation there... $3^{1/5}$ when
4 we have that $1/5$ it means that, how many of these need to be
5 multiplied together to equal just one three because this is a single
6 factor of a three [ENLL, ERPIR], but what type of a factor? Well
7 it's $1/5$ of all those factors...so it takes five $3^{1/5}$ to give me just one
8 three [ENLL, ERPIR].

This excerpt provides evidence of Ben's ability to learn and implement new ways of thinking about exponents. In previous discussions, he relied upon his procedural understanding of fractional exponents as representing roots. In line 3 of Excerpt 41, Ben states that he can now provide what he perceived as a "better explanation" for describing fractional exponents. This explanation involved thinking about $3^{1/5}$ as representing the factor, which multiplied by itself five times, produced a full factor of 3^1. This way of thinking prompted Ben to consider the quantity of $3^{1/5}$ as a partial factor and thus he was able to use this justification for determining that $\left(3^{1/5}\right)^5 = 3$ [ENLL, ERPIR].

The researcher ask Ben to comment on his thoughts about whether this way of thinking about fractional exponents would be helpful for his students. In Excerpt 42, Ben provides additional information on his new thinking of fractional exponents.

Excerpt 42

Ben: Yeah, that's definitely going to help because especially now since we are going into radicals and the roots... So this definitely preps me a little bit better for that...I am sure they've seen this before now I can hopefully give them a better explanation of it instead of just going, ok this equals three to the one half equals the square root of three. Well why? Well ah, it just does [mimicking what he used to say to his students].

In lines 4-6, Ben commented that his new explanation would allow him to describe fractional exponents in another way as opposed to emphasizing the rule that $a^{1/n} = \sqrt[n]{a}$ for all positive values of a and n. Line 6 of this excerpt provides evidence that he previously was unable to describe fractional exponents to his students and thus, in this case, relied upon his (meaningless) justification of fractional exponents.

Table 26 provides a summary of the analysis of Ben's progress as he worked through the Properties of Exponents activity.

Table 26

Ben's Reasoning Model for the Properties of Exponents

Code	Framework Description	Analysis of Ben
ENLN ●	Use *notation* to represent b^x as x factors of b for rational values of x.	Ben used exponential notation to represent $3^2 \cdot 3^5 = 3^7$, $3^{1/2} \cdot 3^{1/5} = 3^{7/10}$ and $3^{1/2} \cdot 3^{1/2} = 3^1$
ERLL ●	Use *language* to describe b^x as x factors of b for rational values of x.	Ben used language to describe his exponential notation. He stated "$3^{1/2}$ is a number that when I multiply by itself by another $3^{1/2}$ gives me just 3."
ERPIR ◍	Use *partial-interval reasoning* to describe exponential behavior	Through probing by the researcher, Ben described $3^{1/2}$ as a partial factor of the whole factor of 3^1. However, it was not clear whether Ben would be able to apply this on his own in another context.
Evidence of *no* understanding ○	Evidence of *weak* understanding ◍	Evidence of *strong* understanding ●

Possessing a strong ability to describe fractional exponents, such as $3^{1/2}$, in meaningful ways provided the foundation for building more robust covariational, multiplicative and partial-interval reasoning abilities. Through probing questions, Ben was able to describe $3^{1/2}$ as a partial factor of the whole factor of 3^1 [ERPIR]. However, it was not clear whether Ben would be able to apply this idea on his own in another context. Therefore, the next activity focuses on Ben's thinking as he attempted to apply the idea of partial factors in filling in the blanks of the provided tables in the activity. The presentation of the activity is followed by a conceptual analysis of Ben's thinking relative to this activity.

The Filling in the Missing Pieces Using Tables Activity

The next activity extended the idea of b^x as representing x factors of b and emphasized partial-interval reasoning in the context of exponential growth and decay data tables. The activity probed Ben to find the missing values in the tables below and to explain his reasoning while working through the task. The activity also prompted Ben to describe the changes in the output values compared with changes in the input values using language of covariation (e.g., for every increase of 0.5 in the input, the output increases by a factor of $2^{1/2}$).

Filling in the Missing Pieces Using Tables

The following tables provide data for linear and exponential functions. Use the data provided to fill in the blanks. Justify your reasoning. Write an algebraic function for each table.

Input	Output
-2.0	-7.0
-1.5	
-1.0	-4.0
-0.5	
0.0	-1.0
0.5	
1.0	2.0
1.5	
2.0	5.0

Input	Output
-4.0	¼
-3.0	
-2.0	½
-1.0	
0.0	1.0
1.0	
2.0	2.0
3.0	
4.0	4.0

Input	Output
0.0	142.0
0.5	
1.0	47.3
1.5	
2.0	15.7
2.5	
3.0	5.2
3.5	
4.0	1.7
4.5	

The purpose of this activity was to probe Ben's thinking relative to his mathematical decisions for filling in the blanks in the tables. The first table – a linear function – probed Ben to investigate *how* the output values progressed as the values of the input increased. The second and third tables – both exponential functions – prompted Ben to think about the multiplicative factor and the partial multiplicative factor relative to various increases in the inputs. Table 27 provides the matrix for the Filling in the Missing Pieces Using Tables activity relative to targeted components of the framework.

Table 27

Matrix Mapping of Exponential Function Framework and Filling in the Missing Pieces Using Tables

Code	Framework Description	Filling in the Missing Pieces Using Tables Questions
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.	Construct the multiplicative factor relative to the progression of output values provided in the table.
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.	Describe the multiplicative factor relative to the progression of output values provided in the table.
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.	Describe the progression of output values provided in the table.
ENLE	Use *explicit definition* of exponential function to represent exponential situations explicitly.	Write an algebraic function for each table.
ERCR	Use *covariational reasoning* to describe exponential behavior.	Describe the change in output values relative to changes in the input values.
ERFR	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	Compare multiplicative change provided in the exponential tables with additive change provided in the linear table.
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	Describe the changes in the output values as a multiplicative change by a factor.
ERPIR	Use *partial-interval reasoning* to describe exponential behavior.	Describe the multiplicative change in terms of partial factors representing a full factor.

The first table presented in the activity described a linear function with a rate of change of $\frac{3}{1}$. Ben described this situation as a "bunch of halfway in betweens" and he further articulated his thinking about this table in Excerpt 43.

Excerpt 43

Ben: 1 And since it is a linear function… I am looking at the halfway
2 between the inputs…the output is going to be halfway between the
3 outputs of the two inputs that I looked at.

Ben was able to fill in the blanks of the linear function table by calculating the midpoints between the given output values. Excerpt 44 illustrates Ben's reasoning.

Excerpt 44

Ben: 1 I am adding one half to each input but for the output I am adding
2 one and a half to each output.

Excerpt 44 provides evidence of Ben's thinking about the covariation of the changes in output values in relation to the corresponding changes in the input values. This utterance provides evidence of his ability to successfully coordinate additive change in the independent variable with additive change in the dependent variable. He was able to reason additively by articulating that in order to fill the blanks of the table he added "one and a half to each output" when he added "one half to each input." The evidence presented in Excerpt 43 and 44 suggest that Ben was able to reason through the linear table of values by calculating the arithmetic mean between sets of values. It should be noted that these data connect to Confrey and Smith's (1994) notion of rate as a unit per unit comparison because Ben was comparing the increase of output values as a unit of +1.5 for each increase in the input values of +1.

The second table presented in the activity described an exponential growth function with a growth factor of 2 for every 2 units of increase in the input values. Ben was able to express his thinking about the exponential growth of multiplying the outputs by the same amount as opposed to adding by the same amount as he did with the linear growth table [ERFR2]. Excerpt 45 illustrates Ben's thinking.

Excerpt 45

Ben: I'm not looking to add the same amount to each one. I'm looking to multiply the same amount [ERFR2] well, yeah multiply each one by an amount [ERFR2]. But I didn't increase it by the same amount but I multiplied it by that two [ERFR2]...Here I'm multiplying this amount to get over to this one and that change is going to be different than this change [referring to the linear table, ERFR2].

Despite Ben's ability to describe the operation of multiplication for finding output values in the exponential growth table, he was unsure of how to calculate the missing values. To make sense of the task, he decided to create an exponential function that would allow him to fill in the blanks on the table. Excerpt 46 provides Ben's thinking about his function.

Excerpt 46

Ben: This would be 2^0 is one, 2^1 is two, 2^2 is four. So I was trying to

look at a function like $2^{x/2}$ [ENLN, ENLE1] something like that
because zero over two will still give me zero. Umm, I looked at 2^x,
$2^{x/2}$ [ENLN, ENLE1] as my function of plugging in a -3 there and
so I would have $2^{-3/2}$ [ENLN] and that is the same thing as
saying $2^{-1.5}$, it's $\frac{1}{2^{3/2}}$ so that should be $\frac{1}{\sqrt{8}}$ [ENLN].

Ben later rationalized $\frac{1}{\sqrt{8}}$ as $\frac{\sqrt{2}}{4}$ which became the value he inserted into the first blank. He then proceeded to calculate the value for the next blank by doubling $\frac{\sqrt{2}}{4}$ [ENLI]. He continued doubling each previous blank to calculate each new value in the remaining blanks [ENLI, ERMR2a]. These data suggest that Ben was able to think about the recursive process of maintaining the doubling unit for each increase of 2 units in the input. By creating a function to describe the given data, Ben was able to find the value of the first blank and then he repeatedly multiplied by 2 to fill in the missing values. Ben did not, however, reconstruct the multiplicative unit of 2 for each 2 units of increase in input to accommodate an increase of 1 unit in the input. This reconstruction would have yielded a multiplicative unit of $\sqrt{2}$ for each 1 unit of increase in input. In an effort to probe further, the researcher probed Ben to connect his thinking in the Properties of Exponents activity to this activity. Excerpt 47 shows the dialogue about half factors relative to this exponential growth task.

Excerpt 47

AS: So going back to these properties of exponents that we were doing, we had like $3^{1/2}$. How do we explain that again – what $3^{1/2}$ was in terms of the word *factor*?

Ben: It was a half factor of three [ENLL].

AS: Ok so here in this problem we can use that knowledge to help us here. We knew we were doubling but every two increments in the input. So if we halved that value and went with the one increment then what did we do to the multiplication of two?

Ben: We found the half factor of two, so $2^{1/2}$ or $\sqrt{2}$ [ENLN, ENLL].

The dialogue in Excerpt 47 illuminated Ben's thinking of how the multiplicative unit can be reconstructed into the "half factor of two" or $2^{1/2}$ [ENLN, ENLL]. This way of thinking, however, did not come naturally for Ben as he made sense of this table of values. It was only when the researcher probed Ben that he connected this situation to thinking about partial factors of a full factor.

The third table presented in the activity described an exponential decay function with a decay factor of $\frac{1}{3}$ for every 1 unit of increase in the input values. However, this decay factor was not obvious to Ben when he initially began investigating this table of values. Excerpt 48 displays Ben's initial response to filling in the missing values in this table.

Excerpt 48

Ben: Well it is decreasing at a decreasing rate so before I went from a difference of 94.7 then I went to a difference of 31.6 to a

difference of 10.5 to a difference of 3.5. So the differences
between each of these outputs are getting smaller [ERCR2b].

Excerpt 48 illustrates that Ben was attending to the changing differences in the output values and he determined the differences to be decreasing for increasing input values [ERCR2b]. He also determined that the function was "decreasing at a decreasing rate." With further probing, Ben explained how he could *multiply* the values by a "fraction or decimal" since the values were decreasing. He ultimately decided that $\frac{1}{3}$ was the multiplicative factor between the given output values. Ben found this factor by calculating the ratio of consecutive output values for corresponding input values of interval length one.

Once Ben was able to calculate the multiplicative factor of $\frac{1}{3}$ for every one unit of increase in the input, the researcher probed him to think about the previously discussed idea of half factors. In the previous table, Ben found the value in the first blank by finding a formula and calculating the output for the given input. Then he proceeded to double this value to get the value in the next blank. Therefore, for this table, Ben was prompted to think about filling in the value without using a formula. Excerpt 49 details this discussion.

Excerpt 49

AS: Now see if you can think about this conceptually using this idea of
"$3^{1/2}$ is a ½ of a factor of 3"...What would you have to multiply
the 142 by to get the blank one where the input is 0.5 without

going to the formula?

Ben: By a 6 because I am looking for something that added together so, …I was thinking almost 1/3 as the whole [wrote $b^{1/3}$ on the paper] and I am looking for factors that when I add them together add up to 1/3 and 1/6 plus 1/6. Our two factors of the same base or the same denominator, that when I add them together they equal 1/3.

AS: Now why did you choose to add?

Ben: … I am thinking about it in the terms of the exponents were when we are multiplying I am thinking of well I have some base and say it's that base to the 1/3 [pointed to $b^{1/3}$] so moving for that base now I am looking for ½ of a factor of that 1/3 factor of that base so I was just looking at it in terms of exponents.

Ben's thinking in Excerpt 49 suggests that he viewed the unit of 1/3 as the exponent, rather than as the factor that is repeatedly multiplied for each increase of one unit in the input. The connection of repeated multiplication to exponentiation was absent in Ben's thinking. Furthermore, Ben attempted to find the partial factors of $b^{1/3}$ as $b^{1/6+1/6}$ without consideration for the meaning of the quantity $\frac{1}{3}$ as a multiplicative unit relative to the situation provided in the table[11].

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[11] It is possible that the fraction $\frac{1}{3}$ distracted Ben and thus he proceeded down the wrong path when attempting to make sense of the problem. However, the fact that it became a distraction testifies to his unstable foundation.

To facilitate Ben's thinking, the researcher asked him leading questions by probing him to think about the *factor* that related two output values for each one unit of increase. Excerpt 50 presents this discussion.

Excerpt 50

AS: Ok so that constant multiple of 1/3 does that go in the exponent? Is that the number of factors that I am multiplying my base out? Or does that 1/3 go in the base and I some how multiply a certain number of factors together?

Ben: That number goes in my base cuz here 47.3 is a third of 142.0 [ERMR2a] and 15.7 is gonna be not 2/3, but the combination of those thirds which multiplied together is 1/9 [ERMR2b, ERPIR1]. So if I took 142 and divided that by 9 I get my 15.7 repeating [ERMR2a].

AS: Exactly.

Ben: So, I'm taking my 1/3 to the power of whatever here [ENLL].

In Excerpt 50, Ben realized that 1/3 represented the multiplicative factor, which related the output value of 142 together with 47.3. This prompted him to reconsider the partial factor of $\left(\frac{1}{3}\right)^{1}$ as a way for calculating the missing value in the table for an input value of 0.5. However, Ben experienced additional difficulty in conceptualizing the idea of half factor of $\frac{1}{3}$ as representing the factor that multiplied twice equals the full factor of $\frac{1}{3}$. The discussion continues in Excerpt 51.

Excerpt 51

Ben: Half of one third [he wrote 1/6 on the paper].

AS: So keep in mind that we decided that $3^{1/2}$, we would say that's ½ a factor of 3, so…

Ben: Oh, ok so 1/3 and that half of…so right here I am looking, all of these increments are just, so that's to the first [ERMR2b]. And then not going straight back down, but from there this [he points to 142] to this [he points to 47.3] is $\left(\frac{1}{3}\right)^{1}$ [ERMR2a], and from this [he points to 47.3] to this [he points to 15.7] $\left(\frac{1}{3}\right)^{1}$ [ERMR2b]. So I'm now looking for the $\left(\frac{1}{3}\right)^{1/2}$ [ENLN, ENLL, ERPIR1]. Ok…so I take my 142.0 and multiply it by $\frac{\sqrt{3}}{3}$ [ERMR2a, ERPIR1]…if I am going half increments, half increments. So instead of going that full zero to one zero to half and then I multiply this, I am trying, well I am thinking…how I want to say this because it maybe made since in my head but it didn't come out right…I am gonna be in increments of $\frac{\sqrt{3}}{3}$ [ERMR2a, ERPIR1]. So instead of multiplying by $1/3$ to get down to here I need to multiply it by $\frac{\sqrt{3}}{3}$

to get to here [ERMR2a, ERPIR1] and then by another $\frac{\sqrt{3}}{3}$ to get to here and $\frac{\sqrt{3}}{3}$ to get to here and then going down that way [ERMR2b, ERPIR1].

Ben's thinking in Excerpt 51 reveals his ability to finally conceptualize the role of the half factors of the full factor $\left(\frac{1}{3}\right)^{1}$. In lines 16-19, Ben explained the necessity to multiply the output value by $\frac{\sqrt{3}}{3}$ to obtain the missing value for the blank as he increased by increments of 0.5 in the input[12]. These utterances suggest Ben's ability to think about fractional exponents as partial factors of a full factor [ERPIR1]. His ability to repeatedly multiply the outputs by the half factor (i.e., $\frac{\sqrt{3}}{3}$) as he progressed in half increments of the inputs illustrates Ben's multiplicative reasoning ability for describing the progression of outputs as a recursive process.

After Ben found the missing values in the table, he reflected on his initial thinking about the role of the $\frac{1}{3}$ as an exponent rather then the multiplicative factor. Excerpt 52 describes his thinking.

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[12] Ben spontaneously decided to represent $\left(\frac{1}{3}\right)^{1/2}$ as $\frac{\sqrt{3}}{3}$. It can be argued that the representation of $\left(\frac{1}{3}\right)^{1/2}$ or $\sqrt{1/3}$ better preserves the structure that helps bring meaning to this situation. It seems that the tendency to convert to the root structure and then rationalize the denominator only handicaps teachers and students when reasoning through this type of situation.

Excerpt 52

Ben: Well here [he pointed to $b^{1/3}$ on the paper] I was still thinking of the third as being my exponent because I am looking for some type of multiplication of an exponent to another to that exponent. So the same base but to different power to get me back to 1/3.

AS: Ok, as opposed to keeping your 1/3 as the actual multiplier?

Ben: Yeah, so I was up here I am not even thinking of the 1/3 as being that base, as being what's going to a power, but being the power itself. But then I am trying to find half of that power, which is where a half factor of a third factor of the base came from, right there.

AS: Right, when it should just be a half factor of...

Ben: Of a third.

Ben's utterances suggested that he realized his thinking was inconsistent with the multiplicative behavior inherent in the situation. With probing by the researcher, Ben was able to make sense of the behavior by connecting the notion of half factors to the notion of a partial factor which multiplied by itself twice resulted in a full factor. In line 9, Ben reiterated his thinking about "half factor of a third factor of the base" and realized that he should have been thinking about *half factor of* $\frac{1}{3}$. His utterances revealed his inconsistent use of the term *factor* and thus it seemed that Ben had not fully appropriated the term *factor* for representing multiplication. After reflecting on his thinking, Ben was able to articulate his inconsistent thinking.

The final task in this activity prompted Ben to write an algebraic formula to describe the multiplicative behavior of the function described in the table of values. Ben's initial function was $f(x)=\left(\frac{1}{3}\right)^{x/2}+141$. His reasoning for adding 141 – as opposed to 142 – was "if I put in the zero that gives me a zero up here, $\left(\frac{1}{3}\right)^{0}$ is just one plus 141 gives you 142" [ERFR0]. This reasoning illustrated Ben's inappropriate use of additive reasoning when describing an exponential situation [ERFR0]. It also suggested that he did not consider the connection between the multiplicative behavior illuminated in the table and the general form of an exponential function. The researcher prompted Ben to test whether his function worked for other input values besides zero. As a result, Ben revised his function to $f(x)=142\left(\frac{\sqrt{3}}{3}\right)^{2x}$. His thinking follows in Excerpt 53.

Excerpt 53

Ben: Here I am multiplying by just a factor of one square root of three
over three (pointing from 142 to the next blank of $142\left(\frac{\sqrt{3}}{3}\right)$).
Here's two square root of three over threes (pointing from $142\left(\frac{\sqrt{3}}{3}\right)$
to 47.3). Then if I am only inputting one…I need to have two of
these being multiplied together [ERMR2a, ERPIR1] so if I am
putting in a one there that gives me the two to get from here to here
and then again that means…that we have a total of four square root

of three over threes multiplied together to get down to here [ERMR2a, ERPIR1]. So two multiplied into here gives me those four square root of three over threes [ERPIR1], and then just starting at zero if I put in a two times a zero that gives me a zero, square root of three over three to the zero equals one and one forty-two times that one equals that one forty-two [ENLE1].

The data in Excerpt 53 illuminates the advances in Ben's thinking for considering the multiplicative decay of the values illustrated in the table. Previously, Ben had experienced difficulty with conceptualizing partial factors as representing a full factor. The data revealed that while he was able to think successfully about exponential behavior in terms of partial factors for some situations, he was not successful in all situations. The discussion in Excerpt 53 served as a turning point for Ben as he was able to successfully articulate in lines 4-5 and 7-8 that he (a) needed multiple factors of $\frac{\sqrt{3}}{3}$ as determined by the exponent (i.e., an exponent of 1 represented two factors of $\frac{\sqrt{3}}{3}$ and an exponent of 2 represented four factors of $\frac{\sqrt{3}}{3}$); and (b) needed to revise the exponent of his initial function to capture the doubling of his partial factor of $\frac{\sqrt{3}}{3}$. Ben summarized this situation as "going from one step to the next you are multiplying by a square root of three over three each time."

Table 28 provides a summary of the analysis of Ben's thinking as he worked through the Filling in the Missing Pieces Using Tables activity.

Table 28

Ben's Reasoning Model for Filling in the Missing Pieces Using Tables

Code	Framework Description	Analysis of Ben
ENLN ●	Use *notation* to represent b^x as x factors of b for rational values of x.	Ben connected the half factor of two to the exponential notation of $2^{1/2}$ or $\sqrt{2}$.
ENLL ●	Use *language* to describe b^x as x factors of b for rational values of x.	Ben described the multiplicative behavior in terms of half factors.
ENLI ◍	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben described the output values as a process of multiplying the previous output by a multiplicative factor.
ENLE ●	Use *explicit definition* of exponential function to represent exponential situations explicitly.	Ben wrote and justified the exponential functions associated with the provided table of values.
ERCR ◍	Use *covariational reasoning* to describe exponential behavior.	Ben demonstrated limited ability to reason covariationally by describing how the differences in the output values of the exponential decay table were decreasing.
ERFR ●	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	Ben described exponential growth of multiplying the outputs by the same amount as opposed to adding by the same amount as with linear growth table.
ERMR ●	Use *multiplicative reasoning* to describe exponential behavior.	Ben described the output values as a process of multiplying the previous output by a multiplicative factor.
ERPIR ●	Use *partial-interval reasoning* to describe exponential behavior.	Ben described the situations in term of the multiplicative factor of the partial intervals (i.e., to multiply by $\frac{\sqrt{3}}{3}$ for every increase of 0.5 in input).

Evidence of *no* understanding ○ Evidence of *weak* understanding ◍ Evidence of *strong* understanding ●

Episode 3: Linear, Quadratic and Exponential Growth

The third teaching episode included two activities: (a) Filling in the Missing Pieces Using Graphs activity and (b) Comparing Linear, Quadratic, and Exponential functions activity. The goals for this episode focused on (a) extending Ben's notion of multiplicative unit to the graphical representation and (b) further developing his notion of exponential behavior by exploring the contrast between multiplicative rate of change with linear and quadratic rate of change. In Episode 1, Ben provided his image of exponential function using a concept map. This concept map and follow-up discussion illuminated Ben's thinking about quadratic function as representing a subset of exponential function. Therefore, the activities of Episode 3 provided opportunities for Ben to enhance his understanding of exponential function as possessing a multiplicative rate of change, while contrasting this behavior with rate of change of polynomial functions (e.g., linear and quadratic functions).

The Filling in the Missing Pieces Using Graphs Activity

This activity prompted Ben to think about the corresponding changes in the outputs relative to changes in the inputs, specifically for non-integer, positive values of the input. Thinking about the multiplicative unit, relative to the graphical representation of an exponential function, prompted Ben to describe his thinking about decimal exponents (i.e., $3^{1.1}$). This activity also promoted conversation about continuous functions, which provided insight into Ben's thinking about the meaning of connecting discrete points on an exponential graph.

Filling in the Missing Pieces Using Graphs

The following table and graph provide data for a tripling exponential function. Use the data provided to fill in the blanks. Justify your reasoning. Write an algebraic function for the data presented.

x	2	3	4
$f(x)$	9		81

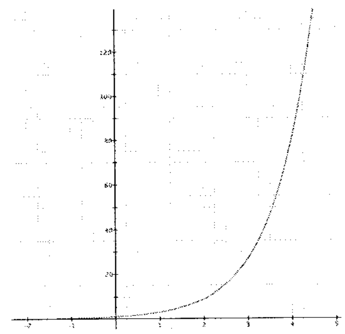

Table 29 provides the matrix for the Filling in the Missing Pieces Using Graphs activity relative to targeted components of the exponential function framework.

Table 29

Matrix Mapping of Exponential Function Framework and Filling in the Missing Pieces Using Graphs

Code	Framework Description	Filling in the Missing Pieces Using Graphs Questions
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.	Use the table and graph provided to represent the multiplicative growth of the function.
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.	Describe the tripling situation in terms of the previous output values as they relate to the next output value.
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	Describe the tripling situation in terms of the multiplicative factor.
ERPIR	Use *partial-interval reasoning* to describe exponential behavior.	How do you know what the in-between values (e.g., for inputs of 2.5 and 3.5) are when looking at the table and/or the graph provided?

Episode 3 began by reviewing the second activity from Episode 2. This discussion prompted Ben to think about the connections between filling in the missing values using only data tables and filling in the missing values using a *graph*. He was able to articulate his thinking for the previous activity and utilize his ideas to correctly fill in the blank provided in the Episode 3 activity. He was able to explain that the missing value was 27, which he obtained by "taking the 9 and tripling it to get to the next value of 27 which then gets tripled to get to 81." This evidence shows Ben's thinking about the patterns of the data.

Ben also discussed his thinking about the graphical representation provided in the activity, which illuminated his understanding of continuous functions. This conversation between Ben and the researcher, as detailed in Excerpt 54, revealed his thoughts about decimal and fractional exponents relative to the graph.

Excerpt 54

Ben: And the graph itself is just showing me a reading in between those two points where the amount of change is between each little section of those two points.

AS: Now how do we know that this graph is defined in between…all those four points that you have plotted there. How do we know that we can actually connect all those points?

Ben: Well based off just the information we don't know for sure. Right now we are just dealing with these whole numbers but I mean in terms of just numbers that we are going to deal with in order to get

> from 1 to 2 if you are thinking in terms of decimal places, well before you can get to 1.2 to count it you have to go to 1.1, the space that you are going to in terms of the graph to get to that spot so I just slowly go up from one to two decimal spots or the fractions if you want to think of that, something that's still affecting what's happening to the graph, it's not just well we are counting in between there, we get to 2 and then finally we have a change in the graph. There's always a change going on between those two points [ERCR2a]. Because there's no three to the 1.1 power [he notes that the table does not show this value], it doesn't suddenly not exist...that $3^{1.1}$ has to be somewhere and its not quite 3^2 but it's definitely bigger than 3^1 power [ERPIR].

In lines 17-18, Ben articulated his thinking about how the graph illuminated the changing values between two points. He was able to recognize that the function situation, promoted by the graph, represented a continuous change of values rather than discrete changes (as exhibited in lines 15-17). Furthermore, lines 20-21 demonstrate Ben's thinking of the quantity of $3^{1.1}$ as a value between 3^1 and 3^2. His utterance in line 21, however, suggested that he believed $3^{1.1}$ to be larger than 3^1, but he did not provide information on why he held this idea. Thus, the researcher continued the conversation by probing his thinking about decimal exponents. Excerpt 55 shows a portion of this conversation.

Excerpt 55

AS: Now how do we know it's [referring to $3^{1.1}$] bigger than 3^1?

Ben: ...one three is less than two three's being multiplied together [ENLN, ENLL] and now I don't just have one three, I have three and just a little bit more of its value...and that little bit more is all you need to get larger than the previous [ENLL].

At this point in the discussion, Ben only vaguely described why $3^{1.1}$ was larger than 3^1 and later explained that it was "mathematically bigger." He was able to articulate his thinking about the quantity of 3^1 as representing one factor of three multiplied and then 3^2 as representing two factors of three multiplied together [ENLN, ENLL]. The conversation that continued is provided in Excerpt 56.

Excerpt 56

AS: Is there a way that we could write maybe $3^{1.1}$ as something that has a fractional exponent or includes a fractional exponent?

Ben: Yeah like a 0.1 is equal to 1/10th, so you could write this as $3^{11/10}$ [ENLN] and if you want to break that down you have $3^{10/10} \cdot 3^{1/10}$ or $3^1 \cdot 3^{1/10}$ [ENLN] and $3^{1/10}$, anytime you have that fraction it means that's the root of three we are looking for, so it's $3^1 \cdot \sqrt[10]{3}$.

Ben's utterances in lines 5-6 revealed once again his persistent thinking of fractional exponents in terms of the operation of root. When prompted to connect his ideas to the previous discussion on fractional exponents, Ben was able to offer another way of thinking as illuminated in Excerpt 57.

Excerpt 57

Ben: …it goes to how many threes are being multiplied together to form products [ENLL] so here I have a one so I'm just multiplying one three [ENLL] but here I have 1/10th which means I'm multiplying 1/10th of a three which isn't $\frac{3}{10}$ but…a factor of 3 that when multiplied 10 times together [ENLL, ERPIR2]. So it's whatever this 1/10th of 3 is times 1/10th of 3 times 1/10th of 3 and when I multiply it out 10 times it equals 3 [ERPIR2]. It's not $3 \cdot \frac{1}{10}$ or $\frac{3}{10}$. It's 1/10th value of 3 that when multiplied together 10 times will end up being 3 [ERPIR2] and the single one is just 1/10th of that value, not the whole [ERPIR2].

AS: Okay so if 3^1 conceptually is one factor of 3 then we could say $3^{\frac{1}{10}}$ is how many factors of 3?

Ben: Is 1/10th factor [ENLL]…it makes more sense to say 1/10th factor of that is a number that's going to be…multiplied by itself 10 times in order to get to the whole number we want [ERPIR2].

Ben's utterances in Excerpt 57 illuminate the complexity of thinking about b^x as x factors of b when the value of x is a fraction. On numerous occasions, he referred to this idea as "1/10th of 3" indicating the operation of multiplication of the quantities 1/10 and 3; yet at the same time, he articulated that he was not thinking about the *product* of these two numbers, but rather about the *number of factors* of the base number. His utterances (such as in lines 14-15) provided clear evidence that he was able to think about the exponent as representing the number of factors multiplied to obtain the "whole" factor. In other words, he had the ability to view fractional exponents as partial factors of a full factor [ERPIR2]. Excerpt 58 provides additional evidence of Ben's thinking about partial factors using the table of values provided in the second activity of Episode 3. In this conversation, Ben described his thoughts about the input values of 2.5 and 3.5 relative to the corresponding output values.

Excerpt 58

Ben: …so if I'm looking for a 2.5 I'm looking for essentially $3^{2.5}$ [ERCR1]…so I write 3^2 times my $\sqrt{3}$ [ENLN] so if I'm looking for half [referring to the decimal exponent] I'm multiplying in a certain amount of a $\sqrt{3}$'s [ENLN] and by that I mean…I'm looking at multiplying in one $\sqrt{3}$ by going up to the next one I'm going up a half, I need to multiply in another $\sqrt{3}$ [ERCR3, ERMR2a] so now I'm multiplying $\sqrt{3}$ times $\sqrt{3}$ [ENLI]. Going up another half I got three $\sqrt{3}$'s being multiplied together [ENLI, ERMR2a]…

The discussion in Excerpt 58 provides further evidence, yet again, of Ben's thinking about fractional exponents as representing the operation of root. In this instance, however, he was able to demonstrate for the first time his thinking about the multiplicative relationship among the output values by saying, "going up to the next one I'm going up by a half, I need to multiply in another $\sqrt{3}$" [ERCR3, ERMR2a]. This utterance suggests Ben's ability to reason covariationally at Level 3 based on his thinking that for each ½ increase in input, the output changed by a multiplicative factor of $\sqrt{3}$. His thinking also revealed evidence for his ability to view the exponential function as a recursive process [ENLI].

The Filling in the Missing Pieces Using Graphs activity provided the opportunity for Ben to explore the role of the multiplicative factor when analyzing the graphical representation of the function. Ben articulated that the graph revealed the tripling behavior when considering the size of the output values relative to the input values. Excerpt 59 illustrated his thinking.

Excerpt 59

Ben: The 1 is at the 9, that 2 is at the 27. How many times bigger is 27 than 9? It's 3 times, so between these two, I've increased it 3 times its size or 3 times it's amount…so we took this previous term and tripled it to get to my next full amount unit [ENLI, ERMR2a].

Ben's utterances in Excerpt 59 revealed his thinking for the multiplicative factor of 3 for every one unit increase in the input. The next discussion prompted Ben to think about how the graph illuminated the role of the multiplicative factor of $\sqrt{3}$ that Ben had used previously to describe the half-intervals. Excerpt 60 reveals his understanding.

Excerpt 60

AS: Where on the graph is the square root of three?

Ben: The $\sqrt{3}$ exists halfway in between these two input values [ERPIR2]. So I have to put in one-half in order to get that square.

And in terms of the output values, its going to… If I look at the straight line between these points, it's going to exist in between them. But its overall value is not quite halfway in between them [ERFR, ERPIR].

AS: Why not?

Ben: Well halfway in between 27 and 81 is 59, which is right about there[13] [points to the graph]…that's the actual midpoint of 3.5 [ERFR]…and that's my $27 \cdot \sqrt{3}$ [ENLN, ERMR2], but this, in terms of the curved distance between, or if I had a straight line here that's the distance between these two points [ERFR]. It occurs at the value of 59, which if I plotted it on the graph is definitely higher than $27 \cdot \sqrt{3}$ is at [ERFR]…and so, we're not so much looking for just a multiple of that 3. So maybe like a 6 or a 9, 12 something like that. Or maybe a 1.5, a 3/2 [ERFR]. But we're looking at a number of threes that get multiplied together [ENLL, ERMR2]. And since we're going by a square root, that's not half of the 3 [ERFR], that's a term that when multiplied by itself equals 3 [ENLL, ERPIR]. One half of a 3 is 1.5, but 1.5 multiplied together is 2.25 [ERFR].

---

[13] The correct value between 27 and 81 is 54, not 59. However, Ben's thinking is relevant despite his incorrect calculation.

As illustrated in this excerpt, Ben was able to contrast the multiplicative behavior of increasing by a factor of $\sqrt{3}$ with additive behavior of finding the arithmetic mean between the output values of 27 and 81 [ERFR]. Ben's ability to reason through the exponential function situation, from a multiplicative perspective, facilitated his understanding of the progression of output values. He was able to demonstrate his thinking that the output values of an exponential graph fall below the output values of a linear graph; thus, according to Ben, the midpoint between two output values would not be an accurate calculation for describing the partial factor obtained from the exponential situation.

Table 30 provides a summary of the analysis for Ben's thinking as he worked through the Filling in the Missing Pieces Using Graphs activity.

Table 30

Ben's Reasoning Model for Filling in the Missing Pieces Using Graphs

Code	Framework Description	Analysis of Ben
ENLM ◍	Use *multiple representations* to represent the multiplicative processes of exponential function.	While Ben utilized the table and graph provided, he demonstrated weak evidence of his ability to view the connectedness of these representations.
ENLI ●	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben described the situation as multiplying by $\sqrt{3}$ for each half unit increased in the inputs.
ERMR ●	Use *multiplicative reasoning* to describe exponential behavior.	Ben described the tripling behavior as "we took this previous term and tripled it to get to my next full amount unit."
ERPIR ●	Use *partial-interval reasoning* to describe exponential behavior.	Ben described $3^{1/10}$ as $1/10^{th}$ factor of 3 while also describing this quantity as the number multiplied by itself 10 times to get the "whole number" of 3^1.

Evidence of *no* understanding ○ Evidence of *weak* understanding ◍ Evidence of *strong* understanding ●

The next activity focuses on Ben's ability to compare and contrast between linear, quadratic, and exponential functions. Following the presentation of this activity is a conceptual analysis of Ben's thinking relative to comparing these functions.

The Comparing Linear, Quadratic and Exponential Functions Activity

The second activity of Episode 3 advanced Ben's thinking about rate of change of linear and quadratic functions relative to rate of change of exponential functions by engaging him in analyzing tables of values for each of these functions. This activity prompted Ben to grapple with constant and changing rates of change. The last question of this activity probed Ben's thinking about the growth of a polynomial function relative to the growth of an exponential function.

Comparing Linear, Quadratic and Exponential Functions

In a previous session, we discussed the Cheerios problem and the Pennies on a Chessboard activity which both produced an exponential function of the form $f(x)=2^{x-1}$. Let's compare the data table of this function with the data table for the linear function, $f(x)=3x+2$, and the data table for the quadratic function, $f(x)=x^2$.

Linear Function

	x	fx
1	1	5
2	2	8
3	3	11
4	4	14
5	5	17
6	6	20
7	7	23
8	8	26
9	9	29
10	10	32

Quadratic Function

	x	fx
1	1	1
2	2	4
3	3	9
4	4	16
5	5	25
6	6	36
7	7	49
8	8	64
9	9	81
10	10	100

Exponential Function

	x	fx
1	1	1
2	2	2
3	3	4
4	4	8
5	5	16
6	6	32
7	7	64
8	8	128
9	9	256
10	10	512

- Using the tables provided, answer the following questions:
- How would you go about analyzing the data for each of these functions?
- What analysis strategies would you emphasize with your own students if you were to use these tables?
- What kinds of questions would you ask your students to facilitate their analysis?
- Who wins?: Consider the following two functions: $f(x)=x^n$ and $f(x)=n^x$ where $n>1$. Which one of these functions grows faster over time? Provide a convincing argument to justify your response.

Previously in Episode 1, Ben had asserted his thinking about quadratic function as representing a subset of exponential function because "anything that has a power of something" constitutes an exponential function (see Excerpt 21 lines 1-2). Therefore, the activities in Episode 3 prompted Ben to reflect on the various rates of change using graphical, tabular and algebraic representations of the functions as a tool for comparing various function families. Table 31 provides the matrix for the Comparing Linear, Quadratic, and Exponential Functions activity relative to targeted components of the exponential function framework.

Table 31

Matrix Mapping of Exponential Function Framework and Comparing Linear, Quadratic, and Exponential Functions

Code	Framework Description	Comparing Linear, Quadratic, and Exponential Functions Questions
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.	Utilize table of values to create graphical and algebraic representation.
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.	Describe which function grows faster: $f(x) = x^n$ or $f(x) = n^x$ where $n > 1$. Describe the exponential function (table, graph, and algebraic form) using multiplicative reasoning.
ERFR	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	Compare multiplicative change provided in the exponential table with additive change provided in the linear table and with linear change provided in the quadratic table.
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	How do you know what the in between values are when looking at the table and/or the graph provided?

In this activity, Ben utilized computer software, called Fathom™ by Key Curriculum Press, to facilitate his analysis of the provided data. Ben was encouraged to analyze the data using differences and ratios of output values as a tool for finding patterns among the data. Ben began the analysis with the linear function table. He quickly recognized that the output values increased by the same amount for each unit added in the input. With probing by the researcher, Ben recognized that the second differences were always zero. Ben defined this function in the following way: "A linear function means it's always going to have that same amount of differences between each point. It's not changing the amount." He also demonstrated constant rate of change using the familiar finger tool to visualize the constant amount of increases in output values for equal increases in the input values.

After analyzing the linear function table of values, Ben transitioned to the quadratic function table where he recognized the pattern emerging from these outputs. Excerpt 61 reveals Ben's initial thoughts about the quadratic function table.

Excerpt 61

Ben: Well, the distances between all the points are increasing [ERCR2]. But they're increasing at almost a …wait no I can't say that. Yeah, they're increasing at a constant rate [ERCR3]…I'm adding two to the previous distance [ERFR]. So from the first term to the second I just have a distance of three. But from the second to the third term I add on a two to that. So now I have a distance of five. So to find the next term I just add two to the difference of the previous two…I always add two to the previous distance [ERFR].

In Excerpt 61, Ben provided his thinking about the output values for the quadratic function as increasing differences by articulating, "I always add two to the previous difference" [ERCR2, ERFR]. At first glance, his utterance in line 3 seems to suggest that he viewed the output values of the quadratic function as increasing at a constant rate

[ERCR3]. However, he provided additional evidence that he viewed the *differences of the outputs* – not the output values – as increasing at a constant rate [ERCR3]. Upon further reflection, Ben noticed that he could describe the differences of the outputs of the quadratic function as representing a linear function [ERFR]. He also used the finger tool to visualize the changing output values relative to their changing differences. Excerpt 62 reveals Ben's thinking of the finger tool for this quadratic function.

Excerpt 62

Ben: So from 0 to 1, I just increased it by that one…here I'm increasing it more than just that one value. See I'm not going from adding one anymore, see I just added on here [ERFR]. Now I'm adding 3 up in there because I had a difference of one here and a difference of 2 there, so now I have a difference of 3. The next point I don't have a difference of 3, I have a difference of 5 because although the output is going up and changing, the differences are only increasing by 2 each time [ERCR2]. So now I'm increasing by 5, by 7. I'm just talking my best guess to show that the differences are increasing for each point [ERCR2, ERFR].

Ben articulated how the output values increased using his fingers to gesture the amount of increase for each new output value as the input values increased in equal increments. In lines 9-10 of Excerpt 62, he stated again that the "differences are increasing for each point" [ERCR2, ERFR]. The data suggested that he was able to reason covariationally for the linear and quadratic functions because he coordinated changes of output values with the corresponding changes in the input values.

The next portion of the conversation focused on analyzing the exponential function table. Ben immediately noticed that the output values were "double the previous" value and he recognized the doubling pattern in the differences of the outputs – for both first and second differences. However, the data suggested that Ben held a procedural understanding of the function concept because he consistently made sense of the situation by first calculating values and then describing the emerging pattern. He did not utilize covariational reasoning or conceptual understanding of function to holistically describe function characteristics, such as rate of change. Excerpt 63 provides an example of Ben's typical reasoning pattern when describing the increasing output values for quadratic and exponential functions.

Excerpt 63

Ben: So for the quadratic I put in a one, and that gave me a one back out. I put in a 2 and that gave me a 4 out, instead of back to that one. Same with the exponential function, I put in a one there and now I got a 2, I put in a 2 there… actually if I put a one in there I still get out a one, I put a 2 in there I get out that 2. The output values are going up in both those [ERFR].

Ben also compared the graphs for both the quadratic function, $f(x) = x^2$, and the exponential function, $f(x) = 2^{x-1}$ [ENLM]. He used Graphing Calculator by Pacific Tech to view the graphs. After comparing the graphs for a few integer values of inputs, Ben decided that while both graphs started out by growing slowly, the exponential graph demonstrated "a much bigger jump up…it's going much higher up there" [ERFR]. This utterance was consistent with Ben's thinking about "jumps" illustrated in the exponential graph when discussing the Population Task in Excerpt 2 (see chapter 6). The data suggest that he viewed the increases in the output values of an exponential function as "jumps" rather than a continuous progression of output values. Confrey (1994) discovered the same finding in her case study of Dan. As Dan made sense of an exponential situation, she noticed that he "conceptualized the interval as a 'jump' and then worked to build jumps of greater and greater size" (p. 322).

As shown in Excerpt 64, Ben continued in his descriptions of the quadratic and exponential graphs.

Excerpt 64

Ben: 1 Well, because in the quadratic we're only just changing the
2 distance between those points by the same amount, so I'm only
3 increasing it by 2 each time. Whereas here, I'm not increasing it by
4 2 but I'm doubling that previous amount [ERFR, ENLI, ERMR2a].
5 So when you're doubling a number you're doing more to it than
6 just increasing it by a certain amount [ERMR2a].

Ben's utterances in lines 2-4 demonstrated his thinking about the distinction between quadratic behavior and exponential behavior. His view of the exponential situation was consistent with his previous utterances; that is, he viewed exponential behavior as a recursive procedure of multiplying the previous output by a factor.

Ben's thinking of the differences as a tool for making sense of the function situation evolved during this activity. As shown in Excerpt 65, he commented on his new insights to reasoning about exponential behavior.

Excerpt 65

Ben: Well I've always known that the exponent is something that multiplies these together, but I liked seeing those difference things going on because, now I can visually see why that exponential function is not going to come down to zero and hit that zero mark [referring to the idea that the differences of an exponential function will never be zero]. Now I have a pretty good idea why, it's because of that change going on…it is an exponential function…no matter how far down the line I go the differences will always double themselves. It's not going to end…it's neat how that worked out, I've never really looked at the differences of an exponential function, much less the differences from there.

Ben's comments suggested that he made important mathematical connections by comparing the tables for the quadratic and exponential situations. The difference tables provided an opportunity for him to consider the rates of change as a powerful tool for making sense of each situation. When prompted to further explain his thinking about the connection of quadratic and exponential functions displayed in his initial concept map, Ben provided the following thoughts (see Excerpt 66).

Excerpt 66

Ben: ...they still kind of tie into each other because I mean if I have a two, a 2^2 is part of that quadratic. It is part of that exponential. But the exponential is going to keep going, I mean its going to go on and on in a completely different direction than the quadratic does [ERFR]. Whereas for the quadratic, it's fairly simple to determine the next term. And the exponential sometimes you start getting some real big numbers. It's like, what am I changing this by? I'm just getting to big numbers much quicker with that exponential [ERFR].

Despite the activities emphasized during this teaching experiment, Ben continued to hold on to his deep-seated notion that quadratic function represents a subset of exponential function. (It is interesting to note that the teaching experiment conducted with Andy during the Exploratory Study also revealed a similar finding.) In lines 2-3, Ben stated that the quantity 2^2 is part of an exponential function and part of the quadratic function. He recognized that both functions contain the point $(2, 2^2)$, but he did not attempt to look beyond that fact to parse out distinguishing characteristics of functions (e.g., rate of change) that serve to identify one class of function from another. This particular conceptual obstacle is a critical finding in this study, and it illuminates the complexities associated with understanding function conceptually.

Table 32 provides a summary of the analysis for Ben's thinking as he worked through the Comparing Linear, Quadratic, and Exponential Functions activity.

Table 32

Ben's Reasoning Model for Comparing Linear, Quadratic, and Exponential Functions

Code	Framework Description	Analysis of Ben
ENLM ◍	Use *multiple representations* to represent the multiplicative processes of exponential function.	Ben utilized the graph and table to discover the changes in output values for each function.
ENLI ●	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben demonstrated strong evidence for his thinking about exponential function as a recursive situation: "double the previous value."
ERFR ◍	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	Ben demonstrated his observations about the changing output values for linear, quadratic, and exponential functions; yet, he did not provide well-connected notions of exponential behavior as separate from quadratic (polynomial) functions.
ERMR ◍	Use *multiplicative reasoning* to describe exponential behavior.	Ben described exponential behavior as a doubling of the previous value with no indication of an understanding of multiplicative rate of change.
Evidence of *no* understanding ○	Evidence of *weak* understanding ◍	Evidence of *strong* understanding ●

Episode 4: Exponential Decay Through Experimentation

The fourth teaching episode entailed the Light Intensity activity that was designed to both promote and assess Ben's ways of thinking that were explored in Episodes 1-3. The primary goals for this episode include: (a) experimenting with exponential decay through a real-world context of light intensity by conjecturing, collecting data, analyzing data, and extending ideas of exponential behavior; (b) reinforcing multiplicative, covariational, and partial-interval reasoning using a real-world context; and (c) using multiple representations as a tool for making sense of the function situation.

The Light Intensity Activity

This activity focused on exponential decay emerging from a real-world experiment of measuring light intensity relative to the number of layers of window tint. The activity emphasized multiplicative reasoning through (a) analyzing the table of values for the data collected in the experiment, (b) crafting a discrete graphical representation of the data, (c) calculating the exponential regression function, and (d) exploring various extension questions relative to the light intensity context.

The Light Intensity Activity

Part A: Making Conjectures

- What happens to light intensity when you tint your windows?
- Generate a list of conjectures regarding the relationship between light intensity and window tint.
- Sketch a graph of your conjecture about the nature of light intensity with respect to an increase in layers of tint. Label the axes appropriately.

Part B: Collecting Data

It's time to test your conjectures about the impacts of tint on light intensity. Use the following materials to answer the questions below.

- CBL Unit with Light Sensor Probe
- Tinted squares
- Light source (indirect sunlight is best!)

Connect the Light Sensor to the CBL Unit. The CBL Unit will automatically detect the light sensor and will revert to the default setting for measuring light. Calculate the light intensity on a given source of light. Now, add one layer of tinted material (directly over the light sensor) and record the new light intensity. Continue adding layers of tinted material (directly over the light sensor) and record your findings below.

# of layers	Light Intensity
0	
1	
2	
3	
4	
5	
6	
7	
8	

Part C: Data Analysis

Open Fathom and fill in a table of values that represent the number of layers and the corresponding light intensity value obtained from the CBL.

- Analyze your data in Fathom using each of the following:
- Table of values
- Difference Columns
- Ratio Columns
- Graph
- Formula (using sliders for each parameter)
- Explanation of each parameter
- Units
- Describe your findings. Consider the following questions as you complete your description.
    - o How is light intensity related to the amount of tint used?
    - o Is the rate-of-change increasing, decreasing, or constant?
    - o Do you have any data that is inconsistent with your other findings?
    - o How does this graph differ from linear behavior?

Part D: Extension

- If each sheet of window tinting blocks 15% of the light, what is the smallest number of layers of window tinting needed to block at least 50% of the light?
- What if you can buy window tinting that is 2.5 times darker than the tint used in the light experiment. How does this situation alter your function developed in your experiment? How will the light intensity be changed as a result of the increased amount of tint? How does this value compare to the graph of your data?
- What if you can buy window tinting that is 30 times darker than the tint used in the light experiment. How does this situation alter your function developed earlier? How will the light intensity be changed as a result of the increased amount of tint? How does this value compare to the graph of your data? Is it possible to completely block all light using tinted squares? Explain.

The conjecturing portion of the activity prompted Ben to anticipate the effects of adding layers of window tint on light intensity. The next part of the activity focused on collecting data (values of light intensity measured in mW/cm^2) using the TI-Calculator Based Laboratory (CBL), light probe and cellophane sheets of window tint, each one uniformly shaded the same. The data analysis section emphasized techniques and strategies for analyzing data gathered from the CBL. These techniques and strategies involved finding differences and ratios of the light intensity values (outputs), graphing the data and writing an algebraic formula to represent the situation. The final portion of the activity offered extension questions to advance Ben's thinking relative to constant percent decay and partial factors.

Table 33 provides the matrix for the Light Intensity activity relative to targeted components of the exponential function framework.

Table 33

Matrix Mapping of Exponential Function Framework to Light Intensity Activity

Code	Framework Description	Light Intensity Questions
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.	Represent the decay factor and number of layers of tint using exponential notation.
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.	Describe the exponential situation using the number of layers of tint as the number of factors of the decay factor.
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.	Create a table, graph, and algebraic representation for light intensity relative to the number of tinted layers.
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.	Describe the pattern emerging from the light intensity data (i.e., discuss the ratio column).
ENLE	Use *explicit definition* of exponential function to represent exponential situations explicitly.	Write an algebraic representation to model the experiment data.
ERCR	Use *covariational reasoning* to describe exponential behavior.	Describe how the light intensity decreases relative to equal increases in the number of layers of tint.
ERFR	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	Compare the exponential behavior emerging from the light intensity context with linear behavior.
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	Use multiplicative rate of change to describe changes in light intensity with respect to increases in tinted layers.
ERPIR	Use *partial-interval reasoning* to describe exponential behavior.	Describe partial layers of tint relative to the multiplicative factor.

The activity began with Ben conjecturing about the effects on light intensity when adding successive layers of window tint. Ben stated, "As you keep adding the window tint you're going to get less intense light." With prompting by the researcher, Ben conjectured about how the light intensity was changing for each successive layer of tint added. Excerpt 67 illuminates Ben's conjecture.

Excerpt 67

Ben: 1 I think like with every shade you put on it doubles how much less
2 intensity goes on so that first one is just the initial shade and then
3 however how much less you get from that you put on the second
4 one and its going to be like…double because now you have
5 doubled your amount of tinting you have over there so you've
6 doubled the amount of block, the light intensity coming through.

Ben's utterances in lines 1-2 suggested that he viewed the light intensity context as a multiplicative process of doubling the light blocked for each additional layer of window tint. Yet, in lines 4-6 he maintained that the doubling idea emerged from doubling the amount of tinting. This utterance implied Ben's thinking of the light intensity values as a direct proportional quantity of the number of layers of window tint (i.e., two layers of window tint yield twice as much light blocked). Ben also thought about each layer of tint as representing a percent value of decrease for each layer of tint added. Excerpt 68 reveals Ben's thinking.

Excerpt 68

AS: 1 So let's say…I put one square [layer of tint] on, what would you say is the amount of light coming through now?

Ben: Maybe 80% of the light?

AS: Okay and then what happens when I put then another layer?

Ben: There would be 60%, one layer takes off yeah about 60% of the light getting through.

AS: Are you thinking of each tinted square being a percentage number?

Ben: Yeah like taking off 20% each time you put a layer on…decreasing at 20% so that's decreased at 40% there [referring to having 2 layers of tint].

AS: And then like if I did a 3rd one then I would say a 3rd layer of tint would be…

Ben: 60%, now you've taken away 60% so there's only 40% of the original amount of light coming through.

Excerpt 68 reveals that Ben did not consider the effects of percentage change as the number of layers increased. According to him, the outputs (represented as percents) decreased by a constant 20% amount for each layer of tint added. Ben failed to see the light intensity remaining as a function of the number of layers of window tint used. Instead, he related the percentage of decay as a function of the number of tinted layers.

To justify further, Ben explained his thinking using a graph of the percent of light as a function of the number of tinted layers. Figure 14 illustrates Ben's graph.

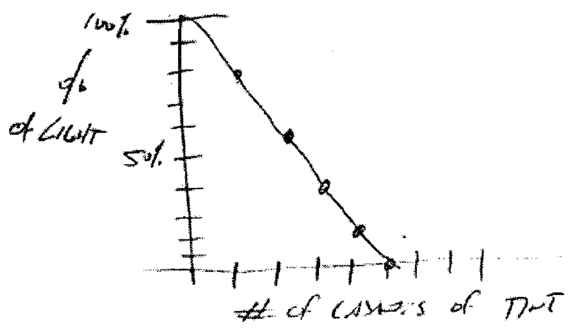

Figure 14: Ben's light intensity graph.

Figure 14 provides evidence of Ben's additive thinking in relating the number of layers of window tint with the percent of light. For every one layer of window tint, the percent of light decreased 20%. Excerpt 69 reveals Ben's thinking.

Excerpt 69

AS: 1 Okay so it would be blocking all the light?

Ben: 2 Yeah.

AS: 3 Okay so what kind of general pattern would that be?

Ben: 4 Ah, pretty linear pattern [ERFR0], it's going straight down like the
5 difference between each of those points isn't increasing
6 [ERFR0]…like from here to here is 20% of the light and from here
7 to here is 20% of the light, 20%, 20%, and 20% [ERFR0].

Ben's utterances in Excerpts 68-69 illustrate Ben's inappropriate use of additive reasoning when attempting to make sense of the exponential situation [ERFR0]. However, these initial conjectures provided the baseline of thinking for Ben as he progressed to conducting the experiment by collecting his own data to verify these ideas.

It is interesting to note that the Light Intensity activity elicited Ben's inconsistent thinking of the effects of decaying by a constant percent (he thought the amount of light would lower by additive decreases from 100% to 80%, then to 60%, then to 40%, etc.), whereas the Salary Problem and the Half of the Half-Life problem elicited a different kind of response. In the latter problems (see chapter 6), he was able to reason about multiplicative decrease. This important finding will be discussed in the final chapter.

The next portion of the activity focused on collecting and analyzing data for the experiment. Ben collected data (values of light intensity measured in mW/cm^2) using the TI-Calculator Based Laboratory (CBL), light probe, layers of window tint, and indirect outdoor sunlight. Figure 15 illustrates Ben's experimental data collected.

	tint	light
1	0	0.9126
2	1	0.5365
3	2	0.3461
4	3	0.2208
5	4	0.1527
6	5	0.0954
7	6	0.0663
8	7	0.0449
9	8	0.0323

Figure 15: Ben's light intensity data.

Ben analyzed the data by first calculating the differences of the outputs. After considering the pattern of values in the difference column, he decided that the tint was "blocking a percentage of whatever the previous amount was" [ENLI]. With prompting by the researcher, he calculated the ratios among the output values as a way to determine the percentage of light blocked. Figure 16 shows the difference and ratio columns that he constructed.

	tint	light	diffoflight	ratio
1	0	0.9126	0	inf
2	1	0.5365	-0.3761	0.587881
3	2	0.3461	-0.1904	0.645107
4	3	0.2208	-0.1253	0.637966
5	4	0.1527	-0.0681	0.691576
6	5	0.0954	-0.0573	0.624754
7	6	0.0663	-0.0291	0.694969
8	7	0.0449	-0.0214	0.677225
9	8	0.0323	-0.0126	0.719376

Figure 16: Ben's light intensity data illustrating differences and ratios.

Ben recognized that the ratios were not consistent throughout the column: he saw that the ratios varied from 58.7% to 71.9%. He attributed the fluctuations of the ratios to experimental error in collecting data, such as movement of the light sensor or other objects outside. In spite of the fluctuations, he determined that a graph would most accurately reveal the trend of the data. The left graph in Figure 17 represents Ben's graph for light intensity as a function of the number of layers of tint. The right graph in Figure 17 represents Ben's graph for differences of light intensity as a function of the number of layers of tint.

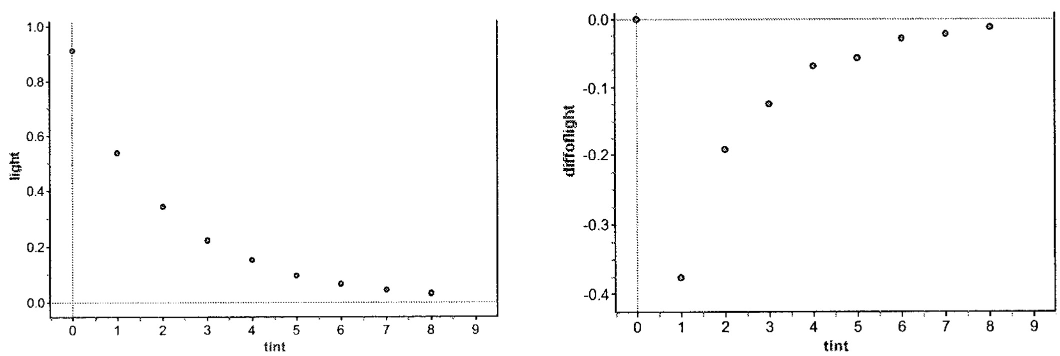

Figure 17: Ben's graph of light intensity vs. number of tinted layers (left) and his graph of differences of light intensity vs. number of tinted layers (right).

Excerpt 70 illustrates Ben's thinking relative to the graphs in Figure 17.

Excerpt 70

Ben: The first graph [Figure 17, left] it's just showing the amount of light going through with each layer of the window tint going on so with each layer there's less and less and less amount of light coming through [ERCR2b] but that second graph [Figure 17, right] is showing with those less and less amounts it's showing the difference between those amounts that those difference between those amounts gets closer or is getting closer and closer together so the amounts are instead of being starting off with the first one where we're you know pretty far apart now those points are getting a little bit closer together and little bit closer and little bit closer which is what the tail end of the difference of light [referring to Figure 17, right] is showing you know they're getting closer and closer together there's very little difference between the amount of light from one layer of tint to the next [ERCR2b].

Ben relied on information from these graphs to further warrant his thinking about the trends of the data for the light intensity values [Figure 17, left] and the differences of the light intensity values [Figure 17, right]. In lines 3-4, he stated that for "each layer there's less and less and less amount of light coming through." This utterance suggested that Ben was thinking about the amounts of change for both the number of layers of tint (input) and the corresponding values of light intensity (output). His covarying of the two quantities showed evidence of reasoning at Level 2 [ERCR2b] because his utterances included the notion of decreasing differences. Lines 12-14 indicate that he thought about the differences of light intensity as getting less and less with increases in layers of tint. He articulated that eventually very little differences of light intensity would emerge from adding layers of tint. This utterance also entailed reasoning about the amounts of change of these two quantities [ERCR2b].

The activity prompted Ben to write an algebraic formula to express an approximation of his experimental data. The writing of this function did not come naturally for Ben. His thinking of the function traced through multiple attempts to arrive at the exponential function structure. The next several transcript excerpts illuminate Ben's thinking of how to create the algebraic formula to represent the light intensity as a function of the number of tinted layers. Excerpt 71 provides Ben's initial formula and subsequent thinking.

Excerpt 71

Ben: Well the structure generally is going up at an increasing rate [ERCR2a, referring to his idea of the structure of a typical exponential function] and now [referring to the light intensity situation] we're going down at a decreasing [ERCR2b]...[long pause]...I almost wanted to say like something to the $1/t$ but that wouldn't work because plugging in zero for t you can't do that because you can't have zero in the denominator [ERCR1].

Ben's initial thinking about the algebraic formula focused on the exponent as representing the reciprocal of the number of tinted layers, $1/t$. This preliminary thinking was devoid of conceptions of the multiplicative decay factor. He articulated the decreasing values for light intensity [ERCR2b] and decided that his conjecture of the fractional exponent, $1/t$, incorrectly described this situation because "you can't have zero in the denominator" [ERCR1].

Upon realizing that the exponent for this situation was not $1/t$, Ben reconsidered his thinking about the algebraic structure. Excerpt 72 provides his new thinking.

Excerpt 72

Ben: Maybe if…I kept the exponent a whole number but now instead of having it be a whole number to that exponent power it's like a fraction to that exponent power because fraction would keep getting smaller [ENLN, ENLL, ERMR2] because ½ so say it's $\left(\frac{1}{2}\right)^x$ so $\left(\frac{1}{2}\right)^1$ or $\left(\frac{1}{2}\right)^0$ is one but $\left(\frac{1}{2}\right)^1$ is ½ of that it's less than the previous amount [ENLI], $\left(\frac{1}{2}\right)^2$ is ¼ and that's a smaller amount than ½ [ENLI] and that's going down and down and down [ERMR2] so something like that maybe [pause] and I'm just trying to think of what type of fraction to start it at because it needs to be something…like to the power of $t+1$ cuz otherwise we'd have to start it at the number one for zero if it's going to be any type even if it's going to be a regular exponential or a decreasing because anything to the zero power is one and we're not starting at one so say we start at like... $(0.9126)^{t+1}$.

This excerpt reveals that Ben believed that the multiplicative factor had to be a fraction in order for the output values to decrease. In lines 5-7, he considered the progression of values resulting from the expression $\left(\frac{1}{2}\right)^x$ for values of $x = 0, 1, 2$ because these output values went "down and down and down" [ENLN, ENLL, ERMR2]. These utterances suggest Ben's ability to reason multiplicatively using recursive procedures to justify his thinking. After conjecturing $t+1$ as the exponent, he settled on the expression $(0.9126)^{t+1}$ to describe the light intensity values relative to the number of tinted layers. Ben's utterances in this excerpt suggested that he did not conceptualize the situation in terms of multiplicative decay nor did he conceptualize the decay factor as representing the ratio of two successive output values. Furthermore, he ignored the role of the initial value relative to the algebraic structure and multiplicative process of the exponential situation. This impoverished reasoning led Ben to consider the expression $(0.9126)^{t+1}$ as representing the light intensity values with no consideration for the expression as representing $(\text{initial value})^{\#\text{ layers of tint }+1}$ (i.e., he did not see that 0.9126 represented the initial value and not the decay factor).

Ben used the graph of $f(t) = (0.9126)^{t+1}$ together with experimental data points (presented in Figure 15] to determine the accuracy of his expression. Figure 18 illustrates his revised graph.

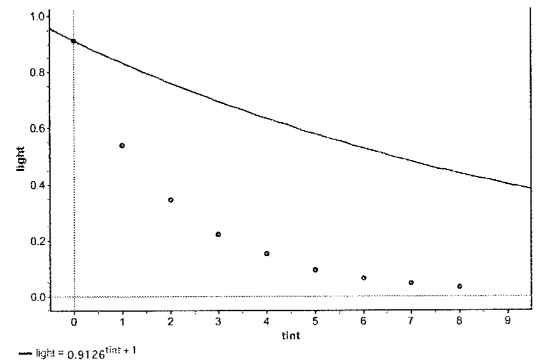

Figure 18: Ben's graph for light intensity $= (0.9126)^{\text{tint}+1}$.

Once Ben realized from seeing the plot that his expression was highly inaccurate, he decided to return to his exponent of $t+1$ because "that's the part that's taking it [the value of the expression] down telling you how many times that's being put together [the number of factors of the decay factor] to decrease that number." His utterances suggested that he realized the theoretical output values modeled by his function were not decreasing as rapidly as the experimental data. Again, he ignored the meaning of the quantity 0.9126 as representing the initial value of light intensity found during the experiment. It was unclear why Ben confused this value of light intensity with the multiplicative decay factor.

At this point, the conversation turned to direct instruction by the researcher to help Ben develop connections and attach meanings to the quantities represented in the problem. The motivation here was to have Ben realize that the initial value of 0.9126 was not the decay factor. The discussion focused on the general formula of an exponential function of the form $y = a \cdot b^x$. The researcher reminded Ben of the meaning of the parameters for this situation (that is, the meaning of a as the initial value and b as the decay factor). Upon further reflection, Ben finally decided that the 0.9126 was the initial value. Then he said, "I'm still thinking that 0.9126 is going to be multiplied by something to that exponent power." With probing by the researcher, Ben decided to consider the values in the ratio column for the decay factor. Excerpt 73 describes Ben's thinking.

Excerpt 73

Ben:	1	Maybe that ratio cuz if I just kept it just to the tint power and I put
	2	in that ratio where the next amount would be the tint...I'd have
	3	to multiply it by 0.58 because that would give me 58% of that
	4	previous amount [ENLI, ERMR2a].

Ben's utterances in lines 3-4 demonstrated his thinking of the ratio 0.58 as representing 58% of the previous light intensity value[14]. In addition, these utterances suggested that Ben was able to think about the exponential situation in terms of recursive procedures [ENLI, ERMR2a]. Furthermore, he decided to alter the power of $t+1$ to t and changed his function to $\text{light intensity} = 0.9126(0.587881)^{\text{tint}}$. Figure 19 (left) reveals his revised graph for the decay factor of 0.587881. Ben continued to adjust his graph by changing the values of the decay factor. He settled on 0.635 as the decay factor that provided the most accurate approximation of his experimental data [see Figure 19, right].

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[14] The value of 0.58 is a truncated version of the first quantity listed in Ben's ratio column.

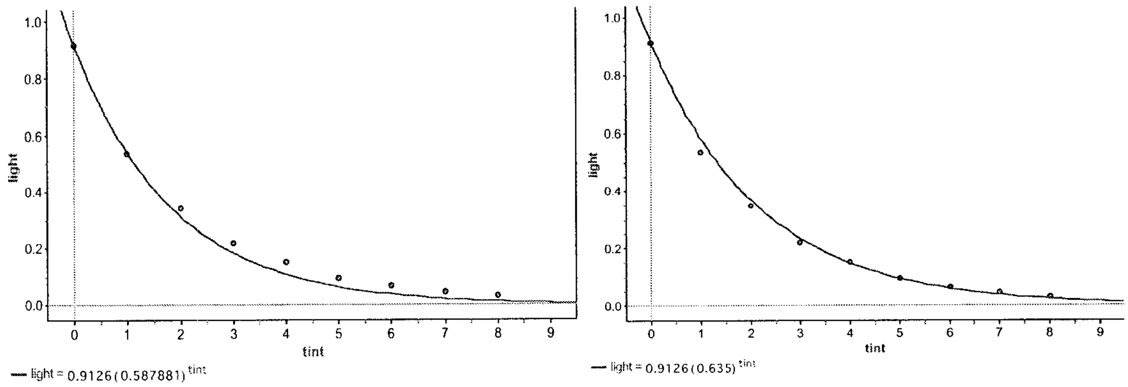

Figure 19: Ben's graph for light intensity = $0.9126(0.587881)^{\text{tint}}$ and his graph for light intensity = $0.9126(0.635)^{\text{tint}}$.

Excerpt 74 provides Ben's thinking relative to the decay factor of 0.635 provided in Figure 19 (right).

Excerpt 74

Ben: It's the percentage of light being allowed through of the starting amount so starting if my whole amount of light is 0.9126 then each layer of tint is taking away…it's just the percentage of the amount of light allowed to come through with each layer of tint [ERCR3b] and that percentage is changing each time I add a layer of tint cuz 0.635 to the 2nd power is a different number than 0.635 to the 3rd power…it's 63% of 63% [ENLN, ENLL, ERMR2a] then 63% of 63% of 63% [ENLN, ENLL, ERMR2b] so you're taking a percentage of a smaller number and then of another smaller number [ERMR2b].

After creating several iterations of an algebraic formula to model the data, Ben refined his thinking about the role of the decay factor and the meaning of this quantity relative to the light intensity values. In lines 3-4, he stated that the value of 0.635 is the "percentage of the amount of light allowed to come through with each layer of tint." This utterance suggested that Ben was able to eventually reason about the situation in terms of the covarying relationship between the number of layers of tint and the corresponding light intensity values [ERCR3b]. Furthermore, Ben applied multiplicative reasoning when considering *how* the quantity of $(0.635)^2$ represents $63\% \cdot 63\%$ [ERMR2a]. Correspondingly, Ben provided evidence of his thinking about the quantity $(0.635)^3$ as representing $63\% \cdot 63\% \cdot 63\%$ [ERMR2a]. These data also suggested that Ben viewed the exponent as indicating the number of factors of 63.5 [ENLN, ENLL].

The next portion of the discussion focused on connecting Ben's initial conjectures of linear decay to his final discovery of the exponential function that modeled the data. He explained that a linear function increases or decreases "at the same rate." The example he provided, however, utilized *percentage* decrease to substantiate his claim of decreasing at the same rate. Excerpt 75 illuminates his thinking.

Excerpt 75

Ben: 1 In this case of a decreasing model let's say we have the 0.9126 and
2 that it would be minus 0.635 times x [referring to the linear
3 function $\text{light intensity} = 0.9126 - 0.635x$]...so it's going to be
4 63% difference between each of those points.

The data suggested that Ben was thinking about 63% as the rate of change of the linear function he created. Thus, in Ben's mind, the 63% is the additive change in light intensity accumulated for each layer of tint added. This way of thinking parallels Ben's ideas on the Salary Problem (see Excerpt 9 in chapter 6) where he stated that one pitfall for students is thinking that the decrease – calculated by taking the percentage times the *original* amount – is constant throughout the situation. Ben's utterances in Excerpt 75 suggested that he was subject to the very same pitfall. Ben continued his discussion in Excerpt 76.

Excerpt 76

Ben: Whereas here [referring to the exponential situation] it's 63% just of the previous amount [ENLI, ERMR2] and not the original and so that's the difference there [ERFR1] that the 63% of a previous amount is less [ERCR2b, ERMR2]...it's going to be a little bit less than the amount before that so it's like I was saying you taking a percentage of a smaller amount you get a smaller amount [ERMR2a] so that amount of change is just getting smaller and smaller [ERMR2b] whereas the linear it's just going to be a certain amount of 63% away [ERFR1].

Ben's argument here focused on distinguishing between linear and exponential behavior [ERFR1]. His notion of exponential behavior as a percent of the previous amount illustrated his ability to conceptualize the recursiveness of exponential behavior [ENLI, ERMR2]. Yet, his explanations in Excerpts 74-75 for comparing both of these functions involved the notion of percent. These data provided evidence for his thinking about linear behavior as constant amount of change versus exponential behavior as constant percent change.

Ben returned to his initial conjecture – which he had made before data collection – where he surmised that each layer of tint would reduce the light intensity by 20%. Excerpt 77 provides Ben's thinking of the conceptual difference between thinking about 20% decrease of light intensity for each layer of tint versus 20% decrease of light intensity based on the previous value. In Ben's view, there were two different ways to think about the light intensity situation.

Excerpt 77

Ben: Here [referring to his conjectured linear graph illustrated in Figure 14], we are decreasing 20% and now we are decreasing another 20% and now we are decreasing another 20% and now we are decreasing another it's always decreasing by 20% [ERFR0] whereas here [referring to the exponential behavior of the experimental data and theoretical model equation] yeah it's decreasing, well here it's decreasing by 20% but 20% from whatever we started with [ENLI] so I've added on two layers of tint here so I've added on two layers of tint that means it's decreasing two 20% differences…if 100% of something is 100%, 20% of 100 is going to be this [refers to 80%] but maybe 20% of 80%…so instead of taking 20% of the original light coming through now I'm only taking 20% of the 80% coming through [ERMR2] and that's going to be…less than however much I was taking away before [ERCR2b].

As evidenced in Excerpt 77, Ben revised his initial conjecture about the linear behavior of light intensity with respect to increases in window tint: he decided the light intensity would decrease by different amounts. Excerpt 77 exposes the prominence in Ben's thinking of linear behavior as decreasing values of the output by a constant percent for incremental increases in the input. His justification for this thinking was that the constant percent decreases are based upon the *original* amount. He contrasted this behavior with constant percent decreases based upon the *previous* amount, which he categorized as exponential behavior. The Light Intensity activity provided the opportunity to investigate (a) Ben's understanding of exponential decay through experimentation, and (b) his thinking of exponential decay relative to linear decay. The data from this activity revealed his conceptual difficulties when grappling with the parameters and variables of the exponential function in the form $y = a \cdot b^x$.

Further analysis revealed some inconsistencies in Ben's ways of thinking. In the Salary Problem, Ben was able to articulate the constant percent growth of 7% as exponential growth for each year that elapsed. Additionally, he was able to contrast this constant percent growth of 7% with constant amount growth of $2450 for each year that elapsed. Yet, in the Light Intensity activity, Ben concentrated on whether the percent was calculated based on the previous amount (exponential) *or* the original amount (linear). The data suggested that even though Ben was able to view a percentage growth process as exponential in nature, he did not conceptualize percentage decay as inherently exponential.

Table 34 provides a summary of the analysis for Ben’s thinking as he worked through the Light Intensity activity.

Table 34

Ben's Reasoning Model for the Light Intensity Activity

Code	Framework Description	Analysis of Ben
ENLN ◍	Use *notation* to represent b^x as x factors of b for rational values of x.	Ben used notation to represent the effects of multiple layers of tint as $(½)^x$.
ENLL ●	Use *language* to describe b^x as x factors of b for rational values of x.	Ben described two layers of tint as 63% of 63% and three layers of tint as 63% of 63% of 63%.
ENLM ●	Use *multiple representations* to represent the multiplicative processes of exponential function.	Ben relied upon the table of values (including the difference and ratio column), graph, and context to generate the algebraic representation.
ENLI ●	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben articulated that the exponential behavior represented constant percent decay based on the previous amount.
ENLE ◍	Use *explicit definition* of exponential function to represent exponential situations explicitly.	Ben experienced difficulty in creating the algebraic formula to model the experimental data. After exploration, Ben found a reasonable model.
ERCR ◍	Use *covariational reasoning* to describe exponential behavior.	Ben reasoned about the covarying quantities of light intensity and number tinted layers (i.e., for each layer of tint, the light intensity decreased by 63%).
ERFR ◍	Use *exponential function reasoning* to compare and contrast exponential function with other functions.	Ben described exponential decay as constant percent decay of the previous amount and linear behavior as constant percent decay of the original amount.
ERMR ●	Use *multiplicative reasoning* to describe exponential behavior.	Ben reasoned through the multiplicative situation using recursive procedures and processes.
ERPIR ○	Use *partial-interval reasoning* to describe exponential behavior.	No evidence of Ben's ability to think about 2.5 layers of tint as partial layers.

Evidence of *no* understanding ○ Evidence of *weak* understanding ◍ Evidence of *strong* understanding ●

Episode 5: Investments and Compound Interest

The fifth teaching episode included the Investment activity that was designed to focus on compound interest as a tool for developing deeper understanding of exponential growth. The primary goals of this activity include (a) advance Ben's thinking of exponential growth relative to compound interest, (b) develop Ben's language fluency with the compound interest formula, and (c) promote Ben's understanding of the effects of increasing compounding periods on future values. Next, the activity is presented along with a conceptual analysis of Ben's thinking as he worked through this activity.

The Investment Activity

The activity prompted Ben to extend his notion of exponentiation beyond static images of recursion – that is, to extend from recursive procedures to recursive processes. This activity focused on exponential growth emerging from the context of compound interest. It also emphasized multiplicative reasoning through (a) analyzing the table of values for the data provided, (b) crafting an algebraic representation of the data, and (c) exploring various extension questions relative to investments and compounding periods. Furthermore, the activity prompted Ben's to use the process of exponentiation to calculate unknown values, such as the initial value for $t = 0$ where t represents the number of compounding periods. The presentation of the activity follows.

The Investment Activity

Part A: Observations and Representations

The table below illustrates the growth of an investment over a 5-year period. Each value in the future value column represents the value of the investment at the end of the compounding period *t*.

t	Future Value
12	5630.48
13	6024.61
14	6446.34
15	6897.58
16	7380.41

- What type of relationship do you think is represented by the table?
- Extend the table to express (in function notation) each future value as a function of the previous future value. Also, express (in function notation) each future value as a function of the initial investment value. Use exponents when necessary.

t	Future Value	Future Value as a function of previous Future Value	Future Value as a function of initial investment value
12	5630.48		
13	6024.61		
14	6446.34		
15	6897.58		
16	7380.41		

- Represent symbolically (in function notation) an investment growth function that matches as closely as possible the data in the table.
- Use Fathom to represent graphically your investment growth function.
- What kind of function is represented by your graph? Is it discrete or continuous?
- What is the *y*-intercept? What does it represent?
- Describe two ways to determine when (in compounding periods) the investment will reach $1,000,000.

- Generalize your function to represent an investment, invested at $r\%$, as a function of t compounding periods.

Part B: Relationships

- Describe mathematically the connection between the interest rate of an investment and its growth factor.
- Based on your extended table above, how does the output change when the input is increased by 1 compounding period?
- How will the output change when the input is increased by 6 compounding periods? 12 compounding periods?
- Based on your response from the previous questions, express the future value at the end of compound period 20 as a function of the future value at the end of compound period 14.
- Generalizing from your observations so far, describe the effect on the output when time T is added to any input t.
- Discuss how applying the law of exponents directly to the function $f(t) = a \cdot b^t$ results in the formula $f(t+T) = b^T \cdot f(t)$, where b represents the growth factor and a represents the initial value.

The activity was designed to begin with a table of values for compounding periods 12-16 and to probe Ben's ability to analyze the data in determining the pattern emerging from the future values. This data analysis component was intended to prompt Ben to think about the multiplicative growth pattern resulting from the data (i.e., multiplicative factor of 1.07). Furthermore, the activity provided Ben the opportunity to explore the components of the compound interest formula relative to the contextual situation of investments.

Table 35 provides the matrix for the Investment activity relative to targeted components of the exponential function framework.

Table 35

Matrix Mapping of Exponential Function Framework Investment Activity

Code	Framework Description	Investment Questions
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.	Use exponential notation to represent increasing compounding periods (i.e., an increase of t compounding periods is represented as b^t.
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.	Use language to describe increasing compounding periods as an increase in the number of times a growth factor is multiplied by itself.
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.	Represent investments in terms of a table, graph, algebraic formula and contextual representation.
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.	Represent investments using descriptions of multiplicative growth of the previous output.
ENLE	Use *explicit definition* of exponential function to represent exponential situations explicitly.	Represent investments using the algebraic representation of the situation.
ERCR	Use *covariational reasoning* to describe exponential behavior.	Use covariational reasoning to describe the growth of the future value of the investment as the number of compounding periods increases.
ERMR	Use *multiplicative reasoning* to describe exponential behavior.	Describe the context of investments as multiplicative growth by a constant factor.

Ben began the activity by entering the data into a table using the Fathom program. He immediately computed the differences of the future values to assess the underlying trend in the data. As a result of computing these differences, he noticed that "each time it's increasing more and more." Ben determined that the differences of the values seemed to be increasing by approximately 30 each time. Thus, he computed the second differences to validate his conjecture. Figure 20 provides Ben's calculations for these values.

	t	FV	dFV	ddFV
1	12	5630.48	0	0
2	13	6024.61	394.13	0
3	14	6446.34	421.73	27.6
4	15	6897.58	451.24	29.51
5	16	7380.41	482.83	31.59

Figure 20: Ben's Investment activity table of values with differences.

Ben decided that the data revealed an exponential function because "the differences are continually increasing." When asked to further analyze the data, he found the ratios of the *first differences* (rdFV). This prompted him to also find the ratios of the *future values* (rFV). Figure 21 illustrates the ratio values.

	t	FV	dFV	ddFV	rdFV	rFV
1	12	5630.48	0	0	0	0
2	13	6024.61	394.13	0	0	1.07
3	14	6446.34	421.73	27.6	1.07003	1.07
4	15	6897.58	451.24	29.51	1.06997	1.07
5	16	7380.41	482.83	31.59	1.07001	1.07

Figure 21: Ben's Investment activity table of values with ratios.

Once Ben found the ratios of the future values, he confirmed his conjecture that the data illustrated an exponential function. He distinguished this exponential growth from linear growth as illustrated in Excerpt 78.

Excerpt 78

Ben: So from year to year it's increasing by 100 [referring to a linear situation] whereas by here it's increasing by a different amount each time, an increasing amount [ERCR2a, referring to the Investment activity data]...I think it's exponential just because the ratio of these future values is the same, it gives us the same value each time [ERFR2, ERMR2a].

In this excerpt, Ben used covariational reasoning to describe the increasing amount for increases in time [ERCR2a]. He was able to differentiate between increasing by constant amounts and increasing by increasing amounts [ERFR2, ERMR2a], although he did not articulate *how* the amounts were increasing.

The next portion of the discussion focused on writing an algebraic representation of the future value as a function of the previous future value. Ben found the future value as a function of the number of compounding periods. Excerpt 79 provides Ben's thinking.

Excerpt 79

Ben: Well starting at 5630.48 I'm increasing it 1.07 [ERMR2] so the one allows me to keep the original value and then I'm increasing it by 0.07...that's going to give me too big of numbers it should be $t-12$ because that way if I put in 12 for t I get zero.

In Excerpt 79, Ben determined that the exponent for his function would be $t-12$. This exponent, according to Ben, provided a zero in the exponent when the value of t equaled 12. Thus, he obtained the future value of $5630.48 when $t=12$ (the future value after 12 compounding periods). In line 1, Ben mentioned that he increased 5630.48 by 1.07, which indicated his thinking that 1.07 represented the multiplicative factor. When probed to find the initial value of the investment (i.e., the value after zero compounding periods), Ben stated that he "took the 5630.48 and divided it by $(1.07)^{12}$." Ben reveals his reasoning in Excerpt 80.

Excerpt 80

Ben: If I'm looking to go backwards that means I'm taking away the t's
and if I want to go back to my initial investment when t is zero
that means I have to take away 12 t's [ENLL, ERMR2] and since
t is in the exponent…it would be $5630.48(1.07)^{-12}$ but the negative
I take the reciprocal of whatever it's to the power of that negative
and puts the 1.07 in the denominator [ENLN, ERMR2].

Ben's utterances suggested his thinking about negative exponents as representing repeated *division* by the growth factor, where the number of division factors is specified by the negative exponent. These data revealed Ben's ability to reason about the recursive change by a constant multiplicative factor [ERMR2]. Furthermore, his ideas in lines 3 and 5-6 provide evidence of his thinking about the role of the exponent as representing the number of factors of the growth factor [ENLN, ENLL].

Ben persisted with this process of finding the initial value, which he calculated to be $2500. He maintained that he could check this value by calculating $2500(1.07)^{12}$ where he "took the 2500 and multiplied it by 1.07, took that number and multiplied it by another 1.07, took that number and multiplied it by another 1.07, and I do that 12 times."

These utterances suggested that Ben was able to think about recursion as a process of moving forward (increasing input values) and backward (decreasing input values) within the set of data [ENLI, ERMR2].

To further probe Ben's understanding of the recursive process, the researcher probed him to describe his thinking about a strategy for finding the future value for the 50^{th} compounding period. Excerpt 81 reveals his thinking.

Excerpt 81

Ben: 1 You could either keep multiplying 1.07 to whatever your previous
2 amount is [ENLI, ERMR2] or just take your initial multiply it by
3 1.07 to the 50^{th} [ENLN] power meaning you're multiplying 1.07
4 into that number 50 times [ENLL].

Ben's thinking in Excerpt 81 revealed an important evolution of his ability to fluently describe the process of exponentiation using both notation and language to represent b^x as x factors of b [ENLN, ENLL]. In the beginning of this research investigation, Ben continually relied on his thinking about recursion as a static image of procedures used to carry about the multiplication by a factor. In the Salary Problem (discussed in chapter 6), for example, Ben did not provide evidence of his ability to extrapolate output values of an exponential function when the intermediate values were not provided. Ben did not recognize that he could continue his multiplication process to find the intermediate values (for integer values of the inputs) as a tool for extrapolating future output values. Now, in the context of the Investment activity, he demonstrated his ability to extrapolate to the 50^{th} compounding period using two conceptual strategies: (a) repeatedly multiply previous value by 1.07 [ENLI, ERMR2], and (b) multiply initial value by $(1.07)^{50}$ [ENLN, ENLL]. These two different, yet connected, conceptions of exponential function provided meaningful ways of thinking for Ben as he made sense of the investment context in terms of compounding periods.

The next portion of the discussion concentrated on the mathematical relationship between an interest rate of an investment and its growth factor. Excerpt 82 provides Ben's thinking about this comparison.

Excerpt 82

Ben: Well they are both telling me how much my investment is going to grow by so I mean if I am multiplying it by 1.07 then you know I'm taking that initial and adding in that 0.07 which…is my interest rate…so the interest rate tells me how much I want it to grow by and the growth factor allows me to multiply that in and allow it to grow.

In lines 4-6, he provided specific descriptions of his thinking about the interest rate as a *growth indicator* and the growth factor as a *growth enabler*. In other words, he articulated his understanding that the interest rate as indicates *how much* something will grow while the growth factor *enables* this something to grow.

As an extension of these ideas, the researcher probed Ben to investigate the effects of increasing the number of compounding periods for any investment. First, Ben reviewed the compound interest formula, $A = P\left(1+\frac{r}{k}\right)^{kt}$ and articulated his thinking about the quantities of r/k in terms of an investment. Excerpt 83 provides Ben's justification for this quantity.

Excerpt 83

Ben: The r/k still represents the interest rate, it's just looking at what that interest rate will be if I just look at it specific times of the year so say I wanted to compound it monthly so I have my annual rate r but I'm not going to apply that to every single month because it's an annual rate meaning it only applies to one single year so you divide that rate into 12 so that each individual month has just a certain amount to that rate going into it that way after 12 of them hit then you've gotten your full amount of...whatever your annual interest rate is so it takes k amount of, yeah it takes k amount in order to get to your annual interest rate and whatever your k is whether it be 12 months, 6 months, two months, whatever.

After thinking about the components of the compound interest formula, Ben articulated his ideas that the quantity r/k represented the partial interest rate obtained by dividing the annual interest rate by the number of compounding periods, k. These data suggested that Ben was able to consider the partitioning of one year for an investment by the number of compounding periods. This notion of splitting – or slicing – a yearly investment into smaller and smaller segments of time (i.e., increasing the number of compounding periods per one year) provided the basis for thinking about the effects of increasing the number of compounding periods of the future value of the investment.

The next segment explored the idea of increasing compounding periods for an investment of $1000 compounded yearly, quarterly, monthly, daily, and hourly at 8% interest. Ben utilized the Fathom program for creating the formulas to compute the values within each column. Figure 22 provides the table of values.

	t	yearly	quart	month	daily	hourly
1	1	1080	1082.43	1083	1083.28	1083.29
2	2	1166.4	1171.66	1172.89	1173.49	1173.51
3	3	1259.71	1268.24	1270.24	1271.22	1271.25
4	4	1360.49	1372.79	1375.67	1377.08	1377.13
5	5	1469.33	1485.95	1489.85	1491.76	1491.82
6	6	1586.87	1608.44	1613.5	1615.99	1616.07
7	7	1713.82	1741.02	1747.42	1750.57	1750.67
8	8	1850.93	1884.54	1892.46	1896.35	1896.48
9	9	1999	2039.89	2049.53	2054.27	2054.43
10	10	2158.92	2208.04	2219.64	2225.35	2225.53

Figure 22: Ben's compound interest table of values.

After reflecting on the pattern emerging from the data, Ben articulated his thoughts about the increases in values. Excerpt 84 illustrates Ben's thinking.

Excerpt 84

Ben: 1 Yearly is the least amount so after 10 years yearly gives you the
2 least amount, quarterly gives you a little bit more, monthly gives
3 you a little more, daily gives you a little more and hourly gives you
4 a little more so...whichever one you chose you get a little more
5 than the previous [ERCR2].

Ben recognized that the future value of the investment increased as the number of compounding periods increased [ERCR2a]. Ben was able to recognize that the values were increasing with respect to the number of compounding periods [ERCR2a]. He stated, "the more amount of time you compound the more amount that money is going to increase the amount of output is going to increase."

Table 36 provides a summary of analysis for Ben's thinking as he worked through the Investment activity.

Table 36

Ben's Reasoning Model for the Investment Activity

Code	Framework Description	Analysis of Ben
ENLN ●	Use *notation* to represent b^x as x factors of b for rational values of x.	Ben represented the multiplicative factor for the 50th compounding period as $(1.07)^{50}$.
ENLL ●	Use *language* to describe b^x as x factors of b for rational values of x.	Ben described $(1.07)^{50}$ as 50 factors of 1.07.
ENLM ●	Use *multiple representations* to represent the multiplicative processes of exponential function.	Ben utilized the graph, algebraic formula, table and context to make sense of the investment situation.
ENLI ●	Use *implicit definition* of exponential function to represent exponential situations recursively.	Ben described his thinking about recursion forward (increase inputs) and recursion backward (decreasing inputs) as multiplying/dividing by growth factors of the previous output value.
ENLE ●	Use *explicit definition* of exponential function to represent exponential situations explicitly.	Ben used the algebraic formula to describe negative exponents and multiplication of growth factors.
ERCR ◍	Use *covariational reasoning* to describe exponential behavior.	While Ben described that increasing compounding periods yielded an increase in future value, he did not provide evidence for *how* the values were increasing.
ERMR ●	Use *multiplicative reasoning* to describe exponential behavior.	Ben described the situation in terms of multiplicative recursion by a constant factor.

Evidence of *no* understanding ○   Evidence of *weak* understanding ◍   Evidence of *strong* understanding ●

Concept Map: Post-Assessment

At the end of the teaching experiment, Ben revised his initial concept map (shown again in Figure 23) to reflect his new ways of thinking generated by working through the collection of exponential growth/decay tasks in this project. Figure 24 illustrates Ben's revised concept map of his understanding of exponential function.

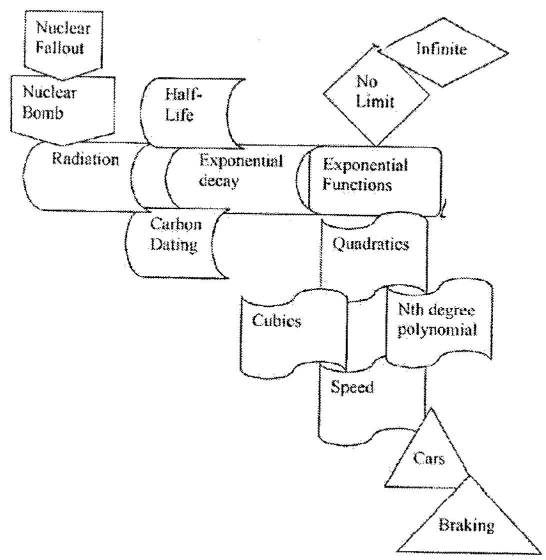

Figure 23: Ben's initial concept map of exponential function.

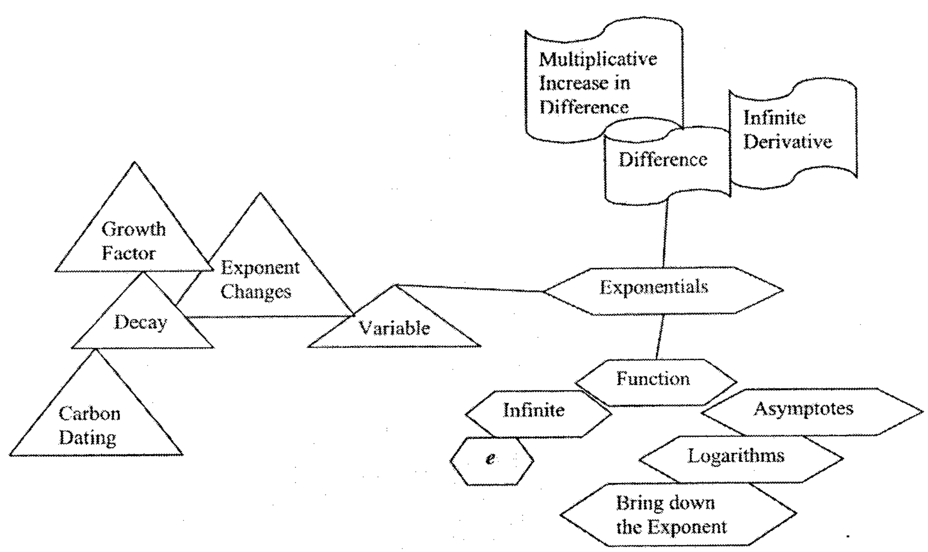

Figure 24: Ben's final concept map of exponential function.

Ben's revised concept map of exponential function provides a visual representation of his thinking after completing the teaching experiment activities. A comparison of the two concept maps reveals new ideas and concepts that were not previously included in his original map: namely, the connection of quadratic function as a subset of exponential function no longer appears on the map. The revised map and subsequent utterances provide emphasis on variable as representing changes in the exponent, which strengthens the distinction between exponential and quadratic functions. Furthermore, the notion of using differences of outputs as a tool for making sense of exponential behaviors appears in the new map. Finally, Ben's idea of "multiplicative increase in difference" suggests his newly constructed image of the multiplicative pattern emerging from comparing the differences of the output values.

The last section of this chapter focuses on the PCA post-test scores and analysis.

PCA Post-Test Results and Analysis

This section presents a comparison of pre-test and post-test data along with results of the teachers' conceptual change on the individual exponential function assessment items. Once the post-test assessments were completed, the researcher analyzed the *Functions* course teachers' overall performance on the post-test as compared with the overall performance on the pre-test to measure potential conceptual growth. Calculations of the normalized gain index (Hake, 1998) for each teacher provided the basis for analysis of change. The overall normalized gain index $< g >$ was calculated using the formula $\frac{(post - pre)}{(25 - pre)}$ where *post* is the individual teacher's post-test score on the PCA and *pre* is the individual teachers' pre-test score. The exponential item normalized gain index $< eg >$ was calculated using the formula $\frac{(post - pre)}{(6 - pre)}$. Table 37 displays the results of the pre- and post-test, including the overall normalized gain index $< g >$ and the exponential normalized gain index $< eg >$.

Table 37

Results of the PCA Pre- and Post-Test

Teacher	Primary Discipline	Overall PCA Pre-Test Score	Overall PCA Post-Test Score	Overall Normalized Gain Index $< g >$	Exponential Pre-Test Score	Exponential Post-Test Score	Exponential Normalized Gain Index $< eg >$
T1	Science	21	24	0.75	6	6	N/A
T2	Math	20	24	0.80	4	6	1.00
T3	Math	19	22	0.50	3	4	0.33
T4	Science	18	20	0.29	3	6	1.00
T5	Math	16	21	0.56	4	4	0.00
T6	Math	15	21	0.60	4	4	0.00
T7	Math	15	16	0.10	3	1	-0.67
Ben	Math	14	16	0.18	3	4	0.33
T9	Science	13	21	0.67	2	4	0.50
T10	Science	10	19	0.60	1	6	1.00
T11	Science	10	14	0.27	1	4	0.60
T12	Science	10	11	0.07	4	1	-1.50
T13	Science	9	13	0.25	2	3	0.25
T14	Science	8	12	0.24	0	2	0.33
T15	Science	6	17	0.58	1	3	0.40
T16	Science	6	6	0.00	1	0	-0.20

Ben completed the PCA post-test after he completed all sessions of the teaching experiment. His PCA pre-test score of 14 ranked him as eighth among the teachers in this section of the *Functions* course. However, his PCA post-test score of 16 dropped him in rank to a tied for 10th place. His overall post-test score increased by two points out of 11 points possible, resulting in an overall Hake gain score of 0.18. His exponential post-test score increased by one point out of three points possible, giving him an overall Hake gain score of 0.33. Ben correctly answered two of the three exponential growth questions on the post-test[15]. However, he correctly answered two of the three exponential decay questions on the post-test; these questions were ones that Ben answered incorrectly on the pre-test. These results suggest that Ben improved in his ability to respond to exponential decay items on the PCA test[16].

Chapter Summary

During teaching experiment, Ben demonstrated positive gains in his ability to make sense of and describe exponential function situations. In Episode 1, Ben provided an image of his concept map, which diagrammed his thinking relative to the exponential function concept. This map illuminated Ben's incomplete understanding of the distinction between the concepts of quadratic function and exponential function. When faced with these two function types, he was unable to explore their rate of change aspects. Furthermore, he was persistent in thinking that quadratic function represented a subset of exponential function due to the existence of an exponent. Finally, Ben was unable to

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[15] On the pre-test, Ben successfully answered all exponential growth questions.

[16] These results reported are merely nuggets of information designed to provide a summary of Ben. These results are not statistically significant due to the small number of participants. The purpose here is to provide Ben's PCA performance as a quantitative measure, not as a statistical measure.

reconstruct his thinking during the experiment, despite direct instruction by the researcher.

Ben grappled with the concept of doubling when working through the Cheerios problem and the Pennies activity in Episode 1. He recognized the doubling pattern in both situations and he utilized this pattern to make sense of the task in finding values for the table. The Pennies activity prompted Ben to explore fractional exponents as partial factors of a full factor using the chessboard as a physical tool to make sense of the situation. Ben reasoned about the value of $(2^7)^{1/2}$, for example, by conceptualizing this expression as a partial factor that multiplied by itself twice resulted in a full factor of 2^7.

Episode 2 highlighted the properties of exponents as a means for building stronger conceptions of fractional exponents to facilitate an understanding of various exponential tasks. This activity emphasized the notion of conceptualizing b^x as x factors of b. By the end of the activity, Ben used language to describe $3^{1/2}$ as the "number that when I multiply by itself by another $3^{1/2}$ gives me just 3." This way of thinking provided the motivation for designing the Filling in the Missing Pieces Using Tables activity in which Ben reasoned about the multiplicative factor for input intervals of less than one unit. The data revealed that thinking about fractional exponents as partial factors emerged as a powerful tool for Ben when reasoning through exponential situations.

Episode 3 focused on comparing linear, quadratic, and exponential functions from a rate of change and covariation perspective. The activity emphasized the process of analyzing output values represented in tables using first differences, second differences and ratios. Ben demonstrated his observations about the changing output values for linear, quadratic, and exponential functions; yet, he still did not provide well-connected notions of exponential behavior as distinct from quadratic (polynomial) functions.

Episode 4 incorporated an experimental approach to investigating exponential decay through empirical data collection of light intensity as a function of the number of layers of tint. Ben's utterances in this activity illuminated persistent conceptual obstacles of reasoning about exponential decay relative to linear decay. He described exponential decay as decreasing by a constant percent based on the *previous* value. Yet, he described linear decay as decreasing by a constant percent based on the *original* value. He was unable to articulate that linear decay is a process of decreasing by constant *amounts* that can be calculated as a percent of the original value. This subtle, but crucial distinction, however, was not clear in Ben's arguments of linear decay compared to exponential decay. The primary findings from this activity were that (a) Ben held an incomplete understanding of percent decay, and (b) Ben's conceptions of exponential growth did not transfer to robust conceptions of exponential decay.

Episode 5 engaged Ben in connecting his thinking of exponential growth to the context of compound interest and investments. This activity provided opportunities for him to analyze data using differences and ratios to determine the trend. He provided evidence of his ability to think about negative exponents as representing repeated *division* by the growth factor for the number of factors as specified by the exponent. He described his thinking about *recursion forward* (increase inputs) and *recursion backward* (decreasing inputs) as multiplying/dividing by growth factors of the previous output value. He also explored the effects of increasing the number of compounding periods on the future value of an investment. He described the effects of compounding money as "the more often it gets compounded the faster the total amount of money grows."

Table 38 provides a final summary of the analysis for all interview tasks (discussed in chapter 6) and teaching experiment activities (discussed in the present chapter). This table reveals the chronological order of the interview tasks and teaching experiment activities employed in this study, along with the analysis of Ben's progress as he completed these tasks. The process of constructing this table provided an opportunity to conduct a final review of the analysis and it served to further warrant the assertions generated from the data.

The table reveals that Ben made advances in his thinking and communicating of ideas as he worked through the instructional unit. For example, when looking across the rows, the row displaying ENLI: Implicit Definition illuminates Ben's strong ability to think about exponential behavior using a recursive model (i.e., $\text{NEW} = \text{NOW} \cdot b$). The ENLL: Language for b^x row also reveals that Ben's ability to describe b^x as x factors of b improved during the study. Ben demonstrated his ability to reason multiplicative as evidenced by the number of black dots in the row of ERMR: Multiplicative Reasoning. The row of ERPIR: Partial-Interval Reasoning illuminates the progression of Ben's ability to utilize ideas about partial change and partial factors due to the increase in partially shaded dots and black dots as the study elapsed. In contrast, the row of ERCR: Covariational Reasoning revealed that Ben's use of covariation was limited as he grappled with the tasks and activities.

Comparing the columns in the table provides another perspective on Ben's ability to use various components of the framework relative to specific tasks and activities. The three white and three partially shaded dots in the Population Growth task column, for example, illustrates Ben's limited notation, language, and reasoning abilities exhibited when working through this task. Whereas, the six black dots and one partially shaded dot in the Investment Activity column demonstrated Ben's improved ways of thinking relative to notation, language, multiple representations, implicit definition, explicit definition, and multiplicative reasoning.

Table 38

Summary of Ben's Analysis for Interviews and Teaching Experiment Activities

	Interview Tasks				Teaching Experiment Activities							
	Population Growth Task	The Salary Problem	Half of the Half-Life	What is the Half-Life?	Cheerios Problem	Pennies on a Chessboard	Properties of Exponents	Filling in Using Tables	Filling in Using Graphs	Compare functions	Light Intensity Activity	Investment Activity
Notation & Language												
ENLN Notation for b^x		○	●	◍	◍	◍	●	●			◍	●
ENLL Language for b^x		○	○	●	◍	○	●	●			●	●
ENLM Multiple Representations	◍	●	◍	◍	●				◍	◍	●	●
ENLP Parametric Changes				◍		●						
ENLI Implicit Definition	○	●	●	◍	●	●		◍	●	●	●	●
ENLE Explicit Definition				●	◍	●		●			◍	●
Reasoning Abilities												
ERCR Covariational Reasoning	◍	◍	●	●	◍	◍		◍			◍	◍
ERFR Exp. Function Reasoning	◍	●	●					●		◍	◍	
ERMR Multiplicative Reasoning	○	◍	●	●	●	●		●	●	◍	●	●
ERPIR Partial-Interval Reasoning	○		◍			◍	◍	●	●		○	

Blank = Not applicable | Evidence of *no* understanding ○ | Evidence of *weak* understanding ◍ | Evidence of *strong* understanding ●

Finally, the analysis of Ben's revised concept map and his PCA post-test score revealed his somewhat improved, but still limited, understanding of exponential function. The revised concept map, which did not include polynomial function, provided a glimpse of Ben's conceptual advances relative to his ability to distinguish exponential behavior from quadratic behavior. His PCA post-test score of 16 – with Hake gain of 0.33 for the exponential items – illuminated Ben's incomplete conceptual understanding of the function concept.

The final chapter provides a discussion of the major findings of this investigation relative to the revised exponential function framework and plans for future research. The chapter concludes with the implications of this study for curriculum and instruction as well as teacher development.

CHAPTER 8: DISCUSSION AND CONCLUSIONS

This chapter provides a background summary of the dissertation investigation and then summarizes the general conclusions as supported by the evidence presented in the previous two chapters. Next, the chapter provides the specific findings of the study relative to the three research questions. A final presentation of – and reflections on – the framework follow, along with the limitations of the study and implications for curriculum and instruction. Lastly, the chapter concludes with directions for future research.

Background Summary of the Investigation

The purpose of this investigation was to explore a secondary mathematics teacher's ways of thinking about exponential function as he worked through a collection of tasks and activities focused on exponential behavior. A conceptual analysis was conducted to investigate the nature of the teacher's knowledge of the concept of exponential function. In addition, the study set forth an exponential function framework designed to describe the notation, language and reasoning abilities for knowing exponential function. The study extended previous research on multiplicative reasoning by (a) focusing on a secondary teacher as a case study and (b) investigating a teacher's thinking and understanding of exponential function. The existing body of research was devoid of this context and emphasis.

The investigation began with a pre-test assessment followed by task-based interviews and an in-depth teaching experiment focused on the concept of exponential function. The analysis of this data facilitated the development and refinement of an exponential function framework that provided a lens for exploring teacher ways of thinking about exponential behavior and for developing meaningful curricula intended to promote multiplicative reasoning. This dissertation provides details of the framework that guided the implementation and analysis of the study.

The subject participant for this study was a secondary mathematics teacher, Ben, who enrolled in a graduate-level mathematics education course, *Functions: Mathematical Tools for Science*, at a large public southwestern university. This course was the first in a sequence of four courses designed to promote mathematics and science teachers' understanding of the function concept with emphasis on rate of change and covariation.

General Conclusions and Discussion

Analysis of the data from this investigation yielded three general findings: (a) understanding and characterizing exponential growth patterns is complex and multifaceted; (b) an understanding of linear behavior provides a *conceptual springboard* for developing a mature understanding of exponential behavior; and (c) understanding exponential function as recursion necessitates a process view of the recursive structure of exponential behavior. What follows is a summary discussion of each of these general findings.

Initially, Ben's weak understanding of exponential growth patterns was revealed in his inability to interpret and characterize the *meaning* of exponential notation. During the course of the study, and as a result of the instructional intervention, Ben's understanding of exponential growth patterns improved, as did his ability to interpret exponential notation. His initial weaknesses are largely attributed to his weak understanding of function and his impoverished covariational reasoning ability. His initial inability to distinguish between $f(x) = b^x$ and $g(x) = x^2$ suggests that he was not imagining how the output values varied while imagining changes in the input. Furthermore, he had difficulty crafting algebraic representations of the function for decay contexts (see Light Intensity activity) due to his impoverished ability to reason about the context of decay. Investigating Ben's language revealed weaknesses in his ability to represent multiplicative behavior, which also revealed his impoverished understanding relative to various characteristics of exponential function (e.g., difficulty representing exponential decay algebraically).

In the context of fractional exponents such as in the expression $b^{1/x}$, Ben was unable to provide a meaningful description of the quantities in the formula. His thinking appeared to be rooted in a procedural view of fractional exponents as roots; he often described expressions of the form $b^{1/x}$ as the $\sqrt[x]{b}$ without providing meaningful explanations of the quantity represented by $b^{1/x}$. However, once Ben became comfortable with the notion of b^x as x factors of b, he was also able to view fractional exponents as partial factors of b. For example, conceiving of $\left(\frac{1}{2}\right)^1$ as one full factor of ½ provided the opportunity for Ben to consider fractional exponents, such as $\left(\frac{1}{2}\right)^{1/5}$, as representing partial factors of the base number. In this situation, the full factor of ½ is obtained from multiple partial factors of ½ and in this case the number of partial factors multiplied together would be five to create a full factor of ½. By promoting mature conceptions of multiplicative units of exponential function for intervals not equal to one, this new way of thinking for Ben facilitated his ability to build a more robust understanding of exponential behavior. Lamon (1994) also found that interpreting and constructing reference units of the situation "appears critical to the development of increasingly sophisticated mathematical ideas" (p. 92). In summary, simultaneously holding in mind the dual concepts of (a) fractional exponents as partial factors of the base and (b) the factor itself was found to be important in developing the necessary language of exponential behavior.

In the context of exponential decay, Ben experienced increased difficulty in articulating his thinking relative to how the output values decreased as time increased. For example, Ben expressed his thinking of half-life as the number of years times the value of the half-life (see Excerpt 11). His thinking of half-life did not utilize rate of change characteristics to describe changes of the amount of substance in relation to changes of time. Ben also struggled in articulating his thinking about percent change. During the Light Intensity activity, he expressed percent decay for the exponential situation as a change by a percent of the *previous* amount. He contrasted this behavior with constant percent decreases based upon the *original* amount, and he characterized this behavior as linear (see Excerpt 77). His use of imprecise language to describe the function situations revealed his use of inconsistent descriptions of the mathematics (even when he was using appropriate notation).

The NCTM *Standards* (2000) state that students should be able to analyze change in multiple contexts as well as compare and contrast characteristics of function families (e.g., rates of change and graphical behavior). For example, one of the everyday examples that it presents where students need to recognize patterns and analyze change specifically in the context of exponential function, is the following: Each year the population grows by 2%. While many students (and teachers) interpret this growth in population to be linear, the behavior is of course exponential. However, in the area of exponential function, the NCTM *Standards* (2000) focuses attention almost exclusively on exponential growth with little or no attention to exponential decay. The present study illuminated the importance of strengthening the ability to speak meaningfully in the context of exponential decay (decay by constant percent) alongside improving conceptions of exponential growth.

While Ben was able to make mathematical sense of many of the exponential concepts, he was often unable to fluently articulate his understanding. One could imagine that this impoverished ability to clearly and concisely verbalize his thinking could have a negative impact on his classroom teaching performance and on his students' learning. Despite Ben's ability to reason covariationally and multiplicatively at an intermediate level, his language revealed his inconsistent and incoherent understanding of the concept of exponential function.

Past research studies have reported that function notation and language are obstacles for students as they develop and refine their understanding of the function concept (Carlson, 1998). Carlson (1998) states, "Gaining an understanding of the many components of the function concepts is complex. It requires acquisition of a language for talking about its many features and the ability to translate that language into several different representations" (p. 137). Her study found that even high-performing students do not understand the concept of function as a relationship between two quantities and thus do not possess the necessary language to adequately describe function relationships. The present study found that Ben's communication abilities revealed his somewhat inconsistent and disconnected understanding of the exponential function concept.

A second major finding of the study entails the notion of utilizing an understanding of linear function as a conceptual springboard for building stronger conceptions of exponential behavior. For example, Ben often referred to linear behavior as a strategy for explaining his thinking about exponential behavior. When describing exponential behavior, he often explained the trend if the data were linear and then used that explanation to justify his thinking about exponential behavior. The frequency and success of this strategy provided evidence that a well-developed understanding of linear behavior facilitates an understanding of exponential behavior through comparing and contrasting these two function behaviors.

An emphasis on rate of change of exponential functions can facilitate the development of students' understanding of this topic and provide an avenue for developing fluency among various function families (Carlson, 1998; Carlson, Smith, & Persson, 2003; Thompson, 1994a, 1994b, 1994c). Comparing rate of change of linear functions with that of exponential functions can make explicit both the differences and similarities between these two families of functions. This study found that building an understanding of exponential function involves the ability to distinguish between exponential and linear behavior.

In Ben's case, he relied upon his understanding of additive behavior to make sense of fractional exponents. For example, when grappling with the idea of progressing by a fractional square (half square), he first considered that the value would be halfway between the output values. Once Ben abandoned the idea that the quantity of $2^{3.5}$ was "halfway" (i.e., the arithmetic mean) between the quantities 2^3 and 2^4, he was able to think about the multiplicative relationship inherent in the doubling pattern. His utterances demonstrated that he was able to take a path of multiplicative thinking by first using linear behavior as a conceptual springboard into multiplicative reasoning.

A third major finding of the present study involved thinking about exponential function as recursion, which necessitates a process view of the recursive structure of exponential behavior. Thinking about the structure of an exponential function as a recursive process facilitates the ability to mentally imagine the indefinite process of actions placed on the recursive objects along a continuum of values. This way of thinking leads naturally to notions of covariation and multiplicative change. The ability to fluently carry out recursive processes on the recursive objects entails recursion (Thompson, 1985). The ability to act upon a finite number of recursive objects does not translate to the ability to reason multiplicatively throughout the problem situation. Instead, considering the multiplicative progression of output values as representing an indefinite sequence determined by multiplying the previous output value to obtain the next output value is a recursive process.

The present study revealed two levels of recursion for contexts involving an exponential function: (a) recursive *action* and (b) recursive *process*. Ben's thinking of exponential function was deeply rooted in his image of the recursive structure of the function. Throughout the study, he referred to the multiplicative behavior as multiplying the previous output value by a factor to get the next output value. In doing so, Ben often computed the individual output values for each increment in the input by multiplying the previous output by some factor. This action orientation of recursion was the principal model for Ben's thinking throughout this investigation. Once he was able to make sense of the situation by calculating several output values, he then was able to generalize the pattern emerging from these values.

It is important to note that Ben's thinking of recursion evolved over the course of the study. In the early stages, he experienced difficulty in carrying out the procedure of repeatedly multiplying by a factor in order to extrapolate to future values. Analysis of the data from the Salary Problem revealed Ben's static image of recursion in the sense that he was unable to progress the process forward and calculate the salary in year 10 when he knew only the values in years zero through two. He maintained that in order to calculate the salary in year 10, he would need to know the salary in year nine. Thus, since he did not know the intermediate salaries for years three through nine, he claimed that he could not find the salary in year 10. Thus, his view of recursion was action-oriented and static in the sense that he did not see that one might carry out the repeated multiplication to extrapolate the salary for year 10.

Ben's notion of recursion progressed as he continued in the investigation. For example, he was able to express his thinking of exponential behavior as a sequence of recursive objects that enabled him to extrapolate a future output value. In the Pennies activity, Ben was able to find the number of pennies on the 10^{th} square by starting on the first square and doubling until he arrived at the 10^{th} square. He did not connect this process of repeated multiplication, however, to notions of exponentiation. Instead, he imagined each individual output value (i.e., the number of pennies on each square), which provided additional evidence of his action-orientation of recursion.

Ultimately, Ben's image of recursion advanced to a more process-oriented view of recursion. In the Investment activity, he fluently described the future value of an investment for the 50^{th} compounding period even though he knew only the values for compounding periods 12-16. His utterances promoted ideas of exponentiation where he was able to describe the future value of the 50^{th} compounding period as the initial value times the growth factor to the 50^{th} power (i.e., $2500(1.07)^{50}$). It was at this stage of the investigation that, for the first time, Ben was able to articulate his thinking of exponentiation without first describing or finding the intermediate values leading up to the 50^{th} compounding period.

In summary, the three general conclusions presented in this study focus on (a) inability to interpret the meaning of exponential notation and inability to use language to convey the meaning of exponential growth/decay patterns, (b) linear functions as conceptual springboards, and (c) exponential behavior as recursive action versus recursive process. The ability to express the *meaning* of exponential notation and the *process* of exponentiation provides the foundation for building a mature understanding of exponential function. Furthermore, well-formed conceptions of linear function act as conceptual springboards for advancing the primitive understanding of exponential function. Lastly, viewing exponential behavior as a recursive process, rather than as recursive actions, leads naturally to ideas of covariation and multiplicative growth (e.g., exponentiation). The next section provides a summary of the specific findings of the study with respect to the three research questions laid out in chapter 1.

Summary of the Specific Findings

This section discusses (a) the specific findings relative to the three research questions of the investigation and (b) their connections to previous research in the field.

Research Question One: A Teacher's Conceptions of Exponential Function

The purpose of this question was to investigate the ways of thinking about exponential function from the perspective of a secondary mathematics teacher. Even though Ben's ways of thinking are generally categorized as limited and procedural, his utterances nonetheless provided a rich story about the nature of a teacher's knowledge of exponential function. The next sections discuss the specific findings of Ben's ways of thinking.

Covariational Reasoning

Initially, Ben did not rely on covariational reasoning as a tool for making sense of some of the exponential tasks and activities. His inability to reason about *how* the output values changed – relative to corresponding changes in the input – restricted his ability to provide a complete understanding of the exponential behavior. This inability presented a barrier in advancing towards a more fully-developed conceptual understanding of exponential function.

Throughout the study, Ben was persistent in demonstrating his predominate thinking that since the change in output values of an exponential function were *different* for equal increments in the input, the behavior was categorized by him as exponential. In many instances, he was able to articulate his thoughts about the output values as increasing (or decreasing) as the input values increased. Thus, his covariational reasoning ability was categorized as impoverished. Occasionally, Ben was able to describe the exponential situation using constant percent change language, but this way of thinking was infrequent. In summary, he was able to progress from describing amounts of change of the covarying quantities (i.e., utterances of increasing/decreasing differences in outputs with changes in the exponent) to coordinating the constant percent change for the covarying quantities.

Past research has shown that the ability to reason covariationally is a powerful tool for developing robust conceptions of function (Carlson *et al.*, 2002; Confrey & Smith, 1995; Cottrill *et al.*, 1996; Rizzuti, 1991; Thompson, 1994a, 1994b; Zandieh, 2000). Confrey and Smith (1995) argue that covariation offers a more qualitative approach to describing in general the changes of one quantity in relation to another quantity. The present study found that Ben was able to qualitatively describe two changing quantities of an exponential situation using amounts of change as a dominant way of thinking.

Exponential Function Reasoning

Another finding that played a role in understanding Ben's thinking entailed his ability to make sense of exponential functions by distinguishing this function's characteristics from linear functions and quadratic functions. In the case of linear function, Ben was able to use linear behavior as a conceptual springboard for advancing his thinking about exponential function. For example, Ben described linear functions as having the same amount of differences between each point. He successfully distinguished this aspect about linear functions from the differences of output values for an exponential function by saying that the exponential values were increasing (or decreasing) and thus not remaining constant. Towards the end of the experiment, however, Ben had difficulty distinguishing constant percent change as indicating an exponential function. He confused the issue of constant percent change of the previous amount with constant percent change of the original amount. He classified the latter as a linear function and displayed an algebraic linear representation for his thinking (i.e., he wrote $\text{light intensity} = 0.9126 - 0.635x$ where 0.635 represented 63.5% decay).

In the context of quadratic function, Ben was unable to fully distinguish this function as representing non-exponential behavior. The concept map revealed Ben's thinking about quadratic function as representing a subset of exponential function because he classified "anything that has a power" as exponential. His initial reasoning did not involve rate of change or covariation as a tool for distinguishing these two functions. His thinking was rooted in the notion that both functions contained an exponent.

Once Ben was able to analyze tables of values for these function families, he was able to recognize the growth pattern emerging from quadratic functions as different from the pattern emerging from exponential functions. He calculated first and second differences of the output values for both a quadratic and exponential table of values. His analysis revealed that the quadratic table contained first differences that grew linearly where the differences always increased by a constant number. He was able to distinguish this growth from exponential growth by articulating that the differences of the exponential table did not behavior in the same way. Ben did not attempt to parse out the distinguishing characteristics of function that set them apart from other functions – such as rate of change and covariation. This finding illuminates the complexities associated with understanding function conceptually. Ben's impoverished ability to reason through function situations using rate of change and covariation contributed in part to his inability to fully differentiate exponential function as representing a family of functions that do not contain quadratic – or polynomial – function.

Multiplicative Reasoning

Understanding exponential function requires the ability to reason multiplicatively throughout the situation. An important finding of the present study concerned Ben's ability to reason multiplicatively using mostly a recursive model of conceptualizing exponential behavior. His thinking was deeply rooted in his image of exponential behavior as a *recursive action* built by multiplying the previous output by a factor to obtain the next output value (i.e., $\text{NEXT} = \text{NOW} \cdot b$). While Ben was able to demonstrate his knowledge of multiplicative growth/decay by a constant factor, his image of this progression involved first calculating a few output values and then generalizing the pattern. His generalizations of the pattern provided glimpses into his thinking about recursion as a process. Yet, he was unable to provide this evidence consistently.

The complexities of constant percent (or factor) change were evident in Ben's approach to reasoning about exponential function. He experienced difficulty in moving beyond the recursive model of thinking to thinking about multiplicative rate of change. The ability to conceptualize multiplicative rate of change involves many facets of exponential function, including the ability to unpack ideas about recursion, percent change, and increasing rates. Ben's limited notion of multiplicative rate of change encompassed an understanding of amounts of change with minimal evidence for *how* the amounts were changing.

Developing an understanding of exponential function hinges upon the ability to reason multiplicatively when grappling with the notion of rate of change (Confrey, 1994; Confrey & Smith, 1994). Previous investigations by Confrey and Smith (1995) led them to view rate of change of exponential functions as a gateway for students as they developed mature conceptions of exponential behavior. In the present study, Ben rarely relied on thinking about rate of change. Instead, he focused on the progression of output values without consideration for the covarying of input and output quantities.

Partial-Interval Reasoning and Partial Factors

In most cases, Ben was able to describe the multiplicative behavior as doubling or halving for increments of one unit in the input when the context elicited this type of behavior. However, he experienced difficulty when attempting to describe multiplicative behavior for input increments not equal to one unit (i.e., either less than or more than one unit). The teaching experiment emphasized thinking about fractional (partial) exponents for the expression b^x as x factors of b for non-integer rational values of x. Once Ben embraced this way of thinking, he was able to provide richer and more detailed qualitative descriptions for the meaning of these quantities. Furthermore, thinking about fractional exponents in this manner also facilitated his thinking about multiplicative units and the exponentiation process.

The ability to reason by using partial-intervals encompasses the ability to reason both covariationally and multiplicatively for intervals smaller than one unit (e.g., half-intervals, third-intervals, etc.). The fact that conventional curriculum typically ignores this ability may explain why students generally have difficulties with fractional exponents. Lacking the ability to connect fractional exponents to contextual situations and physical models, students (and teachers) suffer from impoverished notions of the meaning of fractional exponents relative to fractional factors.

Multiple Representations

The use of multiple representations facilitated Ben's ability to make sense of exponential behavior. One major finding relative to multiple representations of exponential function concerned Ben's ability to use the graphical representation as the primary tool for recognizing exponential patterns. In the context of the graph, he described the "jumps" in the output values. This finding was consistent with Confrey's (1994) subject, Dan, and his thinking of the graph as building "jumps of greater and greater size" (p. 322). While this view provides the groundwork for building covariation language, this narrow view of function lead him to view function as a static relationship between inputs and outputs, rather than as a dynamic *process* that accepts inputs and produces outputs.

Ben also viewed the graph as illuminating the changing rate of change of an exponential function, whereas when the table of values was provided he did not immediately recognize the pattern emerging from the data. However, Ben was able to use the context of exponential situations as a tool for calculating data and this proved to be extremely useful in facilitating his ability to recognize patterns in the data.

Past research studies and theories have illuminated the importance of infusing multiple representations when learning the function concept (Carlson, Oehrtman, & Thompson, 2007; Lobato & Bowers, 2000, April; Rizzuti, 1991). Carlson *et al.* (2007) argue that students should be provided the opportunities to "make and compare judgments about functions across multiple representations" (p. 161). The present study found that Ben's thinking of exponential behavior was more closely connected when he used more than one representation to make sense of the function situation.

Research Question Two: Effectiveness of Exponential Function Framework

The purpose of this question was to investigate the effectiveness of the current attributes of the developed exponential function framework in describing the ways of understanding exponential function. In addition, the present study provided empirical data for further refinements of the framework. The Exploratory Study and past research guided the initial framework for studying a teacher's conceptions of exponential function. A revised framework guided the implementation and analysis of the present study. The next section discusses the major findings of the initial and revised frameworks.

Effectiveness of Initial Exponential Function Framework

Analysis of the Exploratory Study data revealed that the initial exponential function framework lacked sufficient organization of components in assisting the categorization of utterances for understanding of exponential function. The initial framework included three categories: (a) Notation and Language, (b) Reasoning Abilities, and (c) Conceptual Understandings. However, the analysis process revealed that separating reasoning abilities from conceptual understandings did not provide a powerful approach for coding specific utterances. Instead, the coding process illuminated the overlap of the components detailed in the conceptual understanding section of the framework with the components described in the reasoning abilities section. Therefore, these two sections were consolidated and revised to improve the framework structure.

Another significant modification to the framework included the addition of the partial-interval reasoning component. This reasoning ability emerged in the Exploratory Study as important in building strong conceptions of exponential function, especially when connected to notions of fractional exponents. Furthermore, this reasoning ability requires the ability to think about the expression b^x as x factors of b.

Effectiveness of Revised Exponential Function Framework

Data analysis from the present study data revealed that the revised exponential function framework provided a powerful lens for coding and categorizing utterances about exponential behavior. The focus of the revised framework included (a) Notation and Language and (b) Reasoning Abilities. The condensed version of the framework provided a useful and organized approach for coding Ben's utterances and capturing his thinking. The revised framework allowed the researcher to encapsulate a broad range of utterances that the initial – and somewhat limited – framework did not afford (see Table 39 for Revised Exponential Function Framework).

The data from the Exploratory Study illuminated the importance of reasoning about exponential function using other functions – especially linear functions – to compare and contrast with exponentials. Thus, the component of exponential function reasoning was added under the Reasoning Abilities category to capture such utterances.

Table 39

Revised Exponential Function Framework

	Notation and Language
ENLN	Use *notation* to represent b^x as x factors of b for rational values of x.
ENLL	Use *language* to describe b^x as x factors of b for rational values of x.
ENLM	Use *multiple representations* to represent the multiplicative processes of exponential function.
ENLP	Use *parametric changes* to alter representations of an exponential function.
ENLI	Use *implicit definition* of exponential function to represent exponential situations recursively.
ENLE	Use *explicit definition* of exponential function to represent exponential situations explicitly.
	Reasoning Abilities
ERCR	Use *covariational reasoning* to describe exponential behavior by attending to incremental changes in the independent variable. ERCR1: Output Changes ERCR2: Amounts of Change ERCR3: Constant Percent Change ERCR4: Multiplicative Rate of Change
ERFR	Use *exponential function reasoning* to compare and contrast exponential function with other functions (i.e., linear function). ERFR0: Inappropriate Use of Additive Reasoning ERFR1: Compare Amount of Change with Percent Change ERFR2: Compare Constant Rate with Changing Rate ERFR3: Compare Constant Rate with Multiplicative Rate
ERMR	Use *multiplicative reasoning* to describe exponential behavior. ERMR1: Proportional Parameters ERMR2: Recursive Change by Constant Factor ERMR3: Constant Proportionality of Outputs
ERPIR	Use *partial-interval reasoning* to describe exponential behavior. ERPIR1: Partial Change ERPIR2: Partial Factors

Research Question Three: Effectiveness of Instructional Unit

The purpose of this question was to investigate the ways in which an instructional unit – focused on exponential function with an emphasis on multiplicative reasoning – facilitated the development of a secondary mathematics teacher's understanding of exponential function. The Exploratory Study provided a setting to test all interview items and teaching experiment activities. Analysis of the data from the Exploratory Study suggested further revisions to the instructional unit.

Analysis of the instructional unit on exponential function yielded important insights for infusing these activities into curriculum and instruction. First, the present study found that exponential decay was more difficult to describe and represent than exponential growth. Thus, a paralleled emphasis on exponential decay provided an opportunity for Ben to explore exponential decay behavior in the midst of exponential growth situations. As a result, Ben was able to make connections about multiplicative decay in the context of constant percent change (see Light Intensity activity). Second, the emphasis on providing contextualized situations enhanced Ben's ability to make sense of the situation. He was able to grapple with the scenario and unpack his ideas about the behavior illustrated in the problem. Real-world contexts for exponential growth and decay facilitated the development of Ben's ways of thinking about exponential behavior. Third, activities – such as the Salary Problem, Light Intensity activity, and the Investment activity – emerged as powerful activities for Ben in exploring the characteristics of exponential behavior. These activities connected exponential behavior with linear behavior as a tool for creating meaning of the multiplicative process. This emphasis provided an avenue for Ben to describe his thinking with specific utterances of constant percent change connected to constant amount change. Lastly, an emphasis on multiplicative reasoning and partial-interval reasoning facilitated in building a mature understanding of exponential function by emphasize the changing rate of change as a multiplicative change from a contextual perspective.

The conceptual analysis of Ben's thinking illuminated positive shifts in his understanding of the exponential function concept. The teaching experiment trajectory provided opportunities for Ben to explore various aspects of exponential behavior, such as rate of change, covariation and constant percent change (see Table 39 for a summary of the interview and teaching experiment analysis of Ben). Ben was able to shift his limited, static thinking of recursion (see Salary Problem, Excerpts 5-10) towards a more dynamic image of a set of actions that collectively form a process of exponentiation (see Light Intensity activity, Excerpts 67-77). Nevertheless, Ben's final assessment on the PCA post-test revealed that his understanding – as evidence through the teaching experiment analysis – was not internalized. His limited gain on the PCA post-test illuminates the complexity of advancing one's *thinking in the moment* to contexts outside of the immediate experiment.

The instructional unit of this dissertation study focused on exploring Ben's ways of thinking from multiple perspectives using multiple tasks and activities. Certainly, it is important to note that the instructional unit provided here is not directly transferable to the classroom because of the inordinate length of the unit. However, this study provides empirical data over a broad spectrum of tasks and this information will contribute to the refinement of additional curricular materials for teaching the concept of exponential functions.

Limitations of the Study

Even though this investigation was successful in accomplishing the goals of exploring a teacher's ways of thinking about exponential function, caution should be taken when interpreting the results of this study. This study has at least three limitations directly related to methodology and design of the study.

First, the subject of this study had minimal experience in teaching mathematics prior to the start of this investigation. This teacher was also participating in a nationally funded research project designed to enhance secondary teachers' understanding of mathematics and science. Due to the small number of participants ($n = 1$), the results of this study cannot be generalized to all secondary mathematics teachers. Yet, this case study demonstrates a piece of a larger picture illustrating teachers' ways of thinking and understanding of exponential function. This teacher was not selected randomly; instead, *extreme case sampling* was employed for the selection of the participant (Morrow & Smith, 2000). This type of purposeful sampling (defined by Patton, 1990), provided an opportunity to select a subject participant that represented a rich variety of interesting characteristics. These characteristics include....

A second limitation of this study involved the research instruments, namely the interview tasks and teaching experiment activities. The data revealed that additional prompts should be included in the tasks and activities that might reveal other ways of thinking of the subject. Furthermore, the teaching experiment activities should be revised to include more scaffolding in places where Ben experienced difficulty in communicating his understanding. Although these instruments had been piloted in the Exploratory Study and some of the items had been included in past research, these instruments will undoubtedly benefit from future testing on larger populations of teachers and students.

The limitations mentioned here are important for framing the findings of this investigation. Although these limitations do not diminish the importance of the findings from this study, an awareness of these limitations is nonetheless important for moving forward with future research plans.

Implications for Curriculum and Instruction

The results of this study provide several implications for both curriculum and instruction. First, mathematics teachers and curriculum designers need to be aware of the challenges for knowing and learning exponential function when designing curricular and instructional materials. The concept of exponential function encompasses complex mathematical behaviors that require sophisticated ways of thinking to make sense of exponential behavior. These ways of thinking include multiplicative reasoning, recursive processes, notation and language, covariational reasoning, partial-interval reasoning, and the ability to reason about exponential function relative to other function families. Specifically, the following implications emerge from this study:

- Curriculum and instruction should emphasize constant percent growth/decay (exponential) and distinguish from constant amount growth/decay (linear).
- Mathematics teachers need to be aware of the ways of thinking for possessing a mature understanding of exponential function.

- Curriculum and instruction should emphasize notions of exponential decay along with exponential growth.
- Curriculum and instruction should emphasize meaningful ways of thinking about fractional exponents to foster their development to recognize the covarying relationships between the exponent (input) and the output value of an exponential function.
- Exponentiation as a recursive structure must be considered as a developmental process where recursion is first conceived as recursive actions. These recursive actions then collectively construct a recursive process.
- In developing students' (and teachers') understanding of exponential growth, it is vital to emphasize that exponential growth is a multiplicative process where the size of the growing object is multiplied by an equal amount in each unit time interval (Confrey & Smith, 1994). The ability to distinguish between linear behavior and exponential behavior facilitates an understanding of the exponential function concept.
- Infusing opportunities for students to compare and contrast function families using tables of values can facilitate students' development of exponential behavior by comparing rates of change for such functions. Analyzing tabular representations of functions provides a powerful mechanism for developing an understanding of function and covariation.

Directions for Future Research

The present study suggests several ideas for future research in the area of knowing and learning exponential function. Some of these ideas include the following:

- A continued exploration of teacher and student understanding of exponential decay and continuous variation. This dissertation study uncovered potential difficulties and pitfalls in understanding exponential decay. The study also illuminated ways of thinking about the effects of increasing compounding periods with extension into continuous variation. Both of these topics should be explored in more detail to provide empirical data for (a) developing deeper understanding of the complexities of exponential decay and (b) developing the ways of thinking to support continuous variation and strong conceptions of the number e.
- A longitudinal study conducted on teachers' development of the concept of exponential function. This type of study could begin with secondary mathematics teachers who, while new to the field of teaching, are progressing along in stages of their teacher development. For example, assessing their knowledge using semi-structured interviews and observing their classroom instruction on the topic of exponential function could be fruitful. An examination of their course materials, namely student handouts and assessment items, should be included in the data corpus. A teaching experiment with the teacher and researcher could be conducted and a follow-up set of interviews and classroom teaching observations could conclude the

investigation. Such a study may illuminate the conceptual and pedagogical development of the selected mathematics teachers.

- A replicable and comparable study conducted with other secondary mathematics teachers to investigate the effectiveness and robustness of the exponential function framework in describing various ways of thinking about exponential function. It would also be worthwhile to conduct such a study to compare and contrast the findings and results with those of this present investigation.
- A cross-sectional investigation conducted to investigate ways of thinking about exponential function on the part of students, teachers, mathematicians, and/or mathematics education researchers. Such a study would provide powerful evidence of the various ways of thinking about exponential function by a diverse group of participants.

Future research investigations focused on the concept of exponential function ought to be highly useful in contributing knowledge to the field of mathematics education. An increased emphasis on developing both frameworks and curricular materials that are research-based – and that focus on improving teachers' and students' ability to reason multiplicatively in the context of exponential function – will undoubtedly provide valuable knowledge to the field of mathematics education research.

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APPENDIX A

INTERVIEW PROTOCOL

Teacher Interview Protocol

Functions Course Cohort 3

Interviewee Name:____

Interviewee Signature:____________________________________

Interview Date:____

Interviewer:____

Begin the session by reading the following paragraph to the interviewee:

Thank you for agreeing to participate in this interview. Data are being collected to assist us in identifying aspects of the Functions Course (Course 1) that are most effective and to assist us in understanding how to improve the course in subsequent semesters. Your name will not be used in any of our reporting. As we progress through the interview, we ask that you verbalize all your thoughts so that we can gather information about your thinking on various tasks. If at any time you wish to stop the interview, please let me know. We want you to be as comfortable as possible.

As part of our research we are interested in gaining knowledge about how you approach various tasks with your students and some of the misconceptions you believe students have when learning such contexts. Throughout this interview, we will present various mathematical tasks and will ask you questions to get at your thinking on a deeper level.

TASK #1

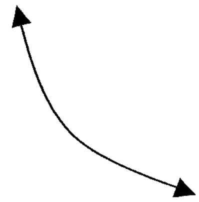
I would like to begin our interview session with a video clip from a professional learning community of teachers who are discussing ideas about half-life. Before we watch the video, can you tell me how you think about half-life? How would you approach the concept of half-life with your students?

As you watch this video, I would like you to listen to how the teachers interact with each other and wrestle with this mathematical concept. Reflect on your own concept of half-life and how you would engage in this conversation.

- PLAY VIDEO FILE: FN2_CR_TEM_D_PLC_12_V1_06-04-19.MP4
- Timestamp: 9:15 to 14:45 (video will be clipped into another file ready to show)
- What are your initial thoughts about the discussion we just saw on the video?
- Which teacher do you connect with the most? Tell me why you feel you connect with this teacher.
- If the teacher does not feel they connect with any teacher from the video clip, ask them to describe why they do not see themselves in any of the PLC teachers. The point here is to get the teacher to open up about their own teaching and ways of thinking about both content and pedagogical knowledge.
- What would you have added to this discussion if you were present with these teachers?
- How would you teach this topic?
- Why would you teach it this way?
- Thinking about the graph presented at the end of the video, could she have drawn the graph differently? Explain.
- The current graph shown in the video only illustrates the values for $t = 0, 3, 6, 9, \ldots$. Is it possible to draw a graph that illustrates how much of a substance is left at time $t = 1$ or $t = 2$? If so, what does this graph look like?

TASK #2

Two students are having a debate about a function whose graph looks like the figure below. One student declares the function is "decreasing at a decreasing rate" while the other says it is "decreasing at an increasing rate." Which student do you think is correct? Explain.



- What is complex about understanding language like this? What are some of the common misconceptions that students have with this language?
- What is the meaning of the second use of "decreasing" in the given situation? What is the meaning of the first usage?
- What do you believe is mathematically relevant to understand language like "decreasing at a decreasing rate"? What does this phrase mean to you?
- What would you have students focus on to promote discussion? Why that?
- Can you provide a context in your discipline (mathematics or science) that might behave in this way?
- Using the graph, can you explain how the phenomenon is changing within your context?
- Follow-Up Questions:
- If they describe the picture above as decreasing at an decreasing rate, then ask "Can you draw additional graphs of situations that are:
- Increasing at a decreasing rate?" Increasing at an increasing rate?" "Decreasing at an increasing rate?"
- If they describe the picture above as decreasing at an increasing rate, then ask "Can you draw additional graphs of situations that are:
- Increasing at a decreasing rate?" "Increasing at an increasing rate?" "Decreasing at an decreasing rate?"

TASK #3

The volume of a box that is formed by cutting out squares from each corner of an 8.5" x 11" piece of paper is defined as $V(x) = (8.5 - 2x)(11 - 2x)(x)$. What is the meaning of *x* in this context? What is the meaning of *V*? What is the meaning of *V(x)*?

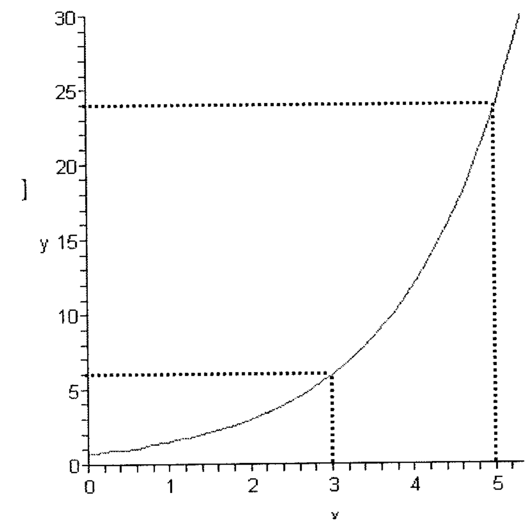
Follow-Up Questions:

- How do you think about *x* in this context? What are the possible values that *x* can represent?
- How do you think about *V* in this context?
- How do you think about *V(x)* in this context? What are the possible values that *V(x)* can represent?
- Can you draw a picture of the piece of paper and label the appropriate information from this context?
- Describe how the volume changes with respect to the size of the square cut from each corner of the paper. Can you describe this in terms of a graph?

TASK #4

Imagine that you provided both a table and a graph illustrating population growth as a function of time (measured in hours) and you ask a student to find how fast the function is changing (growing) between $x = 3$ and $x = 5$. The student responds 18. What might the student been thinking?

x	0	1	2	3	4	5	6
$f(x)$	0.75	1.5	3	6	12	24	48



Follow-Up Questions:

- How would you like your students to interpret the table? Graph?
- What difficulties will students have with this question?
- Finding the correct answer?
- Knowing what the answer means in terms of the growing population?
- Interpreting the table of values properly in order to find the appropriate values to use in the computation?
- Understanding what the answer means in terms of the graph or using the graph to find the appropriate values to use in the computation?
- What must a student need to understand to answer this question?
- What would you initially say to the student?
- How would you guide the students? Why would you guide that way?
- What would the population be at 8 hours?
- What would be different about your explanation to the student if you wanted to find out how fast the function was changing between $x = 1$ and $x = 3$?
- How would you describe to the student how fast the function is changing between $x = 3$ and $x = 3.5$?

TASK #5

A car went from San Diego to El Centro, a distance of 93 miles, at an average speed of 40 miles per hour. At what average speed would it need to return to San Diego if it were to have an average speed of 65 miles per hour over the round trip?

Follow-Up Questions:

- How would you verify your answer?
- Does your answer seem reasonable? Explain.
- What does the phrase "average speed" mean in the context of this problem?
- If they are having trouble:
- Is this problem confusing to you? If so, what is your confusion?
- How fast would you imagine the car would travel on the return trip in order to have an average speed of the total trip be 65 mph? Would the speed be a little more than 40 miles per hour or a lot more than 40 miles per hour? Would you expect the speed to double? Triple? Quadruple? How do you know?
- If they answer 90 mph:
- Can you explain how you arrived at your answer?
- If they answer correctly:
- Most people will find the average speed on the return trip to be 90 miles per hour. However, average speed cannot be computed by averaging the individual speeds to obtain the final average speed. Why is this true?
- The distance is arbitrary when finding average speed. Using 40 miles per hour for the first trip and the desired average speed of 65 miles per hour for the entire trip, can you explain why the average speed for the return trip is the same no matter the distance traveled?

TASK #6

Jack plants a 5 cm beanstalk in his back yard. It grows by about 15% per day for the next month.

Imagine that you have given this problem to five of your students and you ask them to write a formula that represents the height of the beanstalk as a function of the number of days since it was planted. Each student answered the problem differently. Can you describe what each student is thinking and how he or she might have arrived at their answer?

Casey : $H(t) = 5 + 0.15t$

Daniel : $H(t) = 5.15t$

Sarah : $H(t) = 5(1.15)^t$

Mike : $H(t) = (5.15)^t$

Heather : $H(t) = 5 + 1.15t$

- Which of the five students is correct? Why do you believe this is the correct response?
- For the other four students, what kinds of misconceptions did they have when attempting to write this situation in function form?
- What instructional advice would you give to these four students to help them understand this situation?
- Did the students responses make sense in terms of units they used?
- Follow-up Questions:
- What does 5 cm represent in this context?
- What does 15% represent in this context?
- If they suggest that Casey, Daniel or Heather's response is correct, then ask "why did you decide that this situation described linear behavior as opposed to exponential behavior?"
- If they suggest that Sarah or Mike's response is correct, then ask "why did you decide that this situation described exponential behavior as opposed to linear behavior?"

TASK #7

A radioactive substance decays according to the function $A(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5}}$ where t is measured in years and A is the amount of the unstable portion of the substance in micrograms. How long does it take for half of the substance to decay? In other words, what is its half-life?

- If the participant just states that the half-life is 5, probe for more information. Ask how they arrived at their answer. We want to know if the participant just memorized some procedure for locating the half-life in this function or whether they understand something deeper about this function. Can they unpack the structure of the formula in a conceptual way or do they rely on procedural knowledge?
- Will it take as long to lose half of its substance when it is 35kg as when it is 0.25 kg? Why?
- What difficulties do students encounter when identifying half-life?
- How do you help students understand the concept of half-life?

TASK #8

Sheena just graduated from college and recently received two job offers. The first job, an economist at a local firm, has a salary of $45,000 per year with a guaranteed raise of $1000 every year. The second job, a teaching position at a local college, has a salary of $35,000 per year with a guaranteed raise of 7% every year. Which position is the best job offer? Explain.

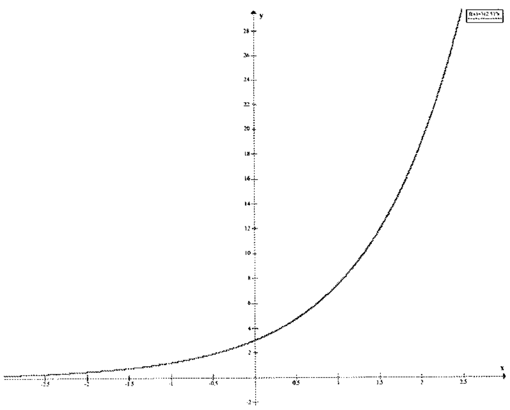
Follow-Up Questions:

- What is the difference between these growth values?
- Describe the meaning of $1000 as a rate of increase.
- Prompt for $1000 growth as repeated addition of m
- Describe the meaning of 7% as a rate of increase.
- Prompt for 7% as repeated multiplication
- Can you provide a graph for each of the two situations described?
- Prompt for understanding of slope and in terms of change in y per unit of change in x
- In what year will the two salaries be the same?
- Describe how the economist's salary is changing over time.
- Describe how the teacher's salary is changing over time.
- What difficulties do students have in understanding and making sense of these two situations?
- How do you help students understand these ideas?

TASK #9

The following representations illustrate three different ways of expressing the same function. How do you think about the notion of *variable* in each of these different representations? What is the meaning of x? What is the meaning of $f(x)$?

Representation #1:



Representation #2:

x	f(x)
-2.50	0.30
-2.00	0.48
-1.50	0.75
-1.00	1.20
-0.50	1.89
0.00	3.00
0.50	4.74
1.00	7.50
1.50	11.85
2.00	18.75
2.50	29.64

Representation #3

$$f(x) = 3(2.5)^x$$

- What does the term "variable" mean to you?
- How would you describe the concept of variable to a student? What is necessary for something to be a variable?
- What are the most important things to keep in mind when working with variables?
- How does the idea of variable relate to the concept of function?
- Which of the three representations do you feel most comfortable with? Why?
- In each of the three representations, describe how *f(x)* changes with respect to changes in *x*.
- When you look at the graph in Representation #1 what aspects, if anything, do you see varying?

TASK #10

If math teacher:

You are planning an introductory lesson on exponential functions (such as $y = ab^x$) in a second year algebra course...

If biology teacher:

You are planning an introductory lesson on population models which uses exponential functions of the form $y = ab^x$

If physics teacher:

You are planning an introductory lesson on radioactive decay which uses exponential models of the form $y = ab^x$

If earth science teacher:

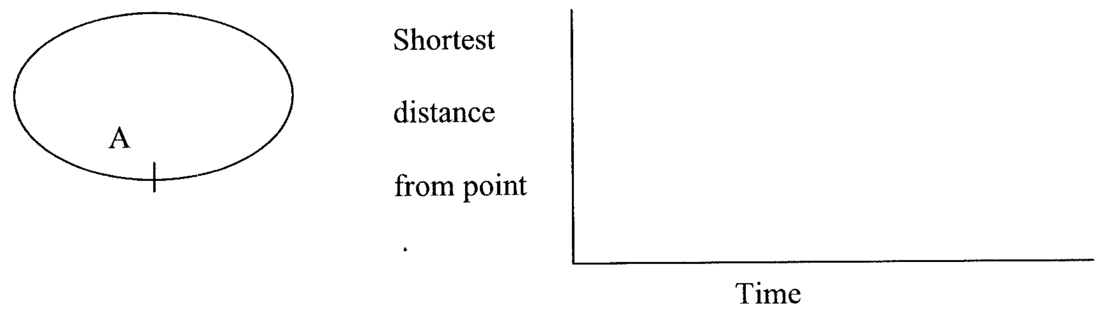
You are planning an introductory lesson on radio-carbon dating which uses exponential models of the form $y = ab^x$

What important aspects of knowing (and/or understanding) exponential functions will you include in your lesson? How will you introduce and build these aspects of exponential functions?

- What difficulties do students have in understanding exponential functions?
- How will you adjust your lesson to help students overcome these obstacles?
- What is the meaning of 'a' in $y = ab^x$?
- How does 'a' relate to the context of your lesson? Describe what happens to the context and the y-values when 'a' changes.
- What is the meaning of 'b' in $y = ab^x$?
- How does 'b' relate to the context of your lesson? Describe what happens to the context and the y-values when 'b' changes?
- If they refer to 'b' as a "rate", probe for more. What does rate mean? What is the difference between rate in linear functions to rate in exponential functions?
- What is the meaning of 'x' in $y = ab^x$?
- How does 'x' relate to the context of your lesson? Describe what happens to the context and the y-values when 'x' changes.

TASK #11

A person is running around an oval race track at a constant speed. Construct a rough sketch showing the *shortest* distance between the runner and point *A* as a function of time. (Please talk through your thinking.)



- Once the respondent has completed the question:
- If the graph components are curved, ask "Should the graph contain curves or should it be straight? Explain"
- If some components are linear, say "Tell me why you made some of the pieces straight."
- Label the important points on the graph and diagram with corresponding letters (A, B, C,…). Make sure to mark all extrema. How do these points you labeled correspond to the situation? How do you know this?
- Follow-Up Questions:
- If the teacher has difficulty answering the question:
- Try to get the interviewee to understand that "the shortest distance between the runner and point A" refers to the distance along a straight line from point A to the point where the runner is at any given time.
- Can you think of any alternative ways to draw a graph of the shortest distance?

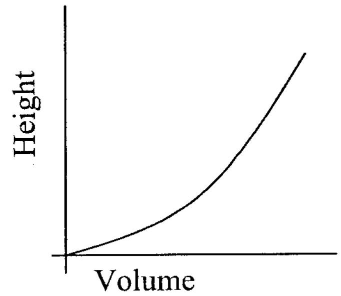
TASK #12

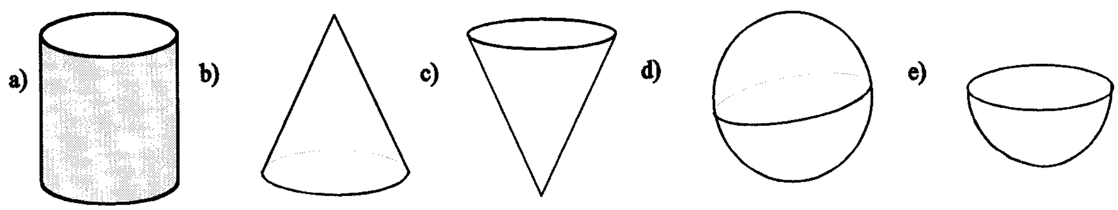
The half-life of a radioactive substance is 1 hour. In 30 minutes, how much of the substance will decay?

- If the teacher needs help, pose the following prompt:
- Do you think it will be less than ¼? More than ¼? Or exactly ¼?
- How do you know?
- Can you explain that using a graph? How?
- If they struggle, suggest they draw a graph of the function, and use that to try to answer the question. If they cannot draw the graph, draw an exponential graph, label the axes, make sure they understand what it refers to and ask if they can use it to answer the question.
- Can you explain it using the idea of repeated multiplication? How?
- If they struggle, ask over what time period do we repeatedly multiply by ½. If they cannot answer 1 hour, help them. Then ask how much you would have to repeatedly multiply by every 30 minutes to obtain multiplication by ½ in an hour. Ask them to show you this repeated multiplication yielding ½.

TASK #13

The following graph represents the height of water as a function of volume as water is poured into a container. Which container is represented by this graph?



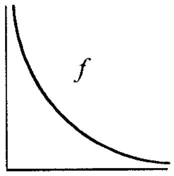


- Describe why the remaining four containers do not produce the shape of this graph when filled with water.
- What mathematical ideas does this problem promote to help students build an understanding of function?

APPENDIX B

SAMPLE OF PRECALCULUS CONCEPT ASSESSMENT (PCA) QUESTIONS

1. A function f is defined by the following graph. Which of the following describes the behavior of f?



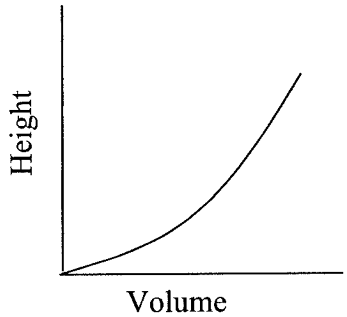
I. As the value of x approaches 0, the value of f increases.

II. As the value of x increases, the value of f approaches 0.

III. As the value of x approaches 0, the value of f approaches 0.

2. The model that describes the number of bacteria in a culture after t days has just been updated from $P(t) = 7(2)^t$ to $P(t) = 7(3)^t$. What implications can you draw from this information?

3. The following graph represents the height of water as a function of volume as water is poured into a container. Which container is represented by this graph?



4. Jack plants a 5 cm beanstalk in his back yard. It grows by about 15% per day for the next month. Which formula represents the height of the beanstalk as a function of the number of days since it was planted?

5. The half-life of a radioactive substance is 1 hour. In ½ hour, how much of the substance will decay?

6. A radioactive substance decays according to the function $A(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{5}}$ where t is measured in years and A is the amount of the unstable portion of the substance in micrograms. How long does it take for half of the substance to decay? In other words, what is its half-life?

APPENDIX C

LETTER OF CONSENT

Project Pathways: Strom Research Project

Letter of Consent

Dear Teacher:

I am a graduate student in the College of Education at Arizona State University working with the Center for Research on Education, Science, Mathematics, Engineering and Technology (CRESMET) under the direction of Dr. Marilyn Carlson, Principal Investigator for *Project Pathways*. I am conducting a research study to investigate the learning of important concepts in algebra and the impact of new curricular materials being developed by the CRESMET research team.

I am requesting your participation, which may involve some or all of the following: (1) taking brief assessments; (2) participating in videotaped interviews, surveys, and observations as agreed upon; (3) teaching experiment sessions; and (4) allowing all of your written work to be duplicated for research purposes. The results of this research study may be published or presented at research conferences, but your name will not be used. You will be compensated at the rate of $35/hr for each hour you participate in this research. This funding will be paid by the NSF Project Pathways grant.

Your participation in this study is voluntary. If you choose not to participate, there will be no penalty, and it will not affect your grade in any ASU class. You may choose to withdraw from the study at any time.

If you have any questions concerning the research study, please call me at (480) 727-8884 or email me at april.strom@asu.edu.

Sincerely,
April D. Strom

I am at least 18 years of age:

By signing below you are giving consent to participate in the above study.

____________________	____________________	______
Signature	Printed Name	Date

If you have any questions about your rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the Chair of the Human Subjects Institutional Review Board, through the ASU Research Compliance Office, at (480) 965-6788.

APPENDIX D

HUMAN SUBJECTS APPROVAL LETTER

ARIZONA STATE UNIVERSITY
Research Compliance Office
Office for Research & Sponsored Projects Administration
P.O. Box 873503
Tempe, AZ 85287-3503
Phone (480) 965-6788
Facsimile (480) 965-7772

To: Marilyn Carlson
UC 206C

From: Mark Roosa, Chair
Institutional Review Board

Date: 08/03/2006

Committee Action: **Expedited Approval**

Approval Date: 08/03/2006

Review Type: Expedited F6 F7

IRB Protocol #: 0607001000

Study Title: SecondaryMathematics Teachers' Understanding of Exponential Functions

Expiration Date: 08/02/2007

The above-referenced protocol was approved following expedited review by the Institutional Review Board.

It is the Principal Investigator's responsibility to obtain review and continued approval before the expiration date. You may not continue any research activity beyond the expiration date without approval by the Institutional Review Board.

Adverse Reactions: If any untoward incidents or severe reactions should develop as a result of this study, you are required to notify the Institutional Review Board immediately. If necessary a member of the IRB will be assigned to look into the matter. If the problem is serious, approval may be withdrawn pending IRB review.

Amendments: If you wish to change any aspect of this study, such as the procedures, the consent forms, or the investigators, please communicate your requested changes to the Institutional Review Board. The new procedure is not to be initiated until the IRB approval has been given.

Please retain a copy of this letter with your approved protocol.